Research Article

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Periodic measures of fractional stochastic discrete wave equations with nonlinear noise

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Abstract: The primary focus of this work lies in the exploration of the limiting dynamics governing fractional stochastic discrete wave equations with nonlinear noise. First, we establish the well-posedness of solutions to these stochastic equations and subsequently demonstrate the existence of periodic measures for the considered equations.

Keywords: stochastic discrete wave equations, fractional discrete Laplacian, nonlinear noise, periodic measure

MSC 2020: 35B40, 35B41, 37L30

1 Introduction

The aim of this study is to establish the existence of periodic measures for a fractional stochastic discrete wave equation with nonlinear noise on $\mathbb{Z}$

$$\begin{cases} \dfrac{\mathrm{d}\dot{u}_i}{\mathrm{d}t} + \alpha\dot{u}_i + (-\Delta_d)^s u_i + \lambda u_i = f_i(u_i) + a_i(t) + \displaystyle\sum_{j=1}^{\infty}(\sigma_{i,j}\hat{g}_{i,j}(u_i) + b_{i,j}(t))\dfrac{\mathrm{d}W_j}{\mathrm{d}t}, \quad t > 0, \\ u_i(0) = u_{i,0}, \dot{u}_i(0) = \dot{u}_{i,0}, \end{cases} \tag{1.1}$$

where $\alpha, \lambda > 0$, $\dot{u}_i$ denotes the first-order time-derivative of u_i, $(-\Delta_d)^s$ is the fractional discrete Laplacian, $s \in (0,1)$, $a = (a_i)_{i\in\mathbb{Z}}$ and $b = (b_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}}$ are two random sequences depending on time t, $\sigma = (\sigma_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}}$ is given in ℓ^2, $f_i, \hat{g}_{i,j} : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, and $(W_j(t))_{j\in\mathbb{N}}$ is a sequence of mutually independent two-sided real-valued Wiener processes, defining on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

The discrete partial differential equations (PDEs) are commonly derived from spatial discretizations of continuum PDEs defined on unbounded domains, which have extensive applications in modeling real problems involving random phenomena in physics, biology, and chemistry [1,2]. The investigation of traveling wave solutions for such equations has been conducted by researchers in [3–6]. The examination of chaotic properties in the solutions has been carried out by scholars in [7,8] and references therein. For a comprehensive investigation into the random attractors of discrete PDEs, we recommend consulting the literature on first-order equations in [9–14] and second-order equations in [15–17]. Currently, in order to effectively handle stochastic equations with nonlinear noise, the concept of weak pullback mean random attractors was

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introduced by Kloeden and Lorenz [18] and Wang [19,20]. Subsequently, this concept has been extensively applied in numerous studies on stochastic equations by various scholars [21–39].

The fractional discrete Laplacian, extensively investigated in previous studies [40–42], explores the fractional powers of the discrete Laplacian. In [42], the examination of discrete equations involving the fractional discrete Laplacian led to the derivation of pointwise nonlocal formulas and various properties associated with this operator. Furthermore, Schauder estimates were established in discrete Hölder spaces, ensuring the existence and uniqueness of solutions for the considered system. The theories of analytic semigroups and cosine operators successfully established existence and uniqueness of solutions to Schrödinger, wave, and heat systems with the fractional discrete Laplacian in [43]. Recent research has primarily focused on investigating the existence, uniqueness, and upper semi-continuity of random attractors for fractional stochastic discrete equations with either linear or nonlinear multiplicative noise [12,44].

Our objective is to obtain a periodic measure for equation (1.1) in the presence of time-dependent functions that exhibit periodicity. Periodic measures serve as counterparts to invariant measures for dynamical systems and can be utilized to characterize the long-term periodic behavior of stochastic systems. A probability measure μ on the natural function class for equation (1.1) is referred to as a periodic measure if its initial probability distribution, equal to μ, generates time-periodic probability distributions of the solution. Conversely, it is called an invariant measure if it yields time-invariant probability distributions of the solution. An invariant measure can be derived by projecting the periodic measure onto a cylinder and considering its average over one period. Extensive investigations on the periodic measures of stochastic differential equations have been conducted by numerous experts in [26,27,45–49]. In particular, a study was carried out in [46] to examine the existence of periodic measures for a stochastic delay reaction-diffusion lattice system with globally Lipschitz continuous nonlinear drift and diffusion terms.

The main challenge of this study lies in proving the weak compactness in $\ell^2 \times \ell^2$ of a specific set of distribution laws for solutions to equation (1.1) defined on the unbounded integer set $\mathbb{Z}$, which is analogous to the case of stochastic PDEs on unbounded domains where Sobolev embedding is no longer compact, as discussed in [49–52]. Following the approach used in [20–25] for invariant measures of lattice systems, we will demonstrate the desired weak compactness of distributions for solutions to equation (1.1) in $\ell^2 \times \ell^2$ by employing Krylov-Bogolyubov's method along with Feller property, Markov property, T-periodicity, and uniform tail estimates.

The study is organized as follows: Section 2 introduces some basic concepts, assumptions, and lemmas and discusses the well-posedness of equation (1.1). Section 3 gives essential uniform estimates of solutions, which play a pivotal role in demonstrating the main findings in Section 4. Section 4 focuses primarily on investigating the existence of periodic measures for equation (1.1) in space $\ell^2 \times \ell^2$. Finally, we provide a concluding remark in the last section.

2 Preliminaries

In this section, we will investigate the well-posedness of the fractional stochastic discrete wave equation (1.1). We denote by $\ell^p(1 \le p \le \infty)$ the space of sequences $(u_i)_{i\in\mathbb{Z}}$ with the norm

$$\|u\|_p^p := \sum_{i\in\mathbb{Z}} |u_i|^p < \infty, \quad 1 \le p < \infty, \quad \|u\|_\infty := \sup_{i\in\mathbb{Z}}|u_i|, \quad p = \infty.$$

In particular, ℓ^2 is a Hilbert space with the inner product and norm given by

$$(u, v) = \sum_{i\in\mathbb{Z}} u_i v_i, \quad \|u\|^2 = (u, u), \ \ u, v \in \ell^2.$$

For $0 \le s \le 1$, define ℓ_s by

$$\ell_s = \left\{ u : \mathbb{Z} \to \mathbb{R} \,\middle|\, \|u\|_{\ell_s} := \sum_{i\in\mathbb{Z}} \frac{|u_i|}{(1 + |i|)^{1+2s}} < \infty \right\}.$$

Obviously, $\ell^m \subset \ell^n \subset \ell_s$ if $1 \le m \le n \le \infty$ and $0 \le s \le 1$.

The fractional discrete Laplacian $(-\Delta_d)^s$ simplifies to the standard discrete Laplacian $-\Delta_d$ if $s = 1$. For $i \in \mathbb{Z}$, the discrete Laplacian $-\Delta_d$ is defined by

$$-\Delta_d u_i = 2u_i - u_{i-1} - u_{i+1}.$$

For $0 < s < 1$ and $u_j \in \mathbb{R}$, the fractional discrete Laplacian $(-\Delta_d)^s$ is defined by the semigroup method in [53] as

$$(-\Delta_d)^s u_j = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_d} u_j - u_j) \frac{\mathrm{d}t}{t^{1+s}}, \tag{2.1}$$

where $\Gamma(-s) = \int_0^\infty (e^{-r} - 1) \frac{\mathrm{d}r}{r^{1+s}} < 0$, and $v_j(t) = e^{t\Delta_d} u_j$ is the solution of the semidiscrete heat equation

$$\begin{cases} \partial_t v_j = \Delta_d v_j, & \text{in } \mathbb{Z} \times (0, \infty), \\ v_j(0) = u_j, & \text{on } \mathbb{Z}. \end{cases} \tag{2.2}$$

The solution of equation (2.2) can be expressed by

$$e^{t\Delta_d} u_j = \sum_{i \in \mathbb{Z}} G(j - i, t) u_i = \sum_{i \in \mathbb{Z}} G(i, t) u_{j-i}, \quad t \ge 0, \tag{2.3}$$

where $G(i, t)$ is defined as $e^{-2t} I_i(2t)$, I_i represents the modified Bessel function of order i.

The subsequent presentation provides the pointwise formula for $(-\Delta_d)^s$.

Lemma 2.1. [42, Lemma 2.3] *Let* $0 < s < 1$ *and* $u = (u_i)_{i \in \mathbb{Z}} \in \ell_s$. *Then, we have*

$$(-\Delta_d)^s u_i = \sum_{j \in \mathbb{Z}, j \ne i} (u_i - u_j) \tilde{K}_s(i - j),$$

where the discrete kernel $\tilde{K}_s$ *is given by*

$$\tilde{K}_s(j) = \begin{cases} \dfrac{4^s \Gamma\left(\frac{1}{2} + s\right)}{\sqrt{\pi}\, |\Gamma(-s)|} \cdot \dfrac{\Gamma(|j| - s)}{\Gamma(|j| + 1 + s)}, & j \in \mathbb{Z} \backslash \{0\}, \\ 0, & j = 0. \end{cases}$$

In addition, there exist positive constants $\check{c}_s \le \hat{c}_s$ *such that for any* $j \in \mathbb{Z} \backslash \{0\}$,

$$\frac{\check{c}_s}{|j|^{1+2s}} \le \tilde{K}_s(j) \le \frac{\hat{c}_s}{|j|^{1+2s}}.$$

In addition, by Lemma 2.1, we can obtain that $(-\Delta_d)^s u$ is a nonlocal operator on $\mathbb{Z}$ and $(-\Delta_d)^s u$ is a well-defined bounded function wherever $u \in \ell^p (1 \le p \le \infty)$. In particular, for $0 < s < 1$ and $u \in \ell^2$, then

$$(-\Delta_d)^s u \in \ell^2 \ \text{ satisfying } \ \|(-\Delta_d)^s u\| \le 4^s \|u\|. \tag{2.4}$$

Moreover, we assume that f_i, $\hat{g}_{i,j}$ in equation (1.1) are locally Lipschitz continuous uniformly with respect to $i \in \mathbb{Z}$ and $j \in \mathbb{N}$; i.e., for any bounded interval $I \subseteq \mathbb{R}$, there exist $L_n = L_n(I) (n = 1, 2)$ such that for all $z_1, z_2 \in I$,

$$|f_i(z_1) - f_i(z_2)| \le L_1 |z_1 - z_2|, \quad i \in \mathbb{Z}, \tag{2.5}$$

$$|\hat{g}_{i,j}(z_1) - \hat{g}_{i,j}(z_2)| \le L_2 |z_1 - z_2|, \quad i \in \mathbb{Z}, j \in \mathbb{N}. \tag{2.6}$$

We also assume that for all $z \in \mathbb{R}$, $i \in \mathbb{Z}$, and $j \in \mathbb{N}$,

$$|f_i(z)| \le \phi_{1,i} |z| + \phi_{2,i}, \phi_1 = (\phi_{1,i})_{i \in \mathbb{Z}} \in \ell^\infty, \quad \phi_2 = (\phi_{2,i})_{i \in \mathbb{Z}} \in \ell^2, \tag{2.7}$$

$$|\hat{g}_{i,j}(z)| \le \varphi_{1,i} |z| + \varphi_{2,i}, \varphi_1 = (\varphi_{1,i})_{i \in \mathbb{Z}} \in \ell^\infty, \quad \varphi_2 = (\varphi_{2,i})_{i \in \mathbb{Z}} \in \ell^2. \tag{2.8}$$

In addition, we assume that $\sigma = (\sigma_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}}$ satisfies:

$$c_\sigma = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{Z}}|\sigma_{i,j}|^2 < \infty. \tag{2.9}$$

Define the operators $f, g_j : \ell^2 \to \ell^2$ by

$$f(u) = (f_i(u_i))_{i\in\mathbb{Z}} \quad \text{and} \quad g_j(u) = (\sigma_{i,j}\hat{g}_{i,j}(u_i))_{i\in\mathbb{Z}}, \quad \forall u = (u_i)_{i\in\mathbb{Z}} \in \ell^2.$$

By (2.7) and (2.8), we obtain

$$\|f(u)\|^2 = \sum_{i\in\mathbb{Z}}|f_i(u_i)|^2 \le 2\|\phi_1\|_\infty^2\|u\|^2 + 2\|\phi_2\|^2$$

and

$$\sum_{j\in\mathbb{N}}\|g_j(u)\|^2 = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{Z}}|\sigma_{i,j}\hat{g}_{i,j}(u_i)|^2 \le 2\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{Z}}|\sigma_{i,j}|^2(|\varphi_{1,i}|^2|u_i|^2 + |\varphi_{2,i}|^2) \le 2c_\sigma\|\varphi_1\|_\infty^2\|u\|^2 + 2c_\sigma\|\varphi_2\|^2.$$

Hence, f and g_j are well-defined. We assume that $a(t) = (a_i(t))_{i\in\mathbb{Z}}$ and $b(t) = (b_{i,j}(t))_{i\in\mathbb{Z},j\in\mathbb{N}}$ satisfy that for all $t \in \mathbb{R}$,

$$\|a(t)\|^2 = \sum_{i\in\mathbb{Z}}\|a_i(t)\|^2 < \infty \quad \text{and} \quad \sum_{j\in\mathbb{N}}\|b_j(t)\|^2 = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{Z}}|b_{i,j}(t)|^2 < \infty. \tag{2.10}$$

Moreover, we will establish the periodic measures of equation (1.1) for which we assume that all given time-dependent functions are T-periodic in $t \in \mathbb{R}$ for some $T > 0$; this is, for all $t \in \mathbb{R}$,

$$a(t + T) = a(t) \quad \text{and} \quad b(t + T) = b(t).$$

If $\zeta : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function, we denote

$$\bar{\zeta} = \max_{0\le t\le T}\zeta(t).$$

Using the above notation, we can rewrite equation (1.1) in ℓ^2 as follows:

$$\begin{cases}\dfrac{\mathrm{d}\dot{u}}{\mathrm{d}t} + \alpha\dot{u} + (-\Delta_d)^s u + \lambda u = f(u) + a(t) + \displaystyle\sum_{j=1}^{\infty}(g_j(u) + b_j(t))\frac{\mathrm{d}W_j}{\mathrm{d}t}, \quad t > 0,\\ u(0) = u_0, \dot{u}(0) = \dot{u}_0.\end{cases} \tag{2.11}$$

Let $\delta > 0$ be a constant, and we denote

$$\Phi(t) := \begin{pmatrix}u(t)\\ v(t)\end{pmatrix} \quad \text{with } v(t) := \dot{u}(t) + \delta u(t). \tag{2.12}$$

Then, we rewrite equation (2.11) as the following equation:

$$\begin{cases}\mathrm{d}\Phi(t) = F(\Phi(t))\mathrm{d}t + \displaystyle\sum_{j=1}^{\infty}G_j(\Phi(t))\mathrm{d}W_j, \quad t > 0,\\ \Phi(0) = \Phi_0 = \begin{pmatrix}u_0\\ v_0\end{pmatrix},\end{cases} \tag{2.13}$$

where $u_0 = (u_{i,0})_{i\in\mathbb{Z}}$, $v_0 = (\dot{u}_{i,0} + \delta u_{i,0})_{i\in\mathbb{Z}}$,

$$F(\Phi(t)) = \begin{pmatrix}v(t) - \delta u(t)\\ -(\lambda + \delta^2 - \alpha\delta)u(t) - (\alpha - \delta)v(t) - (-\Delta_d)^s u(t) + f(u(t)) + a(t)\end{pmatrix},$$

and

$$G_j(\Phi(t)) = \begin{pmatrix}0\\ g_j(u(t)) + b_j(t)\end{pmatrix}.$$

Let δ be a fixed positive constant such that

$$\alpha - \delta > 0 \quad \text{and} \quad \lambda + \delta^2 - \alpha\delta > 0. \tag{2.14}$$

For convenience, we write

$$\kappa = \min\{\delta, \alpha - \delta\}. \tag{2.15}$$

In addition, we assume

$$\|\phi_1\|_\infty \le \frac{\kappa(\lambda + \delta^2 - \alpha\delta)}{2} \wedge \frac{\kappa}{4} \quad \text{and} \quad \|\varphi_1\|_\infty^2 \le \frac{\kappa(\lambda + \delta^2 - \alpha\delta)}{8c_\sigma}. \tag{2.16}$$

Let $\Phi_0 \in L^2(\Omega, \ell^2 \times \ell^2)$ be $\mathcal{F}_0$-measurable. Then, a continuous $\ell^2 \times \ell^2$-valued $\mathcal{F}_t$-adapted stochastic process $\Phi(t)$ is called a solution of equation (2.13) if $\Phi(t) \in L^2(\Omega, C([0,T], \ell^2 \times \ell^2))$ for all $T > 0$ and for almost all $\omega \in \Omega$,

$$u(t) = u_0 + \int_0^t (v(r) - \delta u(r))\mathrm{d}r,$$

$$v(t) = v_0 + \int_0^t (-(\lambda + \delta^2 - \alpha\delta)u(r) - (\alpha - \delta)v(r) - (-\Delta_d)^s u(r) + f(u(r)) + a(r))\mathrm{d}r + \sum_{j=1}^\infty \int_0^t (g_j(u(r)) + b_j(r))\mathrm{d}W_j(r)$$

in $\ell^2 \times \ell^2$ for all $t \ge 0$.

By (2.5)–(2.8) and the theory of the functional differential equation from [54], we can obtain that for any $\Phi_0 \in L^2(\Omega, \ell^2 \times \ell^2)$, equation (2.13) has local solutions $\Phi(t) \in L^2(\Omega, C([0,T], \ell^2 \times \ell^2))$ for every $T > 0$. Moreover, similar to [36], we can obtain that the local solutions are also global solutions.

The subsequent lemma will be repeatedly utilized in various estimations of solutions to equation (2.13).

Lemma 2.2. [12, Lemma 2.3] *Let $u, v \in \ell^2$. Then, for every $s \in (0,1)$,*

$$((-\Delta_d)^s u, v) = \left((-\Delta_d)^{\frac{s}{2}} u, (-\Delta_d)^{\frac{s}{2}} v\right) = \frac{1}{2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}, j \ne i} (u_i - u_j)(v_i - v_j)\tilde{K}_s(i - j).$$

Section 3 establishes uniform estimates for the solutions to equation (2.13), which play a pivotal role in substantiating the existence of periodic measures.

3 Uniform estimates

Lemma 3.1. *Suppose* (2.5)–(2.10) *and* (2.14)–(2.16) *hold. Let* $\Phi_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in L^2(\Omega, \ell^2 \times \ell^2)$ *be the initial data of equation* (2.13), *then solution* $\Phi(t, 0, \Phi_0) = \begin{pmatrix} u(t,0,u_0) \\ v(t,0,v_0) \end{pmatrix}$ *of equation* (2.13) *satisfies*

$$\begin{aligned} &\mathbb{E}\left[\|u(t)\|^2 + \|v(t)\|^2 + \left\|(-\Delta_d)^{\frac{s}{2}} u(t)\right\|^2\right] \\ &\le M_1\left(\mathbb{E}\left[\|u_0\|^2 + \|v_0\|^2 + \left\|(-\Delta_d)^{\frac{s}{2}} u_0\right\|^2\right] + \|\phi_2\|^2 + \|\bar{a}\|^2 + \|\varphi_2\|^2 + \sum_{j=1}^\infty \|\bar{b}_j\|^2\right), \end{aligned} \tag{3.1}$$

where M_1 is a positive constant independent of u_0 and v_0.

Proof. By (2.13) and Itô's formula, we obtain that for all $t \ge 0$,

$$d\|u\|^2 = 2(u, v)\mathrm{d}t - 2\delta\|u\|^2\mathrm{d}t$$

and

$$
\begin{aligned}
& d\|v\|^2 + 2(\lambda + \delta^2 - \alpha\delta)(v, u)\mathrm{d}t + 2(\alpha - \delta)\|v\|^2\mathrm{d}t + 2((-\Delta_d)^s u, v)\mathrm{d}t \\
& \quad = 2(f(u), v)\mathrm{d}t + 2(a(t), v)\mathrm{d}t + \sum_{j=1}^{\infty}\|g_j(u) + b_j(t)\|^2\mathrm{d}t + 2\sum_{j=1}^{\infty}(g_j(u) + b_j(t), v)\mathrm{d}W_j.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[(\lambda + \delta^2 - \alpha\delta)\|u\|^2 + \|v\|^2 + \left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right] + 2\delta(\lambda + \delta^2 - \alpha\delta)\mathbb{E}[\|u\|^2] + 2(\alpha - \delta)\mathbb{E}[\|v\|^2] \\
& \quad + 2\delta\mathbb{E}\left[\left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right] \\
& \quad = 2\mathbb{E}[(f(u), v)] + 2\mathbb{E}[(a(t), v)] + \sum_{j=1}^{\infty}\mathbb{E}[\|g_j(u) + b_j(t)\|^2].
\end{aligned}
\tag{3.2}
$$

By (2.7) and (2.16), we obtain

$$
\begin{aligned}
2\mathbb{E}[(f(u), v)] &\le 2\|\phi_1\|_\infty\mathbb{E}[\|u\|\|v\|] + 2\mathbb{E}[\|\phi_2\|\|v\|] \\
&\le \|\phi_1\|_\infty\mathbb{E}[\|u\|^2 + \|v\|^2] + \frac{\kappa}{4}\mathbb{E}[\|v\|^2] + \frac{4}{\kappa}\|\phi_2\|^2 \\
&\le \frac{1}{2}\kappa(\lambda + \delta^2 - \alpha\delta)\mathbb{E}[\|u\|^2] + \frac{\kappa}{2}\mathbb{E}[\|v\|^2] + \frac{4}{\kappa}\|\phi_2\|^2.
\end{aligned}
\tag{3.3}
$$

Note that

$$
2\mathbb{E}[(a(t), v)] \le \frac{\kappa}{2}\mathbb{E}[\|v\|^2] + \frac{2}{\kappa}\mathbb{E}[\|a(t)\|^2]. \tag{3.4}
$$

For the last term of (3.2), by (2.8) and (2.16), we obtain

$$
\begin{aligned}
\sum_{j=1}^{\infty}\mathbb{E}[\|g_j(u) + b_j(t)\|^2] &\le 2\sum_{j=1}^{\infty}\mathbb{E}[\|g_j(u)\|^2] + 2\sum_{j=1}^{\infty}\mathbb{E}[\|b_j(t)\|^2] \\
&\le 4c_\sigma\|\varphi_1\|_\infty^2\mathbb{E}[\|u\|^2] + 4c_\sigma\|\varphi_2\|^2 + 2\sum_{j=1}^{\infty}\mathbb{E}[\|b_j(t)\|^2] \\
&\le \frac{1}{2}\kappa(\lambda + \delta^2 - \alpha\delta)\mathbb{E}[\|u\|^2] + 4c_\sigma\|\varphi_2\|^2 + 2\sum_{j=1}^{\infty}\mathbb{E}[\|b_j(t)\|^2].
\end{aligned}
\tag{3.5}
$$

It follows from (2.15) and (3.2)–(3.5) that

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[(\lambda + \delta^2 - \alpha\delta)\|u\|^2 + \|v\|^2 + \left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right] + \kappa\mathbb{E}\left[(\lambda + \delta^2 - \alpha\delta)\|u\|^2 + \|v\|^2 + \left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right] \\
& \quad \le \frac{4}{\kappa}\|\phi_2\|^2 + \frac{2}{\kappa}\|\bar{a}\|^2 + 4c_\sigma\|\varphi_2\|^2 + 2\sum_{j=1}^{\infty}\|\bar{b}_j\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \mathbb{E}\left[(\lambda + \delta^2 - \alpha\delta)\|u(t)\|^2 + \|v(t)\|^2 + \left\|(-\Delta_d)^{\frac{s}{2}}u(t)\right\|^2\right] \\
& \quad \le e^{-\kappa t}\mathbb{E}\left[(\lambda + \delta^2 - \alpha\delta)\|u_0\|^2 + \|v_0\|^2 + \left\|(-\Delta_d)^{\frac{s}{2}}u_0\right\|^2\right] + \frac{1}{\kappa}\left(\frac{4}{\kappa}\|\phi_2\|^2 + \frac{2}{\kappa}\|\bar{a}\|^2 + 4c_\sigma\|\varphi_2\|^2 + 2\sum_{j=1}^{\infty}\|\bar{b}_j\|^2\right).
\end{aligned}
$$

This completes the proof. □

The next step entails acquiring uniform estimations on the tails of solutions to equation (2.13), which is pivotal in establishing the compactness of a family of solution distributions. For this purpose, we choose a differentiable function $\vartheta(r)$ that adheres $0 \le \vartheta(r) \le 1$ for all $r \in \mathbb{R}^+$, and

$$\vartheta(r) = \begin{cases} 0, & 0 \le r \le 1, \\ 1, & r \ge 2. \end{cases}$$

Moreover, given $s \in (0,1)$, by Lemma 3.3 of [11], we obtain that for all $i \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\sum_{j\in\mathbb{Z}, j\neq i} \left| \vartheta\left(\frac{|i|}{k}\right) - \vartheta\left(\frac{|j|}{k}\right) \right|^2 \tilde{K}_s(i-j) \le \frac{L_s^2}{k^{2s}}. \tag{3.6}$$

Lemma 3.2. *Suppose* (2.5)–(2.10) *and* (2.14)–(2.16) *hold. For compact subset* $\mathcal{K} \in \ell^2 \times \ell^2$, *let* $\Phi_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in L^2(\Omega, \ell^2 \times \ell^2)$ *be the initial data of equation* (2.13), *then the solution* $\Phi(t,0,\Phi_0) = \begin{pmatrix} u(t,0,u_0) \\ v(t,0,v_0) \end{pmatrix}$ *of equation* (2.13) *satisfies*

$$\lim_{k\to\infty} \sup_{t\ge 0} \sup_{\Phi_0 \in \mathcal{K}} \sum_{|i|\ge k} \mathbb{E}[|u_i(t,0,u_0)|^2 + v_i(t,0,v_0)|^2] = 0.$$

Proof. For $k \in \mathbb{N}$, set $\vartheta_k = \left(\vartheta\left(\frac{|i|}{k}\right)\right)_{i\in\mathbb{Z}}$, $\vartheta_k u = \left(\vartheta\left(\frac{|i|}{k}\right)u_i\right)_{i\in\mathbb{Z}}$, and $\vartheta_k v = \left(\vartheta\left(\frac{|i|}{k}\right)v_i\right)_{i\in\mathbb{Z}}$. By (2.13), we have

$$\mathrm{d}\vartheta_k\Phi(t) = \vartheta_k F(\Phi(t))\mathrm{d}t + \sum_{j=1}^{\infty} \vartheta_k G_j(\Phi(t))\mathrm{d}W_j,$$

which along with Itô's formula implies that

$$d\|\vartheta_k u\|^2 = 2(u, \vartheta_k^2 v)\mathrm{d}t - 2\delta\|\vartheta_k u\|^2 \mathrm{d}t$$

and

$$\begin{aligned} & d\|\vartheta_k v\|^2 + 2(\lambda + \delta^2 - \alpha\delta)(u, \vartheta_k^2 v)\mathrm{d}t + 2(\alpha - \delta)\|\vartheta_k v\|^2 \mathrm{d}t + 2((-\Delta_d)^s u, \vartheta_k^2 v)\mathrm{d}t \\ &= 2(f(u), \vartheta_k^2 v)\mathrm{d}t + 2(a(t), \vartheta_k^2 v)\mathrm{d}t + \sum_{j=1}^{\infty} \|\vartheta_k g_j(u) + \vartheta_k b_j(t)\|^2 \mathrm{d}t + 2\sum_{j=1}^{\infty} (g_j(u) + b_j(t), \vartheta_k^2 v)\mathrm{d}W_j. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[(\lambda + \delta^2 - \alpha\delta)\|\vartheta_k u\|^2 + \|\vartheta_k v\|^2] + 2\delta(\lambda + \delta^2 - \alpha\delta)\mathbb{E}[\|\vartheta_k u\|^2] \\ &\quad + 2(\alpha - \delta)\mathbb{E}[\|\vartheta_k v\|^2] + 2\mathbb{E}[((-\Delta_d)^s u, \vartheta_k^2 \dot{u})] + 2\delta\mathbb{E}[((-\Delta_d)^s u, \vartheta_k^2 u)] \\ &= 2\mathbb{E}[(f(u), \vartheta_k^2 v)] + 2\mathbb{E}[(a(t), \vartheta_k^2 v)] + \sum_{j=1}^{\infty} \mathbb{E}[\|\vartheta_k g_j(u) + \vartheta_k b_j(t)\|^2]. \end{aligned} \tag{3.7}$$

By Lemma 2.2, we have

$$\begin{aligned} -2((-\Delta_d)^s u, \vartheta_k^2 \dot{u}) &= -2\left((-\Delta_d)^{\frac{s}{2}} u, (-\Delta_d)^{\frac{s}{2}} \vartheta_k^2 \dot{u}\right) \\ &= -\sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}, j\neq i} (u_i - u_j)\left(\vartheta^2\left(\frac{|i|}{k}\right)\dot{u}_i - \vartheta^2\left(\frac{|j|}{k}\right)\dot{u}_j\right)\tilde{K}_s(i-j) \\ &= -\sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}, j\neq i} (u_i - u_j)(\dot{u}_i - \dot{u}_j)\vartheta^2\left(\frac{|i|}{k}\right)\tilde{K}_s(i-j) - \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}, j\neq i} (u_i - u_j)\left(\vartheta^2\left(\frac{|i|}{k}\right)\right. \\ &\quad \left. - \vartheta^2\left(\frac{|j|}{k}\right)\right)\dot{u}_j \tilde{K}_s(i-j). \end{aligned} \tag{3.8}$$

By Lemma 2.2 and (3.6), we obtain

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i}\left|(u_i-u_j)\left(\vartheta^2\left(\frac{|i|}{k}\right)-\vartheta^2\left(\frac{|j|}{k}\right)\right)\dot{u}_j\tilde{K}_s(i-j)\right| \\
&\le 2\|\dot{u}\|\left[\sum_{i\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z},j\neq i}\left|\vartheta\left(\frac{|i|}{k}\right)-\vartheta\left(\frac{|j|}{k}\right)\right|^2\tilde{K}_s(i-j)\right)\left(\sum_{j\in\mathbb{Z},j\neq i}|u_i-u_j|^2\tilde{K}_s(i-j)\right)\right]^{\frac{1}{2}} \\
&\le \frac{2L_s}{k^s}\|\dot{u}\|\left[\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i}|u_i-u_j|^2\tilde{K}_s(i-j)\right]^{\frac{1}{2}} \\
&\le \frac{\sqrt{2}L_s}{k^s}\left(\|v-\delta u\|^2+\left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right) \\
&\le \frac{\sqrt{2}L_s}{k^s}\left(2\|v\|^2+2\delta^2\|u\|^2+\left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right),
\end{aligned} \tag{3.9}$$

which along with (3.8) implies that

$$-2\mathbb{E}[((-\Delta_d)^s u,\vartheta_k^2\dot{u})]\le-\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\left\|\vartheta_k(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right]+\frac{\sqrt{2}L_s}{k^s}\mathbb{E}\left[2\|v\|^2+2\delta^2\|u\|^2+\left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right]. \tag{3.10}$$

Similarly, by Lemma 2.2, we have

$$\begin{aligned}
-2((-\Delta_d)^s u,\vartheta_k^2 u) &= -2\left((-\Delta_d)^{\frac{s}{2}}u,(-\Delta_d)^{\frac{s}{2}}\vartheta_k^2 u\right) \\
&= -\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i}(u_i-u_j)\left(\vartheta^2\left(\frac{|i|}{k}\right)u_i-\vartheta^2\left(\frac{|j|}{k}\right)u_j\right)\tilde{K}_s(i-j) \\
&= -\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i}|u_i-u_j|^2\vartheta^2\left(\frac{|i|}{k}\right)\tilde{K}_s(i-j)-\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i}(u_i-u_j)\left(\vartheta^2\left(\frac{|i|}{k}\right)-\vartheta^2\left(\frac{|j|}{k}\right)\right)u_j\tilde{K}_s(i-j).
\end{aligned} \tag{3.11}$$

By Lemma 2.2 and (3.6), we obtain

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i}\left|(u_i-u_j)\left(\vartheta^2\left(\frac{|i|}{k}\right)-\vartheta^2\left(\frac{|j|}{k}\right)\right)u_j\tilde{K}_s(i-j)\right| \\
&\le 2\|u\|\left[\sum_{i\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z},j\neq i}\left|\vartheta\left(\frac{|i|}{k}\right)-\vartheta\left(\frac{|j|}{k}\right)\right|^2\tilde{K}_s(i-j)\right)\left(\sum_{j\in\mathbb{Z},j\neq i}|u_i-u_j|^2\tilde{K}_s(i-j)\right)\right]^{\frac{1}{2}} \\
&\le \frac{2L_s}{k^s}\|u\|\left[\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i}|u_i-u_j|^2\tilde{K}_s(i-j)\right]^{\frac{1}{2}} \\
&\le \frac{\sqrt{2}L_s}{k^s}\left(\|u\|^2+\left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right).
\end{aligned} \tag{3.12}$$

Then, it follows from (3.11) and (3.12) that

$$-2\delta\mathbb{E}[((-\Delta_d)^s u,\vartheta_k^2 u)]=-2\delta\mathbb{E}\left[\left\|(-\Delta_d)^{\frac{s}{2}}\vartheta_k u\right\|^2\right]+\sqrt{2}\,\delta\frac{L_s}{k^s}\mathbb{E}\left[\|u\|^2+\left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right]. \tag{3.13}$$

By (2.7) and (2.16), we obtain

$$\begin{aligned}
2\mathbb{E}[(f(u),\vartheta_k^2 v)] &\le 2\|\phi_1\|_\infty\mathbb{E}[\|\vartheta_k u\|\|\vartheta_k v\|]+2\mathbb{E}[\|\vartheta_k\phi_2\|\|\vartheta_k v\|] \\
&\le \|\phi_1\|_\infty\mathbb{E}[\|\vartheta_k u\|^2+\|\vartheta_k v\|^2]+2\mathbb{E}[\|\vartheta_k\phi_2\|\|\vartheta_k v\|] \\
&\le \frac{1}{2}\kappa(\lambda+\delta^2-\alpha\delta)\mathbb{E}[\|\vartheta_k u\|^2]+\frac{\kappa}{2}\mathbb{E}[\|\vartheta_k v\|^2]+\frac{4}{\kappa}\sum_{|i|\ge k}|\phi_{2,i}|^2.
\end{aligned} \tag{3.14}$$

Note that

$$2\mathbb{E}[(a(t),\vartheta_k^2 v)] \le \frac{\kappa}{2}\mathbb{E}[\|\vartheta_k v\|^2] + \frac{2}{\kappa}\sum_{|i|\ge k}|a_i(t)|^2. \tag{3.15}$$

For the last term of (3.7), by (2.8) and (2.16), we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\mathbb{E}[\|\vartheta_k g_j(u)+\vartheta_k b_j(t)\|^2] &\le 2\sum_{j=1}^{\infty}\mathbb{E}[\|\vartheta_k g_j(u)\|^2] + 2\sum_{j=1}^{\infty}\mathbb{E}[\|\vartheta_k b_j(t)\|^2]\\ &\le 4c_\sigma\|\varphi_1\|_\infty^2\mathbb{E}[\|\vartheta_k u\|^2] + 4c_\sigma\|\vartheta_k\varphi_2\|^2 + 2\sum_{j=1}^{\infty}\|\vartheta_k b_j(t)\|^2\\ &\le \frac{1}{2}\kappa(\lambda+\delta^2-\alpha\delta)\mathbb{E}[\|\vartheta_k u\|^2] + 4c_\sigma\sum_{|i|\ge k}|\varphi_{2,i}|^2 + 2\sum_{j=1}^{\infty}\sum_{|i|\ge k}|b_{i,j}(t)|^2.\end{aligned} \tag{3.16}$$

It follows from (2.15), (3.7), (3.10), and (3.13)–(3.16) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[(\lambda+\delta^2-\alpha\delta)\|\vartheta_k u(t)\|^2 + \|\vartheta_k v(t)\|^2 + \left\|\vartheta_k(-\Delta_d)^{\frac{s}{2}}u(t)\right\|^2\right]\\ &\quad+\kappa\mathbb{E}\left[(\lambda+\delta^2-\alpha\delta)\|\vartheta_k u(t)\|^2 + \|\vartheta_k v(t)\|^2 + \left\|\vartheta_k(-\Delta_d)^{\frac{s}{2}}u(t)\right\|^2\right]\\ &\le \frac{\sqrt{2}L_s}{k^s}\mathbb{E}\left[2\|v\|^2 + (\delta+2\delta^2)\|u\|^2 + (1+\delta)\left\|(-\Delta_d)^{\frac{s}{2}}u\right\|^2\right] + \frac{4}{\kappa}\sum_{|i|\ge k}|\phi_{2,i}|^2 + \frac{2}{\kappa}\sum_{|i|\ge k}|\bar{a}_i|^2 + 4c_\sigma\sum_{|i|\ge k}|\varphi_{2,i}|^2\\ &\quad+2\sum_{j=1}^{\infty}\sum_{|i|\ge k}|\bar{b}_{i,j}|^2,\end{aligned} \tag{3.17}$$

which implies that

$$\begin{aligned}&\mathbb{E}\left[(\lambda+\delta^2-\alpha\delta)\|\vartheta_k u(t)\|^2 + \|\vartheta_k v(t)\|^2 + \left\|\vartheta_k(-\Delta_d)^{\frac{s}{2}}u(t)\right\|^2\right]\\ &\le e^{-\kappa t}\mathbb{E}\left[(\lambda+\delta^2-\alpha\delta)\|\vartheta_k u_0\|^2 + \|\vartheta_k v_0\|^2 + \left\|\vartheta_k(-\Delta_d)^{\frac{s}{2}}u_0\right\|^2\right]\\ &\quad+\frac{\sqrt{2}L_s}{k^s}\int_0^t e^{\kappa(r-t)}\mathbb{E}\left[2\|v(r,0,v_0)\|^2 + (\delta+2\delta^2)\|u(r,0,u_0)\|^2 + (1+\delta)\left\|(-\Delta_d)^{\frac{s}{2}}u(r,0,u_0)\right\|^2\right]\mathrm{d}r\\ &\quad+\frac{1}{\kappa}\left(\frac{4}{\kappa}\sum_{|i|\ge k}|\phi_{2,i}|^2 + \frac{2}{\kappa}\sum_{|i|\ge k}|\bar{a}_i|^2 + 4c_\sigma\sum_{|i|\ge k}|\varphi_{2,i}|^2 + 2\sum_{j=1}^{\infty}\sum_{|i|\ge k}|\bar{b}_{i,j}|^2\right).\end{aligned} \tag{3.18}$$

By (2.4) and the compactness of $\mathcal{K}$, we have that for all $t\ge 0$,

$$\begin{aligned}&\limsup_{k\to\infty}\sup_{\Phi_0\in\mathcal{K}} e^{-\kappa t}\mathbb{E}\left[(\lambda+\delta^2-\alpha\delta)\|\vartheta_k u_0\|^2 + \|\vartheta_k v_0\|^2 + \left\|\vartheta_k(-\Delta_d)^{\frac{s}{2}}u_0\right\|^2\right]\\ &\le(\lambda+\delta^2-\alpha\delta+1+4^{\frac{s}{2}})\limsup_{k\to\infty}\sup_{\Phi_0\in\mathcal{K}}\mathbb{E}\left[\sum_{|i|\ge k}(|u_{0,i}|^2+|v_{0,i}|^2)\right]=0.\end{aligned} \tag{3.19}$$

By Lemma 3.1, for all $t\ge 0$, $\Phi_0\in\mathcal{K}$ and $k\to\infty$, we obtain

$$\begin{aligned}&\frac{\sqrt{2}L_s}{k^s}\int_0^t e^{\kappa(r-t)}\mathbb{E}\left[2\|v(r,0,v_0)\|^2 + (\delta+2\delta^2)\|u(r,0,u_0)\|^2 + (1+\delta)\left\|(-\Delta_d)^{\frac{s}{2}}u(r,0,u_0)\right\|^2\right]\mathrm{d}r\\ &\le\frac{\sqrt{2}L_s}{\kappa k^s}\sup_{r\ge 0}\mathbb{E}\left[2\|v(r,0,v_0)\|^2 + (\delta+2\delta^2)\|u(r,0,u_0)\|^2 + (1+\delta)\left\|(-\Delta_d)^{\frac{s}{2}}u(r,0,u_0)\right\|^2\right]\to 0.\end{aligned} \tag{3.20}$$

By $\phi_2 \in \ell^2$, $\varphi_2 \in \ell^2$, and (2.10), we have

$$\frac{4}{\kappa}\sum_{|i|\geq k}|\phi_{2,i}|^2 + \frac{2}{\kappa}\sum_{|i|\geq k}|\bar{a}_i|^2 + 4c_\sigma\sum_{|i|\geq k}|\varphi_{2,i}|^2 + 2\sum_{j=1}^{\infty}\sum_{|i|\geq k}|\bar{b}_{i,j}|^2 \to 0 \quad \text{as } k \to \infty. \tag{3.21}$$

It follows from (3.18)–(3.21) that

$$\mathbb{E}\left[\sum_{|i|\geq 2k}(|u_i(t,0,u_0)|^2 + |v_i(t,0,v_0)|^2)\right] \leq \mathbb{E}[\|\vartheta_k u(t,0,u_0)\|^2 + \|\vartheta_k v(t,0,v_0)\|^2] \to 0 \quad \text{as } k \to \infty$$

uniformly for $t \geq 0$ and $\Phi_0 \in \mathcal{K}$. This completes the proof. □

4 Existence of periodic measures

The primary objective of this section is to establish the existence of periodic measures for equation (2.13) in $\ell^2 \times \ell^2$. First, we introduce the transition operators associated with the equation and subsequently provide evidence for the convergence and compactness properties exhibited by a family of probability distributions representing solutions to this particular equation.

Suppose $\psi : \ell^2 \times \ell^2 \to \mathbb{R}$ is a bounded Borel function. For $0 \leq r \leq t$, we set

$$(p_{r,t}\psi)(\Phi_0) = \mathbb{E}[\psi(\Phi(t,r,\Phi_0))], \quad \forall \Phi_0 \in \ell^2 \times \ell^2. \tag{4.1}$$

In addition, for $G \in \mathcal{B}(\ell^2 \times \ell^2)$, $0 \leq r \leq t$, and $\Phi_0 \in \ell^2 \times \ell^2$, we set

$$p(r,\Phi_0;\, t, G) = (p_{r,t}1_G)(\Phi_0),$$

where 1_G is the indicator function of G. Then, the probability distribution of $\Phi(t)$ in $\ell^2 \times \ell^2$ can be represented as $p(r,\Phi_0;\, t,\cdot)$. Additionally, for convenience, the transition operator $p_{0,t}$ is denoted as p_t.

Definition 4.1. A probability measure μ of equation (2.13) is called a periodic with period $T > 0$ if

$$\int_{\ell^2\times\ell^2}(p_{0,t+T}\psi)(\Phi_0)\mathrm{d}\mu(\Phi_0) = \int_{\ell^2\times\ell^2}(p_{0,t}\psi)(\Phi_0)\mathrm{d}\mu(\Phi_0), \quad \forall t \geq 0.$$

The next lemma demonstrates the tightness of a family of distributions for solutions to equation (2.13) in $\ell^2 \times \ell^2$. Henceforth, we will employ $\mathcal{L}(\Phi(t,0,\Phi_0))$ to denote the probability distribution of the solution $\Phi(t,0,\Phi_0)$ to equation (2.13).

Lemma 4.1. *Suppose* (2.5)–(2.10) *and* (2.14)–(2.16) *hold. Then, for the given compact subset* $\mathcal{K} \in \ell^2 \times \ell^2$, *we obtain that the family* $\{\mathcal{L}(\Phi(t,0,\Phi_0)) : t \geq 0, \Phi_0 \in \mathcal{K}\}$ *of the distributions of the solutions to equation* (2.13) *is tight on* $\ell^2 \times \ell^2$.

Proof. We write the solution $\Phi(t,0,\Phi_0)$ to equation (2.13) as

$$\Phi(t,0,\Phi_0) = \tilde{\Phi}^n(t,0,\Phi_0) + \hat{\Phi}^n(t,0,\Phi_0), \quad n \in \mathbb{N}, t \geq 0 \tag{4.2}$$

with

$$\tilde{\Phi}^n(t,0,\Phi_0) = (\chi_{[-n,n]}(i)\Phi_i(t,0,\Phi_0))_{i\in\mathbb{Z}} \quad \text{and} \quad \hat{\Phi}^n(t,0,\Phi_0) = ((1-\chi_{[-n,n]}(i))\Phi_i(t,0,\Phi_0))_{i\in\mathbb{Z}},$$

where $\chi_{[-n,n]}$ is the characteristic function of $[-n,n]$. For all $t \geq 0$, by Lemma 3.1, we obtain that there exists a constant $c_1 > 0$ such that for all $t \geq 0$ and $\Phi_0 \in \mathcal{K}$,

$$\mathbb{E}[\|\Phi(t,0,\Phi_0)\|^2_{\ell^2\times\ell^2}] \leq c_1. \tag{4.3}$$

By Lemma 3.2, we obtain that for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an integer $n_m = n_m(\varepsilon, m, \mathcal{K}) \geq 1$ such that

$$\mathbb{E}[\|\hat{\Phi}^{n_m}(t,0,\Phi_0)\|^2_{\ell^2\times\ell^2}] \leq \frac{\varepsilon}{2^{4m}}, \quad \forall t \geq 0 \quad \text{and} \quad \Phi_0 \in \mathcal{K}. \tag{4.4}$$

For every $m \in \mathbb{N}$, let

$$\mathcal{Z}_{1,m} = \{z \in \ell^2 \times \ell^2 : z_i = 0 \text{ for } |i| > n_m \text{ and } \|z\|_{\ell^2\times\ell^2} \leq \frac{2^m\sqrt{C_1}}{\sqrt{\varepsilon}}\}, \tag{4.5}$$

$$\mathcal{Z}_{2,m} = \left\{z \in \ell^2 \times \ell^2 : \|z - \hat{z}\|_{\ell^2\times\ell^2} \leq \frac{1}{2^m}, \text{ for some } \hat{z} \in \mathcal{Z}_{1,m}\right\}. \tag{4.6}$$

By (4.2), (4.5), and (4.6), we obtain

$$\begin{aligned}
&\{\omega \in \Omega : \Phi(t,0,\Phi_0) \notin \mathcal{Z}_{2,m}\}\\
&\subseteq \{\omega \in \Omega : \tilde{\Phi}^{n_m}(t,0,\Phi_0) \notin \mathcal{Z}_{1,m}\} \cup \{\omega \in \Omega : \Phi(t,0,\Phi_0) \notin \mathcal{Z}_{2,m} \text{ and } \tilde{\Phi}^{n_m}(t,0,\Phi_0) \in \mathcal{Z}_{1,m}\}\\
&\subseteq \left\{\omega \in \Omega : \|\tilde{\Phi}^{n_m}(t,0,\Phi_0)\|_{\ell^2\times\ell^2} > \frac{2^m\sqrt{C_1}}{\sqrt{\varepsilon}}\right\} \cup \left\{\omega \in \Omega : \|\hat{\Phi}^{n_m}(t,0,\Phi_0)\|_{\ell^2\times\ell^2} > \frac{1}{2^m}\right\}.
\end{aligned} \tag{4.7}$$

It follows from (4.3) that for all $t \geq 0$ and $\Phi_0 \in \mathcal{K}$, we obtain

$$\mathbb{P}\left(\left\{\omega \in \Omega : \|\tilde{\Phi}^{n_m}(t,0,\Phi_0)\|_{\ell^2\times\ell^2} > \frac{2^m\sqrt{C_1}}{\sqrt{\varepsilon}}\right\}\right) \leq \frac{\varepsilon}{2^m C_1}\mathbb{E}[\|\Phi(t,0,\Phi_0)\|^2_{\ell^2\times\ell^2}] \leq \frac{\varepsilon}{2^{2m}}. \tag{4.8}$$

By (4.4), we obtain that for all $t \geq 0$ and $\Phi_0 \in \mathcal{K}$,

$$\mathbb{P}\left(\left\{\omega \in \Omega : \|\hat{\Phi}^{n_m}(t,0,\Phi_0)\|_{\ell^2\times\ell^2} > \frac{1}{2^m}\right\}\right) \leq 2^{2m}\mathbb{E}[\|\hat{\Phi}^{n_m}(t,0,\Phi_0)\|^2_{\ell^2\times\ell^2}] \leq \frac{\varepsilon}{2^{2m}}. \tag{4.9}$$

Then, by (4.7)–(4.9), we obtain

$$\mathbb{P}(\{\omega \in \Omega : \Phi(t,0,\Phi_0) \notin \mathcal{Z}_{2,\varepsilon}\}) \leq \frac{\varepsilon}{2^{2m-1}}. \tag{4.10}$$

Let $\mathcal{Z}_\varepsilon = \bigcap_{m=1}^{\infty}\mathcal{Z}_{2,m}$, we find that $\mathcal{Z}_\varepsilon$ is a closed and totally bounded in $\ell^2 \times \ell^2$. Then, it is compact in $\ell^2 \times \ell^2$. Given $\varepsilon > 0$, it follows from (4.10) that for all $t \geq 0$ and $\Phi_0 \in \mathcal{K}$,

$$\mathbb{P}(\{\omega \in \Omega : \Phi(t,0,\Phi_0) \notin \mathcal{Z}_\varepsilon\}) \leq \sum_{m=1}^{\infty}\frac{\varepsilon}{2^{2m-1}} < \varepsilon. \tag{4.11}$$

This completes the proof. □

The properties of transition operators $\{p_{r,t}\}_{0\leq r\leq t}$ are now presented as follows.

Lemma 4.2. *Suppose* (2.5)–(2.10) *and* (2.14)–(2.16) *hold. Then, we have*

(*i*) *The family* $\{p_{r,t}\}_{0\leq r\leq t}$ *is Feller; i.e., if* $\psi : \ell^2 \times \ell^2 \to \mathbb{R}$ *is bounded and continuous, then* $p_{r,t}\psi : \ell^2 \times \ell^2 \to \mathbb{R}$ *is bounded and continuous.*

(*ii*) *The family* $\{p_{r,t}\}_{0\leq r\leq t}$ *is T-periodic; i.e.,*

$$p(r,\Phi_0;\, t,\cdot) = p(r+T,\Phi_0;\, t+T,\cdot), \quad \forall r \in [0,t], \Phi_0 \in \ell^2 \times \ell^2.$$

(*iii*) $\{\Phi(t,0,\Phi_0)\}_{t\geq 0}$ *is a* $\ell^2 \times \ell^2$*-valued Markov process.*

Proof. (i) Using a similar approach to Lemma 4.4 in [20], we realize that $\{p_{r,t}\}_{0\leq r\leq t}$ is Feller.

(ii) By (2.13), we have

$$\Phi(t,r,\Phi_0) = \Phi_0 + \int_r^t F(\Phi(s,r,\Phi_0))\mathrm{d}s + \sum_{j=1}^{\infty}\int_r^t G_j(\Phi(s,r,\Phi_0))\mathrm{d}W_j(s). \tag{4.12}$$

We also have

$$\Phi(t+T, r+T, \Phi_0) = \Phi_0 + \int_{r+T}^{t+T} F(\Phi(s, r+T, \Phi_0))\mathrm{d}s + \sum_{j=1}^{\infty} \int_{r+T}^{t+T} G_j(\Phi(s, r+T, \Phi_0))\mathrm{d}W_j(s),$$

which shows that

$$\Phi(t+T, r+T, \Phi_0) = \Phi_0 + \int_r^t F(\Phi(s+T, r+T, \Phi_0))\mathrm{d}s + \sum_{j=1}^{\infty} \int_r^t G_j(\Phi(s+T, r+T, \Phi_0))\mathrm{d}\tilde{W}_j(s), \tag{4.13}$$

where $\tilde{W}_j(s) = W_j(s+T) - W_j(T), j \in \mathbb{N}$, are Brownian motions as well. By (4.12)–(4.13) and Theorem 2.1 of [55], it can be derived that $\Phi(t+T, r+T, \Phi_0)$ have the same distribution law. Consequently, for any $A \in \mathcal{B}(\ell^2 \times \ell^2)$,

$$p(r, \Phi_0; t, A) = p(r+T, \Phi_0; t+T, A), \quad \forall r \in [0, t].$$

(iii) For all $s \geq 0$ and $z \in \ell^2 \times \ell^2$, we will show that the solution $\Phi(t, s, z)$ with $s \geq t$ to equation (2.13) is a $\ell^2 \times \ell^2$-valued Markov process. By the uniqueness of the solutions, we obtain that for every $0 \leq s \leq r \leq t$,

$$\Phi(t, s, z) = \Phi(t, r, \Phi(r, s, z)), \quad \mathbb{P} - \text{a.s.} \tag{4.14}$$

Then, we only need to show that for all bounded and continuous function $\psi : \ell^2 \times \ell^2 \to \mathbb{R}$,

$$\mathbb{E}[\psi(\Phi(t, s, z))|\mathcal{F}_r] = (p_{r,t}\psi)(\tilde{z})|_{\tilde{z}=\Phi(r,s,z)}, \quad \mathbb{P} - \text{a.s.} \tag{4.15}$$

Given $n \in \mathbb{N}$ and $\xi \in L^2(\Omega, \ell^2 \times \ell^2)$, we let $\Phi^n(t, r, \xi)$ be the solution to equation (2.13). Since f satisfies (2.5) and (2.7), g_j satisfies (2.6), (2.8), and (2.9), one can prove that for all bounded and continuous function $\psi : \ell^2 \times \ell^2 \to \mathbb{R}$,

$$\mathbb{E}[\psi(\Phi^n(t, r, z))|\mathcal{F}_r] = \mathbb{E}[\psi(\Phi^n(t, r, \tilde{z}))]|_{\tilde{z}=\xi}, \quad \mathbb{P} - \text{a.s.}, \tag{4.16}$$

$$\lim_{n\to\infty} \Phi^n(t, r, \xi) = \Phi(t, r, \xi), \quad \mathbb{P} - \text{a.s.} \tag{4.17}$$

According to the Lebesgue dominated convergence theorem, as well as (4.16) and (4.17), we can deduce

$$\mathbb{E}[\psi(\Phi(t, r, \xi))|\mathcal{F}_r] = \mathbb{E}[\psi(\Phi(t, r, \tilde{z}))]|_{\tilde{z}=\xi}, \quad \mathbb{P} - \text{a.s.},$$

which along with (4.1) shows that

$$\mathbb{E}[\psi(\Phi(t, r, \xi))|\mathcal{F}_r] = (p_{r,t}\psi)(\tilde{z})|_{\tilde{z}=\xi}, \quad \mathbb{P} - \text{a.s.} \tag{4.18}$$

Consequently, (4.15) can be derived directly from (4.14) and (4.18). This completes the proof. □

Now, the main outcome of this study has been shown by Krylov-Bogolyubov's method.

Theorem 4.1. *Suppose* (2.5)–(2.10) *and* (2.14)–(2.16) *hold. Then, equation* (2.13) *has a periodic measure on* $\ell^2 \times \ell^2$.

Proof. For each $n \in \mathbb{N}$, the probability measure μ_n is given by

$$\mu_n = \frac{1}{n} \sum_{l=1}^{n} p(0, 0; lT, \cdot). \tag{4.19}$$

By Lemma 4.1, we obtain that the sequence $(\mu_n)_{n=1}^{\infty}$ is tight on $\ell^2 \times \ell^2$. Then, there exists a probability measure μ on $\ell^2 \times \ell^2$ and a subsequence (still denoted by $(\mu_n)_{n=1}^{\infty}$) such that

$$\mu_n \to \mu, \quad \text{as } n \to \infty. \tag{4.20}$$

It can be deduced from (4.19) and (4.20) and Lemma 4.2 that for every $t \geq 0$ and every bounded and continuous function $\psi : \ell^2 \times \ell^2 \to \mathbb{R}$,

$$\begin{aligned}
\int_{\ell^2\times\ell^2} (p_{0,t}\psi)(\Phi_0)\mathrm{d}\mu(\Phi_0) &= \int_{\ell^2\times\ell^2}\int_{\ell^2\times\ell^2} \psi(y)p(0,\Phi_0;\, t, \mathrm{d}y)\mathrm{d}\mu(\Phi_0) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2\times\ell^2}\int_{\ell^2\times\ell^2} \psi(y)p(0,\Phi_0;\, t, \mathrm{d}y)p(0,0;\, lT, \mathrm{d}\Phi_0) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2\times\ell^2}\int_{\ell^2\times\ell^2} \psi(y)p(lT,\Phi_0;\, t+lT, \mathrm{d}y)p(0,0;\, lT, \mathrm{d}\Phi_0) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2\times\ell^2} \psi(y)p(0,0;\, t+lT, \mathrm{d}y) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2\times\ell^2} \psi(y)p(0,0;\, t+lT+T, \mathrm{d}y) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2\times\ell^2}\int_{\ell^2\times\ell^2} \psi(y)p(0,\Phi_0;\, t+T, \mathrm{d}y)p(0,0;\, lT, \mathrm{d}\Phi_0) \\
&= \int_{\ell^2\times\ell^2}\int_{\ell^2\times\ell^2} \psi(y)p(0,\Phi_0;\, t+T, \mathrm{d}y)\mathrm{d}\mu(\Phi_0) \\
&= \int_{\ell^2\times\ell^2} (p_{0,t+T}\psi)(\Phi_0)\mathrm{d}\mu(\Phi_0),
\end{aligned}$$

which implies that μ is a periodic measure of equation (2.13). This completes the proof. □

5 Remark

The current focus is on the theoretical proof of the well-posedness of solutions and the existence of periodic measures for fractional stochastic discrete wave equations with nonlinear noise. This objective was achieved through the utilization of uniform tail estimates and Krylov Bogolyubov's method. In future research, our group intends to investigate the Ergodicity of stochastic discrete wave equations possessing a periodic measure. Furthermore, we will employ finite-dimensional numerical approximation methods to address the existence of numerical periodic measures.

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