

Research Article

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The novel quadratic phase Fourier S-transform and associated uncertainty principles in the quaternion setting

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Abstract: In this article, we propose a novel integral transform coined as quaternion quadratic phase S-transform (Q-QPST), which is an extension of the quadratic phase S-transform and study the uncertainty principles associated with the Q-QPST. The Q-QPST possesses some desirable characteristics that are absent in conventional time-frequency transforms, especially for dealing with the time-varying quaternion-valued signals. First, we propose the definition of Q-QPST and then we explore some mathematical properties of the quaternion Q-QPST, including the linearity, modulation, shift, orthogonality relation, and reconstruction formula. Second, we derive the associated Heisenberg's uncertainty inequality and the corresponding logarithmic version for Q-QPST. Finally, an illustrative example and some potential applications of the Q-QPST are introduced.

Keywords: quadratic-phase transform, quaternion quadratic-phase S-transform, modulation, uncertainty principle

MSC 2020: 42A05, 42B10, 46S10, 42C20, 44A35

1 Introduction

Several popular parametric time-frequency analysis tools have revolutionized signal, image, and video processing [1–3] in the modern era. These include the fractional Fourier transform [4], linear canonical transform [5], short-time Fourier transform [6], and Wigner distributions [7]. The quadratic-phase Fourier transform (QPFT), a superior expanded version of the classical Fourier transform, is a new tool added to this set [8]. It is used in radar and other communication systems and handles both stationary and non-stationary signals in an understandable and straightforward manner. Formally, for the arbitrary real parameter set $Y = (A, B, C, D, E)$, $B \neq 0$, the QPFT of any signal $f \in L^2(\mathbb{R})$ is defined by

$$Q_Y[f](u) = \int_{\mathbb{R}} f(x) K_Y(x, u) dx, \quad (1)$$

where $K_Y(x, u)$ is a quadratic-phase kernel and is given by

$$K_Y(x, u) = \frac{1}{\sqrt{2\pi}} e^{i(Ax^2 + Bxu + Cu^2 + Dx + Eu)}. \quad (2)$$

The inversion and Parseval's formulae corresponding to (1) are given by

$$f(x) = Q_Y^{-1}\{Q_Y[f]\}(x) = \int_{\mathbb{R}} Q_Y[f](u) K_{-Y}(x, u) du, \quad (3)$$

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$$\langle f, g \rangle = |B| \langle Q_\mu[f], Q_\mu[g] \rangle. \quad (4)$$

The QPFT has demonstrated its significance in addressing numerous issues in scientific and engineering fields, such as harmonic analysis, sampling, image processing, and many more, thanks to its additional degrees of freedom and global kernel [9,10]. Because the QPFT is insufficient to localize the non-stationary signals' quadratic-phase spectrum content. Various authors have extended the QPFT to create new integral transforms, including the quadratic-phase S-transform (ST), quadratic-phase Wigner distribution, quadratic-phase wavelet and wave packet transforms, and many more, which can be found in [11–14].

The quaternion Fourier transform (QFT) [15–18] is a non-trivial generalization of the classical real and complex Fourier transform. For a long time, mathematicians and physicists have been very interested in this generalization utilizing quaternion algebra. Since QFT has found numerous applications in color image processing, image filtering, watermarking, edge detection, and pattern recognition (see [19–25]), the following extensions of QFT came into existence: the quaternion windowed Fourier transform (QWFT) [26–28]; the fractional quaternion Fourier transform (Fr-QFT) [29,30]; the quaternion linear canonical transform (QLCT) [31,32]; the quaternion offset linear canonical transform [33–36]. Bhat and Dar [37] introduced the extension of the QPFT into the domain of quaternion algebra (Q-QPFT). Later, they introduced several extensions of the Q-QPFT and also studied its vital properties and establish some uncertainty principles. In comparison to QFT, FR-QFT, and QLCT, Q-QPFT is thought to be a crucial tool for non-stationary signal processing due to its extra degrees of freedom and free parameters [38–40].

Authors in [41] introduced the ST, which has garnered significant interest in a number of scientific and engineering domains, including optics, geophysics, oceanology, biomedical imaging, bioinformatics, and signal processing [42–51]. If $\tau(x, w)$ be a window function, then the ST of function $f(x) \in L^2(\mathbb{R})$ with respect to ξ is defined by [52]:

$$S_\tau[f](u, w) = \int_{\mathbb{R}} \xi(u - x, w) e^{-i2\pi w x} f(x) dx, \quad (5)$$

where the window function ψ satisfies the following condition:

$$\int_{\mathbb{R}} \xi(x, w) dx = 1 \quad \forall x \in \mathbb{R} \setminus 0. \quad (6)$$

To provide a time-linear quadratic phase domain representation, authors derived the idea of QPST in [53]. It is proposed that the QPST has several advantageous properties not found in traditional time-frequency transforms.

In the course of our investigation into QPFT extensions, we presented the idea of the novel quaternion quadratic phase Fourier S-transform (Q-QPST), which is itself an extension of the QPST. The Q-QPST is believed to analyze quaternion-valued signals with abilities of multi-angle, multi-resolution, multi-scale and temporal localization. It is particularly suitable for dealing with chirp-like quaternion-valued signals. First, we propose the definition of the novel Q-QPST. The essential characteristics of the suggested transform are then examined, after which certain key conclusions are established, such as the reconstruction formula and the orthogonality relation. Finally, we obtain the related logarithmic uncertainty inequality and the associated Heisenberg's uncertainty inequality for the Q-QPST.

The rest of the article is organized as follows:

We go over some preliminary findings and definitions in Section 2, which are used in the parts that follow. The Q-QPST is introduced in Section 3 and its features, including Modulation, orthogonality, and the reconstruction formula, are studied thereafter. The well-known Heisenberg's uncertainty inequality for the Q-QPST is presented in Section 4. In Section 5, the Q-QPST's possible uses are highlighted. The article's conclusions are presented in Section 6.

2 Preliminary

In this section, we review QPST and Q-QPFT.

2.1 QPST

The QPST is known as a hybrid form of the ST. This transform is an extension of the QPFT and ST, which displays the time and quadratic phase domain-frequency information jointly in the time-frequency plane. It is defined as:

Definition 1. The QPST of any signal $f \in L^2(\mathbb{R})$ with respect to any non-zero window function $\xi \in L^2(\mathbb{R})$ is defined by [53]

$$S_{\xi}^Y[f](u, w) = Q_Y\{\overline{\xi(u-x, w)}f(x)\}(w, u) = \int_{\mathbb{R}} \overline{\xi(u-x, w)}f(x)K_Y(x, u)dx, \quad (7)$$

and its inverse is given by

$$f(x)\overline{\xi(u-x, w)} = Q_{-Y}\{S_{\xi}^Y f(u, w)\} = \int_{\mathbb{R}} S_{\xi}^Y f(u, w)K_{-Y}(x, u)dw, \quad (8)$$

where $K_Y(x, u)$ is given by (2) for $B \neq 0$.

It is worth mentioning that when $Y = (A/2B, -1/B, C/2B, 0, 0)$ and multiplying the RHS of (7) the QPST yields linear canonical ST [54]. For $Y = (\cot\alpha/2, -\csc\alpha/2, \cot\alpha/2, 0, 0)$, $\alpha \neq n\pi$ (7) yields the fractional ST [42] when multiplied its RHS by $\sqrt{1 - i \cot\alpha}$. QPST defined in (7) boils down to classical ST when we take $Y = (0, 1, -1, 0, 0)$.

2.2 Q-QPFT

The Q-QPFT is a recent addition to the class of integral transforms which is generalization of the QPFT in the frame of quaternion algebra. Let $Y_s = (A_s, B_s, C_s, D_s, E_s)$, $s = 1, 2$, be the real parameter sets satisfying $B_s \neq 0$, then the Q-QPFT of a signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined as [37,39]

$$Q_{Y_1, Y_2}^{\mathbb{H}}\{f\}(\mathbf{u}) = \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, u_1)f(\mathbf{x})K_{Y_2}^j(x_2, u_2)d\mathbf{x}, \quad (9)$$

and its inverse is given by

$$f(\mathbf{x}) = Q_{Y_1, Y_2}^{-1}\{Q_{Y_1, Y_2}^{\mathbb{H}}[f]\}(\mathbf{x}) = \int_{\mathbb{R}^2} \overline{K_{Y_1}^l(u_1, x_1)}Q_{Y_1, Y_2}^{\mathbb{H}}[f](\mathbf{u})\overline{K_{Y_2}^j(u_2, x_2)}d\mathbf{u}, \quad (10)$$

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and the kernel signals

$$K_{Y_1}^l(x_1, u_1) = \frac{1}{\sqrt{2\pi}} e^{i(A_1 x_1^2 + B_1 x_1 u_1 + C_1 u_1^2 + D_1 u_1 + E_1 u_1)} \quad (11)$$

and

$$K_{Y_2}^j(x_2, u_2) = \frac{1}{\sqrt{2\pi}} e^{j(A_2 x_2^2 + B_2 x_2 u_2 + C_2 u_2^2 + D_2 u_2 + E_2 u_2)}, \quad (12)$$

respectively.

3 The novel Q-QPST

The QPST is a five-parameter linear integral transformation and includes the traditional Stockwell spectrum, the linear canonical Stockwell spectrum, and the fractional Stockwell spectrum, as its special cases. Many domains, including detection, parameter estimation, filter design, and non-stationary signal representation, use QPST extensively. In this section, we will look at applying the quaternion algebra to generalize the QPST. The Q-QPST is the name given to this extension. Additionally, a number of their fundamental qualities are looked into.

By substituting the kernel of the Q-QPFT for the kernel of the QPFT in the classical definition of QPST, we can derive a definition of the Q-QPST based on the definition of the quadratic phase S-transform (QPST).

Definition 2. For a non-zero quaternion window function $\xi \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$ satisfying

$$\int_{\mathbb{R}^2} \xi(\mathbf{x}, \mathbf{w}) d\mathbf{x} = 1$$

the 2D Q-QPST of quaternion signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ with respect to ξ is defined by

$${}^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) = \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} K_{Y_2}^j(x_2, w_2) d\mathbf{x}, \quad (13)$$

where $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$, $\mathbf{x} = (x_1, x_2)$, $Y_s = (A_s, B_s, C_s, D_s, E_s) \in \mathbb{R}^2$ be a parameter satisfying $B_s \neq 0$ and the kernel signals

$$K_{Y_1}^l(x_1, w_1) = \frac{1}{\sqrt{2\pi}} e^{i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} \quad (14)$$

and

$$K_{Y_2}^j(x_2, w_2) = \frac{1}{\sqrt{2\pi}} e^{j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)}, \quad (15)$$

respectively.

It is noteworthy that certain new time-frequency transformations that are not yet documented in the public literature are created by the Q-QPST:

- For $Y = (A/2B, -1/B, C/2B, 0, 0)$, it yields quaternion linear canonical ST.
- For $Y = (\cot\alpha/2, -\csc\alpha/2, \cot\alpha/2, 0, 0)$, $\alpha \neq n\pi$, we can obtain a novel quaternion fractional ST.
- For $Y_s = (0, 1, -1, 0, 0)$, the Q-QPST boils down to quaternion ST.

From Definition 2, we have

$${}^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) = Q_{Y_1, Y_2}^H \{f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})}\}(\mathbf{w}, \mathbf{u}). \quad (16)$$

Applying inverse Q-QPT to (16), we have

$$f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} = Q_{Y_1, Y_2}^{-1} \{ {}^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) \} = \int_{\mathbb{R}^2} K_{Y_1}^{-l}(x_1, w_1) {}^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) K_{Y_2}^{-j}(x_2, w_2) d\mathbf{w}. \quad (17)$$

Our goal in the follow-up is to examine the essential characteristics of the suggested Q-QPST as stated in (13).

3.1 Properties of the novel Q-QPST

We cover a number of the Q-QPST's fundamental characteristics in this subsection. These characteristics are crucial to the representation of signals. With a few adjustments in the quaternion windowed QPFT, the majority of the quaternion ST's characteristics can be determined in the Q-QPST domain. Nonetheless, it is

evident that the Q-QPST's attributes, such as shift, modulation, and so forth, differ from those of the quaternion short-time quadratic phase Fourier analysis.

Property 1. ${}^{Y_1, Y_2}S_{\xi}^H$ acts as a multiplication operator if $\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})$ independent of $\mathbf{x}$.

Proof. Since $\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})$ independent of $\mathbf{x}$ therefore let $\xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) = H(\mathbf{w})$ then from (13), we have

$$\begin{aligned} {}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) &= \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x}) H(\mathbf{w}) K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= H(\mathbf{w}) \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x}) K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= H(\mathbf{w}) Q_{Y_1, Y_2}^H[f](\mathbf{w}). \end{aligned} \quad (18)$$

So ${}^{Y_1, Y_2}S_{\xi}^H$ is a multiplication operator. $\square$

Note: If we take $\xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) = H(\mathbf{w}) = 1$ then (18) reduces to Q-QPFT as

$${}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) = Q_{Y_1, Y_2}^H[f](\mathbf{w}).$$

Remark 1. If $\xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) = H(\mathbf{x})$, i.e ξ is dependent on $\mathbf{x}$ alone, then from (18), we have

$$\begin{aligned} {}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) &= \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x}) H(\mathbf{x}) K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) G(\mathbf{x}) K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= Q_{Y_1, Y_2}^H[G](\mathbf{w}), \end{aligned}$$

where $G(\mathbf{x}) = f(\mathbf{x})H(\mathbf{x})$; thus, we see Q-QPST reduces to Q-QPFT.

Property 2. For $\xi \in L^2(\mathbb{R}^2, \mathbb{H})$ satisfying $\int_{\mathbb{R}^2} \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) d\mathbf{u} = 1$ then

$$\int_{\mathbb{R}^2} {}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) d\mathbf{u} = Q_{Y_1, Y_2}^H[f](\mathbf{w}).$$

Proof. From (13), we have

$$\begin{aligned} \int_{\mathbb{R}^2} {}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) d\mathbf{u} &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} K_{Y_2}^j(x_2, w_2) d\mathbf{u} d\mathbf{x} \\ &= \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x}) K_{Y_2}^j(x_2, w_2) \int_{\mathbb{R}^2} \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} d\mathbf{u} d\mathbf{x} \\ &= \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x}) K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= Q_{Y_1, Y_2}^H[f](\mathbf{w}), \end{aligned}$$

which completes proof. $\square$

Property 3. (Linearity) Let ξ be a non-zero quaternion window function in $L^2(\mathbb{R}^2, \mathbb{H})$ and $f_n \in L^2(\mathbb{R}^2, \mathbb{H})$, $n \in \mathbb{N}$, then the following holds:

$${}^{Y_1, Y_2}S_{\xi}^H \left[\sum_{n \in \mathbb{N}} a_n f_n \right](\mathbf{u}, \mathbf{w}) = \sum_{n \in \mathbb{N}} a_n {}^{Y_1, Y_2}S_{\xi}^H[f_n](\mathbf{u}, \mathbf{w}), \quad a_n \in \mathbb{H} \quad (19)$$

Proof. We avoid proof as it follows directly from Definition 1. $\square$

Property 4. (Parity) Let $\xi, f \in L^2(\mathbb{R}^2, \mathbb{H})$ where ξ is a non-zero quaternion window function, then we have

$$Y_1, Y_2 S_{P\xi}^H[Pf](\mathbf{u}, \mathbf{w}) = Y'_1, Y'_2 S_{P\xi}^H[f](-\mathbf{u}, -\mathbf{w}), \quad (20)$$

where $P\xi(\mathbf{x}) = \xi(-\mathbf{x})$ and $Y'_s = (A_s, B_s, C_s, -D_s, -E_s)$, $s = 1, 2$.

Proof. From (13), we obtain

$$\begin{aligned} & Y_1, Y_2 S_{P\xi}^H[Pf](\mathbf{u}, \mathbf{w})[Pf](\mathbf{u}, \mathbf{w}) \\ &= \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) Pf(\mathbf{x}) \overline{P\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} f(-\mathbf{x}) \overline{\xi(-\mathbf{u} - (-\mathbf{x}), -\mathbf{w})} \frac{1}{\sqrt{2\pi}} e^{j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i[A_1(-x_1)^2 + B_1(-x_1)(-w_1) + C_1(-w_1)^2 + (-D_1)(-x_1) + (-E_1)(-w_1)]} f(-\mathbf{x}) \overline{\xi(-\mathbf{u} - (-\mathbf{x}), -\mathbf{w})} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j[A_2(-x_2)^2 + B_2(-x_2)(-w_2) + C_2(-w_2)^2 + (-D_2)(-x_2) + (-E_2)(-w_2)]} \\ &= Y'_1, Y'_2 S_{\xi}^H[f](\mathbf{u}, \mathbf{w}), \quad \text{where } Y'_s = (A_s, B_s, C_s, -D_s, -E_s), \quad s = 1, 2. \end{aligned} \quad (21)$$

Property 5. (Shift) Let $f, \xi \in L^2(\mathbb{R}^2, \mathbb{H})$ where ξ is a non-zero quaternion window function, then the following equation follows:

$$Y_1, Y_2 S_{\xi}^H[f(\mathbf{x} - \mathbf{a})](\mathbf{u}, \mathbf{w}) = e^{i(A_1 a_1^2 + B_1 a_1 w_1 + D_1 a_1)} Y_1, Y_2 S_{\xi}^H[F](\mathbf{u} - \mathbf{a}, \mathbf{w}) e^{j(A_2 a_2^2 + B_2 a_2 w_2 + D_2 a_2)}, \quad (22)$$

where $F(\mathbf{t}) = e^{i2A_1 a_1 t_1} f(\mathbf{t}) e^{j2A_2 a_2 t_2}$

Proof. With the help of (13), we have

$$Y_1, Y_2 S_{\xi}^H[f(\mathbf{x} - \mathbf{a})](\mathbf{u}, \mathbf{w}) = \int_{\mathbb{R}^2} K_{Y_1}^l(x_1, w_1) f(\mathbf{x} - \mathbf{a}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} K_{Y_2}^j(x_2, w_2) d\mathbf{x}. \quad (23)$$

Taking the change of variable $\mathbf{t} = \mathbf{x} - \mathbf{a}$ in (23), we obtain

$$\begin{aligned} & Y_1, Y_2 S_{\xi}^H[f(\mathbf{x} - \mathbf{a})](\mathbf{u}, \mathbf{w}) \\ &= \int_{\mathbb{R}^2} K_{Y_1}^l(t_1 + a_1, w_1) f(\mathbf{t}) \overline{\xi(\mathbf{u} - (\mathbf{t} + \mathbf{a}), \mathbf{w})} K_{Y_2}^j(t_2 + a_2, w_2) d\mathbf{t} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i(A_1(t_1 + a_1)^2 + B_1(t_1 + a_1)w_1 + C_1 w_1^2 + D_1(t_1 + a_1) + E_1 w_1)} f(\mathbf{t}) \overline{\xi((\mathbf{u} - \mathbf{a}) - \mathbf{t}, \mathbf{w})} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j(A_2(t_2 + a_2)^2 + B_2(t_2 + a_2)w_2 + C_2 w_2^2 + D_2(t_2 + a_2) + E_2 w_2)} d\mathbf{t} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i(A_1 t_1^2 + B_1 t_1 w_1 + C_1 w_1^2 + D_1 t_1 + E_1 w_1)} e^{i(A_1 a_1^2 + 2A_1 a_1 t_1 + B_1 a_1 w_1 + D_1 a_1)} f(\mathbf{t}) \overline{\xi((\mathbf{u} - \mathbf{a}) - \mathbf{t}, \mathbf{w})} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j(A_2 t_2^2 + B_2 t_2 w_2 + C_2 w_2^2 + D_2 t_2 + E_2 w_2)} e^{j(A_2 a_2^2 + 2A_2 a_2 t_2 + B_2 a_2 w_2 + D_2 a_2)} d\mathbf{t} \\ &= e^{i(A_1 a_1^2 + B_1 a_1 w_1 + D_1 a_1)} Y_1, Y_2 S_{\xi}^H[F](\mathbf{u} - \mathbf{a}, \mathbf{w}) e^{j(A_2 a_2^2 + B_2 a_2 w_2 + D_2 a_2)} \\ &\quad \times \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i(A_1 t_1^2 + B_1 t_1 w_1 + C_1 w_1^2 + D_1 t_1 + E_1 w_1)} \times \{e^{i2A_1 a_1 t_1} f(\mathbf{t}) e^{j2A_2 a_2 t_2}\} \overline{\xi((\mathbf{u} - \mathbf{a}) - \mathbf{t}, \mathbf{w})} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j(A_2 t_2^2 + B_2 t_2 w_2 + C_2 w_2^2 + D_2 t_2 + E_2 w_2)} d\mathbf{t} e^{j(A_2 a_2^2 + B_2 a_2 w_2 + D_2 a_2)} \\ &= e^{i(A_1 a_1^2 + B_1 a_1 w_1 + D_1 a_1)} Y_1, Y_2 S_{\xi}^H[F](\mathbf{u} - \mathbf{a}, \mathbf{w}) e^{j(A_2 a_2^2 + B_2 a_2 w_2 + D_2 a_2)}. \end{aligned}$$

This completes the proof. $\square$

Property 6. (Modulation) Let $\xi \in L^2(\mathbb{R}^2, \mathbb{H})$ be a non-zero quaternion window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then we have

$$\begin{aligned} Y_1, Y_2 \mathcal{S}_\xi^{\mathbb{H}}[\mathcal{M}_y f](\mathbf{u}, \mathbf{w}) &= e^{-i\left[\frac{C_1}{B_1^2}y_1^2 + 2\frac{C_1}{B_1}y_1w_1 + \frac{E_1}{B_1}y_1\right]} \int_{\mathbb{R}^2} K_{Y_1}^i\left(x_1, w_1 + \frac{y_1}{B_1}\right) f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} \\ &\quad \times K_{Y_2}^j\left(x_2, w_2 + \frac{y_2}{B_2}\right) d\mathbf{x} e^{-j\left[\frac{C_2}{B_2^2}y_2^2 + 2\frac{C_2}{B_2}y_2w_2 + \frac{E_2}{B_2}y_2\right]}, \end{aligned} \quad (24)$$

where $\mathcal{M}_y f(\mathbf{x}) = e^{iY_1x_1} f(\mathbf{x}) e^{jY_2x_2}$

Proof. We have from (13) that,

$$\begin{aligned} Y_1, Y_2 \mathcal{S}_\xi^{\mathbb{H}}[\mathcal{M}_y f](\mathbf{u}, \mathbf{w}) &= \int_{\mathbb{R}^2} K_{Y_1}^i(x_1, w_1) e^{iY_1x_1} f(\mathbf{x}) e^{jY_2x_2} \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i(A_1x_1^2 + B_1x_1w_1 + C_1w_1^2 + D_1x_1 + E_1w_1)} e^{iY_1x_1} f(\mathbf{x}) e^{jY_2x_2} \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j(A_2x_2^2 + B_2x_2w_2 + C_2w_2^2 + D_2x_2 + E_2w_2)} d\mathbf{x} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i(A_1x_1^2 + B_1x_1w_1 + C_1w_1^2 + D_1x_1 + E_1w_1 + Y_1x_1)} f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j(A_2x_2^2 + B_2x_2w_2 + C_2w_2^2 + D_2x_2 + E_2w_2 + Y_2x_2)} d\mathbf{x} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i\left[A_1x_1^2 + B_1x_1\left(w_1 + \frac{y_1}{B_1}\right) + C_1w_1^2 + D_1x_1 + E_1w_1\right]} f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j\left[A_2x_2^2 + B_2x_2\left(w_2 + \frac{y_2}{B_2}\right) + C_2w_2^2 + D_2x_2 + E_2w_2\right]} d\mathbf{x} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{i\left[A_1x_1^2 + B_1x_1\left(w_1 + \frac{y_1}{B_1}\right) + C_1\left(w_1 + \frac{y_1}{B_1}\right)^2 + D_1x_1 + E_1\left(w_1 + \frac{y_1}{B_1}\right)\right]} \\ &\quad \times e^{-i\left[\frac{C_1}{B_1^2}y_1^2 + 2\frac{C_1}{B_1}y_1w_1 + \frac{E_1}{B_1}y_1\right]} f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} e^{-j\left[\frac{C_2}{B_2^2}y_2^2 + 2\frac{C_2}{B_2}y_2w_2 + \frac{E_2}{B_2}y_2\right]} \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{j\left[A_2x_2^2 + B_2x_2\left(w_2 + \frac{y_2}{B_2}\right) + C_2\left(w_2 + \frac{y_2}{B_2}\right)^2 + D_2x_2 + E_2\left(w_2 + \frac{y_2}{B_2}\right)\right]} d\mathbf{x} \\ &= e^{-i\left[\frac{C_1}{B_1^2}y_1^2 + 2\frac{C_1}{B_1}y_1w_1 + \frac{E_1}{B_1}y_1\right]} \int_{\mathbb{R}^2} K_{Y_1}^i\left(x_1, w_1 + \frac{y_1}{B_1}\right) f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} \\ &\quad \times K_{Y_2}^j\left(x_2, w_2 + \frac{y_2}{B_2}\right) d\mathbf{x} e^{-j\left[\frac{C_2}{B_2^2}y_2^2 + 2\frac{C_2}{B_2}y_2w_2 + \frac{E_2}{B_2}y_2\right]}. \end{aligned}$$

Hence, this completes the proof. $\square$

The orthogonality relation for Q-QPST will now be developed, and from this, we will derive the reconstruction formula related to Q-QPST, both of which are essential characteristics for signal analysis. Prior to delving into the orthogonality relation, it is important to understand that for the remainder of the study, we shall assume that

$$\int_{\mathbb{R}^2} |\xi(\mathbf{u}, \mathbf{w})|^2 d\mathbf{u} = \Phi_\xi, \quad 0 < \Phi_\xi < \infty. \quad (25)$$

NOTE: The next two theorems require us to take the scalar component of the quaternions due to their non-commutativity, which we denote by $[q]_0$ for any quaternion q .

Theorem 1. (Orthogonality relation) Let ${}^{Y_1, Y_2}S_{\xi}^H[f]$ be the Q-QPST of any signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ with respect to a non-zero quaternion window function $\xi \in L^2(\mathbb{R}^2, \mathbb{H})$, then we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [{}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) \overline{{}^{Y_1, Y_2}S_{\xi}^H[g](\mathbf{u}, \mathbf{w})}]_0 d\mathbf{w} d\mathbf{u} = \frac{1}{B_1 B_2} [\langle \Phi_{\xi} f, g \rangle]_0. \quad (26)$$

Proof. By the definition of Q-QPST, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [{}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) \overline{{}^{Y_1, Y_2}S_{\xi}^H[g](\mathbf{u}, \mathbf{w})}]_0 d\mathbf{w} d\mathbf{u} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{2\pi}} e^{i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} \right. \\ & \quad \times \frac{1}{\sqrt{2\pi}} e^{j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{-i(A_1 t_1^2 + B_1 t_1 w_1 + C_1 w_1^2 + D_1 t_1 + E_1 w_1)} \overline{g(\mathbf{t})} \\ & \quad \times \xi(\mathbf{u} - \mathbf{t}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-j(A_2 t_2^2 + B_2 t_2 w_2 + C_2 w_2^2 + D_2 t_2 + E_2 w_2)} d\mathbf{t} \Big] d\mathbf{x} d\mathbf{w} d\mathbf{u} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{2\pi}} e^{i[A_1(x_1^2 - t_1^2) + D_1(x_1 - t_1)]} f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} \frac{1}{\sqrt{2\pi}} e^{j[A_2(x_2^2 - t_2^2) + D_2(x_2 - t_2)]} \right. \\ & \quad \times \frac{1}{\sqrt{2\pi}} e^{iB_1 w_1(x_1 - t_1)} \overline{g(\mathbf{t})} \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{jB_2 w_2(x_2 - t_2)} d\mathbf{w} \Big] d\mathbf{t} d\mathbf{x} d\mathbf{u} \\ &= \frac{1}{B_1 B_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} e^{i[A_1(x_1^2 - t_1^2) + D_1(x_1 - t_1)]} f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} e^{j[A_2(x_2^2 - t_2^2) + D_2(x_2 - t_2)]} \right. \\ & \quad \times \overline{g(\mathbf{t})} \xi(\mathbf{u} - \mathbf{t}, \mathbf{w}) \delta(\mathbf{t} - \mathbf{x}) d\mathbf{x} \Big] d\mathbf{t} d\mathbf{u} \\ &= \frac{1}{B_1 B_2} \int_{\mathbb{R}^4} [f(\mathbf{x}) \overline{g(\mathbf{t})} \xi(\mathbf{u} - \mathbf{t}, \mathbf{w}) \xi(\mathbf{u} - \mathbf{t}, \mathbf{w})]_0 d\mathbf{t} d\mathbf{u} \\ &= \frac{1}{B_1 B_2} \left[\int_{\mathbb{R}^2} f(\mathbf{x}) \overline{g(\mathbf{t})} d\mathbf{t} \int_{\mathbb{R}^2} |\xi(\mathbf{u} - \mathbf{t}, \mathbf{w})|^2 d\mathbf{u} \right]_0 \\ &= \frac{1}{B_1 B_2} \left[\int_{\mathbb{R}^2} \Phi_{\xi} f(\mathbf{x}) \overline{g(\mathbf{t})} d\mathbf{t} \right]_0 \\ &= \frac{1}{B_1 B_2} [\langle \Phi_{\xi} f, g \rangle]_0. \end{aligned}$$

Thus, the proof is completed. $\square$

Remark 2. If we take $f = g$ in (26), Theorem 1 takes the form

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |{}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} d\mathbf{u} = \frac{\Phi_{\xi}}{B_1 B_2} \|f\|^2. \quad (27)$$

Theorem 2. (Reconstruction formula) Every signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ can be reconstructed from the transformed signal ${}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w})$ by the formula

$$\begin{aligned} f(\mathbf{x}) &= \frac{B_1 B_2}{\Phi_{\xi}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{2\pi}} e^{-i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} {}^{Y_1, Y_2}S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) \right. \\ & \quad \times \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} \Big] d\mathbf{w} d\mathbf{u}, \end{aligned} \quad (28)$$

where $\xi \in L^2(\mathbb{R}^2, \mathbb{H})$ is a quaternion window function that satisfies (25).

Proof. By the virtue of Theorem 1, we can write

$$\begin{aligned}
 \frac{1}{B_1 B_2} [\langle f \Phi_\xi, g \rangle]_0 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [Y_1, Y_2 S_\xi^H[f](\mathbf{u}, \mathbf{w}) \overline{Y_1, Y_2 S_\xi^H[g](\mathbf{u}, \mathbf{w})}]_0 d\mathbf{w} d\mathbf{u} \\
 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[Y_1, Y_2 S_\xi^H[f](\mathbf{u}, \mathbf{w}) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{-i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} \overline{g(\mathbf{x})} \right. \\
 &\quad \times \left. \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} d\mathbf{x} \right] d\mathbf{w} d\mathbf{u} \\
 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[Y_1, Y_2 S_\xi^H[f](\mathbf{u}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} \right. \\
 &\quad \times \left. \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} \overline{g(\mathbf{x})} \right] d\mathbf{w} d\mathbf{u} d\mathbf{x} \\
 &= \left[\left\langle \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Y_1, Y_2 S_\xi^H[f](\mathbf{u}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} \right. \right. \\
 &\quad \times \left. \left. \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} d\mathbf{w} d\mathbf{u}, g \right\rangle \right]_0
 \end{aligned}$$

Since the above equation is valid for every $g \in L^2(\mathbb{R}^2, \mathbb{H})$, we can write

$$\begin{aligned}
 f(\mathbf{x}) \Phi_\xi &= B_1 B_2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{2\pi}} e^{-i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} Y_1, Y_2 S_\xi^H[f](\mathbf{u}, \mathbf{w}) \right. \\
 &\quad \times \left. \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} \right] d\mathbf{w} d\mathbf{u},
 \end{aligned}$$

which yields

$$\begin{aligned}
 f(\mathbf{x}) &= \frac{B_1 B_2}{\Phi_\xi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{2\pi}} e^{-i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1^2 + D_1 x_1 + E_1 w_1)} Y_1, Y_2 S_\xi^H[f](\mathbf{u}, \mathbf{w}) \right. \\
 &\quad \times \left. \xi(\mathbf{u} - \mathbf{x}, \mathbf{w}) \frac{1}{\sqrt{2\pi}} e^{-j(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2^2 + D_2 x_2 + E_2 w_2)} \right] d\mathbf{w} d\mathbf{u}.
 \end{aligned}$$

Thus, the proof is completed. $\square$

4 Uncertainty principles for the novel Q-QPST

The harmonic analysis's classical uncertainty principle asserts that a non-trivial function cannot be sharply localized concurrently with its Fourier transform. An uncertainty principle in quantum mechanics states that it is impossible to know an electron's position and velocity at the same time. Stated differently, an increase in positional knowledge results in a decrease in electron velocity or momentum. Harmonic analysis relies heavily on uncertainty principles because they offer a lower bound on the best simultaneous resolution in both the time and frequency domains. For the Q-QPST, as defined by (13), we will create an analogue of the well-known Heisenberg's uncertainty inequality and the accompanying logarithmic uncertainty principle in this section. First, we prove the following lemma.

Lemma 1. Let $f, \xi \in L^2(\mathbb{R}^2, \mathbb{H})$, where ξ is a non-zero quaternion window function, then we have $f \in L^2(\mathbb{R}^2, \mathbb{H})$, we have

$$\Phi_\xi \int_{\mathbb{R}^2} x_s^2 |f(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x_s^2 |Q_{Y_1, Y_2}^{-1} \{S_\xi^{H_1, H_2}[f](\mathbf{u}, \mathbf{w})\}(\mathbf{x})|^2 d\mathbf{x} d\mathbf{u}, \quad (29)$$

where $s = 1, 2$.

Proof. We avoided the proof as it follows by Theorems 1 and 2. $\square$

Now, we can establish the Heisenberg-type inequalities for the proposed Q-QPST as defined by (13).

Theorem 3. Let ${}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f]$ be the Q-QPST of any signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$, of a signal f with respect to the non-zero quaternion window function $\Psi \in L^2(\mathbb{R}^2, \mathbb{H})$, then we have

$$\left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} w_s^2 |{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{u} d\mathbf{w} \right)^{1/2} \left(\int_{\mathbb{R}^2} x_s^2 |f(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \geq \frac{\sqrt{\Phi_\xi}}{2B_s^2} \|f\|^2 \quad (30)$$

where $s = 1, 2$.

Proof. Heisenberg's inequality for the Q-QPFT can be written as [37]

$$\int_{\mathbb{R}^2} x_s^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} w_s^2 |Q_{Y_1, Y_2}^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} \geq \frac{1}{4B_s^2} \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} \right)^2. \quad (31)$$

Equation (31) can be rewritten as

$$\int_{\mathbb{R}^2} x_s^2 |Q_{Y_1, Y_2}^{-1} \{Q_{Y_1, Y_2}^{H_1, H_2}[f]\}|^2 d\mathbf{x} \int_{\mathbb{R}^2} w_s^2 |Q_{Y_1, Y_2}^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} \geq \frac{1}{4B_s^2} \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} \right)^2. \quad (32)$$

Now, by virtue of Plancherel's theorem for the Q-QPFT, we have from (32)

$$\int_{\mathbb{R}^2} x_s^2 |Q_{Y_1, Y_2}^{-1} \{Q_{Y_1, Y_2}^{H_1, H_2}[f]\}|^2 d\mathbf{x} \int_{\mathbb{R}^2} w_s^2 |Q_{Y_1, Y_2}^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} \geq \left(\frac{1}{2B_s} \int_{\mathbb{R}^2} |Q_{Y_1, Y_2}^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} \right)^2. \quad (33)$$

Since ${}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f] \in L^2(\mathbb{R}^2, \mathbb{H})$, replacing $Q_{Y_1, Y_2}^{H_1, H_2}[f]$ by ${}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f]$, (33) yields

$$\int_{\mathbb{R}^2} x_s^2 |Q_{Y_1, Y_2}^{-1} \{{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{u}, \mathbf{w})\}|^2 d\mathbf{x} \int_{\mathbb{R}^2} w_s^2 |{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} \geq \left(\frac{1}{2B_s} \int_{\mathbb{R}^2} |{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} \right)^2. \quad (34)$$

On taking square root to both sides of (34) and integrating with respect to $d\mathbf{u}$, we have

$$\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} x_s^2 |Q_{Y_1, Y_2}^{-1} \{{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{u}, \mathbf{w})\}|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\mathbb{R}^2} w_s^2 |{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} \right)^{1/2} d\mathbf{u} \geq \frac{1}{2B_s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} d\mathbf{u}. \quad (35)$$

Applying the Cauchy-Schwarz inequality, (35) becomes

$$\left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x_s^2 |Q_{Y_1, Y_2}^{-1} \{{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{u}, \mathbf{w})\}|^2 d\mathbf{x} d\mathbf{u} \right)^{1/2} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} w_s^2 |{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} d\mathbf{u} \right)^{1/2} \geq \frac{1}{2B_s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |{}^{Y_1, Y_2}S_\xi^{H_1, H_2}[f](\mathbf{w})|^2 d\mathbf{w} d\mathbf{u}. \quad (36)$$

Applying Lemma 1 and Remark 2 to the LHS and RHS, respectively of the above inequality, it gives

$$\left(\Phi_{\xi} \int_{\mathbb{R}^2} x_s^2 |f(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\mathbb{R}^2 \mathbb{R}^2} w_s^2 |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} d\mathbf{u} \right)^{1/2} \geq \frac{\Phi_{\xi}}{2B_s^2} \|f\|^2. \quad (37)$$

On further simplifying (37), we obtain

$$\left(\int_{\mathbb{R}^2 \mathbb{R}^2} w_s^2 |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} d\mathbf{u} \right)^{1/2} \left(\int_{\mathbb{R}^2} x_s^2 |f(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \geq \frac{\sqrt{\Phi_{\xi}}}{2B_s^2} \|f\|^2, \quad (38)$$

which completes the proof. $\square$

We now establish the logarithmic uncertainty principle for the Q-QPST as defined by (13).

Theorem 4. For $f, \xi \in S(\mathbb{R}^2, \mathbb{H})$, the Q-QPST $[^{Y_1, Y_2} S_{\xi}^H]$ satisfies the following logarithmic estimate of the uncertainty inequality:

$$|B_1 B_2| \int_{\mathbb{R}^2 \mathbb{R}^2} \ln|\mathbf{w}| |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} d\mathbf{u} + |B_1 B_2| \Phi_{\xi} \int_{\mathbb{R}^2} \ln|\mathbf{x}| |f(\mathbf{x})|^2 d\mathbf{x} \geq (\mathcal{D} - \ln|B_1 B_2|) \Phi_{\xi} \|f\|^2, \quad (39)$$

where $\mathcal{D} = \ln(2\pi^2) - 2\psi(1/2)$ and $\psi = \frac{d}{dt}(\ln(\Gamma(x)))$ and $\Gamma(x)$ is a Gamma function.

Proof. The logarithmic inequality for the Q-QPFT [37] states

$$\int_{\mathbb{R}^2} \ln|\mathbf{w}| |Q_{Y_1, Y_2}^H[f](\mathbf{w})|^2 d\mathbf{w} + \int_{\mathbb{R}^2} \ln|\mathbf{x}| |f(\mathbf{x})|^2 d\mathbf{x} \geq (\mathcal{D} - \ln|B_1 B_2|) \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}. \quad (40)$$

Applying the inversion formula of Q-QPFT to the LHS and Parseval's formula for Q-QPFT to RHS, we obtain from above inequality

$$\begin{aligned} & \int_{\mathbb{R}^2} \ln|\mathbf{x}| |Q_{Y_1, Y_2}^{-1} \{Q_{Y_1, Y_2}^H[f](\mathbf{x})\}|^2 d\mathbf{x} + \int_{\mathbb{R}^2} \ln|\mathbf{w}| |Q_{Y_1, Y_2}^H[f](\mathbf{w})|^2 d\mathbf{w} \\ & \geq (\mathcal{D} - \ln|B_1 B_2|) \int_{\mathbb{R}^2} |Q_{Y_1, Y_2}^H[f](\mathbf{x})|^2 d\mathbf{x}. \end{aligned} \quad (41)$$

Since $Q_{Y_1, Y_2}^H[f]$ and $^{Y_1, Y_2} S_{\xi}^H[f]$ are in $S(\mathbb{R}^2, \mathbb{H})$, we can replace $Q_{Y_1, Y_2}^H[f]$ by $S_{\Psi, Y_1, Y_2}^{\Psi, H}[f]$ on both the sides of (41) to obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \ln|\mathbf{w}| |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} + \int_{\mathbb{R}^2} \ln|\mathbf{x}| |Q_{Y_1, Y_2}^{-1} \{^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})\}(\mathbf{x})|^2 d\mathbf{x} \\ & \geq (\mathcal{D} - \ln|B_1 B_2|) \int_{\mathbb{R}^2} |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w}. \end{aligned} \quad (42)$$

First integrating (42) with respect to $d\mathbf{u}$ and then applying the Fubini theorem, we obtain

$$\begin{aligned} & \iint_{\mathbb{R}^2 \mathbb{R}^2} \ln|\mathbf{w}| |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} d\mathbf{u} + \iint_{\mathbb{R}^2 \mathbb{R}^2} \ln|\mathbf{x}| |Q_{Y_1, Y_2}^{-1} \{^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})\}(\mathbf{x})|^2 d\mathbf{x} d\mathbf{u} \\ & \geq (\mathcal{D} - \ln|B_1 B_2|) \iint_{\mathbb{R}^2 \mathbb{R}^2} |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} d\mathbf{u}. \end{aligned} \quad (43)$$

Now, applying Lemma 1 on LHS and Remark 2 on RHS of (43), we obtain

$$\iint_{\mathbb{R}^2 \mathbb{R}^2} \ln|\mathbf{w}| |^{Y_1, Y_2} S_{\xi}^H[f](\mathbf{u}, \mathbf{w})|^2 d\mathbf{w} d\mathbf{u} + \Phi_{\xi} \int_{\mathbb{R}^2} \ln|\mathbf{x}| |f(\mathbf{x})|^2 d\mathbf{x} \geq (\mathcal{D} - \ln|B_1 B_2|) \frac{\Phi_{\xi} \|f\|^2}{|B_1 B_2|},$$

which completes the proof. $\square$

5 Example and possible applications of the novel Q-QPST

For lucid illustration of the proposed Q-QPST, we present an example:

Example 1. Let us consider a two-dimensional Gaussian quaternion function of the form $f(\mathbf{x}) = e^{-(\tau_1 x_1^2 + \tau_2 x_2^2)}$, for $\tau_1, \tau_2 \in \mathbb{R}$ are positive real constants.

Also, the rectangular window function is

$$\xi(\mathbf{x}) = \begin{cases} 1, & \text{if } |x_1| < \frac{1}{2}, |x_2| < \frac{1}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$

Then, the Q-QPST of f with respect to the above window function is given by

$$\begin{aligned} Y_1, Y_2 S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) &= \int_{\mathbb{R}^2} K_{Y_1}^t(x_1, w_1) f(\mathbf{x}) \overline{\xi(\mathbf{u} - \mathbf{x}, \mathbf{w})} K_{Y_2}^j(x_2, w_2) d\mathbf{x} \\ &= \frac{1}{2\pi} \int_{x_1=-1/2}^{x_1=1/2} \int_{x_2=-1/2}^{x_2=1/2} e^{i(A_1 x_1^2 + B_1 x_1 w_1 + C_1 w_1 + D_1 x_1 + E_1 w_1) - \tau_1 x_1^2} e^{i(A_2 x_2^2 + B_2 x_2 w_2 + C_2 w_2 + D_2 x_2 + E_2 w_2) - \tau_2 x_2^2} dt \\ &= \frac{1}{2\pi} e^{i(C_1 w_1^2 + E_1 w_1)} \int_{x_1=-1/2}^{x_1=1/2} e^{i(A_1 + i\tau_1)x_1^2 + B_1 x_1 w_1 + C_1 w_1 + D_1 x_1} dx_1 \times \int_{x_2=-1/2}^{x_2=1/2} e^{j(A_2 + j\tau_2)x_2^2 + B_2 x_2 w_2 + C_2 w_2 + D_2 x_2} dx_2 \\ &\quad \times e^{j(C_2 w_2^2 + E_2 w_2)}. \end{aligned} \quad (44)$$

For simplicity, we choose $\tau_1 = \iota A_1$ and $\tau_2 = j A_2$, we obtain from (44)

$$\begin{aligned} Y_1, Y_2 S_{\xi}^H[f](\mathbf{u}, \mathbf{w}) &= \frac{1}{2\pi} e^{i(C_1 w_1^2 + E_1 w_1)} \int_{x_1=-1/2}^{x_1=1/2} e^{i(B_1 w_1 + D_1)x_1} dx_1 \int_{x_2=-1/2}^{x_2=1/2} e^{j(B_2 w_2 + D_2)x_2} dx_2 \times e^{j(C_2 w_2^2 + E_2 w_2)} \\ &= \frac{e^{i(B_1 w_1 x_1 + C_1 w_1^2 + D x_1 + E_1 w_1)}}{2\pi(B_1 w_1 + D_1)} \left(e^{i \frac{B_1 w_1 + D_1}{2}} - e^{-i \frac{B_1 w_1 + D_1}{2}} \right) \left(e^{j \frac{B_2 w_2 + D_2}{2}} - e^{-j \frac{B_2 w_2 + D_2}{2}} \right) \frac{e^{j(B_2 w_2 x_2 + C_2 w_2^2 + D x_2 + E_2 w_2)}}{(B_2 w_2 + D_2)}. \end{aligned}$$

The Q-QPST has multiple free parameters and includes the quaternion ST, the quaternion fractional ST, and the quaternion ST as its special cases. The significance of the real parameters Y_s , $s = 1, 2$, used in the construction of the proposed Q-QPST lies in the fact that an appropriate real parameter can be employed to maximize the concentration of the Q-QPST spectrum [12]. The Q-QPST can achieve a reasonable improvement in performance over the current signal processing tools, which include the ST, the linear canonical transform, the fractional ST, and their quaternion counterparts, for representing the LFM signal, which is a crucial non-stationary signal often utilized in radar and sonar systems. This is because the Q-QPST has more degrees of freedom than these existing tools. The Q-QPST can be extremely important in the fields of parameter estimation, filter design, and signal detection.

6 Conclusions

The concept of the Q-QPST, an expansion of the quadratic phase Fourier ST, has been presented, which is believed to analyze quaternion-valued signals with abilities of multi-resolution, multi-angle, multi-scale, and temporal localization. It is particularly suitable for dealing with chirp-like quaternion-valued signals. Prior to establishing certain important conclusions, such as the orthogonality relation and reconstruction formula, we first discussed the fundamental features of the Q-QPST. Most significantly, we obtained the equivalent logarithmic version for the Q-QPST and the stronger version of the accompanying Heisenberg's uncertainty inequality. This will provide room for more research in this field. Our work will contribute to the discovery of stronger and more generalized forms of the inequality, which will transform signal and image processing.

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