Research Article

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Asymptotic study of a nonlinear elliptic boundary Steklov problem on a nanostructure

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Abstract: The present study is related to the existence and the asymptotic behavior of the solution of a nonlinear elliptic Steklov problem imposed on a nanostructure depending on the thickness parameter ε (nano-scale), distributed on the boundary of the domain when the parameter ε goes to 0, under some appropriate conditions on the data involved in the problem. We use epi-convergence method in order to establish the limit behavior by characterizing the weak limits of the energies for the solutions. An intermediate step in the proof provides a homogenization result for the considered structure.

Keywords: asymptotic behavior, homogenization, Steklov condition, epi-convergence method, limit problems

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1 Introduction

In the recent decades, many mathematicians have used the partial differential equations (PDEs) to describe several natural phenomena in different branches of science such as fluid mechanics, biology, medicine and chemistry (see, e.g., [1,2]). Although a large number of mathematical physics methods have been developed to study nonlinear PDEs, the investigation of PDEs with Steklov boundary conditions has attracted much attention due to an efficient description of nonlinear phenomena in fluid mechanics, viscoelasticity, biology, medicine, physics and other areas of science [3,4]. Much efforts have been put in recent years to develop techniques to deal with PDEs at the nanoscale. As a consequence, several ad hoc methods such as Steklov boundary conditions [5,6], Steklov eigenvalue problems [7–10], boundary homogenization [11], also with variable exponent [12], shape optimization method [13,14], multiscale homogenization method [15,16], especially in nanobiology or in nanomedicine [17–21] and other numerical schemes for Steklov problems [7,22] have been formulated.

Bañuolos et al. [7] studied the eigenvalue for the mixed Steklov problem (sloshing phenomena). Zemzemi [18] determined the related inverse problem. Moreover, Girouard et al. [9,10] considered the Large Steklov eigenvalues via homogenization.

However, interested readers can also refer to the following relevant recent papers in the area of nonlinear PDE and their use to describe a series of phenomena in applied sciences either with thin or nanolayer [12,23–28].

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Furthermore, Kuznetsov et al. studied the elliptic nonlinear Steklov problem [29]. It was claimed in [17] that the equation might be relevant to the modeling of nanomedicine. In the last 30 years, the Steklov problem and its various version of eigenvalue were intensively studied in the PDE community due to its many very interesting and remarkable properties: complete homogenization [9], geometric formulations [5,6], and existence of both Neumann and Steklov conditions [7].

Series of interesting results on the homogenization for the Steklov-type equations have been obtained by Girouard and his collaborators [9,10]. To the best of our knowledge, these homogenization results are very interesting (some details can be found in [5,6,11] and references therein). Their results highlight how local structure of the domain affects the solutions of those type of problems. Therefore, several global existence for weak solutions and limit problems to the PDEs was studied by Ait Moussa et al. [30] and oscillating thin layer was presented in [31,32]. On the other hand, the notion of Γ and epi-convergence as a type of convergence for functionals, particularly suitable for the study of variational problems was introduced and studied by De Giorgi [33–36], De Giorgi and Franzoni et al. [37] and references therein [38–44]. This new tool allows us to relate a sequence of minimization problems depending on a parameter ε taken small enough with a limit problem that can possibly have a different nature from the original problems, in terms of energy functionals, functional spaces, physical modelization at nanoscale, etc. Even if there is a difference between the original functional and the limit one in terms of structure, this kind of convergence preserves the notion of minimizers in the limit. In addition, the homogenization of PDEs was studied by some researchers [45,46], and also the homogenization of the p-Laplacian in perforated domains was traited by [47].

Our aim, in this study, is to prove the existence of weak solutions and their limit behavior for the following context: Let us consider a body which occupies a bounded three-dimensional domain, $\Omega \subset \mathbb{R}^3$, with a Lipschitz boundary $\partial\Omega$, composed on a nano-layer B_ε, of mid-surface Σ, where $B_\varepsilon = \{x \in \Omega : |x_3| < \varepsilon^2\}$, and let $\Sigma_\varepsilon = \partial\Omega \cap \partial B_\varepsilon$, $\Gamma_\varepsilon = \partial\Omega \backslash \Sigma_\varepsilon$, and ε be a positive small enough parameter. Let us consider a body occupying the domain Ω, where a very high heat boundary conductivity on Γ_ε is considered. The problem is modeled with the following equations:

$$\begin{cases} -\Delta_p u^\varepsilon = f & \text{in } \Omega, \\ |\nabla u^\varepsilon|^{p-2} \dfrac{\partial u^\varepsilon}{\partial n} + \dfrac{1}{\varepsilon^\alpha} |u^\varepsilon|^{p-2} u^\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ u^\varepsilon = 0 & \text{in } \Gamma_\varepsilon, \end{cases} \tag{P$_\varepsilon$}$$

where the boundary conductivity is expressed by $\frac{1}{\varepsilon^\alpha}$ and the unknown u^ε be the temperature, n be the outward normal to $\partial\Omega$, $p > 1$, $\alpha \geq 0$, Δ_p is p-Laplace operator: $\Delta_p u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\right)$ with $|\nabla u| = \sqrt{\sum_{i=1}^{3} \left|\frac{\partial u}{\partial x_i}\right|^2}$, defined on Sobolev space $W^{1,p}(\Omega)$, and $f \in L^\infty(\Omega)$ (Figure 1).

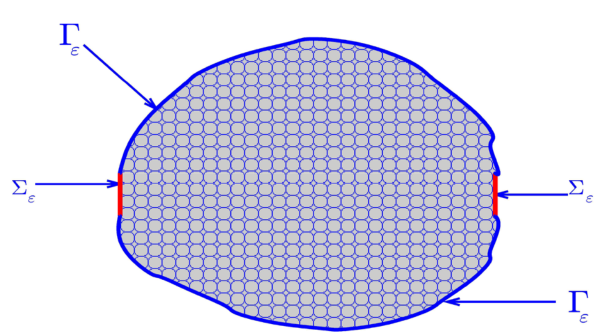

Figure 1: Structure Ω.

The purpose of this work is to describe the asymptotic behavior for constructured sequence when the parameter ε goes to 0. The epi-convergence arguments are used to obtain the limit of this sequence. For the description of this variational convergence well adopted to the asymptotic analysis of minimization problems, we refer to [48], the works proposed by Dal Maso [27,35], Giorgi [33,34], Dal Maso and Longo [36] and Attouch and Picard [23,49].

The remainder of this work is organized as follows. In Section 2, we give some useful notations and some basic definitions. In Section 3, we study the considered problem, the limits behavior of each case are clearly developed, and some special cases are pointed out. Finally, a conclusion is given.

2 Preliminaries

In this section, we will give the notations and some basic notions, that will be used throughout this study. It makes the equations clearer and consequently their interpretation also becomes easier:

- Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$, $\text{meas}(\Omega) > 0$, and $x = (x', x_3)$ where $x' = (x_1, x_2)$,
- $V_\varepsilon = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Gamma_\varepsilon\}$
- $V_0 = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\backslash\partial\Sigma\}$, where $\partial\Sigma$ is the contour of the mid-surface Σ
- C will denote any constant with respect to ε. $\eta(\alpha) = \lim \varepsilon^{1-\alpha}$ where ε tends to 0. (With α is greater than 0).

We will give an efficient notion of operator's sequence convergence, named epi-convergence, which is a special case of the Γ-convergence introduced by De Giorgi (1979) [33]. It is well suited for the asymptotic analysis of sequences of minimization problems.

Definition 1. [48, Definition 1.9] Let $(\mathbb{X}, \tau)$ be a metric space and $(F^\varepsilon)_\varepsilon$ and F be functionals defined on $\mathbb{X}$ and with value in $\mathbb{R} \cup \{+\infty\}$. F^ε epi-converges to F in $(\mathbb{X}, \tau)$, noted $\tau - \text{epilim}_{\varepsilon\to 0} F^\varepsilon = F$, if the following assertions are satisfied

- For all $x \in \mathbb{X}$, there exists x_ε^0, $x_\varepsilon^0 \xrightarrow{\tau} x$ such that $\limsup_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon^0) \leq F(x)$.
- For all $x \in \mathbb{X}$ and all x_ε with $x_\varepsilon \xrightarrow{\tau} x$, $\liminf_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon) \geq F(x)$.

Note the following stability result of the epi-convergence.

Proposition 1. [48, p. 40] *Suppose that F^ε epi-converges to F in $(\mathbb{X}, \tau)$ and that $\Phi : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is τ-continuous. Then, $F^\varepsilon + \Phi$ epi-converges to $F + \Phi$ in $(\mathbb{X}, \tau)$.*

Theorem 2. [48, theorem 1.10] *Suppose that*

(1) *F^ε admits a minimizer on $\mathbb{X}$,*
(2) *The sequence $(\bar{u}^\varepsilon)$ is τ-relatively compact,*
(3) *The sequence F^ε epi-converges to F in this topology τ.*

Then, every cluster point $\bar{u}$ of the sequence $(\bar{u}^\varepsilon)$ minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon'\to 0} F^{\varepsilon'}(\bar{u}^{\varepsilon'}) = F(\bar{u}),$$

if $(\bar{u}^{\varepsilon'})_{\varepsilon'}$ denotes the subsequence of $(\bar{u}^\varepsilon)_\varepsilon$, which converges to $\bar{u}$.

3 Main results

3.1 Study of the problem $(\mathcal{P}^{\varepsilon})$

Note that the problem $(\mathcal{P}^{\varepsilon})$ is equivalent to the minimization problem

$$\inf_{v\in V_0}\{\Phi^{\varepsilon}(v) + G(v)\}, \tag{1}$$

$$\text{where } G(v) = -\int_{\Omega} fv$$

$$\text{and } \Phi^{\varepsilon}(v) = \begin{cases} \frac{1}{p}\int_{\Omega}|\nabla v|^p \mathrm{d}x + \frac{1}{p\varepsilon^{\alpha}}\int_{\Sigma_\varepsilon}|v|^p \mathrm{d}\sigma & \text{if } v \in V_0, \\ +\infty & \text{if } v \in V_\varepsilon \backslash V_0, \end{cases} \tag{2}$$

So, problems $(\mathcal{P}^{\varepsilon})$ and (1) have the same weak solutions in V_ε. Next we will study the minimization problem (1) and the existence of its weak solutions is given in the following proposition.

Proposition 3. *Problem* (1) *admits a unique nontrivial solution* u^{ε} *in* V_0.

The proof of this proposition is based on a classical argument and minmax method, see, e.g., [6,26,50].

In the sequel, we focus on the limit behavior of the solution u^{ε} of problem (1) with respect to the values of α. In fact, we use the epi-convergence method (Definition 1), and to do that, we need to determine the space and the suitable topology. Now, we give in the following lemma, the estimations on ∇u^{ε}.

Lemma 1. *Assuming that there exists a constant* $C > 0$ *and a sequence* $(u^{\varepsilon})_{\varepsilon>0} \subset V_\varepsilon$ *such that* $|\Phi^{\varepsilon}(u^{\varepsilon})| \leq C$. *Then,* $(u^{\varepsilon})_{\varepsilon>0}$ *satisfies to*

$$\int_{\Omega}|\nabla u^{\varepsilon}|^p \mathrm{d}x \leq C, \tag{3}$$

$$\int_{\Sigma_\varepsilon}|u^{\varepsilon}|^p \mathrm{d}\sigma \leq C\varepsilon^{\alpha}. \tag{4}$$

Moreover, u^{ε} *is bounded in* V_0.

Proof. Since u^{ε} satisfies to

$$\frac{1}{p}\int_{\Omega}|\nabla u^{\varepsilon}|^p \mathrm{d}x + \frac{1}{p\varepsilon^{\alpha}}\int_{\Sigma_\varepsilon}|u^{\varepsilon}|^p \mathrm{d}\sigma \leq C.$$

We have (u^{ε}) is a bounded sequence in V_0, if not, we have $||u^{\varepsilon}||_{V_0} \to +\infty$.

Let

$$v_\varepsilon = \frac{u^{\varepsilon}}{||u^{\varepsilon}||_{V_0}},$$

then we have v_ε bounded in V_0, in one hand, there exists $v \in V_0$ such that for a subsequence, noted also v_ε, $v_\varepsilon \rightharpoonup v$ in V_0. In other hand, we have

$$p\frac{\Phi^{\varepsilon}(u^{\varepsilon})}{||u^{\varepsilon}||^p_{V_0}} = \int_{\Omega}|\nabla v_\varepsilon|^p \mathrm{d}x + \frac{1}{\varepsilon^{\alpha}}\int_{\Sigma_\varepsilon}|v_\varepsilon|^p \mathrm{d}\sigma.$$

Now, let us consider the first Steklov eigenvalue of $-\Delta_p$ operator (see for instance [6])

$$\lambda_0 = \inf_{v \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^p \mathrm{d}x}{\int_{\partial\Omega} |v|^p \mathrm{d}\sigma},$$

for a small enough ε, we have $\frac{1}{\varepsilon^\alpha} > \lambda_0$, therefore

$$\int_\Omega |\nabla v_\varepsilon|^p \mathrm{d}x + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} |v_\varepsilon|^p \mathrm{d}\sigma \geq \int_\Omega |\nabla v_\varepsilon|^p \mathrm{d}x + \lambda_0 \int_{\partial\Omega} |v_\varepsilon|^p \mathrm{d}\sigma \tag{5}$$

$$\geq 2\lambda_0 \int_{\partial\Omega} |v_\varepsilon|^p \mathrm{d}\sigma. \tag{6}$$

Since $\Phi^\varepsilon(u^\varepsilon)$ is bounded, we have

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega} |v_\varepsilon|^p \mathrm{d}x = 0,$$

then $v = 0$ on $\partial\Omega$. In other words, we have

$$\int_\Omega |\nabla v_\varepsilon|^p \mathrm{d}x + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} |v_\varepsilon|^p \mathrm{d}\sigma \geq \int_\Omega |\nabla v_\varepsilon|^p \mathrm{d}x. \tag{7}$$

Passing to the limit, we have

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla v_\varepsilon|^p \mathrm{d}x = 0.$$

Consequently,

$$\int_\Omega |\nabla v|^p \mathrm{d}x = 0,$$

then $v = c$ in Ω; however, $v = 0$ on $\partial\Omega$, then $v = 0$. Hence, there is a contradiction with the fact that $||v_\varepsilon||_{V_0} = 1$.

Accordingly, let (u^ε) be a bounded sequence in $W^{1,p}(\Omega)$. Then, we obtain

$$\int_\Omega |\nabla u^\varepsilon|^p \mathrm{d}x \leq C,$$

and

$$\int_{\Sigma_\varepsilon} |u^\varepsilon|^p \mathrm{d}\sigma \leq C\varepsilon^\alpha.$$

□

Remark 1. It is easy to see that the solution of problem (1) satisfies Lemma (1).

According to the real values of α, in the following proposition, we will give some characterization about the behavior of solution $(u^\varepsilon)_\varepsilon$ of problem (1), when ε is close to zero.

Indeed, we need to define an operator, which transforms the functions defined on Σ_ε to the functions defined on Σ, to handle this, we follow the idea of Ait and Messaho [31], let us define the following operator $m^\varepsilon : L^p(\Sigma_\varepsilon) \to L^p(\partial\Sigma)$, by

$$m^\varepsilon u(x') = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(x', x_3)\mathrm{d}x_3, \quad \forall u \in L^p(\Sigma_\varepsilon). \tag{8}$$

Lemma 2. *The operator m^ε defined by* (8) *is linear and bounded to $L^p(\Sigma_\varepsilon)$ in $L^p(\partial\Sigma)$, with norm $\le C\varepsilon^{\frac{-1}{p}}$; moreover, for all $u \in L^p(\Sigma_\varepsilon)$, we have*

$$\int_{\partial\Sigma} |m^\varepsilon u|^p \mathrm{d}\sigma \le C\varepsilon^{-1} \int_{\Sigma_\varepsilon} |u|^p \mathrm{d}\sigma. \tag{9}$$

The proof of this lemma is similar to [31, Lemma 4.2].

Proposition 4. *The solution of the problem* (1), $(u^\varepsilon)_\varepsilon$, *possess a subsequence also denoted by $(u^\varepsilon)_\varepsilon$ weakly convergent toward an element u^* in $W^{1,p}(\Omega)$ satisfying*

$$u^* = 0 \quad on\ \partial\Omega \backslash \partial\Sigma, \tag{10}$$

$$u^*_{|\Sigma} \in L^p(\Sigma), \tag{11}$$

$$\alpha = 1: \ \ u^*_{|\partial\Sigma} \in L^p(\partial\Sigma), \tag{12}$$

$$\alpha > 1: \ \ u^*_{|\partial\Sigma} = 0. \tag{13}$$

Proof. According to remark 1, the sequence u^ε is bounded in $W^{1,p}(\Omega)$, it follows that there exists an element $u^* \in W^{1,p}(\Omega)$ and a subsequence of u^ε, still denoted by u^ε such that $u^\varepsilon \rightharpoonup u^*$ in $W^{1,p}(\Omega)$. Then,

$$u^\varepsilon_{|\partial\Sigma} \rightharpoonup u^*_{|\partial\Sigma} \quad \text{in } L^p(\partial\Sigma),$$

thanks to Lemma 2, we have

$$\int_{\partial\Sigma} |m^\varepsilon u^\varepsilon - u^\varepsilon_{|\partial\Sigma}|^p \mathrm{d}\sigma \le C\varepsilon^p \int_{\Sigma_\varepsilon} |u^\varepsilon|^p \mathrm{d}\sigma,$$

and according to (4), we obtain

$$\int_{\partial\Sigma} |m^\varepsilon u^\varepsilon - u^\varepsilon_{|\partial\Sigma}|^p \le C\varepsilon^{\alpha+p-1}.$$

For $\alpha = 1$, according to estimate (4), the sequence $m^\varepsilon u^\varepsilon$ possesses a subsequence, still denoted by $m^\varepsilon u^\varepsilon$ weakly convergent to an element u^2 in $L^p(\partial\Sigma)$, as $m^\varepsilon u^\varepsilon \rightharpoonup u^*_{|\partial\Sigma}$ in $L^p(\partial\Sigma)$, so one concludes that $m^\varepsilon u^\varepsilon \rightharpoonup u^*_{|\partial\Sigma}$ in $L^p(\partial\Sigma)$ and $u^*_{|\partial\Sigma} = u^2$. Hence, $u^*_{|\partial\Sigma} \in L^p(\partial\Sigma)$.

For $\alpha > 1$, one shows, as in the case $\alpha = 1$ and taking $u^2 = 0$, that $u^*_{|\partial\Sigma} = 0$. □

The limit behavior of problem (1), will be derived and can be proved in the same spirit with the epiconvergence method, (see Definition 1).

3.2 Limit behavior of problem (1)

In this subsection, we will interest, according to the values of α, to find the limit problem of problem (1).

We consider the following energy functional given by

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p + \dfrac{1}{p\varepsilon^\alpha}\int_{\Sigma_\varepsilon} |u|^p, & \forall u \in \mathbb{G}^\alpha \\ +\infty, & \forall u \in W^{1,p}(\Omega)\backslash\mathbb{G}^\alpha. \end{cases} \tag{14}$$

We denote by τ_f the weak topology on $W^{1,p}(\Omega)$ and let

$$V(\partial\Sigma) = \{u \in V_0 : u_{|\partial\Sigma} \in L^p(\partial\Sigma)\},$$
$$V^C(\partial\Sigma) = W_0^{1,p}(\Omega).$$

We show easily that $V(\partial\Sigma)$ is a Banach space endowed with the norm

$$u \mapsto \|u\|_{W^{1,p}(\Omega)}.$$

Let

$$\mathbb{G}^\alpha = \begin{cases} \{u \in V_0 : \eta(\alpha)u\,|_{\partial\Sigma} \in L^p(\partial\Sigma)\} & \text{if } \alpha \le 1, \\ V^C(\partial\Sigma) & \text{if } \alpha > 1. \end{cases}$$

$$\mathbb{D}^\alpha = \begin{cases} \mathcal{D}(\Omega) & \text{if } \alpha \le 1, \\ \{u \in \mathcal{D}(\Omega) : u\,|_{\partial\Sigma} = 0\} & \text{if } \alpha > 1. \end{cases}$$

It is known that $\overline{\mathbb{D}^\alpha} = \mathbb{G}^\alpha$.

Theorem 5. *There exists a functional F^α defined on $W^{1,p}(\Omega)$ with value in $\mathbb{R} \cup \{+\infty\}$ such that $\tau_f - \lim_e \mathcal{F}^\varepsilon = F^\alpha$ in $W^{1,p}(\Omega)$, where the functional F^α is given as follows*:

(1) *If* $0 \le \alpha < 1$:

$$F^\alpha(u) = \frac{1}{p}\int_\Omega |\nabla u|^p, \quad \forall u \in W^{1,p}(\Omega).$$

(2) *If* $\alpha \ge 1$:

$$F^\alpha(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p + \dfrac{2\eta(\alpha)}{p}\displaystyle\int_{\partial\Sigma} |u_{|\partial\Sigma}|^p & \text{if } u \in \mathbb{G}^\alpha, \\ +\infty & \text{if } u \in W^{1,p}(\Omega)\backslash\mathbb{G}^\alpha. \end{cases}$$

Proof. (a) Determination of the upper epi-limit: Let $u \in \mathbb{G}^\alpha \subset W^{1,p}(\Omega)$, there exists a sequence (u^n) in $\mathbb{D}^\alpha$ such that

$$u^n \to u \quad \text{in } \mathbb{G}^\alpha, \text{ when } n \to +\infty.$$

Hence that $u^n \to u$ in $W^{1,p}(\Omega)$.

Let θ be a smooth function satisfying

$$\theta(t) = 1 \text{ if } |t| \le 1, \quad \theta(t) = 0 \text{ if } |t| \ge 2 \quad \text{and} \quad |\theta'(t)| \le 2\ \forall t \in \mathbb{R},$$

and set

$$\theta_\varepsilon(x) = \theta\left(\frac{x_3}{\varepsilon}\right);$$

we define

$$u^{\varepsilon,n} = \theta_\varepsilon(x)u^n{}_{|\partial\Sigma} + (1 - \theta_\varepsilon(x))u^n.$$

It is clear that $u^{\varepsilon,n} \in W^{1,p}(\Omega)$ and $u^{\varepsilon,n} \to u^n$ in $\mathbb{G}^\alpha$, when $\varepsilon \to 0$.

Since

$$\mathcal{F}^\varepsilon(u^{\varepsilon,n}) = \frac{1}{p}\int_\Omega |\nabla u^{\varepsilon,n}|^p + \frac{1}{p\varepsilon^\alpha}\int_{\Sigma_\varepsilon} |u^{\varepsilon,n}|^p,$$

so that

$$\begin{aligned} \mathcal{F}^\varepsilon(u^{\varepsilon,n}) &= \frac{1}{p}\int_{|x_3|>2\varepsilon} |\nabla u^{\varepsilon,n}|^p + \frac{1}{p}\int_{\varepsilon<|x_3|<2\varepsilon} |\nabla u^{\varepsilon,n}|^p + \frac{1}{p\varepsilon^\alpha}\int_{\Sigma_\varepsilon} |u^{\varepsilon,n}|^p \\ &= \frac{1}{p}\int_{|x_3|>2\varepsilon} |\nabla u^n|^p + \frac{1}{p}\int_{\varepsilon<|x_3|<2\varepsilon} |\nabla u^{\varepsilon,n}|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_{\partial\Sigma} |u^n{}_{|\partial\Sigma}|^p. \end{aligned} \tag{15}$$

We check easily that

$$\lim_{\varepsilon\to 0}\left\{\int_{\varepsilon<|x_3|<2\varepsilon}|\nabla u^{\varepsilon,n}|^p\right\}=0. \tag{16}$$

(1) If $\alpha\le 1$: We have $\varepsilon^{1-\alpha}\to\eta(\alpha)$, it follows that

$$\lim_{\varepsilon\to 0}\frac{2\varepsilon^{1-\alpha}}{p}\int_{\partial\Sigma}|u^n{}_{|\partial\Sigma}|^p=\frac{2\eta(\alpha)}{p}\int_{\partial\Sigma}|u^n{}_{|\partial\Sigma}|^p.$$

By passage to the upper limit, one has

$$\limsup_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^{\varepsilon,n})=\limsup_{\varepsilon\to 0}\left(\frac{1}{p}\int_{|x_3|>2\varepsilon}|\nabla u^n|^p+\frac{2\varepsilon^{1-\alpha}}{p}\int_{\partial\Sigma}|u^n{}_{|\partial\Sigma}|^p\right)=\frac{1}{p}\int_\Omega|\nabla u^n|^p+\frac{2\eta(\alpha)}{p}\int_{\partial\Sigma}|u^n{}_{|\partial\Sigma}|^p.$$

(2) If $\alpha>1$: By passage to the upper limit, we then establish

$$\limsup_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^{\varepsilon,n})=\limsup_{\varepsilon\to 0}\left(\frac{1}{p}\int_{|x_3|>2\varepsilon}|\nabla u^n|^p\right)=\frac{1}{p}\int_\Omega|\nabla u^n|^p.$$

Since $u^n\to u$ in $\mathbb{G}^\alpha$, when $n\to+\infty$. Owing to the classical result, diagonalization's lemma [48, Lemma 1.15], there exists a function $n(\varepsilon):\mathbb{R}^+\to\mathbb{N}$ increasing to $+\infty$ when $\varepsilon\to 0$, such that $u^{\varepsilon,n(\varepsilon)}\to u$ in $\mathbb{G}^\alpha$, when $\varepsilon\to 0$. While n approaches $+\infty$, one will have

(1) If $\alpha\ne 1$:

$$\limsup_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^{\varepsilon,n(\varepsilon)})\le\limsup_{n\to+\infty}\limsup_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^{\varepsilon,n})\le\frac{1}{p}\int_\Omega|\nabla u|^p.$$

(2) If $\alpha=1$:

$$\limsup_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^{\varepsilon,n(\varepsilon)})\le\limsup_{n\to+\infty}\limsup_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^{\varepsilon,n})\le\frac{1}{p}\int_\Omega|\nabla u|^p+\frac{2\eta(\alpha)}{p}\int_{\partial\Sigma}|u_{|\partial\Sigma}|^p.$$

If $u\in W^{1,p}(\Omega)\backslash\mathbb{G}^\alpha$, it is clear that, for every $u^\varepsilon\in W^{1,p}(\Omega)$, $u^\varepsilon\rightharpoonup u$ in $W^{1,p}(\Omega)$, one has

$$\limsup_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon)\le+\infty.$$

(b) One is going to determine the lower epi-limit. Let $u\in\mathbb{G}^\alpha$ and (u^ε) be a sequence in $W^{1,p}(\Omega)$ such that $u^\varepsilon\rightharpoonup u$ in $W^{1,p}(\Omega)$, so that

$$\nabla u^\varepsilon\rightharpoonup\nabla u\quad\text{in } L^p(\Omega)^3. \tag{17}$$

(1) If $\alpha\ne 1$: Since

$$\mathcal{F}^\varepsilon(u^\varepsilon)\ge\frac{1}{p}\int_\Omega|\nabla u^\varepsilon|^p.$$

According to (17) and by passage to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon)\ge\frac{1}{p}\int_\Omega|\nabla u|^p.$$

(2) If $\alpha=1$: If $\liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon)=+\infty$, there is nothing to prove, because

$$\frac{1}{p}\int_\Omega|\nabla u|^p+\frac{2\eta(\alpha)}{p}\int_{\partial\Sigma}|u_{|\partial\Sigma}|^p\le+\infty.$$

Otherwise, $\liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence of $\mathcal{F}^\varepsilon(u^\varepsilon)$ still denoted by $\mathcal{F}^\varepsilon(u^\varepsilon)$ and a constant $C > 0$, such that $\mathcal{F}^\varepsilon(u^\varepsilon) \le C$, which implies that

$$\frac{1}{p\varepsilon^\alpha}\int_{\Sigma_\varepsilon}|u^\varepsilon|^p \le C. \tag{18}$$

So u^ε satisfies the hypothesis of the lemma [32, Lemma 3], and according to, and as a result, this last, $m^\varepsilon u^\varepsilon$ is bounded in $L^p(\partial\Sigma)$.

Thus, there exists an element $u_1 \in L^p(\partial\Sigma)$ and a subsequence of $m^\varepsilon u^\varepsilon$, still denoted by $m^\varepsilon u^\varepsilon$, such that $m^\varepsilon u^\varepsilon \rightharpoonup u_1$ in $L^p(\partial\Sigma)$, since $u^\varepsilon{}_{|\partial\Sigma} \rightharpoonup u_{|\partial\Sigma}$ in $L^p(\partial\Sigma)$, and thanks to (9) and (18), we have $m^\varepsilon u^\varepsilon \rightharpoonup u_{|\partial\Sigma}$ in $L^p(\partial\Sigma)$, therefore $u_1 = u_{|\partial\Sigma}$. One has

$$\mathcal{F}^\varepsilon(u^\varepsilon) \ge \frac{1}{p}\int_\Omega|\nabla u^\varepsilon|^p + \frac{1}{p\varepsilon^\alpha}\int_{\Sigma_\varepsilon}|u^\varepsilon|^p \ge \frac{1}{p}\int_\Omega|\nabla u^\varepsilon|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_{\partial\Sigma}|m^\varepsilon u^\varepsilon|^p.$$

Now, by using the sub-differential inequality of

$$v \to \frac{2\varepsilon^{1-\alpha}}{p}\int_{\partial\Sigma}|v|^p, \quad \forall v \in L^p(\partial\Sigma),$$

we obtain

$$\mathcal{F}^\varepsilon(u^\varepsilon) \ge \frac{1}{p}\int_\Omega|\nabla u^\varepsilon|^p + \frac{2\varepsilon^{1-\alpha}}{p}\frac{1}{p}\int_{\partial\Sigma}|u_{|\partial\Sigma}|^p + 2\varepsilon^{1-\alpha}\int_{\partial\Sigma}|u_{|\partial\Sigma}|^{p-2}u_{|\partial\Sigma}(m^\varepsilon u^\varepsilon - u_{|\partial\Sigma}).$$

Consequently, according to (17) and by passage to the lower limit, one obtains

$$\liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon) \ge \frac{1}{p}\int_\Omega|\nabla u|^p + \frac{2\eta(\alpha)}{p}\int_{\partial\Sigma}|u_{|\partial\Sigma}|^p.$$

Let $u \in W^{1,p}(\Omega)\backslash\mathbb{G}^\alpha$ and $u^\varepsilon \in W^{1,p}(\Omega)$, such that $u^\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$:

If $u^\varepsilon \notin G^\alpha$ or $\liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon) = +\infty$, there is nothing to prove.

Now, assume that $u^\varepsilon \in G^\alpha$ and

$$\liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon) < +\infty.$$

So, there exists a constant $C > 0$ and a subsequence of $\mathcal{F}^\varepsilon(u^\varepsilon)$, still denoted by $\mathcal{F}^\varepsilon(u^\varepsilon)$, such that

$$\mathcal{F}^\varepsilon(u^\varepsilon) < C. \tag{19}$$

For $0 \le \alpha < 1$, there is nothing to prove.

Otherwise, one takes the same way as in the case $u \in \mathbb{G}^{\alpha=1}$, we have $m^\varepsilon u^\varepsilon$ bounded in $L^p(\partial\Sigma)$, so there exists an element $u_1 \in L^p(\partial\Sigma)$ and a subsequence of $m^\varepsilon u^\varepsilon$, still denoted by $m^\varepsilon u^\varepsilon$, such that $m^\varepsilon u^\varepsilon \rightharpoonup u_1$ in $L^p(\partial\Sigma)$, since $u^\varepsilon{}_{|\partial\Sigma} \rightharpoonup u_{|\partial\Sigma}$ in $L^p(\partial\Sigma)$, and thanks to (9) and (19), one has $m^\varepsilon u^\varepsilon \rightharpoonup u_{|\partial\Sigma}$ in $L^p(\partial\Sigma)$, so that $u \in \mathbb{G}^\alpha$, which contradicts the fact that $u \notin \mathbb{G}^\alpha$, finally, we obtain

$$\liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon) = +\infty.$$

Hence, the proof of Theorem 5 is complete. □

Remark 2. Note that, from Proposition 1, $\mathcal{F}^\varepsilon + G$ epi-converges to $F^\alpha + G$.

In the following, we are interested in the question of finding the limit problem associated to problem (1), when ε close to zero. From the epi-convergence results (Theorem 2, Proposition 1) and Theorem 5, we have the following result.

Proposition 6. *According to the parameter values of α, there exists $u^* \in W^{1,p}(\Omega)$ satisfying*

$$\begin{aligned} &u^\varepsilon \rightharpoonup u^* \quad \text{in } W^{1,p}(\Omega),\\ &F^\alpha(u^*) + G(u^*) = \inf_{v\in\mathbb{G}^\alpha}\{F^\alpha(v) + G(v)\} \end{aligned}$$

Proof. By applying Lemma 1, the family (u^ε) is bounded in $W^{1,p}(\Omega)$; therefore, it possess a τ_f-cluster point u^* in $W^{1,p}(\Omega)$. And thanks to classical epi-convergence results (Theorem 2), one has u^* is a solution of the limit problem

$$\inf_{v\in W^{1,p}(\Omega)}\{F^\alpha(v) + G(v)\}. \tag{20}$$

Since for $\alpha > 1$, F^α equals $+\infty$ on $W^{1,p}(\Omega)\backslash\mathbb{G}^\alpha$, (20) becomes

$$\inf_{v\in\mathbb{G}^\alpha}\{F^\alpha(v) + G(v)\}. \tag{21}$$

According to the uniqueness of solutions of problem (20), u^ε admits an unique τ_f-cluster point u^*, and therefore $u^\varepsilon \rightharpoonup u^*$ in $W^{1,p}(\Omega)$. □

4 Conclusion

In this study, the epi-convergence method for the Steklov problem on a nanostructure has been successfully applied and employed efficiently to this type of problem. Also, we obtained the limits problem of our equation and we construct the sequence that describes the asymptotic behavior when the parameter ε goes to 0. Therefore, the limit of this sequence is obtained using epi-convergence arguments. From the above analysis, it manifests that the proposed analysis is highly effective and reliable for constructing arguments to generate the corresponding limit problems. In future works, we extend this proposed tool to investigate and examine a nonlinear elliptic problem with Steklov boundary conditions involving the variable, fractional exponent of the P-Laplacian operator.

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