Research Article

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Pseudo compact almost automorphic solutions to a family of delay differential equations

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Abstract: In this article, a family of delay differential equations with pseudo compact almost automorphic coefficients is considered. By introducing a concept of Bi-pseudo compact almost automorphic functions and establishing the properties of these functions, and using Halanay's inequality and Banach fixed point theorem, some results on the existence, uniqueness and global exponential stability of pseudo compact automorphic solutions of the equations are obtained. Our results extend some recent works. Moreover, an example is given to illustrate the validity of our results.

Keywords: delay differential equations, pseudo compact almost automorphic solutions, Bi-pseudo compact almost automorphic, global exponential stability

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1 Introduction

The study of almost periodic type solutions to differential equations have attracted many researchers (see, e.g., [1–10] and the references therein). The space of pseudo compact almost automorphic functions (p.k.a.a.) includes pseudo almost periodic functions (p.a.p.), almost periodic functions (a.p.) and compact almost automorphic functions (k.a.a.). However, as far as we know, there are few results on compact almost automorphic type solutions to differential equations.

In the study by Abbas et al. [11], by applying the properties of p.k.a.a. and bi-almost automorphic (Bi-a.a.), and the Banach fixed point theorem, Abbas et al. obtain the results of p.k.a.a. solutions to the following delay differential equation:

$$\dot{u}(t) = -\alpha(t)u(t) + \sum_{i=1}^{n}\beta_i(t)F_i(\lambda_i(t)u(t-\tau_i(t))) + b(t)H(u(t)), \tag{1.1}$$

where the coefficient function $\alpha : R \to R$ does not consider ergodic perturbation, i.e., α is positive a.a.

Lots of results on equation (1.1) and its analogue equations do not consider the ergodic perturbation of coefficients, such as [2], by applying the properties of p.a.p. and the Banach fixed point theorem, Chérif obtain the results of p.a.p. solutions to Nicholson's blowflies model with mixed delays of the form:

$$\dot{u}(t) = -\alpha(t)u(t) + \sum_{i=1}^{n}\beta_i(t)u(t-\tau_i)e^{-\omega_i(t)u(t-\tau_i)} - b(t)u(t-\sigma) + \beta_0(t)\int_{-\tau}^{0}K(t,s)u(t+s)e^{-u(t+s)}\mathrm{d}s, \tag{1.2}$$

where $\alpha : R \to R$ is positive a.p.

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It is realistic to consider that the coefficients have an ergodic perturbation [12]. Coronel et al. [13] remark that in the context of differential equations with almost automorphic coefficients, the Bi-almost automorphic (Bi-a.a.) property of the Green function is fundamental to prove the existence of solutions of the same class. Naturally, it is necessary to consider the Green function with Bi-compact almost automorphic (Bi-k.a.a.) property and Bi-ergodic components to prove the existence of solutions to differential equations with pseudo compact almost automorphic coefficients. On the other hand, it is meaningful to consider mixed delays with ergodic components.

Motivated by the aforementioned results, we introduce the concept of Bi-pseudo compact almost automorphic functions (Bi-p.k.a.a.) and establish some basic properties for these functions. Then by applying the properties of p.k.a.a. and Bi-p.k.a.a., and the Banach fixed point theorem, we study the p.k.a.a. solutions to the following delay differential equation:

$$\dot{u}(t) = -\alpha(t)u(t) + \sum_{i=1}^{n}\beta_i(t)f_i(t, u(t-\tau_i(t))) + b(t)H(u(t-\sigma)) + \beta_0(t)\int_{-\tau}^{0}K(t,s)f_0(t, u(t+s))\mathrm{d}s, \tag{1.3}$$

where $\alpha : R \to R$ is positive p.k.a.a.

It is clear that (1.3) includes both (1.1) and (1.2). In our results, by using the properties of p.k.a.a and Bi-p.k.a.a., the conditions for some useful results and the existence of p.k.a.a. solutions to equation (1.3) are weaker than those in the existing literature (see Remarks 2.6, 2.15, and 3.7). Moreover, by using Halanay's inequality, the globally exponential stability of the positive p.k.a.a. solution to equation (1.3) does not need to add a condition to the existence conditions as in [9] (Remark 4.6). Therefore, our results extend some known results.

The outline of this article is as follows. In Section 2, we recall the concept and properties of pseudo compact almost automorphic functions, define Bi-pseudo compact almost automorphic functions, and establish some properties of these functions. Sections 3 and 4 contain some results on the existence, uniqueness, and global exponential stability of pseudo compact almost automorphic solution to equation (1.3). In Section 5, we provide an example to illustrate the validity of our results. Finally, a conclusion is drawn in Section 6.

2 Preliminaries

Let $(V, \|\cdot\|)$, $(U, \|\cdot\|)$ be two Banach spaces, and $\mathrm{BC}(R, V)$ (resp. $\mathrm{BC}(R \times U, V)$) denotes the space of bounded continuous functions $f : R \to V$(resp. $f : R \times U \to V$). Denote $\|f\|_\infty := \sup_{t\in R}\|f(t)\|$, $\|f\|_\mu := \sup_{-\mu\le t\le 0}\|f(t)\|$ and $R^+ := [0, \infty)$.

Definition 2.1. [11,14–17] (i) A continuous function $f : R \to V$ is said to be almost automorphic (a.a.) if for any sequence $\{\xi_n'\}_{n=1}^\infty \subset R$, there exists a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi_n'\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} f(t + \xi_n) = \tilde{f}(t)$$

is well defined for all $t \in R$, and

$$\lim_{n\to\infty} \tilde{f}(t - \xi_n) = f(t)$$

for all $t \in R$. Denote by $AA(R, V)$ the space of all such functions. A continuous function $f : R \times U \to V$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in R$ uniformly for all x in any bounded subset of U. Denote by $AA(R \times U, V)$ the space of all such functions.

If the aforementioned limits hold uniformly in compact subsets of R, then f is said to be compact almost automorphic (k.a.a.). Denote by $\mathrm{KAA}(R, V)$, the space of all such functions. A continuous function $f : R \times U \to V$ is said to be compact almost automorphic if $f(t, x)$ is compact almost automorphic in $t \in R$ uniformly for all x in any bounded subset of U. Denote by $\mathrm{KAA}(R \times U, V)$ the space of all such functions.

(ii) A continuous function $f: R \to V$ (resp. $f: R \times U \to V$) is said to be pseudo almost automorphic (p.a.a.) if f is decomposed as follows:

$$f = f_1 + f_2,$$

where $f_1 \in AA(R, V)$ (resp. $AA(R \times U, V)$) and $f_2 \in \mathrm{PAP}_0(R, V)$ (resp. $f_2 \in \mathrm{PAP}_0(R \times U, V)$), which is defined by

$$\mathrm{PAP}_0(R, V) = \left\{ f_2 \in \mathrm{BC}(R, V) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f_2(s)\| \mathrm{d}s = 0 \right\}$$

$$\left(\text{resp. } \mathrm{PAP}_0(R \times U, V) = \left\{ f_2 \in \mathrm{BC}(R \times U, V) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f_2(s, x)\| \mathrm{d}s = 0 \text{ uniformly for all x in any bounded subset of } U \right\} \right).$$

Denote by $\mathrm{PAA}(R, V)$ (resp. $\mathrm{PAA}(R \times U, V)$) the space of all such functions. The functions f_1 and f_2 are, respectively, called the almost automorphic and the ergodic components of f.

(iii) A continuous function $f: R \to V$ (resp. $f: R \times U \to V$) is said to be pseudo compact almost automorphic (p.k.a.a.) if f is decomposed as follows:

$$f = f_1 + f_2,$$

where $f_1 \in \mathrm{KAA}(R, V)$ (resp. $\mathrm{KAA}(R \times U, V)$) and $f_2 \in \mathrm{PAP}_0(R, V)$ (resp. $f_2 \in \mathrm{PAP}_0(R \times U, V)$). Denote by $\mathrm{PKAA}(R, V)$ (resp. $\mathrm{PKAA}(R \times U, V)$) the space of all such functions. The functions f_1 and f_2 are, respectively, called the compact almost automorphic and the ergodic components of f.

Remark 2.2. Let $f = f_1 + f_2 \in \mathrm{PKAA}(R, V)$ with $f_1 \in \mathrm{KAA}(R, V)$ and $f_2 \in \mathrm{PAP}_0(R, V)$. In view of the properties of p.a.a., it follows immediately that f is bounded and $\{f_1(t) : t \in R\} \subset \overline{\{f(t) : t \in R\}}$ since $\mathrm{PKAA}(R, V)$ is a subspace of $\mathrm{PAA}(R, V)$. Moreover, from the definition of k.a.a., we see that $\mathrm{KAA}(R, V)$ is translation-invariant. So the space $\mathrm{PKAA}(R, V)$ is translation-invariant since $\mathrm{PAP}_0(R, V)$ is translation-invariant. See [15,16] for more details on p.a.a.

Lemma 2.3. [11,18] *The following assertions hold*:
(*i*) $\mathrm{PKAA}(R, V)$ *is a Banach space with the supremum norm.*
(*ii*) *The decomposition of a pseudo compact almost automorphic function is unique.*
(*iii*) *Let* $f \in \mathrm{PKAA}(R, R)$ *and* $g \in \mathrm{PKAA}(R, R)$, *then* $fg \in \mathrm{PKAA}(R, R)$.
(*iv*) *A function* $f: R \to V$ *is k.a.a. if and only if it is a.a. and uniformly continuous.*

By [15, Lemma 4.36, Theorem 6.8], we can obtain the following result.

Lemma 2.4. *Let* $f = f_1 + f_2 \in \mathrm{PKAA}(R \times V, V)$ *with* $f_1 \in \mathrm{KAA}(R \times V, V), f_2 \in \mathrm{PAP}_0(R \times V, V)$, *and* $x \mapsto f(t, x)$ *be uniformly continuous in any bounded subset of* V *uniformly for* $t \in R$. *If* $x \in \mathrm{PKAA}(R, V)$, *then* $f(\cdot, x(\cdot)) \in \mathrm{PKAA}(R, V)$.

Lemma 2.5. *If* $f \in \mathrm{PKAA}(R, R)$, $\tau \in \mathrm{PKAA}(R, R) \cap C^1(R, R^+)$ *and* $\tau'(t) \le \tau_* < 1$, *then* $f(\cdot - \tau(\cdot)) \in \mathrm{PKAA}(R, R)$.

Proof. Since $f, \tau \in \mathrm{PKAA}(R, R)$, f and τ can be expressed as $f = f_1 + f_2$ and $\tau = \tau_1 + \tau_2$, respectively, where $f_1, \tau_1 \in \mathrm{KAA}(R, R)$ and $f_2, \tau_2 \in \mathrm{PAP}_0(R, R)$. Denote

$$F_1(t) = f_1(t - \tau_1(t)), \quad F_2(t) = f_2(t - \tau(t)), \quad \text{and} \quad F_3(t) = f_1(t - \tau(t)) - f_1(t - \tau_1(t)).$$

Then

$$f(t - \tau(t)) = F_1(t) + F_2(t) + F_3(t).$$

It follows from [19, Lemma 7] that $F_1 \in \mathrm{KAA}(R, R)$. By the same arguments as in the proof of [11, Lemma 4], it is easy to obtain that $F_2 \in \mathrm{PAP}_0(R, R)$. Now it remains to prove that $F_3 \in \mathrm{PAP}_0(R, R)$. By Lemma 2.3 (iv), we have f_1 is uniform continuity, then for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $t', t'' \in R$, $|t' - t''| < \delta$,

$$|f_1(t') - f_1(t'')| < \frac{\varepsilon}{2}. \tag{2.1}$$

Since $\tau_2 \in \mathrm{PAP}_0(R, R)$, it follows from [20, Lemma 1.1] that

$$\lim_{r\to\infty} \frac{1}{2r} \mathrm{meas}(M_{r,\varepsilon}) = 0,$$

where $\mathrm{meas}(\cdot)$ denotes the Lebesgue measure and $M_{r,\varepsilon} = \{t \in [-r, r] : \|\tau_2(t)\| = \|\tau(t) - \tau_1(t)\| \geq \varepsilon\}$. Particularly, one can find $\delta > 0$ such that $r > \delta$,

$$\left|\frac{1}{2r} \mathrm{meas}([-r, r] \cap M_{r,\delta})\right| < \frac{\varepsilon}{4\|f_1\|_\infty}. \tag{2.2}$$

Thus, we can deduce from (2.1) and (2.2) that

$$\begin{aligned}
\frac{1}{2r}\int_{-r}^{r} |F_3(t)|\mathrm{d}t &= \frac{1}{2r}\int_{-r}^{r} |f_1(t - \tau(t)) - f_1(t - \tau_1(t))|\mathrm{d}t \\
&= \frac{1}{2r}\int_{[-r,r]\cap M_{r,\delta}} |f_1(t - \tau(t)) - f_1(t - \tau_1(t))|\mathrm{d}t + \frac{1}{2r}\int_{[-r,r]\setminus M_{r,\delta}} |f_1(t - \tau(t)) - f_1(t - \tau_1(t))|\mathrm{d}t \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which means $F_3 \in \mathrm{PAP}_0(R, R)$. □

Remark 2.6. Lemma 2.5 shows that the condition "$\tau \in \mathrm{KAA}(R, R) \cap C^1(R, R^+)$" in [11] can be improved by the condition "$\tau \in \mathrm{PKAA}(R, R) \cap C^1(R, R^+)$."

Next we recall and introduce some notations, which will be used to obtain our main results.

Definition 2.7. (i) A continuous function $G : R \times R \to V$ is said to be Bi-almost automorphic (Bi-a.a.) if for any sequence $\{\xi'_n\}_{n=1}^\infty \subset R$, there exists a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi'_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} G(t + \xi_n, s + \xi_n) = \tilde{G}(t, s), \quad \lim_{n\to\infty} \tilde{G}(t - \xi_n, s - \xi_n) = G(t, s)$$

for all $(t, s) \in R^2$. Denote by $BAA(R \times R, V)$ the space of all such functions.

If the aforementioned limits hold uniformly in compact regions of R^2, then G is said to be Bi-compact almost automorphic (Bi-k.a.a.). Denote by $\mathrm{BKAA}(R \times R, V)$ the space of all such functions.

(ii) A continuous function $G : R \times R \to V$ is said to be Bi-pseudo almost automorphic (Bi-p.a.a.) if G is decomposed as follows:

$$G = G_1 + G_2,$$

where $G_1 \in BAA(R \times R, V)$ and $G_2 \in \mathrm{BPAP}_0(R \times R, V)$, which is defined by

$$\mathrm{BPAP}_0(R \times R, V) = \left\{G_2 \in \mathrm{BC}(R \times R, V) : \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|G_2(w + t, w + s)\|\mathrm{d}w = 0 \text{ for all } (t, s) \in R^2\right\}.$$

Denote by $\mathrm{BPAA}(R \times R, V)$ the space of all such functions. The functions G_1 and G_2 are, respectively, called the Bi-a.a. and the Bi-ergodic components of G.

(iii) A continuous function $G : R \times R \to V$ is said to be Bi-pseudo compact almost automorphic (Bi-p.k.a.a.) if G is decomposed as follows:

$$G = G_1 + G_2,$$

where $G_1 \in \mathrm{BKAA}(R \times R, V)$ and $G_2 \in \mathrm{BPAP}_0(R \times R, V)$. Denote by $\mathrm{BPKAA}(R \times R, V)$ the space of all such functions. The functions G_1 and G_2 are, respectively, called the Bi-k.a.a. and the Bi-ergodic components of G.

Remark 2.8.
(I) (i) is given in [21], (ii) is given in [22].
(II) Let $G \in \mathrm{BKAA}(R \times R, V)$. It follows from the definition of Bi-k.a.a. that $\tilde{G}$ is continuous.
(III) Let $G(t,s) = g(t - s)$ for some continuous function $g : R \times R \to V$. Then it is easy to verify that $G \in \mathrm{BKAA}(R \times R, V)$.
(IV) Let $G = G_1 + G_2 \in \mathrm{BPKAA}(R \times R, V)$ with $G_1 \in \mathrm{BKAA}(R \times R, V)$ and $G_2 \in \mathrm{PAP}_0(R \times R, V)$. By the same arguments as in the proof of [22, Proposition 2.8], it is easy to obtain that for every $(t,s) \in R^2$, $\{G_1(t + r, s + r) : r \in R\} \subset \overline{\{G(t + r, s + r) : r \in R\}}$ and the decomposition of $G(t + \cdot, s + \cdot)$ is unique.

Definition 2.9. A continuous function $G : R \times R \to V$ is said to be Bi-uniformly continuous if for any sequences $\{t_n'\}$, $\{t_n''\}$, $\{s_n'\}$ and $\{s_n''\}$ such that $t_n' - s_n' = t_n'' - s_n''$ and $|t_n' - s_n'| \to 0$ as $n \to \infty$, implies

$$\|G(t_n', t_n'') - G(s_n', s_n'')\| \to 0$$

as $n \to \infty$. That is, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $x_1 - x_2 = y_1 - y_2$, and $|x_1 - x_2| < \delta$ implies

$$\|G(x_1, y_1) - G(x_2, y_2)\| < \varepsilon.$$

Remark 2.10.
(i) It is clear that G is Bi-uniformly continuous if $G : R \times R \to V$ is uniformly continuous. But the converse is not true. For instance, it is easy to see that $G(x,y) = x \sin(x - y)$ is Bi-uniformly continuous but not uniformly continuous.
(ii) If $G : R \times R \to V$ is Bi-uniformly continuous, and if $\lim_{m\to\infty} G(t + \xi_m, s + \xi_m) = \tilde{G}(t,s)$ for all $(t,s) \in R^2$ and some $\{\xi_m\}_{m=1}^{\infty} \subset R$, then $\tilde{G}$ is Bi-uniformly continuous. Indeed, by the Bi-uniformly continuity of G, for any sequences $\{t_n'\}$, $\{t_n''\}$, $\{s_n'\}$, and $\{s_n''\}$ such that $t_n' - s_n' = t_n'' - s_n''$ and $|t_n' - s_n'| \to 0$ as $n \to \infty$, implies

$$\|G(t_n', t_n'') - G(s_n', s_n'')\| \to 0$$

as $n \to \infty$. Then

$$\|G(t_n' + \xi_m, t_n'' + \xi_m) - G(s_n' + \xi_m, s_n'' + \xi_m)\| \to 0$$

as $n \to \infty$ uniformly for $\{\xi_m\}$. Thus,

$$\begin{aligned}\lim_{n\to\infty} \|\tilde{G}(t_n', t_n'') - \tilde{G}(s_n', s_n'')\| &= \lim_{n\to\infty}\lim_{m\to\infty} \|G(t_n' + \xi_m, t_n'' + \xi_m) - G(s_n' + \xi_m, s_n'' + \xi_m)\| \\ &= \lim_{m\to\infty}\lim_{n\to\infty} \|G(t_n' + \xi_m, t_n'' + \xi_m) - G(s_n' + \xi_m, s_n'' + \xi_m)\| = 0.\end{aligned}$$

That is, $\tilde{G}$ is Bi-uniformly continuous.

Lemma 2.11. *A function $G : R \times R \to V$ is Bi-k.a.a. if and only if it is Bi-a.a. and Bi-uniformly continuous.*

Proof. Let G is Bi-a.a. and Bi-uniformly continuous. Then for any sequence $\{\xi_n'\}_{n=1}^{\infty} \subset R$, there exists a subsequence $\{\xi_n\}_{n=1}^{\infty}$ of $\{\xi_n'\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} G(t + \xi_n, s + \xi_n) = \tilde{G}(t,s), \quad \lim_{n\to\infty} \tilde{G}(t - \xi_n, s - \xi_n) = G(t,s)$$

for all $(t,s) \in R^2$. Noticing that $\tilde{G}$ is Bi-uniformly continuous since G is Bi-uniformly continuous (Remark 2.10), it is easy to verify that the aforementioned limits hold uniformly for (t,s) in compact regions of R^2. That is, G is Bi-k.a.a.

On the other hand, if G is Bi-k.a.a., then G is Bi-a.a. It remains to show that G is Bi-uniformly continuous. Take sequences $\{t_n'\}$, $\{t_n''\}$, $\{s_n'\}$ and $\{s_n''\}$ such that $t_n' - s_n' = t_n'' - s_n''$ and $|t_n' - s_n'| \to 0$ as $n \to \infty$. Let

$$\alpha_n = \|G(t_n', t_n'') - G(s_n', s_n'')\|,$$

we need to prove $\alpha_n \to 0$ as $n \to \infty$. Now it is sufficient to prove that every subsequence $\hat{\alpha}_n = \|G(\hat{t}_n', \hat{t}_n'') - G(\hat{s}_n', \hat{s}_n'')\|$ of α_n has a subsequence $\tilde{\alpha}_n = \|G(\tilde{t}_n', \tilde{t}_n'') - G(\tilde{s}_n', \tilde{s}_n'')\|$ such that $\tilde{\alpha}_n \to 0$ as $n \to \infty$. Let

$$\{\xi_n\} = \{t_n' - s_n'\} = \{t_n'' - s_n''\},$$
$$\{\hat{\xi}_n\} = \{\hat{t}_n' - \hat{s}_n'\} = \{\hat{t}_n'' - \hat{s}_n''\},$$

where $\{\hat{\xi}_n\} \subset \{\xi_n\}$. Since G is Bi-k.a.a., there exists a subsequence $\{\tilde{\xi}_n\} = \{\tilde{t}_n' - \tilde{s}_n'\} = \{\tilde{t}_n'' - \tilde{s}_n''\}$ of $\{\hat{\xi}_n\}$ such that

$$\lim_{n\to\infty} G(t + \tilde{\xi}_n, s + \tilde{\xi}_n) = \tilde{G}(t, s), \lim_{n\to\infty} \tilde{G}(t - \tilde{\xi}_n, s - \tilde{\xi}_n) = G(t, s) \tag{2.3}$$

uniformly for (t, s) in compact regions of R^2. Meanwhile, since the function $\tilde{G}$ is continuous (Remark 2.8 (II)), and $|\tilde{t}_n' - \tilde{s}_n'| = |\tilde{t}_n'' - \tilde{s}_n''| \to 0$ as $n \to \infty$, we have

$$\|\tilde{G}(\tilde{s}_n', \tilde{s}_n'') - \tilde{G}(\tilde{s}_n' - (\tilde{t}_n' - \tilde{s}_n'), \tilde{s}_n'' - (\tilde{t}_n'' - \tilde{s}_n''))\| \to 0 \tag{2.4}$$

as $n \to \infty$. Then from (2.3) and (2.4), we have

$$\begin{aligned}
\tilde{\alpha}_n &= \|G(\tilde{t}_n', \tilde{t}_n'') - G(\tilde{s}_n', \tilde{s}_n'')\| \\
&\le \|G(\tilde{t}_n' - \tilde{s}_n' + \tilde{s}_n', \tilde{t}_n'' - \tilde{s}_n'' + \tilde{s}_n'') - \tilde{G}(\tilde{s}_n', \tilde{s}_n'')\| + \|\tilde{G}(\tilde{s}_n', \tilde{s}_n'') - \tilde{G}(\tilde{s}_n' - (\tilde{t}_n' - \tilde{s}_n'), \tilde{s}_n'' - (\tilde{t}_n'' - \tilde{s}_n''))\| \\
&\quad + \|\tilde{G}(\tilde{s}_n' - (\tilde{t}_n' - \tilde{s}_n'), \tilde{s}_n'' - (\tilde{t}_n'' - \tilde{s}_n'')) - G(\tilde{s}_n', \tilde{s}_n'')\| \\
&\le \|G(\tilde{\xi}_n + \tilde{s}_n', \tilde{\xi}_n + \tilde{s}_n'') - \tilde{G}(\tilde{s}_n', \tilde{s}_n'')\| + \|\tilde{G}(\tilde{s}_n', \tilde{s}_n'') - \tilde{G}(\tilde{s}_n' - (\tilde{t}_n' - \tilde{s}_n'), \tilde{s}_n'' - (\tilde{t}_n'' - \tilde{s}_n''))\| \\
&\quad + \|\tilde{G}(\tilde{s}_n' - \tilde{\xi}_n, \tilde{s}_n'' - \tilde{\xi}_n) - G(\tilde{s}_n', \tilde{s}_n'')\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Thus, we showed that $\alpha_n = \|G(t_n', t_n'') - G(s_n', s_n'')\| \to 0$ as $n \to \infty$. That is, G is Bi-uniformly continuous. □

We note that the property of $\Psi_\alpha(t, s) = e^{-\int_s^t \alpha(r)\mathrm{d}r}$ is essential to prove the existence of solution to equation (1.3). Similar to [22, Proposition 3.10], we can obtain the following result.

Proposition 2.12. *Let $\alpha = \alpha_1 + \alpha_2 \in \mathrm{PKAA}(R, R)$ with $\alpha_1 \in \mathrm{KAA}(R, R)$ and $\alpha_2 \in \mathrm{PAP}_0(R, R)$. Then Ψ_α is Bi-p.k.a.a., where Ψ_{α_1} is Bi-k.a.a. with $\Psi_{\alpha_1}(t, s) = e^{-\int_s^t \alpha_1(r)\mathrm{d}r}$, and $\Psi_\alpha - \Psi_{\alpha_1}$ is Bi-ergodic.*

Proof. Since $\alpha_1 \in \mathrm{KAA}(R, R)$, then for any sequence $\{\xi_n'\}_{n=1}^\infty \subset R$, there exists a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi_n'\}_{n=1}^\infty$ such that $\lim_{n\to\infty}\alpha_1(r + \xi_n) = \tilde{\alpha}_1(r)$ uniformly for r in compact sets of R. Thus, for all $(t, s) \in R^2$,

$$\begin{aligned}
\lim_{n\to\infty} \Psi_{\alpha_1}(t + \xi_n, s + \xi_n) &= \lim_{n\to\infty} e^{-\int_{s+\xi_n}^{t+\xi_n} \alpha_1(r)\mathrm{d}r} \\
&= \lim_{n\to\infty} e^{-\int_s^t \alpha_1(r+\xi_n)\mathrm{d}r} \\
&= e^{-\int_s^t \tilde{\alpha}_1(r)\mathrm{d}r} \\
&= \Psi_{\tilde{\alpha}_1}(t, s)
\end{aligned}$$

uniformly in compact regions of R^2. Similarly, we can easily obtain that

$$\lim_{n\to\infty} \Psi_{\tilde{\alpha}_1}(t - \xi_n, s - \xi_n) = \Psi_{\alpha_1}(t, s)$$

uniformly in compact regions of R^2. That is, $\Psi_\alpha \in \mathrm{BKAA}(R \times R, R)$. The remains proof is similar to [22, Proposition 3.10 (ii)], and we omit the details here. □

Remark 2.13. Let $\Phi_\alpha(t,s)=e^{\int_s^t \alpha(r)\mathrm{d}r}$ and $\alpha=\alpha_1+\alpha_2\in \mathrm{PKAA}(R,R)$ with $\alpha_1\in \mathrm{KAA}(R,R)$ and $\alpha_2\in \mathrm{PAP}_0(R,R)$, by the same arguments as in the proof of Proposition 2.12, it is easy to obtain that Φ_α is Bi-p.k.a.a., where Φ_{α_1} is Bi-k.a.a. with $\Phi_{\alpha_1}(t,s)=e^{\int_s^t \alpha_1(r)\mathrm{d}r}$, and $\Phi_\alpha-\Phi_{\alpha_1}$ is Bi-ergodic.

Proposition 2.14. *If the function $u\in \mathrm{PKAA}(R,R)$ and there is a Bi-p.k.a.a. function $g: R\times R\to R$ such that*

$$|g(t,s)|\le ce^{-\alpha(t-s)},\quad t\ge s,$$

where c and α are two positive constants, then the function

$$\varrho(t)=\int_{-\infty}^{t} g(t,s)u(s)\mathrm{d}s$$

belongs to $\mathrm{PKAA}(R,R)$.

Proof. Since $u\in \mathrm{PKAA}(R,R)$ and $g\in \mathrm{BPKAA}(R\times R,R)$, u and g can be expressed as $u=u_1+u_2$ and $g=g_1+g_2$, respectively, where $u_1\in \mathrm{KAA}(R,R)$, $u_2\in \mathrm{PAP}_0(R,R)$, $g_1\in \mathrm{BKAA}(R\times R,R)$, $g_2\in \mathrm{BPAP}_0(R\times R,R)$. Denote $\varrho(t)=\varrho_1(t)+\varrho_2(t)+\varrho_3(t)$, where

$$\varrho_1(t)=\int_{-\infty}^{t} g_1(t,s)u_1(s)\mathrm{d}s,\quad t\in R,$$
$$\varrho_2(t)=\int_{-\infty}^{t} g_1(t,s)u_2(s)\mathrm{d}s,\quad t\in R,$$
$$\varrho_3(t)=\int_{-\infty}^{t} g_2(t,s)u(s)\mathrm{d}s,\quad t\in R.$$

Now we can complete the proof by the following two steps.

Step 1. We prove that $\varrho_1\in \mathrm{KAA}(R,R)$. Since $u_1\in \mathrm{KAA}(R,R)$ and $g_1(t,s)$ is Bi-k.a.a., for any sequence $\{t_n'\}_{n=1}^\infty\subset R$, there exists a subsequence $\{t_n\}_{n=1}^\infty$ of $\{t_n'\}_{n=1}^\infty$ such that

$$|u_1(t_n+s)-\tilde{u}_1(s)|\to 0,\ |\tilde{u}_1(s-t_n)-u_1(s)|\to 0 \tag{2.5}$$

as $n\to\infty$ for each $t\in R$. And

$$|g_1(t+t_n,s+t_n)-\tilde{g}_1(t,s)|\to 0,\ |\tilde{g}_1(t-t_n,s-t_n)-g_1(t,s)|\to 0 \tag{2.6}$$

as $n\to\infty$ for each $t,s\in R$. By Remark 2.8 (IV), we have

$$\sup_{r\in R}|g_1(t+r,s+r)|\le \sup_{r\in R}|g(t+r,s+r)|\le ce^{-\alpha(t-s)},\quad t\ge s.$$

Then we have

$$|g_1(t+\cdot,s+\cdot)|\le ce^{-\alpha(t-s)},\quad t\ge s. \tag{2.7}$$

Pose $\tilde{\varrho}_1(t)=\int_{-\infty}^t \tilde{g}_1(t,s)\tilde{u}_1(s)\mathrm{d}s$. Notice that $\|\tilde{u}_1\|_\infty\le\|u_1\|_\infty\le\|u\|_\infty$. Then by Lebesgue's dominated convergence theorem and (2.5)–(2.7), we obtain

$$\begin{aligned}
|\varrho_1(t+t_n)-\tilde{\varrho}_1(t)| &= \left|\int_{-\infty}^{t} g_1(t+t_n,s+t_n)u_1(t_n+s)\mathrm{d}s-\int_{-\infty}^{t}\tilde{g}_1(t,s)\tilde{u}_1(s)\mathrm{d}s\right|\\
&\le \left|\int_{-\infty}^{t}(g_1(t+t_n,s+t_n)-\tilde{g}_1(t,s))\tilde{u}_1(s)\mathrm{d}s\right| + \left|\int_{-\infty}^{t} g_1(t+t_n,s+t_n)(u_1(t_n+s)-\tilde{u}_1(s))\mathrm{d}s\right|\\
&\le \|u\|_\infty\left|\int_{-\infty}^{t}(g_1(t+t_n,s+t_n)-\tilde{g}_1(t,s))\mathrm{d}s\right| + \left|\int_{-\infty}^{t} ce^{-\alpha(t-s)}(u_1(t_n+s)-\tilde{u}_1(s))\mathrm{d}s\right|\\
&\to 0
\end{aligned}$$

as $n \to \infty$ for all $t \in R$. Similarly, we can easily obtain that $|\tilde{\varrho}_1(t - t_n) - \varrho_1(t)| \to 0$ as $n \to \infty$ for all $t \in R$. That is, $\varrho_1 \in AA(R, R)$.

Moreover, we can show that ϱ_1 is uniformly continuous. It follows from Lemma 2.3 (iv) and Lemma 2.11 that $|t_n - s_n| \to 0$ as $n \to \infty$,

$$|g_1(t_n, t_n + r) - g_1(s_n, s_n + r)| \to 0, |u_1(t_n + r) - u_1(s_n + r)| \to 0$$

as $n \to \infty$ for all $r \in R$. Thus,

$$\begin{aligned}
|\varrho_1(t_n) - \varrho_1(s_n)| &= \left| \int_{-\infty}^{t_n} g_1(t_n, s)u_1(s)\mathrm{d}s - \int_{-\infty}^{s_n} g_1(s_n, s)u_1(s)\mathrm{d}s \right| \\
&= \left| \int_{-\infty}^{0} g_1(t_n, t_n + r)u_1(t_n + r)\mathrm{d}r - \int_{-\infty}^{0} g_1(s_n, s_n + r)u_1(s_n + r)\mathrm{d}r \right| \\
&\le \left| \int_{-\infty}^{0} (g_1(t_n, t_n + r) - g_1(s_n, s_n + r))u_1(t_n + r)\mathrm{d}r \right| + \left| \int_{-\infty}^{0} g_1(s_n, s_n + r)(u_1(t_n + r) - u_1(s_n + r))\mathrm{d}r \right| \\
&\le \|u\|_\infty \left| \int_{-\infty}^{0} (g_1(t_n, t_n + r) - g_1(s_n, s_n + r))\mathrm{d}r \right| + \left| \int_{-\infty}^{0} ce^{-ar}(u_1(t_n + r) - u_1(s_n + r))\mathrm{d}r \right| \\
&\to 0
\end{aligned}$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem. That is, $\varrho_1 \in \mathrm{KAA}(R, R)$ by Lemma 2.3 (iv).

Step 2. We prove that $\varrho_2 \in \mathrm{PAP}_0(R, R)$ and $\varrho_3 \in \mathrm{PAP}_0(R, R)$. By applying Fubini theorem, Lebesgue's dominated convergence theorem and the translation invariance property of ergodic function u_2, we have

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varrho_2(t)|\mathrm{d}t &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left| \int_{-\infty}^{t} g_1(t, s)u_2(s)\mathrm{d}s \right| \mathrm{d}t \\
&= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left| \int_{0}^{\infty} g_1(t, t - s)u_2(t - s)\mathrm{d}s \right| \mathrm{d}t \\
&\le \int_{0}^{\infty} ce^{-as} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |u_2(t - s)|\mathrm{d}t\mathrm{d}s \\
&= 0.
\end{aligned}$$

Then by applying Fubini theorem, Lebesgue's dominated convergence theorem and $g_2 \in \mathrm{PAP}_0(R \times R, R)$, we obtain

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\varrho_3(t)|\mathrm{d}t &\le \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left| \int_{-\infty}^{r} g_2(t, s)u(s)\mathrm{d}s \right| \mathrm{d}t \\
&= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left| \int_{-\infty}^{0} g_2(t, t + s)u(t + s)\mathrm{d}s \right| \mathrm{d}t \\
&\le \int_{-\infty}^{0} \|u\|_\infty \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |g_2(t, t + s)|\mathrm{d}t\mathrm{d}s \\
&= 0.
\end{aligned}$$

This means $\varrho_2, \varrho_3 \in \mathrm{PAP}_0(R, R)$. □

Remark 2.15. Proposition 2.14 shows that the condition "there is a Bi-a.a. function $g : R \times R \to R$" in [11, Corollary 1] can be improved by the condition "there is a Bi-p.k.a.a. function $g : R \times R \to R$."

3 Existence and uniqueness of p.k.a.a solution

We consider the delay differential equation (1.3). It is used to model the population growth of species. $u(t)$ denotes the density of the population, $\alpha(t)$ denotes the death rate of the population, $M(t,u) = \sum_{i=1}^{n}\beta_i(t)f_i(t,u(t-\tau_i(t))) + \beta_0(t)\int_{-\tau}^{0}K(t,s)f_0(t,u(t+s))ds$ denotes the birth function, $K(t,s)$ denotes the delay kernel, and $b(t)H(u(t-\sigma))$ means the immigration function or harvesting function, respectively, if $b(t)$ is -nonnegative or nonpositive.

Throughout this article, given a bounded continuous function f defined on R, denote

$$\overline{f} = \sup_{t\in R}\{f(t)\}, \underline{f} = \inf_{t\in R}\{f(t)\},$$

and also we denote

$$\overline{\tau} := \max_{1\le i\le n}\{\overline{\tau_i}\}, \mu = \max\{\overline{\tau}, \tau, \sigma\}.$$

In the population model, only nonnegative solutions are biologically meaningful. We consider the following initial condition

$$u_{t_0} = \varphi, \quad \varphi \in C_0, \tag{3.1}$$

where C_0 defined by

$$C_0 = \{\varphi \in \mathrm{BC}([-\mu, 0], R^+), \varphi(0) > 0\}.$$

Now we make the following assumptions:

(A1) $\alpha, \beta_0, \beta_i, \tau_i : R \to R$ are positive p.k.a.a. with $\underline{\alpha} > 0$, $\underline{\beta_i} > 0$ for $1 \le i \le n$ and $b : R \to R$ is p.k.a.a.

(A2) $K : R \times [-\tau, 0] \to R, (t,s) \mapsto K(t,s)$ is uniformly continuous and positive p.k.a.a. in $t \in R$ for each $s \in [-\tau, 0]$.

(A3) $H : R^+ \to R^+$ is Lipschitz continuous, i.e., there exists a positive constant L_H such that

$$|H(u) - H(v)| \le L_H|u - v|, \text{ for } u, v \in R^+.$$

In addition, we suppose that $H(0) = 0$.

(A4) For all $0 \le i \le n$, $f_i : R \times R \to R$ are nonnegative p.k.a.a., $x \mapsto f_i(t,x)$ is Lipschitzian uniformly for $t \in R$, and f_i reaches its maximum value in $R \times R^+$, i.e., $\overline{f_i} = \sup_{t\in R, x\in R^+} f_i(t,x) = f_i(t_i, n_i^+)$ for some $t_i \in R$, $n_i^+ \in R^+$.

(A5) There exist two positive constants γ_1 and γ_2 such that

$$n^+ < \gamma_1 < \frac{1}{\overline{\alpha}}\left(\sum_{i=1}^{n}\underline{\beta_i}f_{i-} + \underline{b}L_H\gamma_2\right),$$

$$\frac{1}{\underline{\alpha}}\left(\sum_{i=1}^{n}\overline{\beta_i}\overline{f_i} + \overline{\beta_0}\overline{f_0}\,\|K\|_\infty\tau + \overline{b}L_H\gamma_2\right) < \gamma_2,$$

where $n^+ = \max_{1\le i\le n}\{n_i^+\}, f_{i-} = \inf_{t\in R, x\in[n^+,\gamma_2]} f_i(t,x)$.

(A6) For all $0 \le i \le n$, there exist positive constants L_{f_i} such that for all $x, y \in [n^+, \infty)$,

$$|f_i(t,x) - f_i(t,y)| \le L_{f_i}|x - y|, t \in R.$$

Remark 3.1.

(i) Assume that (A2) holds. It is easy to obtain that K is bounded on $R \times [-\tau, 0]$.

(ii) In [11], the nonlinear term $F_i(\lambda_i(t)u) = ue^{-\lambda_i(t)u}$ in the Nicholson model and $F_i(\lambda_i(t)u) = e^{-\lambda_i(t)u}$ in the Lasota-Wazewska model satisfy the following assumption:

(A4′) For all $1 \le i \le n$, $F_i : R^+ \to R^+$ are Lipschitz continuous and reach its maximum value in R^+, i.e., $\overline{F_i} = \sup_{x\in R^+} F_i(x) = F_i(m_i^*)$ for some $m_i^* \in R^+$, and F_i is nonincreasing in $x > m_i^*$.

We note that (A4) is satisfied by lots of functions. For example, $\sin u e^{-\lambda(t)u}$ in the general Nicholson model satisfies (A4), but this function does not satisfy (A4′). Besides, $ue^{-\lambda(t)u}$ and $e^{-\lambda(t)u}$ also satisfy (A4). Consequently, the model we considered includes not only the mixed Nicholson model and the Lasota-Wazewska model but also other useful mixed models, such as the mixed general Nicholson model and the Lasota-Wazewska model.

Lemma 3.2. *Assume that (A2) holds. Let $f \in \mathrm{PKAA}(R \times R, R)$ such that $u \mapsto f(t, u)$ is Lipschitzian uniformly for $t \in R$. If $u \in \mathrm{PKAA}(R, R)$, then the function $\phi(t) = \int_{-\tau}^{0} K(t, s) f(t, u(t + s)) \mathrm{d}s$ belongs to $\mathrm{PKAA}(R, R)$.*

Proof. By using the composition theorem of p.k.a.a. (Lemma 2.4) and translation invariant (Remark 2.2), one can deduce easily that for all $s \in R$ the function $\psi : t \mapsto f(t, u(t + s))$ belongs to $\mathrm{PKAA}(R, R)$. Then, ψ can be expressed as follows:

$$\psi = \psi_1 + \psi_2,$$

where $\psi_1 \in \mathrm{KAA}(R, R)$ and $\psi_2 \in \mathrm{PAP}_0(R, R)$. Since $K : R \times [-\tau, 0] \to R$ is positive p.k.a.a. in $t \in R$ for each $s \in [-\tau, 0]$, i.e., $K(\cdot, s) \in \mathrm{PKAA}(R, R)$ for each $s \in [-\tau, 0]$, K can be expressed as $K = K_1 + K_2$, where $K_1(\cdot, s) \in \mathrm{KAA}(R, R)$ and $K_2(\cdot, s) \in \mathrm{PAP}_0(R, R)$ for each $s \in [-\tau, 0]$. Denote $\phi(t) = \phi_1(t) + \phi_2(t) + \phi_3(t)$, where

$$\phi_1(t) = \int_{-\tau}^{0} K_1(t, s)\psi_1(t)\mathrm{d}s, \quad t \in R,$$
$$\phi_2(t) = \int_{-\tau}^{0} K_1(t, s)\psi_2(t)\mathrm{d}s, \quad t \in R,$$
$$\phi_3(t) = \int_{-\tau}^{0} K_2(t, s)\psi(t)\mathrm{d}s, \quad t \in R.$$

Now we can complete the proof by the following two steps.

Step 1. We prove that $\phi_1 \in \mathrm{KAA}(R, R)$. Since $\psi_1 \in \mathrm{KAA}(R, R)$ and $K_1(\cdot, s) \in \mathrm{KAA}(R, R)$ for each $s \in [-\tau, 0]$, for any sequence $\{t_n'\}_{n=1}^{\infty} \subset R$, there exists a common subsequence $\{t_n\}_{n=1}^{\infty}$ of $\{t_n'\}_{n=1}^{\infty}$ such that

$$|\psi_1(t + t_n) - \tilde{\psi}_1(t)| \to 0, |\tilde{\psi}_1(t - t_n) - \psi_1(t)| \to 0 \tag{3.2}$$

as $n \to \infty$ for all $t \in R$. And

$$|K_1(t + t_n, s) - \tilde{K}_1(t, s)| \to 0, |\tilde{K}_1(t - t_n, s) - K_1(t, s)| \to 0 \tag{3.3}$$

as $n \to \infty$ for all $t \in R$ and for all $s \in [-\tau, 0]$. Pose

$$\tilde{\phi}_1(t) = \int_{-\tau}^{0} \tilde{K}_1(t, s)\tilde{\psi}_1(t)\mathrm{d}s, \quad t \in R.$$

Notice that $\|\tilde{\psi}_1\|_\infty \le \|\psi_1\|_\infty \le \|\psi\|_\infty$ and $\|\tilde{K}_1\|_\infty \le \|K_1\|_\infty \le \|K\|_\infty$ by Remarks 3.1 and 2.2. Then

$$\begin{aligned} |\phi_1(t + t_n) - \tilde{\phi}_1(t)| &= \left| \int_{-\tau}^{0} K_1(t + t_n, s)\psi_1(t + t_n)\mathrm{d}s - \int_{-\tau}^{0} \tilde{K}_1(t, s)\tilde{\psi}_1(t)\mathrm{d}s \right| \\ &\le \left| \int_{-\tau}^{0} (K_1(t + t_n, s) - \tilde{K}_1(t, s))\tilde{\psi}_1(t)\mathrm{d}s \right| + \left| \int_{-\tau}^{0} K_1(t + t_n, s)(\psi_1(t + t_n) - \tilde{\psi}_1(t))\mathrm{d}s \right| \\ &\le \|\psi\|_\infty \left| \int_{-\tau}^{0} (K_1(t + t_n, s) - \tilde{K}_1(t, s))\mathrm{d}s \right| + \|K\|_\infty \left| \int_{-\tau}^{0} (\psi_1(t + t_n) - \tilde{\psi}_1(t))\mathrm{d}s \right| \\ &\to 0 \end{aligned}$$

as $n \to \infty$ for all $t \in R$ by Lebesgue's dominated convergence theorem and (3.2)–(3.3). Similarly, we can easily obtain that $|\tilde{\phi}_1(t - t_n) - \phi_1(t)| \to 0$ as $n \to \infty$ for all $t \in R$. That is, $\phi_1 \in AA(R, R)$.

Moreover, we can show that ϕ_1 is uniformly continuous. By Lemma 2.3 (iv), we have for any sequences $\{t_n\}$ and $\{s_n\}$ such that $|t_n - s_n| \to 0$ as $n \to \infty$,

$$|K_1(t_n, s) - K_1(s_n, s)| \to 0, |\psi_1(t_n) - \psi_1(s_n)| \to 0$$

as $n \to \infty$ for all $s \in [-\tau, 0]$ since $\psi_1 \in \mathrm{KAA}(R, R)$ and $K_1(\cdot, s) \in \mathrm{KAA}(R, R)$ for each $s \in [-\tau, 0]$. Thus,

$$\begin{aligned}|\phi_1(t_n) - \phi_1(s_n)|| &= \left|\int_{-\tau}^{0} K_1(t_n, s)\psi_1(t_n)\mathrm{d}s - \int_{-\tau}^{0} K_1(s_n, s)\psi_1(s_n)\mathrm{d}s\right| \\ &\le \left|\int_{-\tau}^{0}(K_1(t_n, s) - K_1(s_n, s))\psi_1(t_n)\mathrm{d}s\right| + \left|\int_{-\tau}^{0} K_1(s_n, s)(\psi_1(t_n) - \psi_1(s_n))\mathrm{d}s\right| \\ &\le ||\psi||_\infty \left|\int_{-\tau}^{0}(K_1(t_n, s) - K_1(s_n, s))\mathrm{d}s\right| + ||K||_\infty \left|\int_{-\tau}^{0}(\psi_1(t_n) - \psi_1(s_n))\mathrm{d}s\right| \\ &\to 0\end{aligned}$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem. Then $\phi_1 \in KAA\ \ (R, R)$ by Lemma 2.3 (iv).

Step 2. We prove that $\phi_2 \in \mathrm{PAP}_0(R, R)$ and $\phi_3 \in \mathrm{PAP}_0(R, R)$. Since $\psi_2 \in \mathrm{PAP}_0(R, R)$, by applying Fubini theorem and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\phi_2(t)|\mathrm{d}t &= \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\left|\int_{-\tau}^{0}K_1(t, s)\psi_2(t)\mathrm{d}s\right|\mathrm{d}t \\ &\le \int_{-\tau}^{0}||K||_\infty \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\psi_2(t)|\mathrm{d}t\mathrm{d}s \\ &= 0.\end{aligned}$$

Moreover, since $K_2(\cdot, s) \in \mathrm{PAP}_0(R, R)$ for each $s \in [-\tau, 0]$, by applying Fubini theorem and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\phi_3(t)|\mathrm{d}t &\le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\left|\int_{-\tau}^{0}K_2(t, s)\psi(t)\mathrm{d}s\right|\mathrm{d}t \\ &\le \int_{-\tau}^{0}||\psi||_\infty \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|K_2(t, s)|\mathrm{d}t\mathrm{d}s \\ &= 0.\end{aligned}$$

This means $\phi_2, \phi_3 \in \mathrm{PAP}_0(R, R)$. □

Lemma 3.3. *Let* $\Omega_0 = \{\varphi : \varphi \in \mathrm{BC}([-\mu, 0], R^+), \gamma_1 < \varphi(t) < \gamma_2, t \in [-\mu, 0]\}$, *where* γ_1 *and* γ_2 *are given in* (A5). *Assume that* (A1–A5) *hold. Then, for any* $\varphi \in \Omega_0$, *the solution* $u(t, t_0, \varphi)$ *of equation* (1.3) *satisfies*

$$\gamma_1 < u(t, t_0, \varphi) < \gamma_2, t \in [t_0, \zeta(\varphi))$$

and the existence interval of each solution to equation (1.3) *can be extended to* $[t_0, \infty)$.

Proof. The proof is inspired by [9, Lemma 3.1]. Denote $u(t) = u(t, t_0, \varphi)$. Let $[t_0, t^*) \subseteq [t_0, \zeta(\varphi))$. We claim that

$$0 < u(t) < \gamma_2, \quad \forall t \in [t_0, t^*). \tag{3.4}$$

Suppose (3.4) does not hold, then there exists $t_1 \in (t_0, t^*)$ such that

$$u(t_1) = \gamma_2 \quad \text{and} \quad 0 < u(t) < \gamma_2, \quad \forall t \in [t_0 - \mu, t_1).$$

Notice that

$$u'(t_1) = \lim_{t\to t_1} \frac{u(t) - u(t_1)}{t - t_1} \geq 0.$$

On the other hand, in view of (A1)–(A5), we obtain

$$\begin{aligned} u'(t_1) &= -a(t_1)u(t_1) + \sum_{i=1}^{n} \beta_i(t_1) f_i(t_1, u(t - \tau_i(t_1))) + b(t_1)H(u(t_1 - \sigma)) + \beta_0(t_1)\int_{-\tau}^{0} K(t_1, s) f_0(t_1, u(t_1 + s))\mathrm{d}s \\ &\leq -\underline{a}\gamma_2 + \sum_{i=1}^{n} \overline{\beta_i f_i} + \bar{b}L_H\gamma_2 + \overline{\beta_0 f_0}\,\|K\|_\infty \tau \\ &< 0, \end{aligned}$$

which is a contraction and then (3.4) holds.

Next we prove that

$$u(t) > \gamma_1, \quad \forall t \in [t_0, \zeta(\varphi)). \tag{3.5}$$

Suppose (3.5) does not hold, then there exists $t_2 \in (t_0, \zeta(\varphi))$ such that

$$u(t_2) = \gamma_1 \quad \text{and} \quad u(t) > \gamma_1, \quad \forall t \in [t_0 - \mu, t_2).$$

Notice that

$$u'(t_2) = \lim_{t\to t_2} \frac{u(t) - u(t_2)}{t - t_2} \leq 0.$$

On the other hand, in view of (A1)–(A5), we obtain

$$\begin{aligned} u'(t_2) &= -a(t_2)u(t_2) + \sum_{i=1}^{n} \beta_i(t_2) f_i(t_2, u(t_2 - \tau_i(t_2))) + b(t_2)H(u(t_2 - \sigma)) + \beta_0(t_2)\int_{-\tau}^{0} K(t_2, s) f_0(t_2, u(t_2 + s))\mathrm{d}s \\ &\geq -\bar{a}\gamma_1 + \underline{\sum_{i=1}^{n} \beta_i f_{i-}} + \underline{b}L_H\gamma_2 \\ &> 0, \end{aligned}$$

which is a contraction and then (3.5) holds. Thus, it follows from continuation theorem [23, Theorem 2.3.1] that the existence interval of each solution to equation (1.3) can be extended to $[t_0, \infty)$. □

To achieve our main result, we define the operator Γ on PKAA(R, R) by:

$$(\Gamma u)(t) = \int_{-\infty}^{t} e^{-\int_s^t a(\xi)\mathrm{d}\xi} (\mathrm{N}u)(s)\mathrm{d}s,$$

where Nu is defined as follows:

$$(\mathrm{N}u)(s) = \sum_{i=1}^{n} \beta_i(s) f_i(s, u(s - \tau_i(s))) + b(s)H(u(s - \sigma)) + \beta_0(s)\int_{-\tau}^{0} K(s, \xi) f_0(s, u(s + \xi))\mathrm{d}\xi. \tag{3.6}$$

Theorem 3.4. *Assume that* (A1)–(A6) *hold. If*

$$\tau_i \in C^1(R, R^+) \quad \text{and} \quad \tau_i'(t) \leq \tau_* < 1, \quad i = 1, 2, \ldots, n, \tag{3.7}$$

and

$$r = \frac{\sum_{i=1}^{n} \overline{\beta_i} L_{f_i} + \overline{\beta_0}\|K\|_\infty \tau L_{f_0} + \bar{b}L_H}{\underline{a}} < 1, \tag{3.8}$$

equation (1.3) *has a unique solution in* $\Omega = \{u : u \in \mathrm{PKAA}(R, R), \gamma_1 \leq u(t) \leq \gamma_2\}$.

Proof. It follows from Lemma 3.3 that the solution of equation (1.3) with initial condition (3.1) is in Ω, i.e., for all $t \in [t_0, \infty)$, $\gamma_1 < u(t) < \gamma_2$ whenever u satisfies (1.3) and (3.1). Obviously, the solution of equation (1.3) is a fixed point of the mapping Γ in Ω. Let us now prove that

$$\Gamma : \mathrm{PKAA}(R, R) \to \mathrm{PKAA}(R, R).$$

In fact, for any $u \in \mathrm{PKAA}(R, R)$, Lemma 2.5 shows that

$$u(\cdot - \tau_i(\cdot)) \in \mathrm{PKAA}(R, R), \quad i = 1, 2, \ldots, n,$$

since $\tau_i \in \mathrm{PKAA}(R, R) \cap C^1(R, R^+)$ and $\tau'(t) \le \tau_* < 1$. Then it follows from composition Theorem (Lemma 2.4) that

$$f(\cdot, u(\cdot - \tau_i(\cdot))) \in \mathrm{PKAA}(R, R),$$

and we obtain from [11, Theorem1] and Remark 2.2 that

$$H(u(\cdot - \sigma)) \in \mathrm{PKAA}(R, R).$$

We now apply Lemma 2.3 (iii) and Lemma 3.2, and conclude that

$$\beta_i(\cdot) f_i(\cdot, u(\cdot - \tau_i(\cdot))) \in \mathrm{PKAA}(R, R), \quad i = 1, 2, \ldots, n,$$
$$\beta_0(\cdot) \int_{-\tau}^{0} K(\cdot, \xi) f_0(\cdot, u(\cdot + \xi)) \mathrm{d}\xi \in \mathrm{PKAA}(R, R),$$

and

$$b(\cdot) H(u(\cdot - \sigma)) \in \mathrm{PKAA}(R, R),$$

which yield that Nu belongs to $\mathrm{PKAA}(R, R)$. Notice that $e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi}$ is Bi-p.k.a.a. since $\alpha \in \mathrm{PKAA}(R, R)$ by Proposition 2.12. Thus, by Proposition 2.14, it follows that

$$t \mapsto \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi} (\mathrm{Nu})(s) \mathrm{d}s \quad \text{belongs to } \mathrm{PKAA}(R, R)$$

since $e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi} \le e^{-\underline{\alpha}(t-s)}$, $t \ge s$. Thus, $\Gamma : \mathrm{PKAA}(R, R) \to \mathrm{PKAA}(R, R)$.

For any $u \in \Omega$, in view of (A1)–(A5), we obtain

$$\begin{aligned}(\mathrm{Nu})(s) &= \sum_{i=1}^{n} \beta_i(s) f_i(s, u(s - \tau_i(s))) + b(s) H(u(s - \sigma)) + \beta_0(s) \int_{-\tau}^{0} K(s, \xi) f_0(s, u(s + \xi)) \mathrm{d}\xi \\ &\le \left(\sum_{i=1}^{n} \overline{\beta_i f_i} + \overline{\beta_0 f_0} \|K\|_\infty \tau + \bar{b} L_H \gamma_2 \right).\end{aligned}$$

Then,

$$\begin{aligned}(\Gamma u)(t) &= \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi} (\mathrm{Nu})(s) \mathrm{d}s \\ &\le \int_{-\infty}^{t} e^{-\underline{\alpha}(t-s)} (\mathrm{Nu})(s) \mathrm{d}s \\ &\le \frac{\sum_{i=1}^{n} \overline{\beta_i f_i} + \overline{\beta_0 f_0} \|K\|_\infty \tau + \bar{b} L_H \gamma_2}{\underline{\alpha}} \\ &< \gamma_2.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(\mathrm{Nu})(s) &= \sum_{i=1}^{n} \beta_i(s) f_i(s, u(s-\tau_i(s))) + b(s)H(u(s-\sigma)) + \beta_0(s)\int_{-\tau}^{0} K(s,\xi) f_0(s, u(s+\xi))\mathrm{d}\xi \\
&\geq \left(\sum_{i=1}^{n} \underline{\beta_i} f_{i-} + \underline{b} L_H \gamma_2 \right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
(\Gamma u)(t) &= \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi} (\mathrm{Nu})(s)\mathrm{d}s \\
&\geq \int_{-\infty}^{t} e^{-\overline{\alpha}(t-s)} (\mathrm{Nu})(s)\mathrm{d}s \\
&\geq \frac{\sum_{i=1}^{n} \underline{\beta_i} f_{i-} + \underline{b} L_H \gamma_2}{\overline{\alpha}} \\
&> \gamma_1.
\end{aligned}
$$

That is, Γ maps Ω into Ω.

Now for any $u, v \in \Omega$, by (A6), we obtain

$$
\begin{aligned}
|(\mathrm{Nu})(s) - (Nv)(s)| \leq & \left| \sum_{i=1}^{n} \beta_i(s)(f_i(s, u(s-\tau_i(s))) - f_i(s, v(s-\tau_i(s)))) \right| \\
& + \left| \beta_0(s) \int_{-\tau}^{0} K(s,\xi)(f_0(s, u(s+\xi)) - f_0(s, v(s+\xi)))\mathrm{d}\xi \right| \\
& + |b(s)(H(u(s-\sigma)) - H(v(s-\sigma)))| \\
\leq & \left(\sum_{i=1}^{n} \overline{\beta_i} L_{f_i} + \overline{\beta_0} \|K\|_\infty \tau L_{f_0} + \overline{b} L_H \right) \|u-v\|_\infty .
\end{aligned}
$$

Then,

$$
\begin{aligned}
|(\Gamma u)(t) - (\Gamma v)(t)| &= \left| \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi} [(\mathrm{Nu})(s) - (Nv)(s)]\mathrm{d}s \right| \\
&\leq \int_{-\infty}^{t} e^{-\underline{\alpha}(t-s)} |(\mathrm{Nu})(s) - (Nv)(s)|\mathrm{d}s \\
&\leq \frac{\sum_{i=1}^{n} \overline{\beta_i} L_{f_i} + \overline{\beta_0} \|K\|_\infty \tau L_{f_0} + \overline{b} L_H}{\underline{\alpha}} \|u-v\|_\infty .
\end{aligned}
$$

This together with (3.8) implies that

$$
\|(\Gamma u) - (\Gamma v)\|_\infty \leq \frac{\sum_{i=1}^{n} \overline{\beta_i} L_{f_i} + \overline{\beta_0} \|K\|_\infty \tau L_{f_0} + \overline{b} L_H}{\underline{\alpha}} \|u-v\|_\infty = r\|u-v\|_\infty .
$$

By Banach contraction mapping principle, Γ has a unique fixed point in Ω. Hence, equation (1.3) has a unique p.k.a.a. solution in Ω. □

Now we consider equation (1.1), where $f_i(t,u) = F_i(\lambda_i(t)u)$ for $i = 1, 2, \dots, n$. We assume that F_i satisfies (A4′) and the following condition:
(A6′) For all $1 \le i \le n$, there exist positive constants L_{F_i} such that for all $x, y \in [m^*, \infty)$,

$$|F_i(x) - F_i(y)| \le L_{F_i}|x - y|,$$

where $m^* = \max_{1\le i\le n} m_i^*$.
From Theorem 3.4, we obtain the following corollary.

Corollary 3.5. *Assume that* (A1), (A3), (A4′), *and* (A6′) *hold. If*

$$\lambda_i \in \mathrm{PKAA}(R,R) \quad \text{and} \quad \underline{\lambda_i} > 0,\ \tau_i \in C^1(R,R^+) \quad \text{and} \quad \tau_i'(t) \le \tau_* < 1$$

for $i = 1, 2, \dots, n$ *and*

$$r = \frac{\sum_{i=1}^n \overline{\beta_i}\overline{\lambda_i} L_{F_i} + \overline{b} L_H}{\underline{a}} < 1,$$

equation (1.1) *has a unique solution in*

$$\Omega = \{u : u \in \mathrm{PKAA}(R,R), \gamma_1 \le u(t) \le \gamma_2\},$$

where

$$\max_{1\le i\le n} \frac{m_i^*}{\underline{\lambda_i}} < \gamma_1 < \frac{1}{\overline{a}}\left(\sum_{i=1}^n \underline{\beta_i} F_i(\overline{\lambda_i}\gamma_2) + \underline{b} L_H \gamma_2\right),$$
$$\frac{1}{\underline{a}}\left(\sum_{i=1}^n \overline{\beta_i}\overline{F_i} + \overline{b} L_H \gamma_2\right) < \gamma_2.$$

Proof. In view of (A4′), we obtain

$$\overline{f_i} = \overline{F_i} = F_i(m_i^*) \quad \text{with } m_i^* = \lambda_i(t_i)n_i^+ \text{ for some } t_i \in R.$$

So

$$n^+ = \max_{1\le i\le n}\{n_i^+\} \le \max_{1\le i\le n} \frac{m_i^*}{\underline{\lambda_i}}.$$

Moreover, if $\max_{1\le i\le n} \frac{m_i^*}{\underline{\lambda_i}} < \gamma_1$, (A4′) implies that

$$f_{i-} = F_i(\overline{\lambda_i}\gamma_2)$$

since $m_i^* < \underline{\lambda_i} u \le \lambda_i(t)u \le \overline{\lambda_i}\gamma_2$. From (A_6'), we deduce that for all $u, v \in [n^+, \infty)$,

$$|f_i(t,u) - f_i(t,v)| = |F_i(\lambda_i(t)u) - F_i(\lambda_i(t)v)| \le L_{F_i}|\lambda_i(t)(u - v)| \le L_{F_i}\overline{\lambda_i}|u - v|.$$

That is, $L_{f_i} = L_{F_i}\overline{\lambda_i}$. Thus, it follows from Theorem 3.4 that equation (1.1) has a unique positive solution in

$$\Omega = \{u : u \in \mathrm{PKAA}(R,R), \gamma_1 \le u(t) \le \gamma_2\},$$

where

$$\max_{1\le i\le n} \frac{m_i^*}{\underline{\lambda_i}} < \gamma_1 < \frac{1}{\overline{a}}\left(\sum_{i=1}^n \underline{\beta_i} F_i(\overline{\lambda_i}\gamma_2) + \underline{b} L_H \gamma_2\right),$$
$$\frac{1}{\underline{a}}\left(\sum_{i=1}^n \overline{\beta_i}\overline{F_i} + \overline{b} L_H \gamma_2\right) < \gamma_2. \qquad \square$$

Next we consider equation (1.2), where $f_0(t,u) = ue^{-u}$, $f_i(t,u) = ue^{-\lambda_i(t)u}$, $i = 1, 2, \dots, n$, $H(u(t)) = u(t)$. From Theorem 3.4, we obtain the following corollary.

Corollary 3.6. *Assume that* (A1) *and* (A2) *hold. If*

$$\lambda_i \in \mathrm{PKAA}(R,R) \quad \textit{and} \quad \underline{\lambda_i} > 0, \quad i = 1,2,\dots,n,$$

and

$$r = \frac{\sum_{i=1}^{n}\overline{\beta_i} + \overline{\beta_0}\|K\|_\infty\tau + \bar{b}e^2}{e^2\underline{\alpha}} < 1,$$

equation (1.2) *has a unique solution in*

$$\Omega = \{u : u \in \mathrm{PKAA}(R,R), \gamma_1 \le u(t) \le \gamma_2\},$$

where

$$\frac{1}{\min\underline{\lambda_i}} < \gamma_1 < \frac{1}{\overline{\alpha}}\left(\sum_{i=1}^{n}\underline{\beta_i}\gamma_2 e^{-\overline{\lambda_i}\gamma_2} - \bar{b}\gamma_2\right),$$
$$\frac{1}{e\underline{\alpha}}\left(\sum_{i=1}^{n}\overline{\beta_i}\frac{1}{\underline{\lambda_i}} + \overline{\beta_0}\|K\|_\infty\tau\right) < \gamma_2.$$

Proof. Obviously, (A3) and (A4) hold. By a simple computation, we have

$$\overline{f_i} = \inf_{t\in R, s\in R} f_i(t,u) = \frac{1}{\underline{\lambda_i}}e^{-1} \quad \text{with } n_i^+ = \frac{1}{\underline{\lambda_i}},$$
$$n^+ = \max_{1\le i\le n}\{n_i^+\} = \frac{1}{\min\underline{\lambda_i}},$$
$$f_{i-} = \inf_{t\in R, u\in[n^+,\gamma_2]} f_i(t,u) = \gamma_2 e^{-\overline{\lambda_i}\gamma_2}.$$

Notice that for all $u, v \in [1,\infty)$,

$$|ue^{-u} - ve^{-v}| \le \frac{1}{e^2}|u - v|.$$

Then

$$|f_i(t,u) - f_i(t,v)| = |ue^{-\lambda_i(t)u} - ve^{-\lambda_i(t)v}| \le \frac{1}{\lambda_i(t)}\frac{1}{e^2}|\lambda_i(t)(u-v)| \le \frac{1}{e^2}|u - v|.$$

That is, $L_{f_i} = \frac{1}{e^2}$. Moreover, "$-b(t) \le 0$," which present the rate of extraction of the population, it follows from Theorem 3.4 that equation (1.2) has a unique positive solution in

$$\Omega = \{u : u \in \mathrm{PKAA}(R,R), \gamma_1 \le u(t) \le \gamma_2\},$$

where

$$\frac{1}{\min\underline{\lambda_i}} < \gamma_1 < \frac{1}{\overline{\alpha}}\left(\sum_{i=1}^{n}\underline{\beta_i}\gamma_2 e^{-\overline{\lambda_i}\gamma_2} - \bar{b}\gamma_2\right),$$
$$\frac{1}{e\underline{\alpha}}\left(\sum_{i=1}^{n}\overline{\beta_i}\frac{1}{\underline{\lambda_i}} + \overline{\beta_0}\|K\|_\infty\tau\right) < \gamma_2.$$

□

Remark 3.7.

(i) Theorem 3.4 shows that we extend the result of [11]. In [11], the model (without mixed delays) that is a mixture of the Nicholson model and the Lasota-Wazewska model was researched. Other useful population models can be considered in our results, such as mixed general Nicholson model and Lasota-Wazewska model with mixed delays (Example 5.1).

(ii) Theorem 3.4 shows that we extend the result of [2,4,6,8,9,24]. In [2,4,9], the result of p.a.p. solutions to Nicholson's blowflies model was obtained. In [6,8, 24], the result of periodic, a.p. and p.a.p. solutions to the Lasota-Wazewska model was obtained. In our result, we consider the p.k.a.a. solutions to a general model.

(iii) Comparing with condition [11, A1], Corollary 3.5 shows that the condition "$\alpha, \tau_i : R \to R$ are positive k.a.a." can be improved by the condition "$\alpha, \tau_i : R \to R$ are positive p.k.a.a."

(iv) Notice that $n_i^+ \le \frac{m_i^*}{\underline{\lambda}_i} \le \frac{m^*}{\underline{\lambda}}$, where $m^* = \max_{1\le i\le n}\{m_i^*\}$ and $\underline{\lambda} = \min_{1\le i\le n}\{\underline{\lambda}_i\}$. Therefore, $n^+ \le \frac{m^*}{\underline{\lambda}}$. Comparing with condition [11, A5], Corollary 3.5 shows that we assume a lower bound for γ_1. That is, the condition "$\frac{m^*}{\underline{\lambda}} < \gamma_1$" can be improved by the condition "$n^+ < \gamma_1$."

(v) Comparing with [2], Corollary 3.6 shows that the ergodic components of α and delay kernel K can be considered in our result.

4 Global exponential stability and global exponential attractivity of p.k.a.a solution

In this section, we discuss the global exponential stability and global exponential attractivity conditions of the p.k.a.a solution to equation (1.3). Our result is based on Halanay's inequality (Lemma 4.3).

Definition 4.1. [1] Let $u^*(t), u(t)$ be solutions of equation (1.3) with initial condition $u^*(s) = \varphi^*(s)$ and $u(s) = \phi(s)$, $s \in [-\mu, 0]$.

(i) Suppose that there exists constant $\lambda > 0$ and $M_\varphi > 1$ such that

$$|u(t) - u^*(t)| \le M_\varphi \|\varphi - \varphi^*\|_\mu e^{-\lambda t}, \quad t \ge 0,$$

where $\|\varphi - \varphi^*\|_\mu = \sup_{-\mu\le s\le 0}|\varphi(s) - \varphi^*(s)|$. Then u^* is said to be globally exponential stable.

(ii) Suppose that there exists $\varepsilon > 0$ such that

$$e^{\varepsilon t}|u(t) - u^*(t)| \to 0 (t \to +\infty).$$

Then u^* is said to be globally exponential attractive.

Remark 4.2. Global exponential stability implies global exponential attractivity [1].

Lemma 4.3. [25] (Halanay's inequality) *Let t_0 be a real number and μ be a nonnegative number. If $v : [t_0 - \mu, \infty) \to R^+$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) \le -\alpha v(t) + \beta \sup_{s\in[t-\mu,t]} v(s), \quad t \ge t_0,$$

where α and β are constants with $\alpha > \beta > 0$, then

$$v(t) \le |v_{t_0}|_\mu e^{-\eta(t-t_0)}, \quad \textit{for } t \ge t_0,$$

where $|v_{t_0}|_\mu = \sup_{-\mu\le\theta\le 0} v_{t_0}(\theta)$ and η is the unique positive solution of

$$\eta = \alpha - \beta e^{\eta\mu}.$$

Theorem 4.4. *Assume that* (A1)–(A6) *hold, also* (3.7) *and* (3.8) *hold. Then the unique p.k.a.a. solution of equation* (1.3) *in Ω is globally exponential stable.*

Proof. Let $u^*(t)$ be the p.k.a.a. solution of (1.3) given in Theorem 3.4 and $u(t)$ be an arbitrary solution of equation (1.3). Pose

$$z(t) = u(t) - u^*(t),$$

with the initial condition

$$z(s) = \theta(s), \quad s \in [-\mu, 0].$$

Then

$$\dot{z}(t) = -\alpha(t)z(t) + (\mathrm{Nu})(t) - (\mathrm{Nu}^*)(t), \tag{4.1}$$

where Nu is defined as (3.6).

$$\begin{aligned}|(\mathrm{Nu})(s) - (\mathrm{Nu}^*)(s)| &= \Bigg|\sum_{i=1}^{n}\beta_i(t)(f_i(s, u(s-\tau_i(s))) - f_i(s, u^*(s-\tau_i(s)))) + \beta_0(t)\int_{-\tau}^{0}K(s,\xi)(f_0(s, u(s+\xi)) \\ &\quad - f_0(s, u^*(s+\xi)))\mathrm{d}\xi + b(s)H(u(s-\sigma) - H(u^*(s-\sigma)))\Bigg| \\ &\le \left(\sum_{i=1}^{n}\overline{\beta_i}L_{f_i} + \overline{\beta_0}\|K\|_\infty \tau L_{f_0} + \bar{b}L_H\right)\sup_{r\in[s-\mu,s]}|z(r)|.\end{aligned}$$

Then, for $t \ge 0$, we have

$$\begin{aligned}|z(t)| &= \left|z(0)e^{-\int_0^t \alpha(\xi)\mathrm{d}\xi} + \int_0^t e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi}((\mathrm{Nu})(s) - (\mathrm{Nu}^*)(s))\mathrm{d}s\right| \\ &\le |z(0)|e^{-\int_0^t \alpha(\xi)\mathrm{d}\xi} + \int_0^t e^{-\int_s^t \alpha(\xi)\mathrm{d}\xi}|(\mathrm{Nu})(s) - (\mathrm{Nu}^*)(s)|\mathrm{d}s \\ &\le \int_0^t e^{-\underline{\alpha}(t-s)}\left(\sum_{i=1}^{n}\overline{\beta_i}L_{f_i} + \overline{\beta_0}\|K\|_\infty \tau L_{f_0} + \bar{b}L_H\right)\sup_{r\in[s-\mu,s]}|z(r)|\mathrm{d}s + \|\theta\|_\mu e^{-\underline{\alpha}t}.\end{aligned}$$

Thus,

$$|\dot{z}(t)| \le -\underline{\alpha}|z(t)| + \left(\sum_{i=1}^{n}\overline{\beta_i}L_{f_i} + \overline{\beta_0}\|K\|_\infty \tau L_{f_0} + \bar{b}L_H\right)\sup_{r\in[t-\mu,t]}|z(r)|.$$

Notice that $\underline{\alpha} > \sum_{i=1}^{n}\overline{\beta_i}L_{f_i} + \overline{\beta_0}\|K\|_\infty \tau L_{f_0} + \bar{b}L_H > 0$ by (3.8), then it follows from Halanay's inequality that there exists positive constants η such that

$$|z(t)| \le \|\theta\|_\mu e^{-\eta t}, \quad \text{for } t \ge 0,$$

where η is the unique positive solution of

$$\eta = \alpha - \left(\sum_{i=1}^{n}\overline{\beta_i}L_{f_i} + \overline{\beta_0}\|K\|_\infty \tau L_{f_0} + \bar{b}L_H\right)e^{\eta\mu}.$$

That is,

$$|u(t) - u^*(t)| \le \|\theta\|_\mu e^{-\eta t}, \quad \text{for } t \ge 0.$$

Thus, there exists $M > 1$ such that

$$|u(t) - u^*(t)| \le \|\theta\|_\mu e^{-\eta t} \le M\|\theta\|_\mu e^{-\eta t}, \quad \text{for } t \ge 0.$$

Hence, the p.k.a.a. solution $u^*(t)$ of equation (1.3) is globally exponentially stable. □

By Remark 4.2, we can obtain the following corollary.

Corollary 4.5. *Assume that* (A1)–(A6) *hold, also* (3.7) *and* (3.8) *hold. Then the unique p.k.a.a. solution of equation* (1.3) *in* Ω *is globally attractive.*

Remark 4.6. In [9], an additional condition is added to the existence conditions to obtain the global exponential stability by constructing a Lyapunov function. In [2], the globally attractivity of the positive solution is obtained without adding an additional condition to the existence conditions by inequality technique, but the exponential stability is not obtained. In our result, we obtain the globally exponential stability of the positive solution without adding an additional condition to the existence conditions in view of Halanay's inequality.

5 Example

To illustrate our results, we consider a population model of mixed type with mixed delays in this section.

Example 5.1. We consider the following model with harvesting:

$$\dot{u}(t) = -\alpha(t)u(t) + \beta_i(t)\sum_{i=1}^{2} f_i(t, u(t-\tau_i(t))) + b(t)H(u(t-\sigma)) + \beta_0(t)\int_{-\tau}^{0} K(t,s)f_0(t, u(t+s))\mathrm{d}s, \tag{5.1}$$

where

$$\begin{aligned} &\beta_1(t) = 12e^{1.1}, \quad \beta_2(t) = e, \beta_0(t) = 1,\\ &b(t) = -\frac{1}{100e}, \quad H(u(t-\sigma)) = u(t-1),\\ &K(t,s) = e^{-|\cos(t)| - \frac{1}{1+t^2} + s}, \quad \tau = 1. \end{aligned}$$

Let $\alpha_1(n) = \operatorname{sign}(\cos(2\sqrt{2}\pi n))$, $\alpha_1(t)$ is the linear extension of $\alpha_1(n)$ over R. Then

$$\alpha(t) = 14 + \frac{1}{10}\alpha_1(t) + \frac{1}{1+t^2}$$

is p.k.a.a. but not pseudo almost periodic since $\alpha_1(t)$ is k.a.a but not almost periodic [26]. Notice that $\cos(t) + \cos(\sqrt{2}t)$ is almost periodic but not periodic. Then

$$\tau_1(t) = \tau_2(t) = 0.8 + 0.1\cos(t) + 0.1\cos(\sqrt{2}t) + \frac{1}{1+t^2}$$

is p.k.a.a. but not pseudo periodic since the space of k.a.a. functions include almost periodic functions.

Case 1. We consider a model that is a mixture of Nicholson type with $f_1(t,u) = ue^{-1.8u}$ and Lasota-Wazewska type with $f_2(t,u) = e^{-0.7u}$. Obviously, $(A_1) - (A_4)$ hold. By a simple computation, we obtain

$$\begin{aligned} &\tau_1'(t) = \tau_2'(t) = -0.1\sin(t) - 0.1 * \sqrt{2}\sin(t) - \frac{2t}{(1+t^2)^2} < 0.25 + \frac{9}{8\sqrt{3}} < 1.\\ &\overline{\alpha} = 15.1, \quad \underline{\alpha} = 13.9, \quad \overline{\beta_1} = \underline{\beta_1} = 12e^{1.1}, \quad \overline{\beta_2} = \underline{\beta_2} = e, \quad \overline{\beta_0} = 1,\\ &\overline{\tau} = 2, \quad \underline{b} = -\frac{1}{100e}, \quad L_H = 1, \quad \|K\|_\infty = 1.\\ &\overline{f}_0 = e^{-1}, \quad \overline{f}_1 = \frac{1}{1.8}e^{-1}, \quad \overline{f}_2 = 1,\\ &n_1^+ = \frac{1}{1.8}, \quad n_2^+ = 0, \quad m_1^* = 1, \quad m_2^* = 0,\\ &n^+ = \max\left\{\frac{1}{1.8}, 0\right\} = \frac{1}{1.8}, \quad m^* = \max\{1, 0\} = 1, \quad \underline{\lambda} = \min\{1.8, 0.7\} = 0.7. \end{aligned}$$

Notice that for all $u, v \in [\frac{1}{1.8}, \infty)$,

$$|ue^{-u} - ve^{-v}| \le \frac{1}{e^2}|u - v|, \quad |e^{-u} - e^{-v}| \le \frac{1}{e^{\frac{1}{1.8}}}|u - v|.$$

It follows that for all $u, v \in [\frac{1}{1.8}, \infty)$,

$$L_{f_0} = e^{-2}, \quad L_{f_1} = e^{-2}, \quad L_{f_2} = \frac{0.7}{e^{\frac{1}{1.8}}}.$$

Since we consider a model with harvesting ($b(t) \le 0$), we obtain

$$\frac{1}{\underline{\alpha}}\left(\sum_{i=1}^{2}\overline{\beta_i f_i} + \overline{\beta_0 f_0}\,\|K\|_\infty \tau\right) = \frac{12e^{1.1} \times \frac{1}{1.8}e^{-1} + e + e^{-1}}{13.9} \approx 0.7521 < \gamma_2,$$

Now we fix $\gamma_2 = 0.755$, it follows that

$$f_{1-} = 0.755e^{-1.8*0.755} \approx 0.1940, \quad f_{2-} = e^{-0.7*0.755} \approx 0.5895.$$

Thus,

$$0.5556 \approx \frac{1}{1.8} = n^+ < \gamma_1 < \frac{1}{\overline{\alpha}}\left(\sum_{i=1}^{n}\underline{\beta_i} f_{i-} + \underline{b} L_H \gamma_2\right) \approx 0.5690.$$

We fix $\gamma_1 = 0.5557$. That is, condition (A5) holds. Finally, we obtain

$$r = \frac{\sum_{i=1}^{n}\overline{\beta}_i L_{f_i} + \overline{\beta}_0\|K\|_\infty \tau L_{f_0} + \bar{b} L_H}{\underline{\alpha}} \approx 0.4395 < 1$$

Hence, all the conditions of Theorems 3.4 and 4.4 hold. Therefore, the model (5.1) has a unique p.k.a.a. solution that is globally exponentially stable (Figure 1 (a)) in the region

$$\Omega_1 = \{u : u \in \mathrm{PKAA}(R, R), 0.5557 \le u(t) \le 0.755\}.$$

Case 2. We consider a new model that is a mixture of general Nicholson type with $f_1(t, u) = \sin u e^{-1.8u}$ and Lasota-Wazewska type with $f_2(t, u) = e^{-0.7u}$. By a simple computation, we obtain

$$\bar{f}_1 \approx 0.1949, \quad \bar{f}_2 = 1, \quad \text{with } n_1^+ = \arctan\left(\frac{1}{1.8}\right), \quad n_2^+ = 0,$$

$$n^+ = \max\left\{\arctan\left(\frac{1}{1.8}\right), 0\right\} = \arctan\left(\frac{1}{1.8}\right).$$

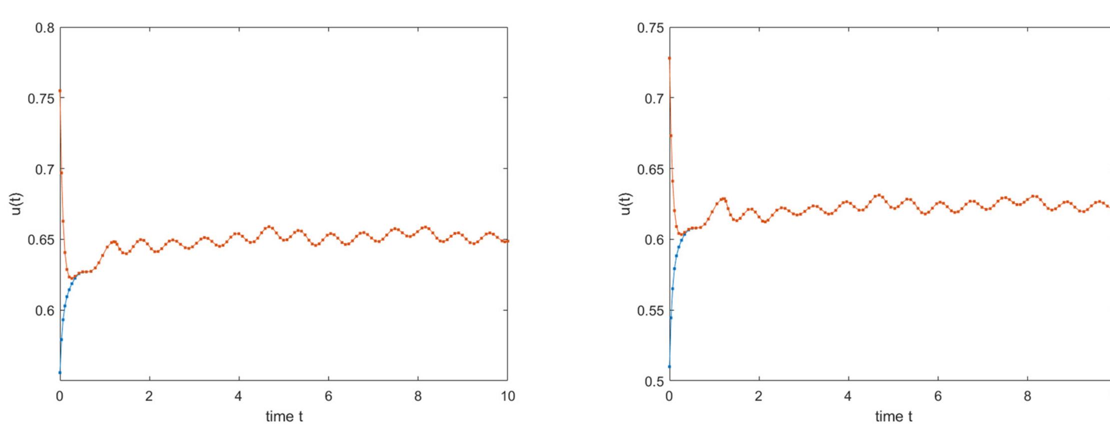

Figure 1: Graphs of model (5.1).

Notice that for all $u, v \in \left[\arctan\left(\frac{1}{1.8}\right), \infty\right)$,

$$|ue^{-u} - ve^{-v}| \le \frac{1}{e^2}|u - v|, \quad |e^{-u} - e^{-v}| \le \frac{1}{e^{\arctan\left(\frac{1}{1.8}\right)}}|u - v|$$

and

$$|\sin(u)e^{-1.8u} - \sin(v)e^{-1.8v}| \le \left|\frac{\cos(\theta) - 1.8\sin(\theta)}{e^{1.8\theta}}\right|_{\theta=\arctan(3.5)}|u - v|.$$

It follows that for all $u, v \in \left[\arctan\left(\frac{1}{1.8}\right), \infty\right)$,

$$L_{f_0} = e^{-2}, \quad L_{f_1} = \left|\frac{\cos(\theta) - 1.8\sin(\theta)}{e^{1.8\theta}}\right|_{\theta=\arctan(3.5)} \approx 0.1611, \quad L_{f_2} = \frac{0.7}{e^{\arctan\left(\frac{1}{1.8}\right)}}.$$

Since we consider a model with harvesting ($b(t) \le 0$), we obtain

$$\frac{1}{\underline{\alpha}}\left(\sum_{i=1}^{2}\overline{\beta_i}\overline{f_i} + \overline{\beta_0}\overline{f_0}\,||K||_\infty\tau\right) = \frac{12e^{1.1} \times 0.1949 + e + e^{-1}}{13.9} \approx 0.7275 < \gamma_2,$$

Now we fix $\gamma_2 = 0.728$, it follows that

$$f_{1-} = \sin(0.728)e^{-1.8*0.728} \approx 0.1801, \quad f_{2-} = e^{-0.7*0.728} \approx 0.6031.$$

Thus,

$$0.5071 \approx \arctan\left(\frac{1}{1.8}\right) = n^+ < \gamma_1 < \frac{1}{\overline{\alpha}}\left(\sum_{i=1}^{n}\underline{\beta_i f_{i-}} + \underline{b}L_H\gamma_2\right) \approx 0.5364.$$

We fix $\gamma_1 = 0.51$. That is, condition (A5) holds. Finally, we obtain

$$r = \frac{\sum_{i=1}^{n}\overline{\beta_i}L_{f_i} + \overline{\beta_0}||K||_\infty\tau L_{f_0} + \overline{b}L_H}{\underline{\alpha}} \approx 0.5102 < 1.$$

Hence, all the conditions of Theorems 3.4 and 4.4 hold. Therefore, model (5.1) has a unique p.k.a.a. solution that is globally exponentially stable (Figure 1 (b)) in the region

$$\Omega_2 = \{u : u \in \mathrm{PKAA}(R, R), 0.51 \le u(t) \le 0.728\}.$$

Remark 5.1.
(i) This example shows that our results are applicable not only to mixed Nicholson model and Lasota-Wazewska model but also to mixed general Nicholson model and Lasota-Wazewska model with mixed delays.
(ii) Comparing with [11], we consider the ergodic components of α and the mixed delays with ergodic components. Moreover, we assume a lower bound for γ_1 in case 1 since $n^+ = \frac{1}{1.8} < \frac{m^*}{\underline{\lambda}} = \frac{1}{0.7}$.

6 Conclusion

We study the existence, uniqueness, and global exponential stability for the pseudo compact automorphic solutions of a family of delay differential equations. By using the properties of p.k.a.a and Bi-p.k.a.a., some novel existence conditions for p.k.a.a. solutions of equations are obtained. Especially, the assumption for coefficients of equations "k.a.a." can be weaken as "p.k.a.a." In addition, by using Halanay's inequality, the conditions for the global exponential stability of pseudo compact automorphic solutions of the equations without adding an additional condition to the existence conditions. The techniques and methods that we use here to study the existence of solutions can be extended to other models. Finally, an example is given to illustrate the validity of our results.

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