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**Research Article**

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# Exact controllability for nonlinear thermoviscoelastic plate problem

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**Abstract:** In this article, we consider a problem of exact controllability in the processes described by a nonlinear damped thermoviscoelastic plate. First, we prove the global well-posedness result for the nonlinear functions that are continuous with respect to time and globally Lipschitz with respect to space variable. Next, we perform a spectral analysis of the linear and uncontrolled problem. Then, we prove that the corresponding solutions decay exponentially to zero at a rate determined explicitly by the physical constants. Finally, we prove the exact controllability of the linear and the nonlinear problems by proving that the corresponding controllability mappings are surjective.

**Keywords:** nonlinear thermoviscoelastic plate, spectral analysis, optimal decay rate, exact controllability

**MSC 2020:** 34D20, 93B05, 81Q10

## 1 Introduction

In the last decades, the theory of control has gained great importance as a discipline for engineers, mathematicians, and other scientists. Examples of control problems range from simple cases, such as driving from heat through a bar, to more complex cases, such as the landing of a vehicle on the Moon, the control of the economy of a nation, and the control of epidemics, among others. There is an extensive literature on control theory for continuous systems.

We study the exact controllability of the following nonlinear damped thermoelastic Kirchhoff-Love plate with controls acting in the whole domain Ω

$$\begin{aligned} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w - \alpha\Delta w_t + \nu\Delta\theta &= u_1 + f_1(t, w, \theta, u_1), \quad \text{in } \Omega \times [0, \infty), \\ \theta_t - \Delta\theta - \nu\Delta w_t &= u_2 + f_2(t, w, \theta, u_2), \quad \text{in } \Omega \times [0, \infty), \end{aligned} \tag{1.1}$$

with Dirichlet boundary conditions

$$w(x, t) = \Delta w(x, t) = \theta(x, t) = 0, \quad x \in \partial\Omega, \ t \geq 0, \tag{1.2}$$

and initial conditions

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^d$ and $\Delta = \sum_{i=1}^{d}\partial_{x_i}^2$ is the Laplace operator.

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The function $w$ represents the vertical displacement of the mid-plane of the plate from its equilibrium position, while $\theta$ the temperature at this middle plane. The vibrations of the plate are described by the Kirchhoff model, which takes into account the rotational inertia through the second term in the first equation of system (1.1). The rotational inertia parameter $\gamma \geq 0$, related to the thickness of the plate, and therefore, it is usually small. The coupling term $\nu\Delta w_t$ ($\nu \neq 0$) takes into account the heat induced by the high-frequency vibrations of the plate, and $\alpha\Delta w_t$ ($\alpha > 0$) is a viscoelastic damping term.

In (1.1), the distributed controls $u_1, u_2 \in L^2(0, \tau; L^2(\Omega))$ and the nonlinear functions $f_i : [0, \tau] \times \mathbb{R}^3 \to \mathbb{R}$, ($i = 1, 2$) are continuous functions on $t$ and globally Lipschitz in the other variables, i.e., there exist constants $l_i > 0$ such that for all $x_1, x_2, y_1, y_2, u_1, u_2 \in \mathbb{R}$, we have

$$\|f_i(t, x_2, y_2, u_2) - f_i(t, x_1, y_1, u_1)\| \leq l_i(\|x_2 - x_1\| + \|y_2 - y_1\| + \|u_2 - u_1\|), \quad t \in [0, \tau].$$

Control problems for system (1.1) have attracted considerable attention in the literature over the past several years. In particular, many efforts have been devoted to study the controllability problem under varying boundary conditions, with different choices of control domain and with different values of $\gamma$.

In the case $\gamma = 0$, which corresponds to neglecting rotational inertia in the vibration of the plate, the null controllability of the linear thermoelastic plate system with hinged boundary conditions was proved by Lasiecka and Triggiani [1] with a single control either on the mechanical component or in the thermal one. The controls are assumed to be supported on the whole set $\Omega$ with regularity $L^2((0, T) \times \Omega)$. These results were extended to other types of boundary conditions in [2]. Some problems were studied by Benabdallah and Naso [3]. They proved that when the control functions act on the whole domain for any $T > 0$ and for any initial data $(w_0, w_1, \theta_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$, there exists a control function $f \in L^2((0, T) \times \Omega)$ such that the problem of linear thermoelastic plate is null controllable. This result was found by Lasiecka and Triggiani [1] by a different method. The same result was obtained when the control function $f$ acts on a small subset $\omega$ of $\Omega$ [3]. The proof of this theorem follows from the procedure developed by Lebeau and Robbiano in [4]. De Terasa and Zuazua [5] proved the exact approximate controllability of linear thermoelastic plate system subject to clamped boundary conditions with interior control acting only on the Kirchoff plate component. Based on the Kalman criterion, Leiva and Zambrano [6] established a necessary and sufficient algebraic condition for the approximate controllability for thermoelastic plate equations with Dirichlet boundary conditions. Chang and Triggiani [7] performed a spectral analysis of an abstract thermoelastic plate equations with different boundary conditions. They provided interesting spectral properties and proved that the resulting semigroup of contractions is neither compact nor differentiable, which contradicts the case $\gamma = 0$. Recently, Triggiani and Zhang [8] proved analyticity, sharp location of the spectrum, and exponential decay with sharp decay rate for a two-dimensional heat-damping viscoelastic plate interaction.

In the case $\gamma > 0$, Lasiecka and Seidman [9] proved observability inequalities for either the thermal component or the elastic one with hinged boundary conditions. The observation is in the whole set $\Omega$, and it is valid for any $T > 0$. Avalos [2] proved the null controllability with a single control acting in the thermal component only under clamped boundary conditions. Another null controllability result was given by Castro and Teresa [10] for linear thermoelastic plate systems with controls written in series form. The result is then obtained by combining the null controllability of these new systems with the convergence of the series. Hansen and Zhang [11] proved that a linear thermoelastic beam may be controlled exactly to zero in a finite time by a single boundary control. Moreover, they showed that the optimal time of controllability becomes arbitrarily small if $\gamma = 0$, as in the case of Euler-Bernoulli beam. Eller et al. [12,13] studied the exact-approximate boundary controllability of thermoelastic plates with variable thermal coefficient, under three boundary controls active in the hinged mechanical/Dirichlet thermal boundary conditions and clamped mechanical/Dirichlet thermal boundary conditions, respectively. Yang and Yao [14] studied the exact-approximate controllability of system (1.1) by the multiplier technique and the Riemannian geometry approach inspired from the work by Avalos and Lasiecka [15].

Because of the smoothing effects associated with analyticity, the null controllability is a more natural question than for the hyperbolic problems. On the contrary, in this article, we consider a hyperbolic model, and we want to show that the nonlinear system (1.1) is exact controllable. The exact controllability is a hard problem even for linear systems; for that reason, there are few results on nonlinear systems. The difficulty here lies in the nonlinear functions that depend on all variables appearing in the considered system, including the control. For exact controllability of non-coupled semilinear equation, one can see, for example, the articles credited by Leiva [16–18]. In this work, we generalize Lieva's works [16–18] on semilinear single equations to coupled thermoviscoelastic plate equations. Our procedure is quite different. In particular, we do not assume that the operator $K$ (defined by (5.25)) is a contraction; but we prove it under a suitable condition, which makes the control holds (see the condition (5.9)). In addition, we show the surjectivity of the controllable mapping differently. In fact, thanks to a result due to Carrasco et al. [19], we prove the exact controllability of the nonlinear problem by showing that the nonlinear controllability mapping $G_F$ (given by (5.3)) is surjective. For this end, we first need to prove that the linear controllability mapping $G$ (given by (5.1)) is surjective. According to Curtain and Zwart [20], this is equivalent to the exact controllability of the linear problem. This is done by proving that the Gramian mapping (defined by (5.4)) is invertible. Nevertheless, our work is very general and can be applied to many systems of coupled partial differential equations. For a more careful review of some known results on analogous problems, we refer to our previous study [21], where the linear thermoelastic model (1.1) is studied for $\alpha = f_1 = f_2 = 0$. When the controls act in the whole domain, we proved the exact and approximate controllability. In the second case, we prove the interior approximate controllability when the controls act only on a subset of the domain. The distributed controls are determined explicitly by the physical constants of the plate in the first case, while this is no longer possible in the second case.

In this article, and through a simple, powerful, and systematic approach based on the spectral analysis of semigroups, we prove the exponential stability, approximate, and exact controllability of the nonlinear thermoviscoelastic plate system given by (1.1)–(1.3). Moreover, we provide the explicit expressions of the optimal decay rate and the distributed controls by the physical constants of the plate. In fact, these expressions have never been given explicitly in the literature. The spirit of this approach is very different from the traditional methods used to study these topics for thermoviscoelastic plate. Our approach is more detailed and complete since it is based on the explicit expressions of the eigenvalues of the corresponding linear operator. The spirit of our approach is very different from the traditional methods used to study these topics for thermoviscoelastic plate. In fact, our spectral analysis gives more precise bounds that allow us to provide the explicit expressions of the optimal decay rate and the distributed controls.

This article is organized as follows. In Section 2, we describe briefly a global well-posedness of the nonlinear problem. In Section 3, we perform the spectral analysis of the linear and uncontrolled problem. In particular, we establish conditions on the physical constants of the plate to guarantee that the roots of the characteristic equation are simple. In Section 4, we prove the exponential stability of the solutions at an optimal rate determined explicitly by the physical constants of the plate. In Section 5, we prove the exact controllability of the considered problem and the explicit expressions of the distributed controls.

## 2 Preliminaries and well-posedness

Before starting the analysis of problems (1.1)–(1.3), let us summarize the main properties of some operators that will be useful to us. Consider the positive operators $A$ and $A^2$ on $X = L^2(\Omega)$ defined by $A\phi = -\Delta\phi$ and $A^2\phi = \Delta^2\phi$ with domains $\mathscr{D}(A) = (H^2 \cap H_0^1)(\Omega)$, $\mathscr{D}(A^2) = \{w \in H^4(\Omega) \text{ with } w = \Delta w = 0 \text{ on } \partial\Omega\}$. The operator $A$ has the following very well-known properties [22].

(a) The spectrum of $A = -\Delta$ consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \infty, \tag{2.1}$$

with finite multiplicity equal to the dimension of the corresponding eigenspace.

(b) The eigenfunctions of $A$ with Dirichlet boundary condition are real analytic functions.

(c) There exists a complete orthonormal set $\{\phi_n\}$ of eigenvectors of $A$.

(d) For all $x \in \mathscr{D}(A)$, we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n x,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $X = L^2(\Omega)$ and

$$E_n x = \langle x, \phi_n \rangle \phi_n.$$

So $\{E_n\}$ is a complete family of orthogonal projections in $X$ and

$$x = \sum_{n=1}^{\infty} E_n x, \quad x \in X. \tag{2.2}$$

(e) The operator $-A$ generates an analytic semigroup $\{e^{-At}\}_{t \geq 0}$ given by

$$e^{-At} x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x.$$

(f) The fractional power spaces $X^r$ are given by

$$X^r = \mathscr{D}(A^r) = \left\{ x \in X, \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty \right\}, \quad r \geq 0,$$

with the norm

$$\|x\|_{X^r} = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r, \tag{2.3}$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x.$$

Setting $\mathcal{Z} = (z_0, z_1, z_2)$, we have

$$z_0 = \Delta w \quad z_1 = \frac{\partial w}{\partial t}, \quad \text{and} \quad z_2 = \theta.$$

Let the state space be

$$Z_\gamma = X \times V_\gamma \times X, \quad \text{where } V_\gamma = \begin{cases} H_0^1(\Omega), & \text{if } \gamma > 0, \\ L^2(\Omega), & \text{if } \gamma = 0, \end{cases}$$

equipped with the inner product

$$\langle \mathcal{Z}_1, \mathcal{Z}_2 \rangle_{Z_\gamma} = \langle y_0, z_0 \rangle_X + \langle y_1, z_1 \rangle_{V_\gamma} + \langle y_2, z_2 \rangle_X, \tag{2.4}$$

where $\mathcal{Z}_1 = (y_0, y_1, y_2)$ and $\mathcal{Z}_2 = (z_0, z_1, z_2)$. In view of (2.4), the corresponding norm in $Z_\gamma$ is given by

$$\left\| \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right\|_{Z_\gamma}^2 = \|z_0\|_X^2 + \|z_1\|_{V_\gamma}^2 + \|z_2\|_X^2. \tag{2.5}$$

Let us introduce the inertia operator $(I - \gamma\Delta)$ with the domain $(H^2 \cap H_0^1)(\Omega)$, where $H_0^1(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \partial\Omega\}$. It is well known that the operator $(I - \gamma\Delta)$ is positive and self-adjoint in $L^2(\Omega)$. Then, one has

$$V_\gamma = \mathscr{D}((I - \gamma\Delta)^{1/2}) = \left\{ z \in X : \sum_{n=1}^{\infty} (1 + \gamma\lambda_n) \|E_n z\|^2 < \infty \right\},$$

endowed with the norm

$$\|z\|_{V_\gamma}^2 = \langle (I - \gamma\Delta)^{1/2} z, (I - \gamma\Delta)^{1/2} z \rangle_X = \sum_{n=1}^{\infty} (1 + \gamma\lambda_n) \|E_n z\|^2. \tag{2.6}$$

System (1.1)–(1.3) can be written, respectively, as a nonlinear evolution equation of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}(t) &= \mathcal{A}\mathcal{Z}(t) + \mathcal{F}(t, \mathcal{Z}, U),\\ \mathcal{Z}(0) &= \mathcal{Z}_0, \quad U = (u_1, u_2),\end{aligned} \tag{2.7}$$

where the nonlinear function $\mathcal{F} : [0, \tau] \times Z_\gamma \times Y \to Z_\gamma$ is given by

$$\mathcal{F}(t, \mathcal{Z}, U) = BU + F(t, \mathcal{Z}, U), \tag{2.8}$$

where $Y = L^2(0, \tau; L^2(\Omega)) \times L^2(0, \tau; L^2(\Omega))$ and the function $F : [0, \tau] \times Z_\gamma \times Y \to Z_\gamma$ is given by

$$F(t, \mathcal{Z}, U) = \begin{pmatrix} 0 \\ J_\gamma f_1(t, \mathcal{Z}, u_1) \\ f_2(t, \mathcal{Z}, u_2) \end{pmatrix}, \tag{2.9}$$

where $J_\gamma = (I - \gamma\Delta)^{-1} : X \to V_\gamma$. We assume that $F$ is a strict contraction on $[0, \tau] \times Z_\gamma \times Y$, i.e., there exists a constant $L < 1$ such that for all $\mathcal{Z}_1, \mathcal{Z}_2 \in Z_\gamma$ corresponding to the controls $U_1, U_2 \in Y$, we have

$$\|F(t, \mathcal{Z}_2, U_2) - F(t, \mathcal{Z}_1, U_1)\| \le L(\|\mathcal{Z}_2 - \mathcal{Z}_1\| + \|U_2 - U_1\|), \quad t \in [0, \tau]. \tag{2.10}$$

The operator $B : X \times X \to Z_\gamma$ in (2.8) is defined by

$$B = \begin{pmatrix} 0 & 0 \\ J_\gamma & 0 \\ 0 & I \end{pmatrix}. \tag{2.11}$$

The linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z_\gamma \to Z_\gamma$ is given by

$$\mathcal{A}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -Az_1 \\ J_\gamma A(z_0 - \alpha z_1 + \nu z_2) \\ -A(\nu z_1 + z_2) \end{pmatrix}, \tag{2.12}$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \begin{cases} H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))^2, & \text{if } \gamma > 0, \\ (H^2(\Omega) \cap H_0^1(\Omega))^3, & \text{if } \gamma = 0. \end{cases}$$

The adjoint operator of $\mathcal{A}$ is easily calculated and given by

$$\mathcal{A}^*\begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -Az_1 \\ J_\gamma A(-z_0 - \alpha z_1 - \nu z_2) \\ A(\nu z_1 - z_2) \end{pmatrix}, \tag{2.13}$$

for $z \in \mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A})$.

Our first result is the following.

**Theorem 1.** *Let the operator $\mathcal{A}$ defined by* (2.12) *generate a $C_0$-semigroup of contractions $(T(t))_{t\ge0}$ on $Z_\gamma$. Suppose that $F : [0, \tau] \times Z_\gamma \times Y \to Z_\gamma$ is a strict contraction satisfying* (2.10)*. Then, for any $\mathcal{Z}_0 \in Z_\gamma$, $U \in Y$, there exists a unique local solution to* (2.7) *such that $\mathcal{Z}(t) \in C(0, \tau; Z_\gamma)$ satisfies*

$$\mathcal{Z}(t) = T(t)\mathcal{Z}_0 + \int_0^t T(t-s)BU(s)\mathrm{d}s + \int_0^t T(t-s)F(s, \mathcal{Z}(s), U(s))\mathrm{d}s, \tag{2.14}$$

*for some $\tau > 0$ and for $t \in [0, \tau)$.*

*Moreover, there exists a unique global solution $\mathcal{Z}(t) \in C(0, \infty; Z_\gamma)$ to* (2.7) *satisfying* (2.14) *with $\tau = +\infty$.*

The proof of Theorem 1 will be completed through several lemmas.

Now, we show that the operator $B$ given by (2.11) is bounded in $Z_\gamma$.

**Lemma 2.1.** *The operator $B$ given by* (2.11) *is bounded in $Z_\gamma$, and satisfies*

$$||B|| \le 1. \tag{2.15}$$

**Proof.** First, we need to show that $J_\gamma : X \to V_\gamma$ is bounded. From $(2.6)_1$, we have

$$||J_\gamma x||^2_{V_\gamma} = \langle (I-\gamma\Delta)^{1/2}(J_\gamma x), (I-\gamma\Delta)^{1/2}(J_\gamma x)\rangle_X = \langle (I-\gamma\Delta)^{-1/2}x, (I-\gamma\Delta)^{-1/2}x\rangle_X = \langle x, J_\gamma x\rangle_X.$$

Since

$$J_\gamma x = (I-\gamma\Delta)^{-1}x = \sum_{n=1}^{+\infty} \frac{1}{1+\gamma\lambda_n} E_n x,$$

we have

$$\begin{aligned} ||J_\gamma x||^2_{V_\gamma} &= \left\langle x, \sum_{n=1}^{+\infty} \frac{1}{1+\gamma\lambda_n} E_n x \right\rangle_X \quad (\text{use } (2.2)) \\ &= \left\langle \sum_{n=1}^{+\infty} E_n x, \sum_{n=1}^{+\infty} \frac{1}{1+\gamma\lambda_n} E_n x \right\rangle_X \\ &= \sum_{n=1}^{+\infty} \frac{1}{1+\gamma\lambda_n} ||E_n x||^2 \le \sum_{n=1}^{+\infty} ||E_n x||^2 = ||x||^2. \end{aligned} \tag{2.16}$$

Therefore, $J_\gamma$ is bounded. Consequently, the operator $B : X \times X \to V_\gamma$ is defined by

$$\begin{aligned} ||By||^2_{Z_\gamma} = \left|\left| \begin{pmatrix} 0 \\ J_\gamma y_1 \\ y_2 \end{pmatrix} \right|\right|^2_{Z_\gamma} &= ||J_\gamma y_1||^2_{V_\gamma} + ||y_2||^2_X \quad (\text{use } (2.6)) \\ &\le ||y_1||^2_X + ||y_2||^2_X = ||y||^2. \end{aligned} \tag{2.17}$$

Thus, the operator $B$ is bounded, and relation (2.15) holds. □

**Remark 2.1.** By the aforementioned argument, one can deduce that the adjoint operator $B^* : V_\gamma \to X \times X$ defined by

$$B^* = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

is bounded and satisfies

$$||B^*|| \le 1. \tag{2.18}$$

The following result shows to be useful to prove Theorem 1.

**Lemma 2.2.** *The operator $\mathcal{A}$ defined by* (2.12) *generates a $C_0$-semigroup of contractions $e^{\mathcal{A}t}$ on $Z_\gamma$.*

**Proof.** Obviously, $\mathcal{D}(\mathcal{A})$ is dense in $Z_\gamma$. We show now that $\mathcal{A}$ is dissipative. For any $\mathcal{Z} \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}\mathcal{Z}, \mathcal{Z}\rangle_{Z_\gamma} &= (-Az_1, z_0)_X + (J_\gamma A(z_0 - \alpha z_1 + \nu z_2), z_1)_{V_\gamma} + (-A(\nu z_1 + z_2), z_2)_X \\ &= (\Delta w_t, \Delta w)_X + (-\Delta(\Delta w - \alpha w_t + \nu\theta), w_t)_X + (\Delta(\nu w_t + \theta), \theta)_X \\ &= (\Delta w_t, \Delta w)_X - (\Delta w, \Delta w_t)_X - \nu(\Delta\theta, w_t)_X + \nu(w_t, \Delta\theta)_X + \alpha(\Delta w_t, w_t)_X + (\Delta\theta, \theta)_X. \end{aligned}$$

Then,

$$\Re e\langle \mathcal{A}\mathcal{Z}, \mathcal{Z}\rangle_{Z_\gamma} = -\alpha||\nabla w_t||^2 - ||\nabla\theta||^2 < 0.$$

Thus, the operator $\mathcal{A}$ is dissipative.

To show that $\mathcal{A}$ is maximal, we need to prove that $I - \mathcal{A} : \mathcal{D}(\mathcal{A}) \to Z_\gamma$ is onto. Let $\mathcal{Z}^* = (z_0^*, z_1^*, z_2^*) \in Z_\gamma$. We must prove that

$$\mathcal{Z} - \mathcal{A}\mathcal{Z} = \mathcal{Z}^* \tag{2.19}$$

has a solution $\mathcal{Z} = (z_0, z_1, z_2)$ in $\mathcal{D}(\mathcal{A})$. This equation leads to the system

$$\begin{aligned} z_0^* &= z_0 + Az_1, \\ z_1^* &= z_1 - J_\gamma A(z_0 - \alpha z_1 + \nu z_2), \\ z_2^* &= z_2 + A(\nu z_1 + z_2). \end{aligned} \tag{2.20}$$

This can be solved by the following many references in the literature [23].

Then, we conclude that operator $\mathcal{A}$ is maximal in $Z_\gamma$. Since $\mathcal{D}(\mathcal{A})$ is densely defined in $Z_\gamma$, the lemma follows from the well-known Lumer-Phillips Theorem (see Theorem 1.4.2 in [24]). □

**Remark 2.2.** It is well known that if $(T(t))_{t\geq 0}$ is a $C_0$- semigroup of contractions with infinitesimal generator $\mathscr{A}$, then $(T^*(t))_{t\geq 0}$ is a $C_0$-semigroup of contractions with infinitesimal generator $\mathscr{A}^*$.

**Proof of Theorem 1.** We need to prove first that the nonlinear function $\mathscr{F} : Z_\gamma \times Y \to Z_\gamma$ defined in (2.8) is locally Lipschitz in $Z_\gamma$. To this end, let us suppose that $\mathcal{Z}_1, \mathcal{Z}_2 \in Z_\gamma$ corresponding to the controls $U_1, U_2 \in Y$, respectively. Then, we have

$$\begin{aligned} \|\mathscr{F}(\mathcal{Z}_2, U_2) - \mathscr{F}(\mathcal{Z}_1, U_1)\| &= \|B(U_2 - U_1) + F(\mathcal{Z}_2, U_2) - F(\mathcal{Z}_1, U_1)\| \quad (\text{use}\,(2.15)\text{ and }(2.10)) \\ &\leq \|U_2 - U_1\| + L(\|\mathcal{Z}_2 - \mathcal{Z}_1\| + \|U_2 - U_1\|) \\ &\leq (L+1)(\|\mathcal{Z}_2 - \mathcal{Z}_1\| + \|U_2 - U_1\|). \end{aligned} \tag{2.21}$$

It follows that $\mathscr{F}$ is locally Lipschitz in $Z_\gamma$. Combining this with Lemma 2.2, we have from a classical result (see [24], Theorem 6.1.4) that the Cauchy problem (2.7) has a unique local mild solution given by (2.14) and defined in a maximal interval $(0, t_{\max})$.

Next, we prove that the solution is global, i.e., $t_{\max} = \infty$. Let $[0, \tau)$ be the maximal interval of existence of the solution of (2.7). Obviously, it suffices to show that $\tau = +\infty$. Assuming $\tau < \infty$, it follows that for $t \in [0, \tau)$,

$$\begin{aligned} \|\mathcal{Z}(t)\|_{Z_\gamma} &\leq \left\| T(t)\mathcal{Z}_0 + \int_0^t T(t-s)BU(s)\mathrm{d}s \right\| + \int_0^t \|T(t-s)\|\|F(s, \mathcal{Z}(s), U(s))\|\mathrm{d}s \quad (\text{use } (2.10)) \\ &\leq \left\| T(t)\mathcal{Z}_0 + \int_0^t T(t-s)BU(s)\mathrm{d}s \right\| + L\int_0^t (\|\mathcal{Z}(s)\| + \|U(s)\|)\mathrm{d}s. \end{aligned} \tag{2.22}$$

By the Cauchy-Schwartz inequality, we have

$$\int_0^t \|U(s)\|\mathrm{d}s \leq \left(\int_0^t \mathrm{d}s\right)^{1/2}\left(\int_0^t \|U(s)\|^2\mathrm{d}s\right)^{1/2} \leq \sqrt{t}\,\|U(s)\|. \tag{2.23}$$

Plugging (2.23) into (2.22), we obtain

$$\|\mathcal{Z}(t)\|_{Z_\gamma} \leq \max_{0\leq t\leq \tau}\left(\left\| T(t)\mathcal{Z}_0 + \int_0^t T(t-s)BU(s)\mathrm{d}s \right\| + L\sqrt{t}\,\|U(s)\|\right) + L\int_0^t \|\mathcal{Z}(s)\|\mathrm{d}s \tag{2.24}$$

which using Gronwall's inequality yields

$$\|\mathcal{Z}(t)\|_{Z_\gamma} \leq \max_{0\leq t\leq \tau}\left(\left\| T(t)\mathcal{Z}_0 + \int_0^t T(t-s)BU(s)\mathrm{d}s \right\| + L\sqrt{t}\,\|U(s)\|\right)e^{L\tau},$$

i.e., $\mathcal{Z}(t)$ is uniformly bounded on $Z_\gamma$ over $[0, \tau]$. Thus, if $\tau < \infty$, similar to the proof of the existence of local solution, we can prove that (2.7) has a unique solution on $[0, \tau + t_0)$ for some $t_0 > 0$. This is a contradiction. Therefore, (2.7) admits a unique global solution. □

# 3 Spectral analysis

In this section, we are interested to study some spectral properties of the operator $\mathcal{A}$ defined by (2.12). This proves to be useful in the following sections, in particular in determining explicitly the optimal decay rate.

Let $\mathcal{Z} \in \mathcal{D}(\mathcal{A})$, computing $\mathcal{A}\mathcal{Z}$ yields

$$\mathcal{A}\mathcal{Z} = \begin{pmatrix} -\sum_{n=1}^{\infty} \lambda_n E_n z_2 \\ \sum_{n=1}^{\infty} \frac{\lambda_n}{1+\gamma\lambda_n} E_n z_1 - \alpha \sum_{n=1}^{\infty} \frac{\lambda_n}{1+\gamma\lambda_n} E_n z_2 + \nu \sum_{n=1}^{\infty} \frac{\lambda_n}{1+\gamma\lambda_n} E_n z_3 \\ -\nu \sum_{n=1}^{\infty} \lambda_n E_n z_2 - \sum_{n=1}^{\infty} \lambda_n E_n z_3 \end{pmatrix}$$

$$= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & -\lambda_n & 0 \\ \frac{\lambda_n}{1+\gamma\lambda_n} & \frac{-\alpha\lambda_n}{1+\gamma\lambda_n} & \frac{\nu\lambda_n}{1+\gamma\lambda_n} \\ 0 & -\nu\lambda_n & -\lambda_n \end{pmatrix} \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{3.1}$$

$$= \sum_{n=1}^{\infty} A_n P_n \mathcal{Z},$$

where $\{P_n\}_{n\geq 1}$ is a complete family of orthogonal projections in $Z_\gamma$

$$P_n = \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix}, \tag{3.2}$$

satisfying

$$P_j P_n = \begin{cases} P_n, & \text{if } j = n \\ 0, & \text{if } j \neq n, \end{cases} \quad \sum_{n\geq 1} P_n = I, \tag{3.3}$$

and

$$A_n = R_n P_n = \begin{pmatrix} 0 & -\lambda_n & 0 \\ \frac{\lambda_n}{1+\gamma\lambda_n} & \frac{-\alpha\lambda_n}{1+\gamma\lambda_n} & \frac{\nu\lambda_n}{1+\gamma\lambda_n} \\ 0 & -\nu\lambda_n & -\lambda_n \end{pmatrix} P_n, \quad n \geq 1. \tag{3.4}$$

We can also easily verify that

$$A_n P_n = P_n A_n, \quad n \geq 1.$$

The characteristic equation of $R_n$ is given by

$$\sigma^3 + \lambda_n\left(1 + \frac{\alpha}{c_n^2}\right)\sigma^2 + \frac{\lambda_n^2}{c_n^2}(1 + \alpha + \nu^2)\sigma + \frac{\lambda_n^3}{c_n^2} = 0, \tag{3.5}$$

where

$$c_n = \sqrt{1 + \gamma\lambda_n}. \tag{3.6}$$

**Remark 3.1.**

(1) The characteristic equation (3.5) is the same obtained in [11,21] for $\alpha = 0$. Moreover, Hansen and Zhang [11] proved that the roots of (3.5) for $\alpha = 0$ are simple if $0 < \nu \leq 1/\sqrt{2}$. Aouadi and Moulahi [21] have improved this result by proving that the roots are simple if $0 < \nu \leq \sqrt{2\sqrt{19} - 8}$.

(2) The eigenvalues of $R_n$ are given by

$$\sigma_i(n) = c_n^{-1}\lambda_n \varrho_i(n), \quad i = 0, 1, 2, \tag{3.7}$$

where $\varrho_i(n)$ are the roots of the following equation:

$$\varrho^3 + c_n(1 + \alpha c_n^{-2})\varrho^2 + (1 + \alpha + \nu^2)\varrho + c_n = 0. \tag{3.8}$$

The roots of (3.8), $\varrho_i(n)$, $i = 0, 1, 2$, are given by the following.

**Proposition 3.1.** *Let us suppose that condition*

$$0 \le \nu < \sqrt{(1 + \alpha)(2\sqrt{19} - 8)}. \tag{3.9}$$

*holds and*

$$\begin{aligned} \delta_0(n) &= c_n^2(1 + \alpha c_n^{-2})^2 - 3(1 + \alpha + \nu^2), \\ \delta_1(n) &= 2c_n^3(1 + \alpha c_n^{-2})^3 - 9c_n(1 + \alpha c_n^{-2})(1 + \alpha + \nu^2) + 27c_n, \\ C(n) &= \sqrt[3]{\frac{1}{2}(\delta_1(n) + \sqrt{\delta_1^2(n) - 4\delta_0^3(n)})}, \end{aligned} \tag{3.10}$$

*where* $\sqrt{\cdot}$ *and* $\sqrt[3]{\cdot}$ *stand for the main branch of complex square and cubic roots, respectively. The spectrum of* (3.8) *consists of a sequence of conjugate pairs* $\{\sigma_1(n)\}_{n=1}^{\infty}$, $\{\sigma_2(n) = \overline{\sigma_1}(n)\}_{n=1}^{\infty}$ *and a real sequence* $\{\sigma_0(n)\}_{n=1}^{\infty}$, *where*

$$\varrho_i(n) = -\frac{1}{3}\left(c_n(1 + \alpha c_n^{-2}) + C(n)e^{\frac{2\mathcal{I}i\pi}{3}} + \frac{\delta_0(n)}{C(n)}e^{\frac{-2\mathcal{I}i\pi}{3}}\right), \quad i = 0, 1, 2, \; n \ge 1, \tag{3.11}$$

*where* $\mathcal{I}$ *is the imaginary unit* ($\mathcal{I}^2 = -1$). *Moreover, we have*

$$\Re e \varrho_i(n) < 0, \quad \textit{for all } i = 0, 1, 2, \; n \ge 1. \tag{3.12}$$

**Proof.** Denote by $\Gamma_n$ the discriminant of the characteristic polynomial (3.8). For $\Gamma_n < 0$, we know that (3.8) possesses three distinct roots: one real and two (non-real) complex conjugate ones [25].

$$\Gamma_n = 18c_n^2(1 + \alpha c_n^{-2})(1 + \alpha + \nu^2) - 4c_n^4(1 + \alpha c_n^{-2})^3 + c_n^2(1 + \alpha c_n^{-2})^2(1 + \alpha + \nu^2)^2 - 4(1 + \alpha + \nu^2)^3 - 27c_n^2. \tag{3.13}$$

If $\Gamma_n < 0$, then (3.8) possesses three distinct roots: one real and two (non-real) complex conjugate ones. Since $ab \le \frac{1}{2}(a^2 + b^2) \le a^2 + b^2$, by a direct computation, we obtain

$$4c_n^2(1 + \alpha c_n^{-2})(1 + \alpha + \nu^2) \le 4c_n^4(1 + \alpha c_n^{-2})^2 + 4(1 + \alpha + \nu^2)^2,$$

with $a = c_n^2(1 + \alpha c_n^{-2})$ and $b = 1 + \alpha + \nu^2$. Using $c_n^2(1 + \alpha c_n^{-2}) > 1$ and $1 + \alpha + \nu^2 > 1$ since $\alpha > 0$, we obtain

$$4c_n^2(1 + \alpha c_n^{-2})(1 + \alpha + \nu^2) \le 4c_n^4(1 + \alpha c_n^{-2})^3 + 4(1 + \alpha + \nu^2)^3.$$

Hence,

$$\Gamma_n \le 14c_n^2(1 + \alpha c_n^{-2})(1 + \alpha + \nu^2) + c_n^2(1 + \alpha c_n^{-2})^2(1 + \alpha + \nu^2)^2 - 27c_n^2.$$

Therefore, to have $\delta(n) < 0$ for $n \ge 1$, it suffices that

$$\wp(1 + \alpha + \nu^2) = (1 + \alpha c_n^{-2})^2(1 + \alpha + \nu^2)^2 + 14(1 + \alpha c_n^{-2})(1 + \alpha + \nu^2) - 27 < 0,$$

or, solving for $1 + \alpha + \nu^2$, if

$$0 \le 1 + \alpha + \nu^2 < \inf_{n\ge 1}\left\{\frac{2\sqrt{19} - 7}{1 + \alpha c_n^{-2}}\right\}, \tag{3.14}$$

where $\frac{\pm 2\sqrt{19}-7}{1+ac_n^{-2}}$ are the roots of $\wp(1+\alpha+\nu^2)=0$. Then, it follows that

$$\inf_{n\geq 1}\left\{\frac{2\sqrt{19}-7}{1+\alpha c_n^{-2}}\right\} = \frac{2\sqrt{19}-7}{\sup_{n\geq 1}\{1+\alpha c_n^{-2}\}} \quad (\text{use } c_n=\sqrt{1+\gamma\lambda_n})$$
$$= \frac{2\sqrt{19}-7}{1+\frac{\alpha}{1+\gamma\inf_{n\geq 1}\{\lambda_n\}}} \quad (\text{use (2.1)})$$
$$= \frac{2\sqrt{19}-7}{1+\frac{\alpha}{1+\gamma\lambda_1}} = \frac{2\sqrt{19}-7}{1+\alpha c_1^{-2}}.$$

Therefore, from (3.14), we infer that

$$0 \leq \nu^2 < \frac{2\sqrt{19}-7}{1+\alpha c_1^{-2}} - (1+\alpha).$$

As $\frac{1}{1+\alpha c_1^{-2}} < 1 < 1+\alpha c_1^{-2} \leq 1+\alpha$, then we have

$$0 \leq \nu < \sqrt{(1+\alpha)(2\sqrt{19}-8)}. \tag{3.15}$$

Hence, we conclude that (3.8) possesses one real and two complex conjugate roots if the condition (3.9) holds. By following the same procedure as in [21,26], we can show that these roots are given by (3.11).

To prove (3.12), we apply the Routh and Hurwitz theorem. From Theorem 2.4 (iii) on p. 33 of [27], the polynomial $p(\varrho)=\varrho^3+a\varrho^2+b\varrho+d$, with real coefficients, is stable if and only if $a>0, b>0, d>0$ and $ab-d>0$. From the polynomial (3.8), we note that $a=c_n(1+\alpha c_n^{-2})>0$, $b=(1+\alpha+\nu^2)>0$, $d=c_n>0$. The last condition $ab-d=c_n[\alpha+\nu^2+\alpha c_n^{-2}(1+\alpha+\nu^2)]>0$ since $\alpha,\nu>0$ and $c_n>0$ for all $n\in\mathbb{N}$.

Since under condition (3.9), the coefficients of the polynomial (3.8) satisfy these inequalities, we obtain $\Re e\varrho_i(n)<0$, $i=0,1,2$, $n\geq 1$. □

It is well known that for Ω a bounded subset of $\mathbb{R}^n$, the eigenvalues of the Dirichlet problem,

$$\begin{cases} -\Delta u = \lambda u, & u\in\Omega,\\ u=0, & u\in\partial\Omega,\end{cases}$$

form an infinite sequence $\{\lambda_n\}$ such that $\lambda_n\to\infty$ to ∞ as $n\to\infty$.

**Lemma 3.1.** *We suppose that condition* (3.9) *holds. The asymptotic expressions of the eigenvalues* $\varrho_i(n)$, $i=0,1,2$, $n\geq 1$, *of* (3.8) *are given by*

$$\begin{aligned}\varrho_0(n) &= -c_n - \frac{2}{3}\left(\nu^2+1-\frac{\alpha}{6}\right)\frac{1}{c_n} + O(c_n^{-2}), \quad \textit{as } n\to\infty,\\ \varrho_1(n) &= -\frac{1}{2}(\nu^2+\alpha)\frac{1}{c_n} + \frac{\mathcal{I}}{3}\left(-2\frac{1}{c_n} + ((\alpha+\nu^2)^2+6\alpha+14\nu^2+4)\frac{1}{c_n^2}\right) + O(c_n^{-2}), \quad \textit{as } n\to\infty.\end{aligned} \tag{3.16}$$

**Proof.** From (3.11), we have

$$\begin{aligned}\varrho_0(n) &= -\frac{1}{3}\left(c_n(1+\alpha c_n^{-2}) + C(n) + \frac{\delta_0(n)}{C(n)}\right),\\ \varrho_1(n) &= -\frac{1}{3}\left[c_n(1+\alpha c_n^{-2}) - \frac{1}{2}\left(C(n)+\frac{\delta_0(n)}{C(n)}\right) + \mathcal{I}\frac{\sqrt{3}}{2}\left(C(n)-\frac{\delta_0(n)}{C(n)}\right)\right].\end{aligned} \tag{3.17}$$

The characteristic polynomial (3.8) $p(\varrho)=\varrho^3+a\varrho^2+b\varrho+d$, we note that $a=c_n(1+\alpha c_n^{-2})>0$, $b=(1+\alpha+\nu^2)>0$, and $d=c_n>0$ and we set $\varrho=\rho-\frac{a}{3}$, (3.8) becomes

$$\rho^3+p_n\rho+q_n=0. \tag{3.18}$$

By setting

$$p_n = -3c_n^2(1 + \alpha c_n^{-2})^2 + 9(1 + \alpha + \nu^2) = -\frac{\delta_0}{3},$$

$$q_n = \frac{1}{27}[2c_n^3(1 + \alpha c_n^{-2})^3 - 9c_n(1 + \alpha c_n^{-2})(1 + \alpha + \nu^2) + 27c_n] = \frac{\delta_1}{27},$$

the discriminant of (3.18) (or equivalently (3.8)) is given by

$$\begin{aligned}\Gamma_n &= -4p_n^3 - 27q_n^2\\ &= -108\left(\frac{q_n^2}{4} + \frac{p_n^3}{27}\right)\\ &= -\frac{1}{27}(\delta_1^2(n) - 4\delta_0^3(n))\\ &= -4c_n^4 + c_n^2[(\alpha + \nu^2)(\alpha + \nu^2 + 8) + 4(3\nu^2 - 2)] + [18\alpha(1 + \alpha + \nu^2) - 12\alpha^2 + 2\alpha(1 + \alpha + \nu^2)^2\\ &\quad - 4(1 + \alpha + \nu^2)^3] + c_n^{-2}[\alpha^2(1 + \alpha + \nu^2)^2 - 4\alpha^3].\end{aligned} \tag{3.19}$$

Using equations $(3.10)_3$ and $(3.19)_3$, we obtain

$$C(n) = \sqrt[3]{\frac{1}{2}(\delta_1(n) + \sqrt{-27\Gamma_n})}, \quad \frac{\delta_0(n)}{C(n)} = \sqrt[3]{\frac{1}{2}(\delta_1(n) - \sqrt{-27\Gamma_n})},$$

which reads

$$\begin{aligned}C(n) + \frac{\delta_0(n)}{C(n)} &= \left(\sqrt[3]{\frac{1}{2}(\delta_1(n) + \sqrt{-27\Gamma_n})} + \sqrt[3]{\frac{1}{2}(\delta_1(n) - \sqrt{-27\Gamma_n})}\right),\\ C(n) - \frac{\delta_0(n)}{C(n)} &= \left(\sqrt[3]{\frac{1}{2}(\delta_1(n) + \sqrt{-27\Gamma_n})} - \sqrt[3]{\frac{1}{2}(\delta_1(n) - \sqrt{-27\Gamma_n})}\right),\end{aligned} \tag{3.20}$$

where $\Gamma_n$ and $\delta_1(n)$ are given by (3.13) and $(3.10)_2$, respectively. Then, the asymptotic expressions of the eigenvalues $\varrho_i(n)$, $i = 0, 1, 2$, are given by the asymptotic expression of $C(n) + \frac{\delta_0(n)}{C(n)}$ and

$$-27\Gamma_n = 108c_n^4(1 - \Phi_n), \tag{3.21}$$

where

$$\begin{aligned}\Phi_n &= c_n^{-2}[(\alpha + \nu^2)(\alpha + \nu^2 + 8) + 4(3\nu^2 - 2)] + c_n^{-4}[18\alpha(1 + \alpha + \nu^2) - 12\alpha^2 + 2\alpha(1 + \alpha + \nu^2)^2 - 4(1 + \alpha + \nu^2)^3]\\ &\quad + c_n^{-6}[\alpha^2(1 + \alpha + \nu^2)^2 - 4\alpha^3].\end{aligned}$$

Since $\Phi_n$ tends to zero as $n \to \infty$, we obtain

$$\sqrt{-27\Gamma_n} = 6\sqrt{3}c_n^2\left(1 - \left(\frac{1}{2}(\alpha + \nu^2)(\alpha + \nu^2 + 8) + 2(3\nu^2 - 2)\right)\frac{1}{c_n^2} + O(c_n^{-3})\right), \tag{3.22}$$

and by $(3.10)_2$,

$$\delta_1(n) = 6\sqrt{3}c_n^2\left[\frac{1}{3\sqrt{3}}c_n + \frac{6 - \alpha - 3\nu^2}{2\sqrt{3}}\frac{1}{c_n} - \frac{\alpha^2 + 3\alpha + 3\alpha\nu^2}{2\sqrt{3}}\frac{1}{c_n^3} + \frac{\alpha^3}{3\sqrt{3}}\frac{1}{c_n^5}\right]. \tag{3.23}$$

From equations (3.22) and (3.23), we obtain

$$\begin{aligned}\delta_1(n) - \sqrt{-27\Gamma_n} &= 2c_n^3\left[1 - 3\sqrt{3}\frac{1}{c_n} + A\frac{1}{c_n^2} + B\frac{1}{c_n^3} + O(c_n^{-3})\right],\\ \delta_1(n) + \sqrt{-27\Gamma_n} &= 2c_n^3\left[1 + 3\sqrt{3}\frac{1}{c_n} + A\frac{1}{c_n^2} - B\frac{1}{c_n^3} + O(c_n^{-3})\right],\end{aligned} \tag{3.24}$$

where

$$A = \frac{3}{2}(6 - \alpha - 3\nu^2), \quad B = \frac{3\sqrt{3}}{2}((\alpha + \nu^2)(\alpha + \nu^2 + 8) + 4(3\nu^2 - 2)). \tag{3.25}$$

From $(3.24)_2$, we note that

$$\delta_1(n) + \sqrt{-27\Gamma} = 2c_n^3[1 + \Psi], \tag{3.26}$$

where

$$\Psi = 3\sqrt{3}\frac{1}{c_n} + A\frac{1}{c_n^2} - B\frac{1}{c_n^3} + O(c_n^{-3}).$$

By setting $n \to +\infty$, we have $\Psi$ that tends to zero.

Hence, by (3.26), note that

$$(1 + \Psi)^{1/3} = 1 + \frac{\Psi}{3} - \frac{\Psi^2}{9} + O(\Psi^2). \tag{3.27}$$

Consequently, from (3.24), we obtain

$$\sqrt[3]{\frac{1}{2}(\delta_1(n) + \sqrt{-27\Gamma})} = c_n\left[1 + \sqrt{3}\frac{1}{c_n} + \left(\frac{A - 9}{3}\right)\frac{1}{c_n^2} - \left(\frac{B + 2\sqrt{3}A}{3}\right)\frac{1}{c_n^3} + O(c_n^{-3})\right] \tag{3.28}$$

and

$$\sqrt[3]{\frac{1}{2}(\delta_1(n) - \sqrt{-27\Gamma})} = c_n\left[1 - \sqrt{3}\frac{1}{c_n} + \left(\frac{A - 9}{3}\right)\frac{1}{c_n^2} + \left(\frac{B + 2\sqrt{3}A}{3}\right)\frac{1}{c_n^3} + O(c_n^{-3})\right]. \tag{3.29}$$

By adding and subtracting equations (3.28) and (3.29), (3.20) becomes

$$\begin{aligned} C(n) + \frac{\delta_0(n)}{C(n)} &= 2c_n + 2\left(\frac{A - 9}{3}\right)\frac{1}{c_n} + O(c_n^{-2}), \\ C(n) - \frac{\delta_0(n)}{C(n)} &= 2\sqrt{3}\frac{1}{c_n} - 2\left(\frac{B + 2\sqrt{3}A}{3}\right)\frac{1}{c_n^2} + O(c_n^{-2}). \end{aligned} \tag{3.30}$$

By plugging $(3.30)_1$ into $(3.17)_1$, we obtain

$$\begin{aligned} \sigma_0(n) &= -\frac{1}{3}\left(c_n(1 + \alpha c_n^{-2}) + C(n) + \frac{\delta_0(n)}{C(n)}\right) \\ &= -c_n - \frac{1}{3}\left[\frac{2}{3}(A - 9) + \alpha\right]\frac{1}{c_n} + O(c_n^{-2}) \\ &= -c_n - \frac{2}{3}\left(\nu^2 + 1 - \frac{\alpha}{6}\right)\frac{1}{c_n} + O(c_n^{-2}). \end{aligned}$$

Similarly, plugging $(3.30)_{1,2}$ into $(3.17)_2$ yields $(3.16)_2$. Hence, the lemma is proved. □

In the following, we use Lemma 3.1 to show that $\Re e\sigma_i(n)$ is strictly monotone.

**Lemma 3.2.** *We suppose that condition* (3.9) *holds. Then,* $\{\Re e\sigma_i(n)\}_{n\geq 1}$ *is strictly decreasing with*

$$\Re e\sigma_i(n) < \Re e\sigma_i(n - 1) < \ldots < \Re e\sigma_i(1) < 0. \tag{3.31}$$

**Proof.** We rewrite (3.5) into the form

$$\sigma^3 + \lambda\sigma^2(1 + \frac{\alpha}{c^2}) + \frac{\lambda^2}{c^2}(1 + \alpha + \nu^2)\sigma + \frac{\lambda^3}{c^2} = 0. \tag{3.32}$$

Actually, $\sigma = c^{-1}(\lambda)\lambda\varrho$ with $c(\lambda) = \sqrt{1 + \gamma\lambda}$ for $\lambda > 0$, i.e., $c$ is a function of $\lambda$ with $c'(\lambda) > 0$.

Let $\sigma_i(\lambda) \in \mathbb{C}$, $i = 1, 2, 3$, denote the roots of (3.32) for arbitrary $\lambda > 0$. To prove that the minimum of $\Re e\sigma_i(\lambda)$ is attained at the smallest $\lambda$, for $i = 1, 2, 3$, we need to show $\frac{\partial \Re e\sigma_i(\lambda)}{\partial\lambda} = \Re e\frac{\partial\sigma_i(\lambda)}{\partial\lambda} \leq 0$. To compute the later derivative, we apply the implicit function theorem to (3.32).

Since the dependence on $\lambda$ in equation (3.32) is quite nontrivial, instead of considering $\frac{\partial\sigma_i(\lambda)}{\partial\lambda}$, one can adopt the equivalent equation (3.8) and prove $\frac{\partial\varrho_i(c)}{\partial c} \leq 0$ for

$$\varrho^3 + c(1 + \alpha c^{-2})\varrho^2 + (1 + \alpha + \nu^2)\varrho + c = 0. \tag{3.33}$$

Differentiating equation (3.33) with respect to $c$ and letting $\varrho' = \frac{\partial\varrho}{\partial c}$, we obtain

$$3\varrho^2\varrho' + (1 - \alpha c^{-2})\varrho^2 + 2c(1 + \alpha c^{-2})\varrho\varrho' + (1 + \alpha + \nu^2)\varrho' + 1 = 0,$$

or, solving for $\varrho'$,

$$\varrho' = -\frac{1 + (1 - \alpha c^{-2})\varrho^2}{3\varrho^2 + 2c(1 + \alpha c^{-2})\varrho + (1 + \alpha + \nu^2)}. \tag{3.34}$$

One can easily verify that the denominator on the right-hand side of (3.34) cannot be 0. Indeed, recalling (3.33), we obtain the algebraic system for "unknown" $\varrho$

$$\begin{aligned} \varrho^3 + c(1 + \alpha c^{-2})\varrho^2 + (1 + \alpha + \nu^2)\varrho + c &= 0, \\ 3\varrho^2 + 2c(1 + \alpha c^{-2})\varrho + (1 + \alpha + \nu^2) &= 0. \end{aligned} \tag{3.35}$$

Equations (3.35) are equivalent to saying that the polynomial $P(\varrho) = \varrho^3 + c(1 + \alpha c^{-2})\varrho^2 + (1 + \alpha + \nu^2)\varrho + c$ has a second-order zero (i.e., $P(\varrho) = 0$, $P'(\varrho) = 0$). But according to Proposition 3.1, we know it is impossible.

Hence, $\varrho'(c)$ is smooth for any $c \geq 1$. Furthermore, $\varrho'(c)$ can only be zero if the numerator on the right-hand side of (3.34) turns zero, which is equivalent to

$$1 + (1 - \alpha c^{-2})\varrho^2 = 0.$$

We conclude

$$\varrho = \pm\frac{\mathcal{I}c}{\sqrt{c^2 - \alpha}},$$

which is impossible according to condition (3.9) ($\Re e\varrho < 0$). Therefore, both $\Re e\varrho$ and $\Im m\varrho$ do not change their sign for $c \geq 1$. Hence, it suffices to prove $\varrho'(1) \leq 0$. Plugging $c = 1$ into (3.33), solving it for $\varrho = \varrho(1)$, and plugging the result into (3.34), we will obtain $\Re e\varrho'(1) \leq 0$. Therefore, $\Re e\varrho(c)$ is monotonically decreasing for $c \geq 1$. Hence, $\Re e\sigma(\lambda)$ is monotonically decreasing as well.

From (3.17) and (3.20), we infer that

$$\sigma_0(n) - \Re e\sigma_1(n) = -\frac{1}{6}(C(n) + \frac{\delta_0(n)}{C(n)}) < 0.$$

Using this together with the aforementioned results on the strictly monotocity of $\sigma_0(\lambda)$ and $\Re e\sigma_1(\lambda)$, we conclude (3.31) (see Lemma 2.7 of [21] for more details). □

**Remark 3.2.** Using same argument as in [21], one can prove that the supremum of $n \longmapsto \max_{i=0,1,2}\Re e\sigma_i(n)$ over $n \geq 1$ is attained at $n = 1$. In fact, from (3.31), one can easily deduce that

$$\sup_{n\geq 1}\max_{i=0,1,2}\Re e\sigma_i(n) = \sup_{n\geq 1}\Re e\sigma_1(n) = \Re e\sigma_1(1) < 0. \tag{3.36}$$

# 4 Exponential decay of linear and uncontrolled problem

In this section, we study the exponential decay of the linear and uncontrolled problem ($u_1 = u_2 = f_1 = f_2 = 0$). In this case, according to Lemma 2.2, the operator $\mathcal{A}$ defined by (2.12) generates a $C_0$-semigroup of contractions $\{T(t)\}_{t\geq 0} = \{e^{-t\mathcal{A}}\}_{t\geq 0}$. In view of (3.1), the $C_0$-semigroup has the following representation:

$$T(t)\mathcal{Z} = \sum_{n=1}^{\infty} e^{A_n t} P_n \mathcal{Z}, \quad t \geq 0, \ \mathcal{Z} \in Z_\gamma, \tag{4.1}$$

where $P_n$ and $A_n$ are defined by (3.2) and (3.4), respectively.

Following the procedure from [28], one can prove

**Lemma 4.1.** *The operator $R_n$ defined by* (3.4) *can be written as*

$$R_n = \sum_{i=0}^{2} \sigma_i(n) q_i^n,$$

*where $\{q_i^n\}_{i=0}^2 \in \mathbb{R}^3$ is a complete family of complementarily projections defined by*

$$\begin{aligned} q_i^n &= \prod_{\substack{k=0\\k\neq i}}^{2} \left(\frac{R_n - \sigma_k(n) I_3}{\sigma_i(n) - \sigma_k(n)}\right) \\ &= \frac{1}{\Upsilon}\begin{pmatrix} \varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2 & -c_n(c_n + \varrho_i) & -\nu \\ c_n^{-1}(c_n + \varrho_i) & \varrho_i^2 + c_n\varrho_i & \nu c_n^{-1}\varrho_i \\ -\nu & -\nu c_n \varrho_i & \varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1 \end{pmatrix}, \end{aligned} \tag{4.2}$$

*where $c_n$ is defined by* (3.6), *$\sigma_i(n)$ is defined by* (3.7), *and*

$$\Upsilon = 3\varrho_i^2 + 2\varrho_i c_n(1 + \alpha c_n^{-2}) + (1 + \alpha + \nu^2). \tag{4.3}$$

*Moreover, we have*

$$e^{R_n t} = \sum_{i=0}^{2} e^{t\sigma_i(n)} q_i^n. \tag{4.4}$$

**Remark 4.1.** From (3.4) and (4.4), the family of linear operators given by (4.1) can be written as

$$e^{A_n t} = \sum_{i=0}^{2} e^{t\sigma_i(n)} P_{ni}, \quad T(t)\mathcal{Z} = \sum_{n=1}^{\infty} \sum_{i=0}^{2} e^{t\sigma_i(n)} P_{ni}\mathcal{Z}, \quad \mathcal{Z} \in Z_\gamma, \tag{4.5}$$

where

$$P_{ni} = q_i^n P_n, \quad i = 0, 1, 2, \tag{4.6}$$

is a complete family of orthogonal projections in $Z_\gamma$.

Here, we are interested in determining explicitly the optimal decay rate of (1.1)–(1.3) when $u_1 = u_2 = f_1 = f_2 = 0$.

**Theorem 2.** *Let condition* (3.9) *holds, and then, the semigroup $\{T(t)\}_{t\geq 0}$ given by* (4.1) *decays exponentially to zero,*

$$\|T(t)\| \leq N e^{\mu_1 t}, \quad t \geq 0, \tag{4.7}$$

*where $N$ is a positive constant and $\mu_1$ is the optimal decay rate given by*

$$\mu_1 = -\frac{\lambda_1}{3\sqrt{1 + \gamma\lambda_1}}\left(\frac{1 + \alpha + \gamma\lambda_1}{\sqrt{1 + \gamma\lambda_1}} - \frac{C(1)}{2} - \frac{\delta_0(1)}{2C(1)}\right) < 0. \tag{4.8}$$

**Proof.** Using $P_n^2 = P_n$, we obtain from (4.5) and (4.6)

$$\|T(t)\mathcal{Z}\|^2 \le \sum_{n=1}^{\infty}\sum_{i=0}^{2}\|e^{\sigma_i(n)t}q_i^n P_n\|_{Z_\gamma}^2 \|P_n\mathcal{Z}\|_{Z_\gamma}^2. \tag{4.9}$$

Consider $\mathcal{Z} = (z_0, z_1, z_2) \in Z_\gamma$, such that $\|\mathcal{Z}\|_{Z_\gamma} = 1$, it follows from (2.6) that

$$\|z_0\|_X^2 = \sum_{j=1}^{\infty}\|E_j z_0\|^2 \le 1, \quad \|z_1\|_{V_\gamma}^2 = \sum_{j=1}^{\infty} c_j^2\|E_j z_1\|^2 \le 1, \quad \text{and} \quad \|z_2\|_X^2 = \sum_{j=1}^{\infty}\|E_j z_2\|^2 \le 1,$$

which immediately implies

$$\|E_j z_0\| \le 1, \quad c_j\|E_j z_1\| \le 1, \quad \|E_j z_2\| \le 1, \quad j \ge 1. \tag{4.10}$$

From (3.2) and (4.2), we have for $i = 1, 2, 3$

$$\|e^{\sigma_i(n)t}q_i^n P_n\mathcal{Z}\|_{Z_\gamma}^2 = \Lambda \left\| \begin{matrix} (\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2)E_n z_0 - c_n(c_n + \varrho_i)E_n z_1 - \nu E_n z_2 \\ c_n^{-1}(c_n + \varrho_i)E_n z_0 + (\varrho_i^2 + c_n\varrho_i)E_n z_1 + \nu c_n^{-1}\varrho_i E_n z_2 \\ -\nu E_n z_0 - \nu c_n\varrho_i E_n z_1 + (\varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1)E_n z_2 \end{matrix} \right\|_{Z_\gamma}^2,$$

where

$$\sigma_i(n) = c_n^{-1}\lambda_n\varrho_i(n) \quad \text{and} \quad \Lambda = \frac{e^{2c_n^{-1}\lambda_n\varrho_i(n)t}}{\Upsilon}. \tag{4.11}$$

Then, using (2.5) yields

$$\begin{aligned} &\|e^{\sigma_i(n)t}q_i^n P_n\mathcal{Z}\|_{Z_\gamma}^2 \\ &\quad = \Lambda\|(\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2)E_n z_0 - c_n(c_n + \varrho_i)E_n z_1 - \nu E_n z_2\|_X^2 \\ &\qquad + \Lambda\|c_n^{-1}(c_n + \varrho_i)E_n z_0 + (\varrho_i^2 + c_n\varrho_i)E_n z_1 + \nu c_n^{-1}\varrho_i E_n z_2\|_{V_\gamma}^2 \\ &\qquad + \Lambda\|-\nu E_n z_0 - \nu c_n\varrho_i E_n z_1 + (\varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1)E_n z_2\|_X^2. \end{aligned}$$

Hence, from (2.3) and (2.6), we obtain

$$\begin{aligned} &\|e^{\sigma_i(n)t}q_i^n P_n\mathcal{Z}\|_{Z_\gamma}^2 \\ &\quad = \Lambda\sum_{j=1}^{\infty}\|E_j((\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2)E_n z_0 - c_n(c_n + \varrho_i)E_n z_1 - \nu E_n z_2)\|^2 \\ &\qquad + \Lambda\sum_{j=1}^{\infty} c_j^2\|E_j(c_n^{-1}(c_n + \varrho_i)E_n z_0 + (\varrho_i^2 + c_n\varrho_i)E_n z_1 + \nu c_n^{-1}\varrho_i E_n z_2)\|^2 \\ &\qquad + \Lambda\sum_{j=1}^{\infty}\|E_j(-\nu E_n z_0 - \nu c_n\varrho_i E_n z_1 + (\varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1)E_n z_2)\|^2. \end{aligned}$$

Using the fact that $\{E_n\}$ is a complete family of orthogonal projections in $Z_\gamma$, i.e.,

$$E_j E_n = \begin{cases} E_n, & \text{if } j = n, \\ 0, & \text{if } j \ne n, \end{cases}$$

we obtain

$$\begin{aligned} &\|e^{\sigma_i(n)t}q_i^n P_n\mathcal{Z}\|_{Z_\gamma}^2 \\ &\quad = \Lambda\|(\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2)E_n z_0 - c_n(c_n + \varrho_i)E_n z_1 - \nu E_n z_2\|^2 \\ &\qquad + \Lambda c_n^2\|c_n^{-1}(c_n + \varrho_i)E_n z_0 + (\varrho_i^2 + c_n\varrho_i)E_n z_1 + \nu c_n^{-1}\varrho_i E_n z_2\|^2 \\ &\qquad + \Lambda\|-\nu E_n z_0 - \nu c_n\varrho_i E_n z_1 + (\varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1)E_n z_2\|^2. \end{aligned}$$

Thus,

$$
\begin{aligned}
&\|e^{\sigma_i(n)t}q_i^n P_n\mathcal{Z}\|_{Z_\gamma}^2 \\
&\leq \Lambda(|\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2|\|E_n z_0\| + |c_n + \varrho_i|c_n\|E_n z_1\| + \nu\|E_n z_2\|)^2 \\
&\quad + \Lambda(|c_n + \varrho_i|\|E_n z_0\| + |\varrho_i^2 + c_n\varrho_i|c_n\|E_n z_1\| + \nu|\varrho_i|\|E_n z_2\|)^2 \\
&\quad + \Lambda(\nu\|E_n z_0\| + \nu|\varrho_i|c_n\|E_n z_1\| + |\varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1|\|E_n z_2\|)^2.
\end{aligned}
$$

From (4.10) and (4.11), the last inequality becomes

$$
\begin{aligned}
\|e^{\sigma_i(n)t}q_i^n P_n\mathcal{Z}\|_{Z_\gamma}^2 \leq &\left[\frac{|\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2| + |c_n + \varrho_i| + \nu}{|3\varrho_i^2 + 2\varrho_i c_n(1 + \alpha c_n^{-2}) + 1 + \alpha + \nu^2|}\right]^2 e^{2c_n^{-1}\lambda_n\varrho_i(n)t} \\
&+ \left[\frac{|c_n + \varrho_i| + |\varrho_i^2 + c_n\varrho_i| + \nu|\varrho_i|}{|3\varrho_i^2 + 2\varrho_i c_n(1 + \alpha c_n^{-2}) + 1 + \alpha + \nu^2|}\right]^2 e^{2c_n^{-1}\lambda_n\varrho_i(n)t} \\
&+ \left[\frac{\nu + \nu|\varrho_i| + |\varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1|}{|3\varrho_i^2 + 2\varrho_i c_n(1 + \alpha c_n^{-2}) + 1 + \alpha + \nu^2|}\right]^2 e^{2c_n^{-1}\lambda_n\varrho_i(n)t}.
\end{aligned} \tag{4.12}
$$

Now, we prove that each term between bracket is bounded by a constant, which does not depend on $n$. From (2.1) and (3.11), we have $|\varrho_i| = C_i(\gamma, \nu, \alpha)\sqrt{\lambda_n} + O(\sqrt{\lambda_n})$ for all $i = 0, 1, 2$. So

$$
\begin{aligned}
|\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2| &= C_i(\gamma, \nu, \alpha)\lambda_n + O(\lambda_n), \\
|c_n + \varrho_i| &= C_i(\gamma, \nu, \alpha)\sqrt{\lambda_n} + O(\sqrt{\lambda_n}), \\
|3\varrho_i^2 + 2\varrho_i c_n(1 + \alpha c_n^{-2}) + (1 + \alpha + \nu^2)| &= C_i(\gamma, \nu, \alpha)\lambda_n + O(\lambda_n),
\end{aligned} \tag{4.13}
$$

where all the constants $C_i$ appearing in (4.13) are generic. Consequently,

$$
\frac{|\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2| + |c_n + \varrho_i| + \nu}{|3\varrho_i^2 + 2\varrho_i c_n(1 + \alpha c_n^{-2}) + 1 + \alpha + \nu^2|} \leq C_i(\gamma, \nu, \alpha) + O(1).
$$

We obtain the same estimation for the second and third terms between brackets in (4.12). Thus,

$$
\|e^{\sigma_i(n)t}q_i^n P_n\mathcal{Z}\|_{Z_\gamma}^2 \leq C_i(\gamma, \nu, \alpha)e^{2c_n^{-1}\lambda_n\varrho_i(n)t}, \quad t \geq 0, \ i = 0, 1, 2.
$$

Therefore,

$$
\sum_{i=0}^{2}\|e^{\sigma_i(n)t}q_i^n P_n\|_{Z_\gamma} \leq Ne^{\mu_1 t}, \quad t \geq 0, \tag{4.14}
$$

where

$$
N = \max_{i=0,1,2}\sqrt{C_i(\gamma, \nu, \alpha)} \geq 0, \tag{4.15}
$$

and from $(4.11)_1$ and Lemma 3.2, we have

$$
\mu_1 = \sup_{n\geq 1}\max_{i=0,1,2}\Re e\ \sigma_i(n) = \max_{i=0,1,2}\Re e\ \sigma_i(1) < 0.
$$

By (3.7) and (3.11), we obtain

$$
\mu_1 = -\frac{\lambda_1}{3\sqrt{1 + \gamma\lambda_1}}\min_{i=0,1,2}\Re e\left(\frac{1 + \alpha + \gamma\lambda_1}{\sqrt{1 + \gamma\lambda_1}} + C(1)e^{\frac{2Ii\pi}{3}} + \frac{\delta_0(1)}{C(1)}e^{\frac{-2Ii\pi}{3}}\right).
$$

According to Remark 3.2, the minimum of the last expression is given by $i = 0$, and consequently, (4.8) follows. We conclude from (4.14) and (4.9) that

$$
\|T(t)\mathcal{Z}\|^2 \leq N^2 e^{2\mu_1 t}\sum_{n=1}^{\infty}\|P_n\mathcal{Z}\|^2.
$$

Finally, following [28], we obtain

$$\sum_{n=1}^{\infty} \|P_n \mathcal{Z}\|^2 = \|\mathcal{Z}\|^2_{Z_\gamma}.$$

Therefore (4.7) follows immediately. □

## 5 Exact controllability of the nonlinear problem

In this section, we show the exact controllability of the nonlinear problem (2.7), where the controls $U = (u_1, u_2) \in Y$ act on the whole domain. In this case, the nonlinear problem (2.7) has a unique mild solution given by (2.14).

Now, we are going to give the definition of the exact controllability [20,22].

**Definition 5.1.** We say that the nonlinear system (2.7) is exact controllable on $[0, \tau]$; if for all $\mathcal{Z}_0, \mathcal{Z}_1 \in Z_\gamma$ there exists a control $U \in Y$ such that the solution $\mathcal{Z}(t)$ given by (2.14) satisfies

$$\mathcal{Z}(\tau) = \mathcal{Z}_1.$$

Following the standard approach (see, e.g., [20]), we define the following concepts:

(a) The linear controllability mapping $G : Y \to Z_\gamma$

$$GU = \int_0^\tau T(\tau - s)BU(s)\mathrm{d}s, \tag{5.1}$$

whose adjoint operator $G^* : Z_\gamma \to Y$, is given by

$$(G^*\mathcal{Z})(s) = B^*T^*(\tau - s)\mathcal{Z}, \quad \forall s \in [0, \tau], \mathcal{Z} \in Z_\gamma. \tag{5.2}$$

(b) The nonlinear controllability mapping $G_F : Y \to Z_\gamma$ given by

$$G_F U = GU + \int_0^t T(t - s)F(s, \mathcal{Z}(s), U(s))\mathrm{d}s, \tag{5.3}$$

where $\mathcal{Z}(t) = \mathcal{Z}(t, \mathcal{Z}_0, U(t))$ is given by (2.14).

(c) The Gramian mapping $W : Z_\gamma \to Z_\gamma$ is given by $W = GG^*$, that is to say

$$W(\tau)\mathcal{Z} = (GG^*\mathcal{Z})(\tau) = \int_0^\tau T(s)BB^*T^*(s)\mathcal{Z}\mathrm{d}s. \tag{5.4}$$

Following the same argument presented in [21], we obtain from (5.4) and the representation (4.1) of $T(t)$ that

$$W(\tau)\mathcal{Z} = \sum_{n=1}^{\infty} W_n(\tau)P_n\mathcal{Z}, \tag{5.5}$$

where $W_n(\tau) : \mathscr{R}(P_n) \to \mathscr{R}(P_n)$ is defined by

$$W_n(\tau) = \int_0^\tau e^{A_n s}BB^*e^{A_n^* s}\mathrm{d}s = P_n\mathrm{B}_n(\tau)P_n, \tag{5.6}$$

while $\mathscr{R}(P_n) = \mathrm{Range}(P_n)$ and

$$\mathrm{B}_n(\tau) = \int_0^\tau e^{R_n s}BB^*e^{R_n^* s}\mathrm{d}s. \tag{5.7}$$

**Remark 5.1.**

(i) Curtain and Zwart [20] proved the equivalence between the exact controllability of the linear problem

$$\frac{d\mathcal{Z}}{dt} = \mathcal{A}\mathcal{Z} + BU, \quad \mathcal{Z}(0) = \mathcal{Z}_0, \tag{5.8}$$

and the surjectivity of the linear controllability mapping $G$ given by (5.1).

(ii) Carrasco et al. [19] generalized this equivalence to the nonlinear case. They proved the equivalence between the exact controllability of the nonlinear problem (2.7) and the surjectivity of the nonlinear controllability mapping $G_F$ given by (5.3).

Recalling that $L < 1$, where $L$ is the Lipschitz constant of the function $F$ (2.10), we state the main result of this section.

**Theorem 3.** *Let condition* (3.9) *hold and* $\tau_0$ *be a positive number satisfying*

$$\sqrt{\tau_0}\, e^{\tau_0} < \frac{1}{L} \tag{5.9}$$

*and* $\tau \le \tau_0$, *then the nonlinear problem* (2.7) *is exact controllable on* $[0, \tau]$.

To show that the nonlinear controllability mapping $G_F$ (defined by (5.3)) is surjective, we need first to prove that the linear controllability mapping $G$ (defined by (5.1)) is surjective. According to Curtain and Zwart [20], this is equivalent to the exact controllability of the linear problem (5.8). Moreover, we have the following result proved in [19,20].

**Lemma 5.1.** *The linear problem* (5.8) *is exact controllable on* $[0, \tau]$ *if and only if the operator* $W$ *given by* (5.4) *is continuous invertible.*

The following result, which can be shown by using the same argument used in [21], proves to be useful in showing the continuous invertibility of $W$ given by (5.4).

**Lemma 5.2.** *Let condition* (3.9) *hold. Then,*

(i) *there exists a positive constant* $\Lambda_2$ *depending on* $\nu$, $\alpha$, *and* $\gamma$ *such that*

$$||W(\tau)|| \le \Lambda_2(\nu, \gamma, \alpha). \tag{5.10}$$

(ii) *Moreover, the operator* $W$ *is invertible in* $Z_\gamma$, *and its inverse* $W^{-1}(\tau) : Z_\gamma \to Z_\gamma$ *is defined by*

$$W^{-1}(\tau)\mathcal{Z} = \sum_{n=1}^{\infty} \frac{1}{\Xi_n} \sum_{i=0}^{2} \begin{pmatrix} b_{i1} & c_n^{-1} b_{i2} & b_{i3} \\ -c_n b_{i2} & b_{i4} & c_n b_{i5} \\ b_{i3} & -c_n^{-1} b_{i5} & b_{i6} \end{pmatrix} P_n \mathcal{Z}, \tag{5.11}$$

*where*

$$\Xi_n = \sum_{i=0}^{2} \frac{e^{2\tau\sigma_i(n)} - 1}{2\sigma_i(n)} (a_{i1}a_{i4}a_{i6} - a_{i1}a_{i5}^2 - a_{i6}a_{i2}^2 - a_{i4}a_{i3}^2 - 2a_{i2}a_{i3}a_{i5}), \quad n \ge 1, \tag{5.12}$$

*and*

$$\begin{aligned} b_{i1} &= a_{i5}^2 + a_{i4}a_{i6}, \quad b_{i2} = a_{i2}a_{i6} + a_{i5}a_{i3}, \\ b_{i3} &= a_{i2}a_{i5} - a_{i3}a_{i4}, \quad b_{i4} = a_{i6}a_{i1} - a_{i3}^2, \\ b_{i5} &= a_{i1}a_{i5} + a_{i2}a_{i3}, \quad b_{i6} = a_{i2}^2 + a_{i1}a_{i4}, \end{aligned} \tag{5.13}$$

$$\begin{aligned} a_{i1} &= (\Theta_{i2}^2 J_y + \Theta_{i3}^2), \quad a_{i2} = (\Theta_{i4}\Theta_{i2}J_y + \Theta_{i5}\Theta_{i3}), \\ a_{i3} &= (-\Theta_{i5}\Theta_{i2}J_y + \Theta_{i6}\Theta_{i3}), \quad a_{i4} = (\Theta_{i4}^2 J_y + \Theta_{i5}^2), \\ a_{i5} &= (\Theta_{i4}\Theta_{i5}J_y - \Theta_{i5}\Theta_{i6}), \quad a_{i6} = (\Theta_{i5}^2 J_y + \Theta_{i6}^2), \end{aligned} \tag{5.14}$$

$$\begin{aligned} \Theta_{i1} &= \frac{1}{\Upsilon}(\varrho_i^2 + c_n\varrho_i(1 + \alpha c_n^{-2}) + \alpha + \nu^2), \quad \Theta_{i2} = -\frac{1}{\Upsilon}(c_n + \varrho_i), \quad \Theta_{i3} = \frac{-\nu}{\Upsilon}, \\ \Theta_{i4} &= \frac{1}{\Upsilon}(\varrho_i^2 + c_n\varrho_i), \quad \Theta_{i5} = \frac{\nu\varrho_i}{\Upsilon}, \quad \text{and} \quad \Theta_{i6} = \frac{1}{\Upsilon}(\varrho_i^2 + \alpha c_n^{-1}\varrho_i + 1). \end{aligned} \tag{5.15}$$

**Remark 5.2.**

(i) From Lemma 5.1, one deduce that the linear problem (5.8) is exact controllable on $[0, \tau]$, which implies that the operator $G$ is surjective (according to Curtain and Zwart [20]).

(ii) For $\tau \geq 0$, the operator $W$ is nonnegative ($W \geq 0$), and hence,

$$\mathcal{R}(\omega, -W) = (\omega I + W)^{-1} \tag{5.16}$$

is the well-defined bounded linear operator for all $\tau \geq 0$ and $\omega > 0$. $\mathcal{R}(\omega, -W)$ is called the resolvent of $-W$. If $W > 0$, then $\mathcal{R}(\omega, -W)$ is defined for $\omega = 0$ as well. Then, for all $\mathcal{Z} \in Z_y$ and $\omega \geq 0$,

$$\langle \mathcal{Z}, (\omega I + W)\mathcal{Z}\rangle \geq (\omega + k)\|\mathcal{Z}\|^2,$$

where $k > 1$ is a constant. Therefore,

$$\|\mathcal{R}(\omega, -W)\| = \|(\omega I + W)^{-1}\| \leq \frac{1}{\omega + k} \leq \frac{1}{k}.$$

We obtain that $\|\mathcal{R}(\omega, -W)\|$ is bounded with respect to $\omega$. Furthermore,

$$\begin{aligned} \|\mathcal{R}(\omega, -W) - W^{-1}\| &= \|(\omega I + W)^{-1} - W^{-1}\| \\ &= \|W^{-1}(W - \omega I - W)(\omega I + W)^{-1}\| \\ &\leq \omega\|W^{-1}\|\|(\omega I + W)^{-1}\| \leq \frac{\omega}{k^2}. \end{aligned}$$

Thus, the operator $\mathcal{R}(\omega, -W)$ converges to the operator $W^{-1}$ as $\omega \to 0$. Consequently, as $\omega \to 0$, we have

$$\|W^{-1}\|_{Z_y} \leq 1. \tag{5.17}$$

We need the following result to prove Theorem 3.

**Lemma 5.3.** *Let* $\tau > 0$ *and* $\mathcal{Z}_1, \mathcal{Z}_2$ *be the solutions to problem* (2.7) *given by* (2.14) *corresponding to the control functions* $U_1, U_2 \in Y$*, respectively. Then, the following estimate holds*:

$$\|\mathcal{Z}_1(t) - \mathcal{Z}_2(t)\|_{Z_y} \leq \Gamma(\tau)\|U_1 - U_2\|_Y, \tag{5.18}$$

*where* $t \in [0, \tau]$ *and*

$$\Gamma(\tau) = (1 + L)\sqrt{\tau}e^{L\tau}. \tag{5.19}$$

**Proof.** Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the solutions to problem (2.7) given by (2.14) corresponding to $U_1$ and $U_2$, respectively. Then,

$$\begin{aligned}
&\|\mathcal{Z}_1(t) - \mathcal{Z}_2(t)\|_{Z_y} \\
&\quad \le \int_0^t \|T(t-s)\|\|B\|\|U_1(s) - U_2(s)\|\mathrm{d}s + \int_0^t \|T(t-s)\|\|F(s, \mathcal{Z}_1(s), U_1(s)) \\
&\quad\quad - F(s, \mathcal{Z}_2(s), U_2(s))\|\mathrm{d}s \quad \text{(use (2.10))} \\
&\quad \le \sup_{0\le s\le t\le\tau} \{\|T(t-s)\|\} \left( \|B\| \int_0^t \|U_1(s) - U_2(s)\|\mathrm{d}s + L\int_0^t \{\|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\| + \|U_1(s) - U_2(s)\|\}\mathrm{d}s \right) \\
&\quad\quad \text{(use } (0 \le t \le \tau)) \\
&\quad \le \sup_{0\le s\le t\le\tau} \{\|T(\tau-s)\|\} \left( (\|B\| + L)\int_0^\tau \|U_1(s) - U_2(s)\|\mathrm{d}s + L\int_0^\tau \|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\|\mathrm{d}s \right).
\end{aligned} \tag{5.20}$$

According to Lemmas 2.1 and 2.2, we obtain

$$\|\mathcal{Z}_1(t) - \mathcal{Z}_2(t)\|_{Z_y} \le (1+L)\int_0^\tau \|U_1(s) - U_2(s)\|\mathrm{d}s + L\int_0^\tau \|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\|\mathrm{d}s. \tag{5.21}$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\|\mathcal{Z}_1(t) - \mathcal{Z}_2(t)\|_{Z_y} &\le (1+L)\left(\int_0^\tau \mathrm{d}s\right)^{1/2}\left(\int_0^\tau \|U_1(s) - U_2(s)\|^2\mathrm{d}s\right)^{1/2} + L\int_0^\tau \|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\|\mathrm{d}s \\
&\le (1+L)\sqrt{\tau}\|U_1 - U_2\| + L\int_0^\tau \|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\|\mathrm{d}s.
\end{aligned}$$

Using Gronwall's inequality, we obtain

$$\|\mathcal{Z}_1(t) - \mathcal{Z}_2(t)\|_{Z_y} \le (1+L)\sqrt{\tau}e^{L\tau}\|U_1 - U_2\|_Y. \qquad \square$$

Now, we are ready to prove the main result of this section.

**Proof of Theorem 3.** Recall that according to the first assertion of Remark 5.2, the operator $G$ is surjective. Thus, given $\mathcal{Z} \in Z_y$, there exists a control $U \in L^2(0, \tau; L^2(\Omega))$ such that

$$GU = \mathcal{Z}. \tag{5.22}$$

Since $W = GG^*$ is invertible, $G^*$ is injective and consequently, the operator $G$ is injective. Hence, $G$ is invertible. Combining (5.2) and (5.4), the inverse of $G$ can be expressed as

$$G^{-1} = G^*(GG^*)^{-1} = G^*W^{-1}. \tag{5.23}$$

Our goal is to prove the surjectivity of $G_F$. First, showing that the following mapping $\widetilde{G}_F : Z_y \to Z_y$ defined by

$$\widetilde{G}_F = G_F \circ G^{-1}$$

is surjective. Using (5.3), the operator $\widetilde{G}_F$ can be written as

$$\widetilde{G}_F\xi = G_F \circ G^{-1}\xi = \xi + K\xi, \quad \text{for } 0 \le t \le \tau \text{ and } \xi \in Z_y, \tag{5.24}$$

where the operator $K : Z_y \to Z_y$ is defined by

$$K\xi = \int_0^\tau T(\tau - s)F(s, \mathcal{Z}_\xi(s), G^{-1}\xi(s))\mathrm{d}s, \tag{5.25}$$

with $\mathcal{Z} = \mathcal{Z}_\xi$ being the solution given by (2.14) corresponding to the control $U$ defined by

$$U(t) = (G^{-1}\xi)(t) = G^*(GG^*)^{-1}\xi = G^*W^{-1}\xi = B^*T^*(\tau - t)W^{-1}\xi, \quad t \in [0, \tau]. \tag{5.26}$$

To show the surjectivity of $\widetilde{G}_F$, we need to show that $K$ defined by (5.25) is a contraction operator. For this, let us suppose that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are the solutions to (2.7) corresponding to the controls $U_1(s) = G^{-1}\xi_1(s)$ and $U_2(s) = G^{-1}\xi_2(s)$, respectively. Then, using (2.10) and Lemma 2.2, we obtain

$$\begin{aligned}\|K\xi_1 - K\xi_2\| &\le \int_0^\tau \|T(\tau - s)\|\|F(s, \mathcal{Z}_1(s), (G^{-1}\xi_1)(s)) - F(s, \mathcal{Z}_2(s), (G^{-1}\xi_2)(s))\|\mathrm{d}s \\ &\le L\int_0^\tau (\|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\| + \|(G^{-1}\xi_1)(s) - (G^{-1}\xi_2)(s)\|)\mathrm{d}s.\end{aligned} \tag{5.27}$$

The last integral in (5.27) can be written as follows:

$$\begin{aligned}&\int_0^\tau (\|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\| + \|(G^{-1}\xi_1)(s) - (G^{-1}\xi_2)(s)\|)\mathrm{d}s \\ &= \int_0^\tau \|\mathcal{Z}_1(s) - \mathcal{Z}_2(s)\|\mathrm{d}s + \int_0^\tau \|U_1(s) - U_2(s)\|\mathrm{d}s \quad \text{(use (5.18))} \\ &\le \tau\Gamma(\tau)\|U_1 - U_2\| + \int_0^\tau \|U_1(s) - U_2(s)\|\mathrm{d}s,\end{aligned} \tag{5.28}$$

where $\Gamma(\tau)$ is given by (5.19). By the Cauchy-Schwartz inequality, the last integral in (5.28) becomes

$$\int_0^\tau \|U_1(s) - U_2(s)\|\mathrm{d}s \le \left(\int_0^\tau \mathrm{d}s\right)^{1/2}\left(\int_0^\tau \|U_1(s) - U_2(s)\|^2\mathrm{d}s\right)^{1/2} \le \sqrt{\tau}\|U_1 - U_2\|. \tag{5.29}$$

Plugging (5.29) and (5.28) into (5.27), we obtain

$$\begin{aligned}\|K\xi_1 - K\xi_2\| &\le L\sqrt{\tau}(\sqrt{\tau}\Gamma(\tau) + 1)\|U_1 - U_2\| \quad \text{(use (5.26))} \\ &\le L\sqrt{\tau}(\sqrt{\tau}\Gamma(\tau) + 1)\|B^*T^*(\tau - s)W^{-1}\|\,\|\xi_1 - \xi_2\| \\ &\le L\sqrt{\tau}(\sqrt{\tau}\Gamma(\tau) + 1)\sup_{0\le s\le t\le\tau}\{\|T^*(t - s)\|\}\|B^*\|\|W^{-1}\|\|\xi_1 - \xi_2\|.\end{aligned} \tag{5.30}$$

As $(T^*(t))_{t\ge0}$ is a semigroup of contractions (Remark 2.2) and using (5.17) and (2.18), (5.30) becomes

$$\|K\xi_1 - K\xi_2\| \le L\sqrt{\tau}(\sqrt{\tau}\Gamma(\tau) + 1)\|\xi_1 - \xi_2\|. \tag{5.31}$$

Therefore, $K$ is the Lipschitz operator with a Lipschitz constant

$$\begin{aligned}L_K &= L\sqrt{\tau}(\sqrt{\tau}\Gamma(\tau) + 1) \quad \text{(use (5.26))} \\ &= \sqrt{\tau}Le^{L\tau}(e^{-L\tau} + \tau(1 + L)).\end{aligned} \tag{5.32}$$

As $\sqrt{\tau}e^\tau < \frac{1}{L}$, we have

$$\sqrt{\tau}Le^{L\tau} < e^{(L-1)\tau}.$$

Using $L < 1$, (5.32) becomes

$$L_K < e^{(L-1)\tau}(e^{-L\tau} + \tau(1 + L)). \tag{5.33}$$

Hence, $L_K < 1$ for $\tau$ satisfying $\sqrt{\tau}e^\tau < \frac{1}{L}$.

Now, we are ready to prove the surjectivity of $\widetilde{G}_F$. For any $y \in Z_\gamma$, we define the mapping $\varphi_y : Z_\gamma \to Z_\gamma$ by

$$\varphi_y\xi = y - K\xi. \tag{5.34}$$

Then, for any $\xi_1, \xi_2 \in Z_\gamma$, we have

$$\|\varphi_y\xi_1 - \varphi_y\xi_2\| = \|K\xi_2 - K\xi_1\| \le L_K\|\xi_1 - \xi_2\|.$$

As $L_K < 1$, then $\varphi_y$ is a contraction, i.e, there exists $\xi_y^* \in Z_y$ such that $\varphi_y \xi_y^* = \xi_y^*$. So, for any $y \in Z_y$, we infer from (5.34) that

$$\varphi_y \xi_y^* = \xi_y^* = y - K\xi_y^*$$

and $y = \xi_y^* + K\xi_y^* = \widetilde{G}_F \xi_y^*$. Thus, for any $y \in Z_y$, there exists $\xi_y^* \in Z_y$, such that $\widetilde{G}_F \xi_y^* = y = \xi_y^* + K\xi_y^*$.

Hence, $\widetilde{G}_F$ is surjective. Using (5.23), the operator $G_F$ can be written as

$$G_F = \widetilde{G}_F \circ G,$$

which implies the surjectivity of $G_F$ (as being composed of two surjective operators), that is to say

$$G_F(L^2(0, \tau; L^2(\Omega))) = \mathscr{R}(G_F) = Z_y.$$

Thus, according to the second assertion of Remark 5.1, the nonlinear problem (2.7) is exact controllable on $[0, \tau]$. □

Finally, we give the explicit form of the distributed controls $U = (u_1, u_2)$ of problem (2.7) acting on the whole domain.

**Theorem 4.** *Under the assumptions of* Theorem 3, *given* $\mathcal{Z} \in Z_y$, *the distributed controls of problem* (2.7) *are given by*

$$U(t) = \sum_{k=1}^{n} \sum_{k=1}^{p} \sum_{i=0}^{2} \frac{e^{(\tau - t)\sigma_i(k)}}{\Xi_k} \begin{pmatrix} d_{i1}^k & d_{i2}^k & d_{i3}^k \\ d_{i4}^k & d_{i5}^k & d_{i6}^k \end{pmatrix} P_k(I + K)^{-1}\mathcal{Z}, \quad t \in [0, \tau], \tag{5.35}$$

*where* $\Xi_k$ *is defined by* (5.12) *and*

$$\begin{aligned} d_{i1}^k &= c_k^{-1}\Theta_{i2}b_{i1} + c_k\Theta_{i4}b_{i2} - c_k^{-1}\Theta_{i5}b_{i3}, \\ d_{i2}^k &= c_k^{-2}\Theta_{i2}b_{i2} + \Theta_{i4}b_{i4} + c_k^{-2}\Theta_{i5}b_{i5}, \\ d_{i3}^k &= c_k^{-1}\Theta_{i2}b_{i3} + c_k\Theta_{i4}b_{i5} - c_k^{-1}\Theta_{i5}b_{i6}, \\ d_{i4}^k &= \Theta_{i3}b_{i1} - c_k^{2}\Theta_{i5}b_{i2} + \Theta_{i6}b_{i3}, \\ d_{i5}^k &= c_k^{-1}\Theta_{i3}b_{i2} + c_k\Theta_{i5}b_{i4} - c_k^{-1}\Theta_{i6}b_{i5}, \\ d_{i6}^k &= \Theta_{i3}b_{i3} + c_k^{2}\Theta_{i5}b_{i5} + \Theta_{i6}b_{i6}. \end{aligned} \tag{5.36}$$

**Proof.** By virtue of Theorem 3, given $\mathcal{Z} \in Z_y$ there exists a control $U \in L^2(0, \tau; L^2(\Omega))$ such that

$$G_F U = \mathcal{Z}. \tag{5.37}$$

We claim that the operator $G_F$ is invertible. Using $(5.24))_1$ it is sufficient to prove that the operator $\widetilde{G}_F$ is invertible. Since we have already proved that $\widetilde{G}_F$ is surjective, it remains to show that $\widetilde{G}_F$ is injective. For this, let us suppose that $\xi_1, \xi_2 \in Z_y$ such that $\widetilde{G}_F\xi_1 = \widetilde{G}_F\xi_2$. Then, using $(5.24)_2$ we obtain $\xi_1 + K\xi_1 = \xi_2 + K\xi_2$, where $K$ is defined by (5.25). This implies

$$\|\xi_1 - \xi_2\| = \|K\xi_2 - K\xi_1\| \le L_K\|\xi_1 - \xi_2\|.$$

Hence, $(1 - L_K)\|\xi_1 - \xi_2\| \le 0$, and as $(1 - L_K) > 0$, then $\xi_1 = \xi_2$. Thus, $\widetilde{G}_F$ is injective, and consequently, the operator $G_F$ is invertible.

Hence, from (5.37), the control $U$ can be expressed as

$$\begin{aligned} U(t) &= G_F^{-1}\mathcal{Z} = (\widetilde{G}_F \circ G)^{-1}\mathcal{Z} \\ &= G^{-1} \circ \widetilde{G}_F^{-1}\mathcal{Z} \quad (\text{use} \quad (5.24) \quad \text{and} \quad G^{-1} = G^*W^{-1}) \\ &= G^*W^{-1}(I + K)^{-1}\mathcal{Z} \quad (\text{use} \quad (5.2)) \\ &= B^*T^*(\tau - s)W^{-1}(I + K)^{-1}\mathcal{Z}. \end{aligned} \tag{5.38}$$

Following the same argument presented in [21], we find that the control $U$ is given by (5.35) and (5.36). □

## 6 Conclusion

In this work, and through a simple, powerful, and systematic approach based on the spectral analysis of semigroups, we prove the exponential stability and the exact controllability for a thermoviscoelastic plate system given by (1.1)–(1.3). Moreover, we provide the explicit expressions of the optimal decay rate and the distributed controls by the physical constants of the system. In fact, these expressions have never been given explicitly in the literature. The spirit of this approach is very different from the traditional methods used to study these topics for thermoviscoelastic plate. But our approach is more detailed and complete since it is based on the explicit expressions of the eigenvalues of the corresponding linear operator. Therefore, our spectral analysis gives more precise bounds and explicit expressions of the optimal decay rate and distributed controls.

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