Research Article

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Signal recovery and polynomiographic visualization of modified Noor iteration of operators with property (E)

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Abstract: This article aims to provide a modified Noor iterative scheme to approximate the fixed points of generalized nonexpansive mappings with property (E) called MN-iteration. We establish the strong and weak convergence results in a uniformly convex Banach space. Additionally, numerical experiments of the iterative technique are demonstrated using a signal recovery application in a compressed sensing situation. Ultimately, an illustrative analysis regarding Noor, SP-, and MN-iteration procedures is obtained via polysomnographic techniques. The images obtained are called polynomiographs. Polynomiographs have importance for both the art and science aspects. The obtained graphs describe the pattern of complex polynomials and also the convergence properties of the iterative method. They can also be used to increase the functionality of the existing polynomiography software.

Keywords: signal recovery, polynomiography, MN-iteration, García-Falset mapping, weak and strong convergence

MSC 2020: 46T99, 47H09, 47H10, 47J25, 49M37, 54H25

1 Introduction and preliminaries

Throughout this article, we will denote C to be a nonempty closed convex subset of a real Banach space X. We will denote the set of fixed points of the operator $T : C \to C$ by $F(T) = \{x \in C : Tx = x\}$. For the sequence $\{x_n\}$ to x in C, the strong convergence and the weak convergence are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. An operator T on C is nonexpansive if, for each $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$, and for any $x \in C$ and $p \in F(T)$,

$$\|Tx - p\| \leq \|x - p\|.$$

A mapping $T : C \to X$ on a subset C of a Banach space X is satisfied

(i) Condition (C) [1]: if $\frac{1}{2}\|x - Ty\| \leq \|x - y\|$, then $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

(ii) Condition (E_μ) [2]: if there exists $\mu \geq 1$ such that

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\| \quad \text{for all } x, y \in C.$$

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Moreover, we say that T satisfies Condition (E) on C whenever T satisfies Condition (E_μ) for some $\mu \geq 1$. Let C be convex. We say that $T : C \to C$ is said to satisfy

(iii) Condition (I) [3]: if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $f(s) > 0, \forall s > 0$, with

$$f(d(x, F(T))) \leq d(x, Tx),$$

$\forall x \in C$, where $d(x, F(T)) = \inf_{x \in F(T)} \|x - p\|$.

Fixed point theory plays a very crucial role in the fields of pure and applied mathematics as well as in many other branches of science, for instance, they are applicable to solving differential equation, classification, regression, signal recovery, and image restoration, see [4–18] and references therein.

Various iterative schemes for fixed points numerical approximation have been introduced by assorted authors. Further, many authors have discussed fixed points of various classes and their generalization to nonlinear maps in a Banach space (see [1,19–23]). Some research works of iterative schemes were originated and generally recognized for estimating fixed points of nonexpansive mappings, in particular [24–29].

In 1890, Picard defined the iteration scheme that bears his name, given by

$$x_{n+1} = Tx_n,$$

which is the simplest iteration scheme and also known as the functional iteration scheme. As many know, the iterative scheme of Picard is not necessarily convergent in the case of nonexpansive operators. Berinde's book [30] provides many interesting and fundamental results on these schemes. The Noor iteration [27] was defined as follows: $x_0 \in C$ and

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $(0, 1)$.

In 2011, Phuengrattana and Suantai [28] introduced the following new three-step iteration process known as the SP-iteration:

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Ty_n, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $(0, 1)$.

In 2018, a generalization for the class of nonexpansive mappings on Banach spaces was proposed by Suzuki [1]. He entitled this property Condition (C) (also labeled the class of Suzuki generalized nonexpansive mappings or simplified the class of Suzuki mappings), but it is appropriately contained into the class of quasi-nonexpansive mappings. With regard to the numerical reckoning of fixed points for Suzuki mappings, some convergence theorems were provided. Furthermore, based on some three-step iteration schemes, the necessary and sufficient conditions for the existence of fixed points were also proved, for example, see [31–33].

In 2011, García-Falset et al. [2] proposed a general class of Suzuki mappings, and Condition (E) is the resulting property. However, it still remains stronger than quasi-nonexpansiveness. The class of mappings satisfying Condition (E) will be henceforward referred to as García-Falset-generalized nonexpansive mappings or García-Falset mappings. Recently, Usurelu et al. [34] provided a consequence regarding the existence of fixed points for García-Falset mappings in the framework of uniformly convex Banach spaces, based on the Thakur et al.'s iterative scheme introduced in [35].

An adjacent direction that went into exceeding nonexpansiveness on Hilbert spaces was the definition of nonspreading mappings by Kohsaka and Takahashi [36] (more precisely, studied in connection with firm type nonexpansive mappings). Subsequently, this class was described using hybrid mappings introduced by Takahashi [37] only for all nonexpansive, nonspreading, and hybrid mappings to be later included by Kocourek et al. [38] into the class of (α, β)-generalized hybrid mappings. However, all these classes of operators are stronger than quasinonexpansive ones whenever a fixed point exists. The study of (α, β)-generalized hybrid mappings, which was first introduced in Hilbert spaces, is investigated in Banach space as follows.

Let X be a Banach space and let C be a nonempty closed convex subset of X. Then, a mapping $T : C \to C$ is called (α, β)-generalized hybrid (see [39]) if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{3}$$

for all $x, y \in C$.

We found that there exists a mapping that satisfies Condition (E) on a subset C but fails to be (α, β)-generalized hybrid mapping. This is the reason why we are interested in studying Condition (E). Indeed, we analyze the following example.

Example 1.1. Consider $X = \mathbb{R}^3$ with the usual Euclidean norm and let

$$C = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

We shall consider the mapping

$$T : C \to C, \quad T : \begin{pmatrix} (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) \\ (1, 0, 0), (0, 0, 0), (0, 0, 1), (0, 1, 0) \end{pmatrix}.$$

Clearly, $\|x - Ty\| \leq \sqrt{2}$ and $\|x - Tx\| \neq 0$ for all $x, y \in C$, so T is a García-Falset mapping with $\mu = \sqrt{2}$. In order to prove that T is not (α, β)-generalized hybrid mapping, we shall take $x = (0, 0, 1)$, and $y = (0, 0, 0)$. It follows that $\|Tx - Ty\|^2 = 2$, $\|Tx - y\|^2 = 1$, $\|Ty - x\|^2 = 2$, and $\|x - y\|^2 = 1$. Evaluating inequality (3) for these values obviously leads to a contradiction; thus, T is not (α, β)-generalized hybrid mapping.

To precede inspirational research, we propose an efficient iterative method, called the MN-iteration process, which generates a sequence $\{x_n\}$ by $x_0 \in C$ and

$$\begin{aligned} z_n &= (1 - e_n)x_n + e_nTx_n, \\ y_n &= (1 - c_n - d_n)x_n + c_nTx_n + d_nTz_n, \\ x_{n+1} &= (1 - a_n - b_n)x_n + a_nTz_n + b_nTy_n, \end{aligned} \tag{4}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{c_n + d_n\}$, and $\{a_n + b_n\}$ are in $(0, 1)$.

The main purpose of this article, by using an iterative scheme (4) for García-Falset mappings in a uniformly convex Banach space, some weak and strong convergence theorems are shown. In Section 2, the weak and strong convergence theorems for our iterative scheme can be obtained by some control assumptions. In Section 3, numerical examples of the iterative technique are demonstrated using a signal recovery application in a compressed sensing situation. Furthermore, we show the use of the proposed method to generate polynomiographs. The effects produced by running the resulting algorithms were visualized via polynomiography. The conclusions are presented in Section 4.

2 Convergence results

In this section, we provide the convergence theorems of MN-iteration (4) for Garcia-Falset mappings in a uniformly convex Banach space.

Now, we prove the weak convergence result.

Throughout this section, let C be a nonempty closed convex subset of a real Banach space X and $T : C \to C$ a quasi-nonexpansive mapping that satisfies Condition (E).

We start with proving the following useful results.

Lemma 2.1. *Suppose that $F(T)$ is nonempty and let $\{x_n\}$ be a sequence defined by the MN-iteration* (4), *where $x_0 \in C$. Then,* $\lim_{n\to\infty}\|x_n - p\|$ *exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$. Since the mapping T is quasi-nonexpansive. Using Proposition 1 [2], we have

$$\|Tz_n - p\| \le \|z_n - p\| \le (1 - e_n)\|x_n - p\| + e_n\|Tx_n - p\| \le \|x_n - p\|. \tag{5}$$

Applying (5), we obtain

$$\begin{aligned} \|Ty_n - p\| \le \|y_n - p\| &= \|(1 - c_n - d_n)(x_n - p) + c_n(Tx_n - p) + d_n(Tz_n - p)\| \\ &\le (1 - c_n - d_n)\|x_n - p\| + c_n\|Tx_n - p\| + d_n\|Tz_n - p\| \\ &\le (1 - c_n - d_n)\|x_n - p\| + c_n\|x_n - p\| + d_n\|z_n - p\| \\ &\le (1 - c_n - d_n)\|x_n - p\| + c_n\|x_n - p\| + d_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{6}$$

Using (5) and (6), we have

$$\begin{aligned} \|x_{n+1} - p\| &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|Tz_n - p\| + b_n\|Ty_n - p\| \\ &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|z_n - p\| + b_n\|y_n - p\| \\ &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|x_n - p\| + b_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{7}$$

Obviously, $\{\|x_n - p\|\}$ is bounded and nonincreasing for all $p \in F(T)$. That is, $\lim_{n\to\infty}\|x_n - p\|$ exists. □

Theorem 2.2. *Let X be a uniformly convex and $\{x_n\}$ be a sequence defined by the MN-iteration* (4), *where $x_0 \in C, \{e_n\}$ is bounded away from* 0 *and* 1 *for all $n \ge 0$. Then, $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Proof. Suppose that $F(T) \ne \emptyset$ and $p \in F(T)$. Then, by Lemma 2.1, there exists $r \ge 0$ such that $r = \lim_{n\to\infty}\|x_n - p\|$ and the sequence $\{x_n\}$ is bounded. Next, we will show that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Taking $\limsup$ in (5), so we obtain

$$r = \limsup_{n\to\infty}\|x_n - p\| \ge \limsup_{n\to\infty}\|z_n - p\|. \tag{8}$$

By the quasinonexpansiveness of T, we have

$$r = \limsup_{n\to\infty}\|x_n - p\| \ge \limsup_{n\to\infty}\|Tx_n - p\|. \tag{9}$$

In addition, using (6), we have

$$\begin{aligned} \|x_{n+1} - p\| &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|Tz_n - p\| + b_n\|Ty_n - p\| \\ &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|z_n - p\| + b_n\|y_n - p\| \\ &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|z_n - p\| + b_n\|x_n - p\| \\ &= (1 - a_n)\|x_n - p\| + a_n\|z_n - p\| \\ &= \|x_n - p\| + a_n(\|z_n - p\| - \|x_n - p\|). \end{aligned}$$

It follows that

$$\|z_n - p\| \ge \|x_{n+1} - p\|.$$

Taking lim inf in the above inequality, we obtain

$$r \ge \limsup_{n\to\infty}\|z_n - p\| \ge \liminf_{n\to\infty}\|z_n - p\| \ge r.$$

That is,

$$r = \lim_{n\to\infty}\|z_n - p\| = \lim_{n\to\infty}\|(1 - e_n)(x_n - p) + e_n(Tx_n - p)\|. \tag{10}$$

Applying (8), (9), and (10) together with Lemma 1.3 [40], we can conclude that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Consequently, assume that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Next, suppose that $p \in A(C, \{x_n\})$. Since T satisfies Condition (E), the following relation is obtained:

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n\to\infty}\|Tx_n - p\| \\ &\le \limsup_{n\to\infty}(\mu\|Tx_n - x_n\| + \|x_n - p\|) \\ &= \limsup_{n\to\infty}\|x_n - p\| = r(Tp, \{x_n\}). \end{aligned}$$

This follows that $Tp \in A(C, \{x_n\})$. By the uniqueness of asymptotic centers, we have $p = Tp$, i.e., $F(T)$ is nonempty. The proof is completed. □

Next, we now prove the weak convergence result.

Theorem 2.3. *Let X be a uniformly convex with Opial's property. Let T and $\{x_n\}$ be the same in Theorem 2.2 and $F(T)$ is nonempty. Then $\{x_n\}$ converges weakly to a point in $F(T)$.*

Proof. First, we have $\{x_n\}$ as a bounded sequence, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$, and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ by Lemma 2.1 and Theorem 2.2. Second, let $\{x_{n_i}\}$ and $\{x_{m_i}\}$ be subsequences of $\{x_n\}$ weakly converging to z_1 and z_2, respectively. Then, $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = \lim_{i\to\infty}\|x_{m_i} - Tx_{m_i}\| = 0$. Then, we obtain $z_1, z_2 \in C$ since C is closed and convex, also by Mazur's theorem. By the demiclosedness at zero of $I - T$ from Theorem 1 [2], we have $z_1, z_2 \in F(T)$. Finally, we can conclude that $z_1 = z_2$ by Lemma 2.7 [41]. Therefore, $\{x_n\}$ converges weakly to a fixed point of T. □

The strong convergence results are discussed in the rest of this section.

Theorem 2.4. *Let X be a uniformly convex Banach space and C be a nonempty, compact, and convex subset of X. Let T and $\{x_n\}$ be the same as in Theorem 2.2. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to a point in $F(T)$.*

Proof. Suppose that $F(T) \neq \varnothing$. According to Theorem 2.2, one has $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. By using the fact that C is compact, $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ that converges strongly to a point $p \in C$. Property (E) gives us the inequality

$$\|x_{n_j} - Tp\| \le \mu\|x_{n_j} - Tx_{n_j}\| + \|x_{n_j} - p\|.$$

Taking the limit in this relation, the right-hand side of this inequality becomes 0, so, by the uniqueness of the limit, $\{x_{n_j}\}$ converges to Tp. This leads to $Tp = p$, so $p \in F(T)$. Finally, by Lemma 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$, so $\{x_n\}$ converges strongly to a fixed point of T. □

Our last convergence result focuses on Garcia-Falset generalized mappings that, in addition, satisfy Condition (I).

Theorem 2.5. *Let X be a uniformly convex Banach space and C a nonempty, closed, and convex subset of X. Define T and $\{x_n\}$ similarly with Theorem 2.2. If T satisfies Condition (I) and $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to a point in $F(T)$.*

Proof. Lemma 2.1 ensures the existence of $\lim_{n\to\infty}\|x_n - p\|$ for any $p \in F(T)$, so $\lim_{n\to\infty}d(x_n, F(T))$ exists. Let us denote $\lim_{n\to\infty}\|x_n - p\| = r \ge 0$. If $r = 0$, the result follows immediately. Assume $r \neq 0$. From Condition (I) and Theorem 2.2, we obtain

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Considering the properties of function f, we obtain

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and $\{y_k\} \subset F(T)$ such that

$$||x_{n_k} - y_k|| < \frac{1}{2^k} \qquad \text{for all } k \in \mathbb{N}. \tag{11}$$

Knowing that $n_{k+1} > n_k$, by repeatedly applying inequality (7), we find

$$||x_{n_{k+1}} - y_k|| \le ||x_{n_k} - y_k|| \le \frac{1}{2^k}.$$

For $k \to \infty$, it follows

$$||y_{k+1} - y_k|| \le ||y_{k+1} - x_{n_{k+1}}|| + ||x_{n_{k+1}} - y_k|| \le \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}} \to 0.$$

Thus, $\{y_k\}$ is a Cauchy sequence in $F(T)$. Conversely, let us note that the set $F(T)$ is closed. Therefore, $\{y_k\}$ converges to a fixed point p of T. This, together with relation (11), implies $x_{n_k} \to p$. Again, using Lemma 2.1, $\lim_{n\to\infty}||x_n - p||$ exists, consequently $x_n \to p$. This completed the proof. □

3 Signal recovery and polynomiography

In this section, we apply our iterative scheme to the problem of recovering the original signal from compressive measurements. We will also show the use of the proposed method to generate polynomiographs.

3.1 Signal recovery

In signal processing, compressed sensing can be modeled as the following under the determinated linear equation system: $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$ are the original signals and the observed data, respectively,

$$y = Ax + \varepsilon, \tag{12}$$

where $A \in \mathbb{R}^{M\times N}(M < N)$, and $\varepsilon \in \mathbb{R}^M$ represents the Gaussian noise. Finding the solution of the previously determinated linear equation system (12) can be seen as solving the following regularized least squares problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}||Ax - y||_2^2 + \zeta||x||_1, \tag{13}$$

where $\zeta > 0$.

The problem (13) can be seen as the fixed point problem through the following settings:

$$Tx = \operatorname{prox}_{\zeta g}(x - \theta\nabla f(x)),$$

where $f(x) = \frac{1}{2}||Ax - y||_2^2$, $g(x) = \zeta||x||_1$, $\nabla f(x) = A^T(y - Ax)$, $\zeta > 0$, and $\theta \in \left(0, \frac{2}{||A||_2^2}\right)$.

In the experiment, y is generated by Gaussian noise with the signal-to-noise ratio (SNR) = 50, A is generated by a normal distribution with mean zero and variance one, and the sparse vector $x \in \mathbb{R}^N$ with m nonzero components to be recovered is generated from a uniform distribution in the interval $[-2, 2]$.

Now, we present a numerical result for problem (13). In particular, we investigate the behavior of the MN-iteration (4) and then compare it with two iterative schemes: the Noor iteration (1) and the SP-iteration (2).

Let $N = 256$ and $M = 128$ be the size of the signal. Assume that the original signal has 20 nonzero elements, then generate the Gaussian matrix A by using the MATLAB routine randn(M, N), $\theta = \frac{1}{||A||_2^2}$ and $\zeta = \frac{1}{10\ ||A||_2^2}$. Next,

Table 1: Numerical comparison of three iterative schemes for $\sigma = 0.01$

	Elapsed time	No. of iterations	SNR
MN-iteraton	1.2160	4,630	11.4818
SP-iteration	2.8754	12,135	11.4776
Noor iteration	18.3084	78,622	11.4772

select $x_0 = A^t y$ as the initial point. For any $n \geq 0$, let $\alpha_n = a_n = c_n = \frac{n+2}{8n+8}$, $\beta_n = \frac{\sqrt{5n+10}}{10\sqrt{15n+14}}$, $\gamma_n = e_n = \sqrt{\frac{n+1}{2n+3}}$, and $b_n = d_n = \frac{5n+6}{6n+12}$

The performance of the tested methods at the nth iteration is measured quantitatively using SNR, which is defined by

$$\mathrm{SNR}(x_n) = 20\log_{10}\left(\frac{\|x_n\|_2^2}{\|x_n - x\|_2^2}\right)$$

and comparing the accuracy between the recovered signals with the mean-squared error

$$\mathrm{MSE}_n = \frac{1}{N}\|x - x_n\|^2 < \sigma,$$

where x_n is the approximated signal of x.

In Table 1, we can find recorded data regarding the elapsed times and the number of iterations for each iterative scheme. As demonstrated below, the MN-iteration typically requires less elapsed time compared to the other two iterative schemes. Similarly, the MN-iteration involves fewer iterations than the others.

Additionally, we present the reconstructed signals after 10,000 iterations in Figure 1 and SNR and mean-squared error in Figures 2 and 3, respectively.

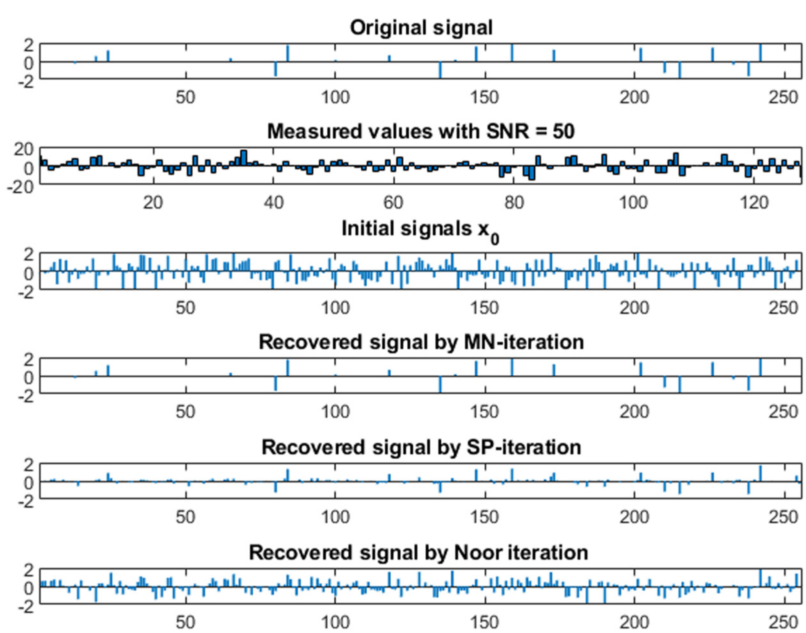

Figure 1: The original signal, measured values with SNR = 50, initial signals x_0, and the recovery signals by the MN-iteration, the SP-iteration, and the Noor iteration.

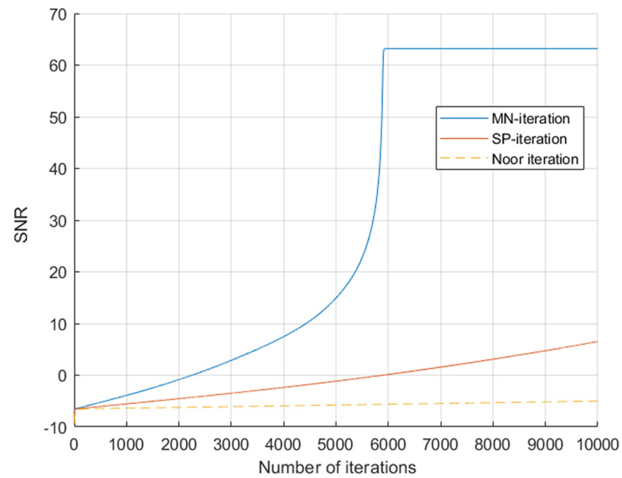

Figure 2: SNR quality versus number of iterations.

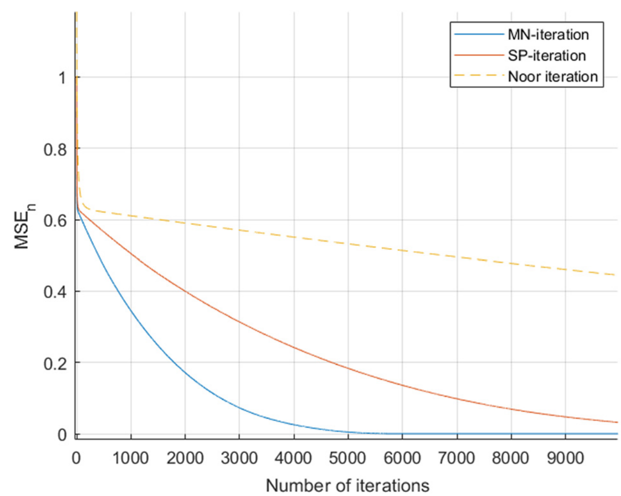

Figure 3: Mean-squared error versus number of iterations.

3.2 Polynomiography

Patterns generated by a single polynomial root-finding method have been recognized since the 1980s and garnered significant attention within the computer graphics community [42–44]. Around the year 2000, a specific term emerged in the literature to describe images created through root-finding techniques. These images were coined *polynomiographs*, and the collective methods for their generation were referred to as *polynomiography*. Both terms were introduced by Kalantari [45]. The precise definition he provided is as follows.

Polynomiography is the art and science of visualizing the approximations of complex polynomial zeros through the creation of fractal (see [46,47]) and non-fractal images, utilizing the mathematical convergence properties of iterative functions. For more information on fractals and iteration schemes, we refer the reader to [48].

To generate a polynomiograph, we define an area of interest in the complex plane denoted as $A \subset \mathbb{C}$. For each point z_0 within this defined area, we employ an iterative root-finding method denoted as $R = T$ over Noor iteration (1), SP-iteration (2), and MN-iteration (4), when $n = 0, 1, \ldots, M$.

We continue the iteration process until either the convergence test criteria are met or the maximum allowed number of iterations is reached. Upon concluding the iteration process, we apply a coloring scheme to the starting point (z_0) using a designated coloring function. Two fundamental coloring functions are commonly used:

(1) Iteration-based coloring: Colors are assigned based on the number of iterations performed, employing a predefined color map.
(2) Basins of attraction coloring: Each root of the polynomial is assigned a unique color, and we determine the color based on the nearest root to the point at which the iteration stops.

In polynomiography, the main element of the generation algorithm is the root-finding method. Many different root-finding methods exist in the literature. Let us recall some of these methods:

(1) The Halley method [49]

$$H(z) = z - \frac{2p'(z)p(z)}{2p'(z)^2 - p''(z)p(z)}.$$

(2) The Newton method [49]

$$N(z) = z - \frac{p(z)}{p'(z)}.$$

(3) The Householder method [50]

$$E(z) = N(z) + \frac{p(z)^2 p''(z)}{2p'(z)^3}.$$

We start with an example presenting the use of an affine combination of the root-finding methods. In the example, we use three root-finding methods: Halley, Newton, and Householder family. And, we generated polynomiographs for three complex polynomials:

(1) $p_2(z) = z^2 + 1$, with roots: $i, -i$.
(2) $p_3(z) = z^3 + 1$, with roots: $-1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$.
(3) $p_5(z) = z^5 + 1$, with roots: $-1, \frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{i\sqrt{2}\sqrt{5-\sqrt{5}}}{4}, -\frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{i\sqrt{2}\sqrt{\sqrt{5}+5}}{4}$.

After each iteration, we proceed with the iteration process till the convergence test is satisfied or the maximum number of iterations is reached. The standard convergence test has the following form:

$$|x_{n+1} - x_n| < \varepsilon,$$

where $\varepsilon > 0$ is the accuracy of the computations. The other parameters used were the following: $\alpha_n = a_n = c_n = \frac{n+2}{8n+8}$, $\beta_n = \frac{\sqrt{5n+10}}{10\sqrt{15n+14}}$, $\gamma_n = e_n = \sqrt{\frac{n+1}{2n+3}}$, and $b_n = d_n = \frac{5n+6}{6n+12}$, $\varepsilon = 0.001$, resolution of 100×100 pixels, and K is the maximum number of iterations.

The metrics used for comparison are defined as follows. The Convergence Area Index (CAI) is calculated using the following formula:

$$\mathrm{CAI} = \frac{N_c}{N},$$

where N_c represents the count of points in the polynomiograph that have achieved convergence, and N denotes the total number of points in the polynomiograph. The CAI's value falls within the range of 0 (indicating that no point has reached convergence) to 1 (signifying that all points have successfully converged).

3.2.1 $p_2(z) = z^2 + 1$

The complex polynomial equation $p_2(z) = z^2 + 1$ features two roots: i and $-i$. Figure 4 showcases the polynomiograph, revealing two separate basins of attraction, each corresponding to one of the polynomial's roots.

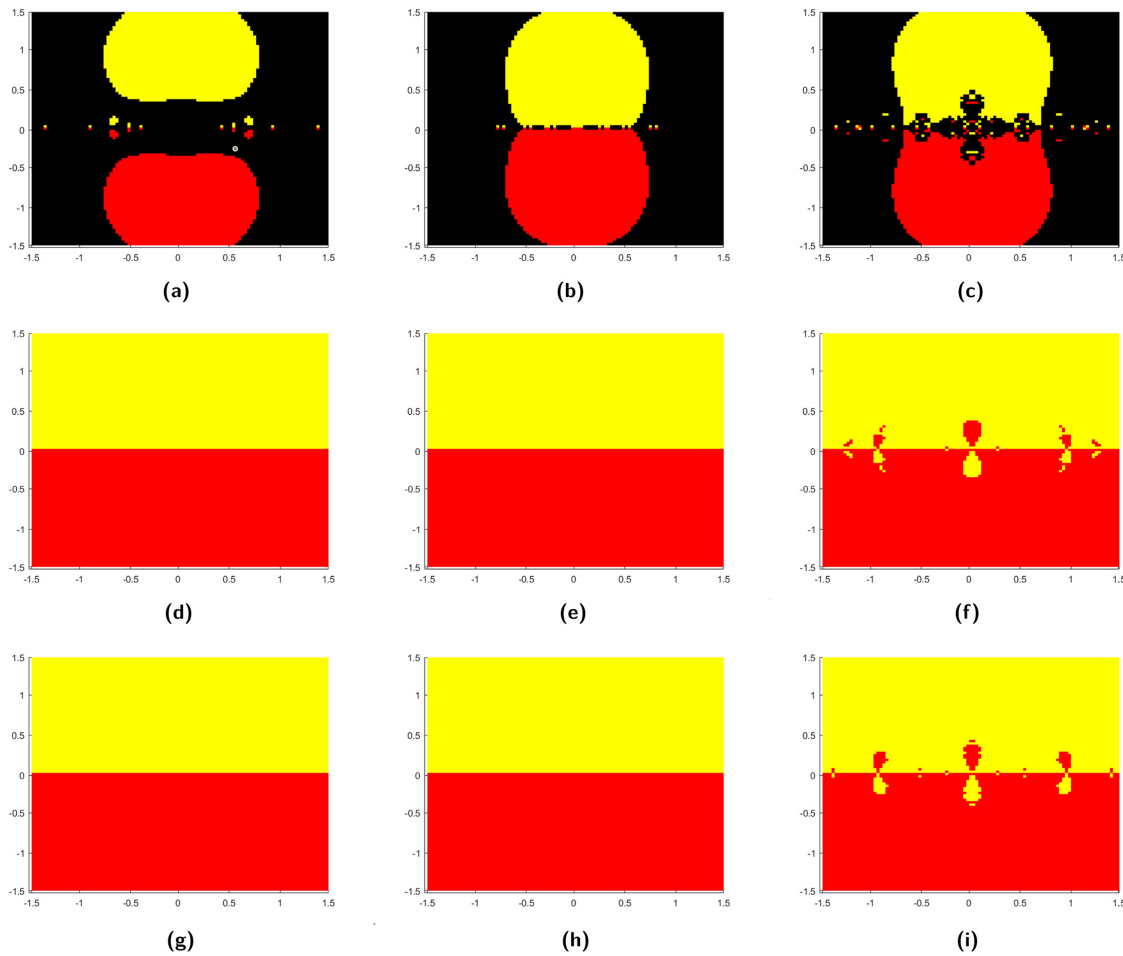

Figure 4: Basins of attraction for $z^2 + 1$ generated using various root-finding methods and black color are divergence for $K = 30$. (a)–(c) come from the use of Noor iteration, (d)–(f) come from the use of SP-iteration, and (g)–(i) come from the use of MN-iteration. (a) Halley method, (b) Newton method, (c) Householder method, (d) Halley method, (e) Newton method, (f) Householder method, (g) Halley method, (h) Newton method, and (i) Householder method.

In Figure 5, we can observe the polynomiographs generated for the $p_2(z)$ polynomial using Halley, Newton, and Householder methods in Noor, SP, and MN iterations, while Figure 4 displays the associated basins of attraction.

Table 2 presents the numerical measures derived from these plots. Upon analyzing the polynomiographs, it becomes evident all methods exhibit exceptional stability in root-finding. This phenomenon is especially pronounced in both MN-iteration and SP-iteration modes, where CAI achieves a pristine score of 1. This outcome signifies the effective partitioning of the complex plane into two equal basins. Points with a real part less than zero reliably converge to −1, while those situated in the other half of the plane converge to 1. However, it is noteworthy that the Noor iteration results in CAI values of 0.1736, 0.2212, and 0.1872 for the Halley, Newton, and Householder methods, respectively.

This indicates that within the considered area, a considerable proportion of starting points do not successfully converge to the desired roots. Conversely, for the other methods, stability diminishes in the vicinity of the vertical line passing through the origin, where black areas represent these non-convergent points.

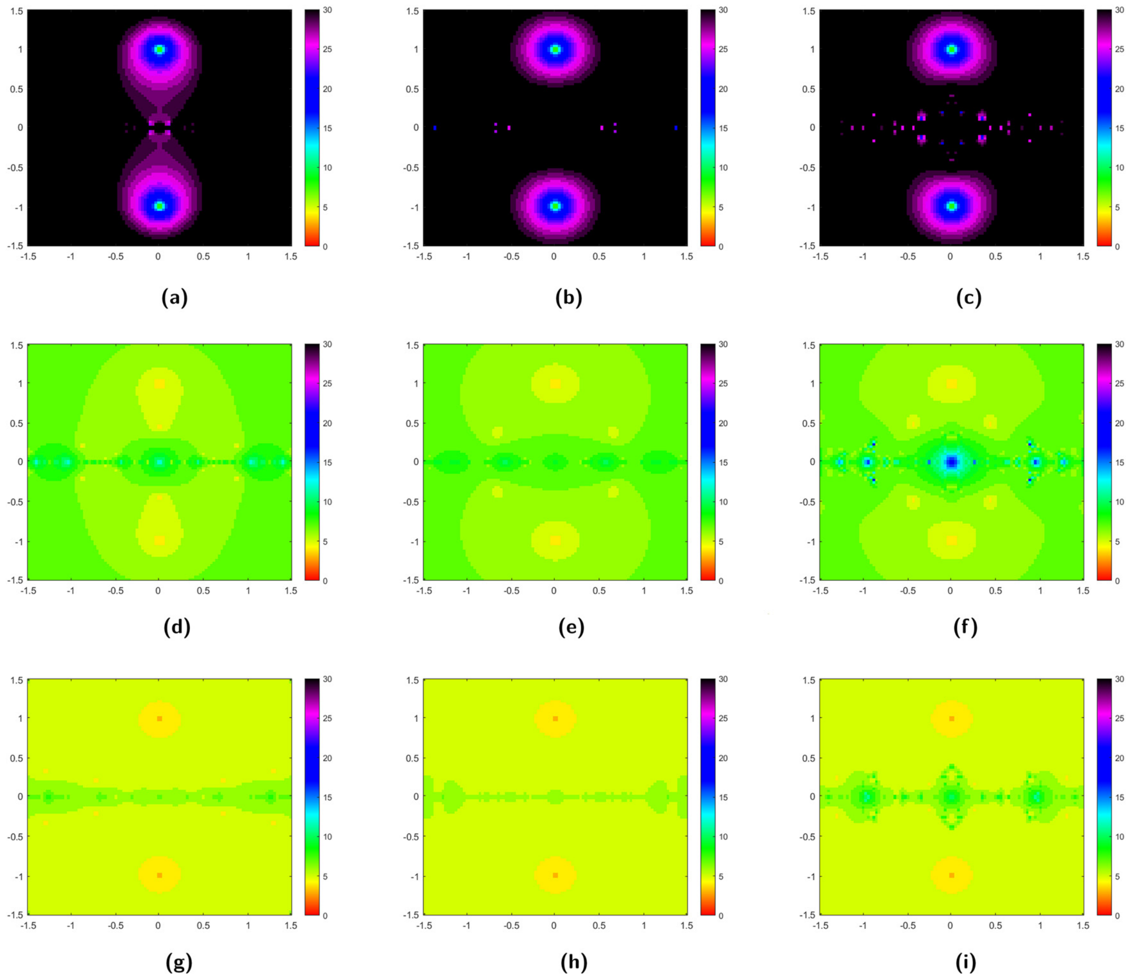

Figure 5: Polynomiographs for $z^2 + 1$ generated using various root-finding methods for $K = 30$. (a)–(c) come from the use of Noor iteration, (d)–(f) come from the use of SP-iteration, and (g)–(i) come from the use of MN-iteration. (a) Halley method, (b) Newton method, (c) Householder method, (d) Halley method, (e) Newton method, (f) Householder method, (g) Halley method, (h) Newton method, and (i) Householder method.

Table 2: $p_2(z) = z^2 + 1$: results obtained for the standard convergence test and the Halley, Newton, and Householder methods come from the use of Noor, SP, and MN iterations for $K = 30$

		CAI
Noor iteration	Halley method	0.1736
	Newton method	0.2212
	Householder method	0.1872
SP-iteration	Halley method	1
	Newton method	1
	Householder method	1
MN-iteration	Halley method	1
	Newton method	1
	Householder method	1

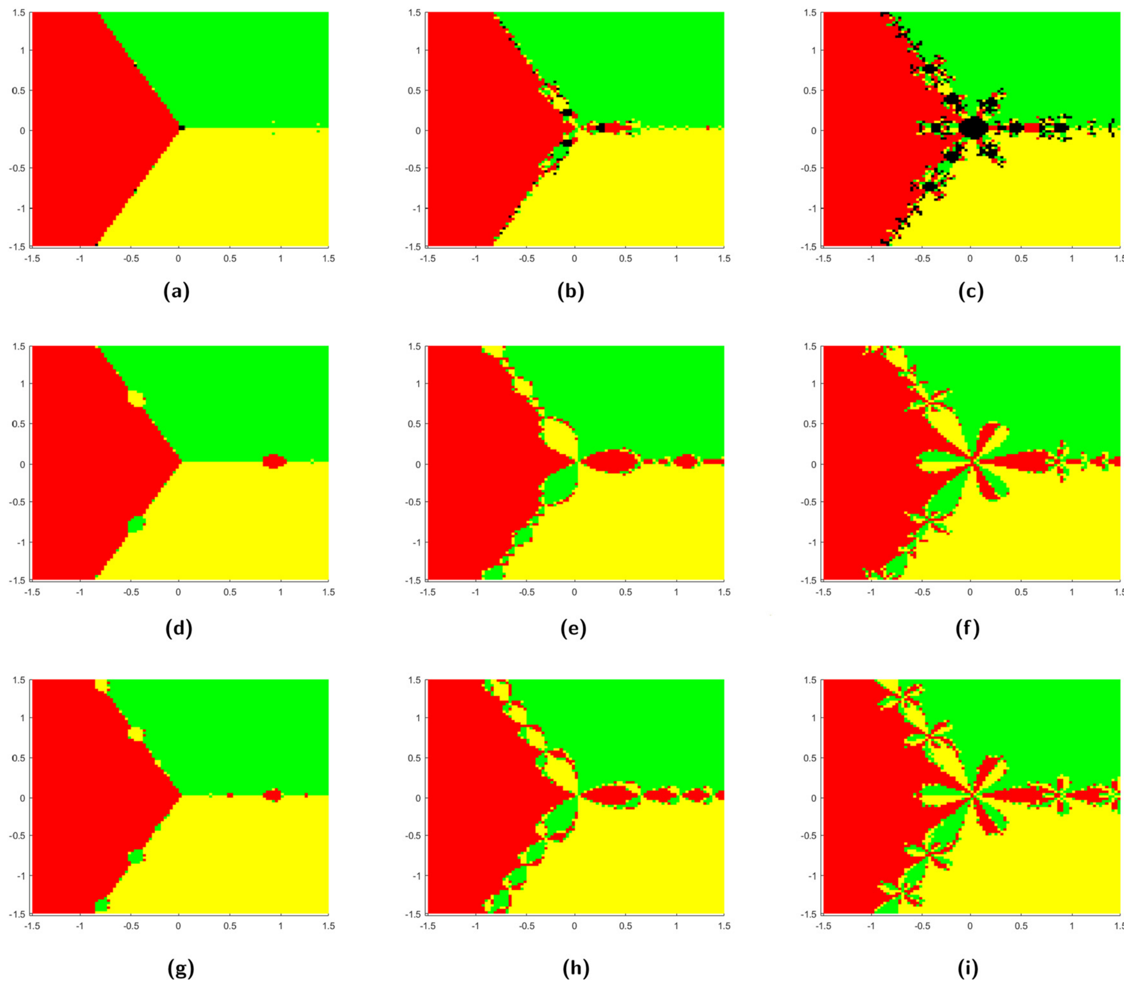

Figure 6: Basins of attraction for $z^3 + 1$ generated using various root-finding methods and black color are divergence for $K = 50$. (a)–(c) come from the use of Noor iteration, (d)–(f) come from the use of SP-iteration, and (g)–(i) come from the use of MN-iteration. (a) Halley method, (b) Newton method, (c) Householder method, (d) Halley method, (e) Newton method, (f) Householder method, (g) Halley method, (h) Newton method, and (i) Householder method.

3.2.2 $p_3(z) = z^3 + 1$

The complex polynomial equation $p_3(z) = z^3 + 1$ features three roots: $-1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$. Figures 6 and 7 showcase the polynomiograph, revealing three separate basins of attraction, each corresponding to one of the polynomial's roots.

The resulting polynomiographs, along with the numerical measures, for the $p_3(z)$ polynomial are presented in Figures 6, 7, and Table 3. In this case, we observe a diverse behavior of the methods. While characteristic braids are visible in each case, their shapes vary among the methods. The interweaving of the basins around the braids is minimal for the Halley method, and the braids appear similar in Noor, SP, and MN iterations. In contrast, the braids formed by the other Halley, Newton, and Householder methods take on various shapes and exhibit different complexities. The most intricate braids are observed with the Householder method, resulting in the largest interweaving of basins. Outside the braided regions, the behavior of the methods is quite similar. Examining the values in Table 3, we find that CAI indicates the best performance for every method in MN-iteration and SP-iteration. It achieves convergence for all starting points within the area, with a CAI value of 1.0. Except in Noor iteration, we see that a small percentage of the starting points did not converge to any of the roots.

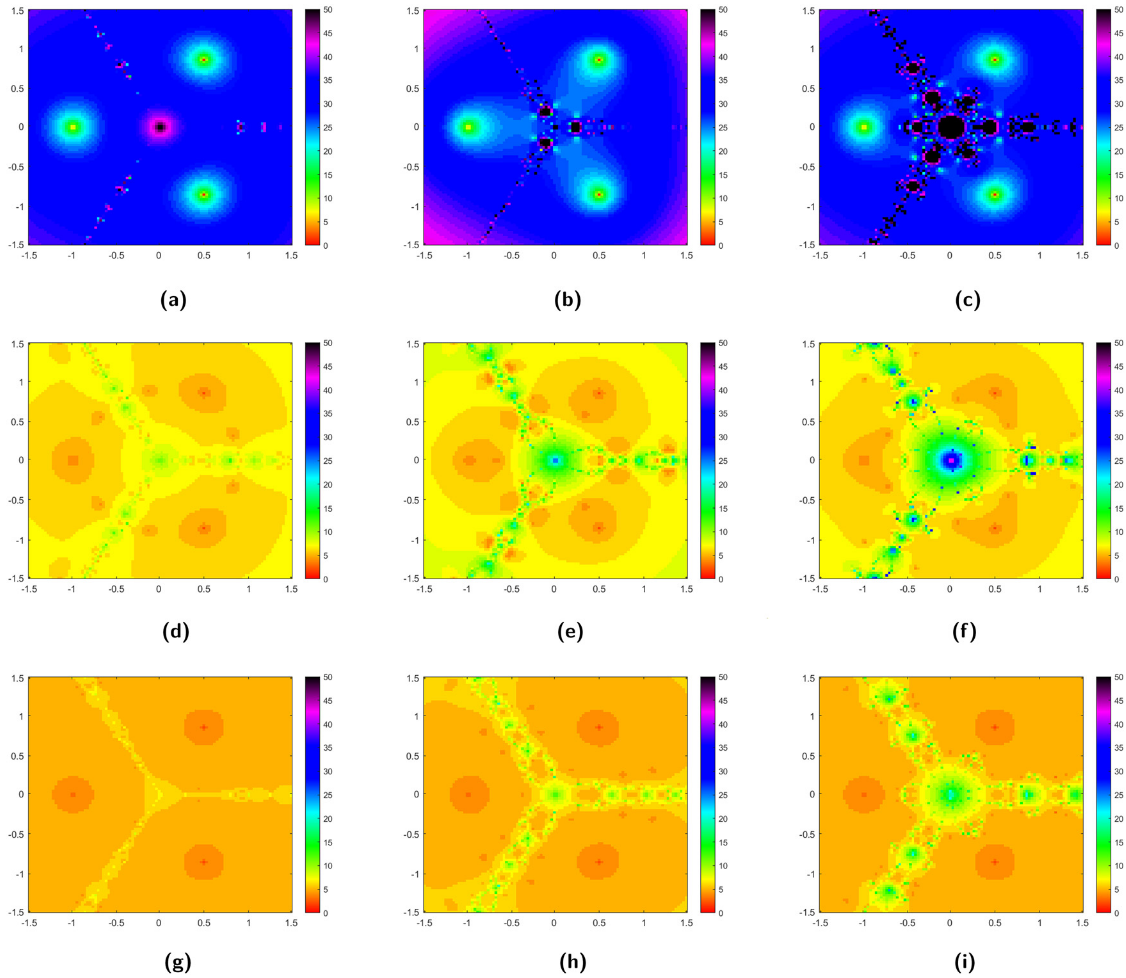

Figure 7: Polynomiographs for $z^3 + 1$ generated using various root-finding methods for $K = 50$. (a)–(c) come from the use of Noor iteration, (d)–(f) come from the use of SP-iteration, and (g)–(i) come from the use of MN-iteration. (a) Halley method, (b) Newton method, (c) Householder method, (d) Halley method, (e) Newton method, (f) Householder method, (g) Halley method, (h) Newton method, and (i) Householder method.

Table 3: $p_3(z) = z^3 + 1$: results obtained for the standard convergence test and the Halley, Newton, and Householder methods come from the use of Noor, SP, and MN iterations for $K = 50$

		CAI
Noor iteration	Halley method	0.9992
	Newton method	0.9936
	Householder method	0.9486
SP-iteration	Halley method	1
	Newton method	1
	Householder method	1
MN-iteration	Halley method	1
	Newton method	1
	Householder method	1

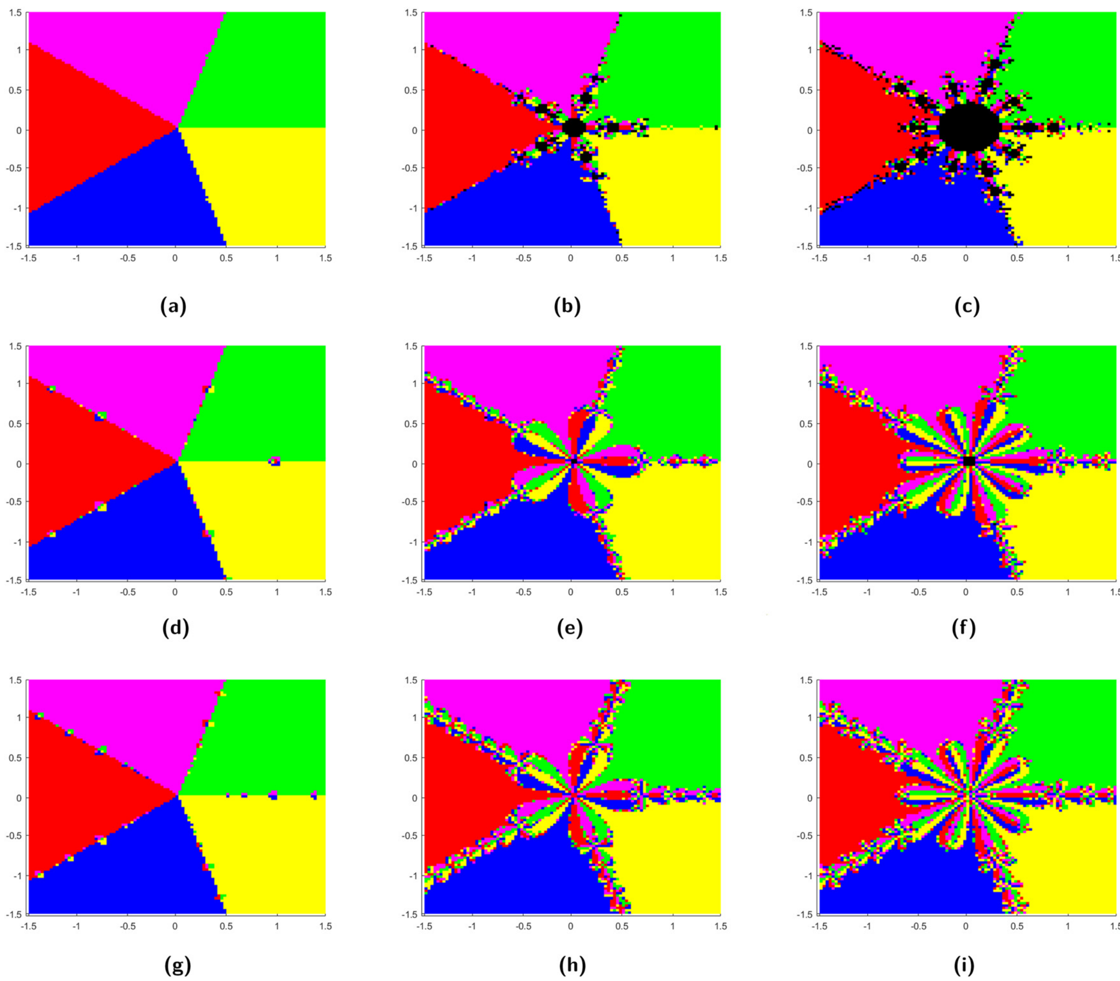

Figure 8: Basins of attraction for $z^5 + 1$ generated using various root-finding methods and black color are divergence for $K = 80$. (a)–(c) come from the use of Noor iteration, (d)–(f) come from the use of SP-iteration, and (g)–(i) come from the use of MN-iteration. (a) Halley method, (b) Newton method, (c) Householder method, (d) Halley method, (e) Newton method, (f) Householder method, (g) Halley method, (h) Newton method, and (i) Householder method.

3.2.3 $p_5(z) = z^5 + 1$

The complex polynomial equation $p_5(z) = z^5 + 1$ features five roots: $-1, \frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{i\sqrt{2}\sqrt{5-\sqrt{5}}}{4}, -\frac{\sqrt{5}}{4} + \frac{1}{4} \pm \frac{i\sqrt{2}\sqrt{\sqrt{5}+5}}{4}$. Figures 8 and 9 showcase the polynomiograph, revealing five separate basins of attraction, each corresponding to one of the polynomial's roots.

In the last case of polynomial, $p_5(z)$, the polynomiographs generated using the considered methods are presented in Figures 8 and 9, and the corresponding numerical measures are summarized in Table 4. This time, we observe significant disparities among the polynomiographs. The Halley method exhibits the most stable behavior, with smaller braids and reduced interweaving of basins compared to the other methods. Conversely, the Householder method displays the least stability. Unlike the other methods, it lacks the characteristic braids and instead reveals a multitude of interwoven basins. This results in the division of basins into numerous smaller regions, creating a highly intricate pattern. Despite its lower stability, the Householder method demonstrated a favorable convergence ratio with CAI values ranging from 0.9142 to 1 for Noor, SP, and MN iterations, we see that this also indicates that there were instances where a small portion of the initial points did not converge to any of the roots. It is noted that although the Halley method gives good CAI values, the Householder method produces beautiful images with more artistic value.

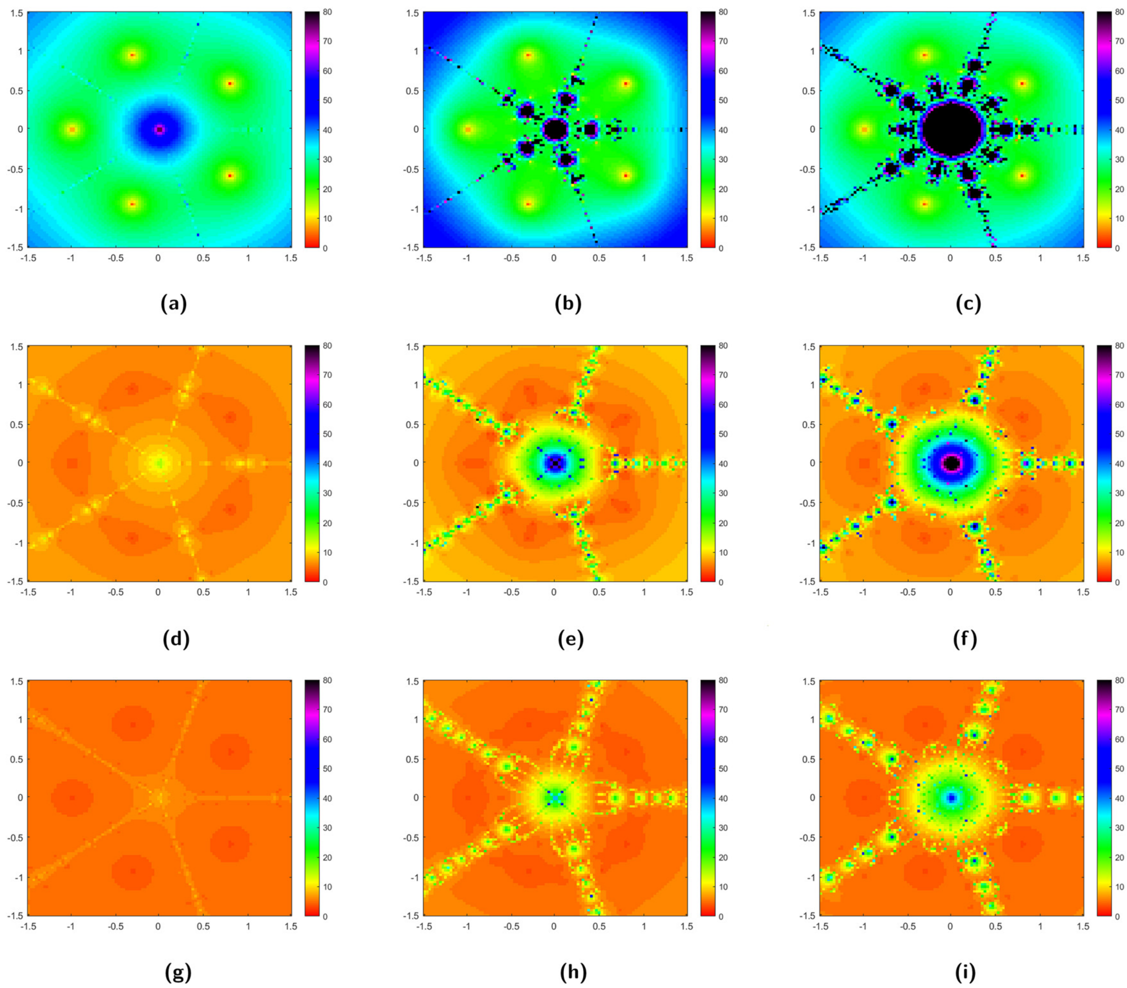

Figure 9: Polynomiographs for $z^5 + 1$ generated using various root-finding methods for $K = 80$. (a)–(c) come from the use of Noor iteration, (d)–(f) come from the use of SP-iteration, and (g)–(i) come from the use of MN-iteration. (a) Halley method, (b) Newton method, (c) Householder method, (d) Halley method, (e) Newton method, (f) Householder method, (g) Halley method, (h) Newton method, and (i) Householder method.

Table 4: $p_5(z) = z^5 + 1$: results obtained for the standard convergence test and the Halley, Newton, and Householder methods come from the use of Noor, SP, and MN iterations for $K = 50$

		CAI
Noor iteration	Halley method	1
	Newton method	0.9760
	Householder method	0.9142
SP-iteration	Halley method	1
	Newton method	0.9994
	Householder method	0.9972
MN-iteration	Halley method	1
	Newton method	1
	Householder method	1

4 Conclusion

In summary, we present a modified Noor iterative method for solving the fixed points of generalized nonexpansive mappings with property (E). Besides, we validate the iterative scheme's weak and strong convergence properties under certain conditions. After that, the iterative scheme is applied to the signal recovery problem in compressed sensing. Furthermore, we show the use of the proposed method to generate polynomiographs. The proposed algorithm has been implemented and tested via numerical simulation in MATLAB. The simulation results show the effectiveness of the proposed algorithm.

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Ethical approval: The conducted research is not related to either human or animals use.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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