Research Article

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Fractional Sturm-Liouville operators on compact star graphs

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Abstract: In this article, we examine two problems: a fractional Sturm-Liouville boundary value problem on a compact star graph and a fractional Sturm-Liouville transmission problem on a compact metric graph, where the orders α_i of the fractional derivatives on the ith edge lie in $(0, 1)$. Our main objective is to introduce quantum graph Hamiltonians incorporating fractional-order derivatives. To this end, we construct a fractional Sturm-Liouville operator on a compact star graph. We impose boundary conditions that reduce to well-known Neumann-Kirchhoff conditions and separated conditions at the central vertex and pendant vertices, respectively, when $\alpha_i \to 1$. We show that the corresponding operator is self-adjoint. Moreover, we investigate a discontinuous boundary value problem involving a fractional Sturm-Liouville operator on a compact metric graph containing a common edge between the central vertices of two star graphs. We construct a new Hilbert space to show that the operator corresponding to this fractional-order transmission problem is self-adjoint. Furthermore, we explain the relations between the self-adjointness of the corresponding operator in the new Hilbert space and in the classical L^2 space.

Keywords: fractional Sturm-Liouville operator, metric graph, transmission condition, fractional-order derivative, star graph

MSC 2020: 26A33, 81Q10, 34B37, 34B45, 34B24

1 Introduction

The well-known Sturm-Liouville differential equation is given by

$$-(p(x)y')' + q(x)y = \lambda w(x)y, \tag{1}$$

and it emerges in many real-world applications [1]. In addition, every second-order linear differential equation can be converted to (1) by choosing an integral factor. For these reasons, the Sturm-Liouville equation has been a crucial topic in mathematics. In particular, the spectral properties of (1) with boundary conditions

$$u_1y(a) + u_2y'(a) = 0,$$
$$v_1y(b) + v_2y'(b) = 0,$$

are well known [1], where p, q, and w are real-valued, continuous functions on a compact interval $[a, b]$ such that $p(x) \neq 0$ and $w(x) > 0$ over $[a, b]$, and u_i and v_i are real numbers for i=1, 2.

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Abel initiated non-integer order calculus while he was studying on tautochrone problem [2,3]. Namely, he proved that the solution of the equation:

$$\frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}\mathrm{d}t = g(x), \tag{2}$$

where $x \in (a,b]$, $0 < \alpha < 1$, exists as a member of the space of integrable functions on $[a,b]$ if and only if the function

$$\frac{1}{\Gamma(1-\alpha)}\int_a^x g(t)(x-t)^{-\alpha}\mathrm{d}t$$

is absolutely continuous on $[a,b]$, and the equation

$$\lim_{x\to a^+}\int_a^x g(t)(x-t)^{-\alpha}\mathrm{d}t = 0$$

holds. Riemann defined a fractional integral using the left-hand side of (2), and then fractional calculus has been developed by many authors [4–12]. In particular, it has been shown that mathematical models based on fractional-order equations provide more accurate results than models based on ordinary differential equations do in many real-world applications [10,13–16].

The fractional counterpart of Sturm-Liouville equation (1) has been introduced in [9]. The results about the eigenvalues and eigenfunctions of (1) are obtained with the help of integration by parts formula. Therefore, to define a fractional analogue of the Sturm-Liouville equation (1), one has to exploit the integration by parts formulas (A1) and (A2). As a result, there are two possible choices to define a fractional counterpart of (1). Namely, one can either use the left Riemann-Liouville derivative with the right Caputo derivative or the right Riemann-Liouville derivative with the left Caputo derivative. Let us choose the first pair to define the fractional Sturm-Liouville equation [9]

$${}^{c}D_{b^-}^{\alpha}(p(x)D_{a^+}^{\alpha}y(x)) + q(x)y(x) = \lambda w(x)y(x), \tag{3}$$

where p, q, and w are real-valued, continuous functions on a compact interval $[a,b]$ such that $p(x) \neq 0$ and $w(x) > 0$ over $[a,b]$. If the order α of the fractional derivative is in $(0,1)$, then the authors imposed the following boundary conditions [9]

$$u_1 I_{a^+}^{1-\alpha}y(a) + u_2 D_{a^+}^{\alpha}y(a) = 0, \tag{4}$$

$$v_1 I_{a^+}^{1-\alpha}y(b) + v_2 D_{a^+}^{\alpha}y(b) = 0, \tag{5}$$

where u_1, u_2, v_1, and v_2 are real numbers such that $u_1^2 + u_2^2 > 0$ and $v_1^2 + v_2^2 > 0$. The authors [9] investigated the eigenvalues and eigenfunctions of the boundary value problem (3)–(5). In particular, they [9] proved that the eigenvalues of the boundary value problem (3)–(5) are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal in the weighted Hilbert space $L_w^2(a,b)$. Other spectral properties of the boundary value problems generated by (3) have been studied in recent years [17–23]. We remark here that $I_{a^+}^{1-\alpha}y(x) \to 0$ as $x \to a^+$ for a bounded, integrable function y over $[a,b]$. Indeed, for $\alpha \in (0,1)$ and a bounded, integrable function y, it follows that [21]

$$\begin{aligned} |I_{a^+}^{1-\alpha}y(x)| &\leq \frac{1}{\Gamma(1-\alpha)}\int_a^x |(x-s)^{-\alpha}y(s)|\mathrm{d}s \\ &\leq \frac{C}{\Gamma(1-\alpha)}\int_a^x (x-s)^{-\alpha}\mathrm{d}s = \frac{C}{\Gamma(1-\alpha)}\frac{(x-a)^{1-\alpha}}{1-\alpha}, \end{aligned}$$

where C is a constant.

Differential operators acting on metric graphs have a long history. There are many reasons for studying such operators. The most important reason is that these operators can be used to model wave propagation in a thin neighborhood of a graph-like structure. A model of this type has been used in chemistry for the first time

[24]. Other models can be observed in quantum chaos, quantum wires, dynamical systems, photonic crystals, scattering theory, nanotechnology, mesoscopic physics, etc. For more models and applications, we refer the reader to the survey [25].

A quantum graph refers to a metric graph equipped with a differential operator. The study of quantum graphs continues to attract the attention of many physicists and mathematicians since it provides a simple but non-trivial model for studying quantum phenomena that are difficult to analyze in higher dimensions. In particular, the number of studies about quantum graphs has increased in the last two decades after Kottos and Smilansky proposed quantum graphs as a model for studying quantum chaos [26]. The interested reader is referred to [27] for a complete review of the applications of quantum graphs in quantum chaos and to the monographs [28,29] for more details about quantum graphs.

Most of the studies about quantum graphs focus on the Laplace operator $-\mathrm{d}^2/\mathrm{d}x^2$ or the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V(x)$ with a potential V as differential operators (Hamiltonians). Both operators are defined by the second-order derivative. To the best of our knowledge, there is no study regarding the differential operators described by fractional derivatives on metric graphs. As many studies [10,13–16] have recently shown that models using fractional derivatives give better results, considering fractional operators on metric graphs will be essential in future applications. Moreover, metric graphs are natural generalizations of intervals and are frequently observed in physical problems where the physical phenomena take place in a network rather than an interval. These motivate us to consider the fractional Sturm-Liouville operator

$$f = (f_i)_{i=1}^n \to ({}^{c}D_{L_i^-}^{\alpha_i}(p_i(x)D_{0^+}^{\alpha_i}f_i(x)) + f_i(x)q_i(x))_{i=1}^n \tag{6}$$

on a compact star graph such that the fractional orders α_i vary over the edges and $\alpha_i \in (0,1)$. Note that if $\alpha_i \to 1$, one gets the second-order Sturm-Liouville operator. We consider a star graph because it is the simplest but non-trivial metric graph, and also, every graph locally (around at a vertex) looks like a star graph.

Many real-world applications can be characterized as classical boundary value problems. However, some of them contain some unusual inner conditions. For instance, a string having the density function $p(x)$ on $[0,1]$, pinned down at $x = 1$, bearing a particle of mass m at $x = 0$, and joined at $x = -1$ by a weightless string can be characterized as follows [30]:

$$\begin{aligned} &-y'' = \lambda p(x)y, \quad x \in [0,1], \\ &y(-1) = y(1) = 0, \\ &\lambda m y(0) = y(0) - y'(0), \end{aligned} \tag{7}$$

where $y'(0)$ is the right-hand derivative of y at $x = 0$ and left-hand derivative of y at $x = 0$ is defined as $y(0) = y'(-1)$. Note that the condition

$$y(0) = y(-1) + y'(-1)$$

holds. The last condition in (7) has the form

$$\lambda m y(0) = y'(-1) - y'(0).$$

Consequently, one has

$$\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\lambda m & 1-\lambda m \end{bmatrix} \begin{bmatrix} y(-1) \\ y'(-1) \end{bmatrix}.$$

Atkinson [30] shared some results for the vector differential equations having symplectic matrices. However, a much more effective approach has been given by Mukhtarov and his colleagues [31–34] for such discontinuous boundary value problems. Namely, they have introduced some special inner products to obtain information about the geometric properties of the Hilbert space. In this work, we will mainly follow this approach [31–36] in the transmission problem on a compact metric graph. Discontinuous boundary value problems incorporating fractional derivatives have been considered in recent years [37–40].

In this study, we investigate two different problems. In the first problem, we consider the fractional Sturm-Liouville operator (6) on a compact star graph with fractional orders $\alpha_i \in (0,1)$. Our primary aim is to determine the vertex conditions that give rise to a self-adjoint fractional-order Hamiltonian. Moreover, we intend to introduce the boundary conditions which reduce to well-known Neumann-Kirchhoff conditions at the central vertex and separated conditions at pendant vertices when $\alpha_i \to 1$. This allows us to generalize the classical (second-order) Sturm-Liouville operator to its fractional counterpart and the fractional Sturm-Liouville operator on a compact interval [9] to metric graphs.

In the second problem, we consider the operator (6) with orders $\alpha_i \in (0,1)$ on a metric graph Ψ which consists of two compact star graphs Ψ_1 and Ψ_2 together with an edge connecting the central vertices of Ψ_1 and Ψ_2. We impose transmission conditions on this mutual edge. This problem is a generalization of a regular boundary value transmission problem on a single interval to a much more complicated setting. On the other hand, this problem also generalizes the second-order boundary value transmission problem on Ψ to its fractional counterpart since $\alpha_i \to 1$ yields the second-order Sturm-Liouville operator. We aim to investigate the self-adjointness of this boundary value transmission problem using two methods. Following the operator-theoretic formulation [31–34], we transfer the transmission problem to a boundary value problem in a special Hilbert space in the first method. For the second method, let us consider the metric graph Ψ' obtained by introducing a new vertex v to the metric graph Ψ at the point of interaction. Following the method in [41], we can transfer the transmission problem on Ψ to a boundary value problem on Ψ' when we consider the transmission conditions as vertex conditions at v. We obtain the necessary condition which makes the Hamiltonian on Ψ' self-adjoint in the usual sense, namely, with respect to the standard inner product (A3). Moreover, motivated by [31–34], we modify the standard inner product to make the Hamiltonian self-adjoint in a new Hilbert space.

Below, we list the notations used in this article.

- $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.
- D denotes the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$.
- $AC[a,b]$ denotes the space of absolutely continuous functions on $[a,b]$.
- For $m = \{1, 2, \ldots\}$, $AC^m[a,b]$ denotes the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $m-1$ on $[a,b]$ such that $f^{(m-1)}(x) \in AC[a,b]$.
- A^T and A^* denote the transpose and adjoint of a matrix A, respectively.
- Γ denotes the Euler gamma function.
- $\mathrm{Re}(z)$ denotes the real part of the complex number z.
- E_v denotes the set of edges including the vertex v.
- d_v denotes the degree of a vertex v.

2 Fractional Sturm-Liouville operator on a compact star graph

We consider a compact star graph $\mathcal{G}$ with a finite edge set $\{e_1, e_2, \ldots, e_n\}$, where the length of the edge e_i is denoted by L_i and $L_i < \infty$ for $i = 1, 2, \ldots, n$. In this section, we define a self-adjoint quantum graph Hamiltonian $\mathcal{H} = (\mathcal{H}_i)_{i=1}^n$ defined by

$$f = (f_i)_{i=1}^n \to \mathcal{H}f \coloneqq (\mathcal{H}_i f_i)_{i=1}^n, \tag{8}$$

where the action of $\mathcal{H}_i$ on the edge e_i is given by

$$\mathcal{H}_i f_i \coloneqq {}^c D_{L_i^-}^{\alpha_i}(p_i(x) D_{0^+}^{\alpha_i} f_i(x)) + f_i(x) q_i(x), \quad x \in [0, L_i] \tag{9}$$

such that the orders of fractional derivatives satisfy $\alpha_i \in (0,1)$. Moreover, for each $i = 1, 2, \ldots, n$, we assume that p_i, q_i are real-valued, continuous functions on $[0, L_i]$ and $p_i(x) \neq 0$ for all $x \in [0, L_i]$. Note that we have dropped the subscript i for the local coordinate $x_i \in [0, L_i]$ since it is more convenient for writing and does not cause confusion.

Remark 2.1. Unlike second-order differential operators given by (A4) and (A5), the fractional-order differential operator $\mathcal{H}_i$ is not symmetric with respect to the reflection

$$(Rf)(x) \coloneqq f(L_i - x),$$

in the interval $[0, L_i]$, since we have

$$RD_{0^+}^{\alpha_i} = D_{L_i^-}^{\alpha_i} R, \quad R({}^c D_{0^+}^{\alpha_i}) = {}^c D_{L_i^-}^{\alpha_i} R.$$

For this reason, we need to consider a directed metric graph to define the Hamiltonian $\mathcal{H}$. We assume that the star graph $\mathcal{G}$ is directed from the pendant vertices v_i $(i = 1, 2, \dots, n)$ towards the central vertex v. The local coordinate of the central vertex corresponds to $x = L_i$ and the local coordinates of the pendant vertices correspond to $x = 0$ on each edge e_i.

Consider the real solutions $u_i(x)$ and $v_i(x)$ of equation

$$\mathcal{H}_i f_i = \lambda f_i, \tag{10}$$

corresponding to the parameter $\lambda = 0$ that satisfy the conditions [23]

$$\begin{aligned} p_i(L_i) D_{0^+}^{\alpha_i} u_i(L_i) = 0, \quad I_{0^+}^{1-\alpha_i} u_i(L_i) = 1, \\ p_i(L_i) D_{0^+}^{\alpha_i} v_i(L_i) = 1, \quad I_{0^+}^{1-\alpha_i} v_i(L_i) = 0. \end{aligned} \tag{11}$$

We note [23] that $u_i(x)$ and $v_i(x)$ are linearly independent on $[0, L_i]$. Let us denote

$$Q(y_i, z_i)(x) \coloneqq p_i(x)[I_{0^+}^{1-\alpha_i} y_i(x) D_{0^+}^{\alpha_i} z_i(x) - D_{0^+}^{\alpha_i} y_i(x) I_{0^+}^{1-\alpha_i} z_i(x)], \quad x \in [0, L_i]. \tag{12}$$

Moreover, consider two solutions $f_i(x, \lambda)$ and $g_i(x, \mu)$ of equation (10) corresponding to the parameters λ and μ, respectively. If $f_i, p_i D_{0^+}^{\alpha_i} f_i \in AC[0, L_i]$ and $g_i, p_i D_{0^+}^{\alpha_i} g_i \in AC[0, L_i]$, then we have the expansion (see Theorem 4.3 in [23])

$$Q(f_i, \overline{g_i}) = Q(f_i, u_i) Q(\overline{g_i}, v_i) - Q(f_i, v_i) Q(\overline{g_i}, u_i), \quad x \in [0, L_i]. \tag{13}$$

At pendant vertices v_i, we shall impose the boundary conditions

$$(\cos \tau_i) Q(f_i, v_i)(0) + (\sin \tau_i) Q(f_i, u_i)(0) = 0, \quad i = 1, 2, \dots, n, \tag{14}$$

where $\tau_i \in \mathbb{R}$. We remark [23] here that the equalities

$$Q(f_i, u_i)(0) = -p_i(0) D_{0^+}^{\alpha_i} f_i(0), \quad Q(f_i, v_i)(0) = I_{0^+}^{1-\alpha_i} f_i(0)$$

hold, and in particular, $Q(f_i, v_i)(0) \to f_i(0)$ and $Q(f_i, u_i)(0) \to -p_i(0) f_i'(0)$ as $\alpha_i \to 1$. Hence, one obtains the separated boundary conditions as $\alpha_i \to 1$.

At the central vertex v, we impose the vertex conditions:

$$I_{0^+}^{1-\alpha_1} f_1(L_1) = I_{0^+}^{1-\alpha_2} f_2(L_2) = \dots = I_{0^+}^{1-\alpha_n} f_n(L_n), \tag{15}$$

$$\sum_{i=1}^{n} p_i(L_i) D_{0^+}^{\alpha_i} f_i(L_i) = 0. \tag{16}$$

Note that the vertex conditions (15)–(16) reduce to Neumann-Kirchhoff conditions when $\alpha_i \to 1$.

Let $D_1(\mathcal{H})$ be the subspace that consists of functions $f = (f_i)_{i=1}^n \in L^2(\mathcal{G}) = \oplus_{i=1}^n L^2(0, L_i)$ such that

(i) $f_i, p_i D_{0^+}^{\alpha_i} f_i \in AC[0, L_i]$ for $i = 1, 2, \dots, n$,

(ii) $\mathcal{H} f \in L^2(\mathcal{G})$,

(iii) f satisfies the vertex conditions (14)–(16).

Theorem 2.2. *The quantum graph Hamiltonian $\mathcal{H} : D_1(\mathcal{H}) \to L^2(\mathcal{G})$ is self-adjoint.*

Proof. Let us first show that $\mathcal{H}$ is symmetric. For $\alpha_i \in (0, 1)$ and $f = (f_i)_{i=1}^n, g = (g_i)_{i=1}^n \in L^2(\mathcal{G})$, we have from (A2) that

$$\langle \mathcal{H} f, g \rangle = \langle f, \mathcal{H} g \rangle + \sum_{i=1}^{n} p_i(x)[D_{0^+}^{\alpha_i} \overline{g_i}(x) I_{0^+}^{1-\alpha_i} f_i(x) - D_{0^+}^{\alpha_i} f_i(x) I_{0^+}^{1-\alpha_i} \overline{g_i}(x)] \Bigg|_{x=0}^{L_i}. \tag{17}$$

By using (12), we can rewrite (17) as follows:

$$\langle \mathcal{H}f, g\rangle - \langle f, \mathcal{H}g\rangle = \sum_{i=1}^{n} Q(f_i, \overline{g_i})(x)\Big|_{x=0}^{L_i}.$$

Suppose $f, g \in D_1(\mathcal{H})$. Then, the vertex conditions (14) imply that

$$Q(f_i, u_i)(0)Q(\overline{g_i}, v_i)(0) - Q(f_i, v_i)(0)Q(\overline{g_i}, u_i)(0) = 0, \quad i = 1, 2, \ldots, n.$$

In view of the relation (13), the last equation is equivalent to

$$Q(f_i, \overline{g_i})(0) = 0, \quad i = 1, 2, \ldots, n.$$

Therefore, it follows

$$\sum_{i=1}^{n} Q(f_i, \overline{g_i})(0) = 0.$$

Moreover, the vertex conditions (15)–(16) yield

$$\sum_{i=1}^{n} Q(f_i, \overline{g_i})(L_i) = 0.$$

Therefore, we have $\langle \mathcal{H}f, g\rangle = \langle f, \mathcal{H}g\rangle$. Let $g = (g_i)_{i=1}^n$, $h = (h_i)_{i=1}^n \in L^2(\mathcal{G})$. It can be shown from (17) that $\langle \mathcal{H}f, g\rangle = \langle f, h\rangle$ for every $f \in D_1(\mathcal{H})$ iff g satisfies the vertex conditions (14)–(16) and $g \in D_1(\mathcal{H})$. This completes the proof. □

Remark 2.3. If $\alpha_i \to 1$, it follows that the fractional quantum graph Hamiltonian $\mathcal{H}$ coincides with the usual second-order Sturm-Liouville operator and the vertex conditions (14)–(16) reduce to their standard forms.

3 Fractional boundary value transmission problem on a metric graph

Let us consider the compact metric graph Ψ which consists of two star graphs Ψ_1 and Ψ_2 with central vertices v_1 and v_2, respectively, and an edge e_m connecting v_1 and v_2 (Figure 1). The edges of Ψ_1 are labeled as $e_1, e_2, \ldots, e_{m-1}$ and the edges of Ψ_2 are labeled as $e_{m+1}, e_{m+2}, \ldots, e_n$. All edges of Ψ_1 and Ψ_2 are directed towards their central vertices, and e_m is directed from v_1 to v_2. The length of the edge e_i is denoted by L_i and $L_i < \infty$ for $i = 1, 2, \ldots, n$.

In this section, we shall consider fractional Sturm-Liouville operator $\mathcal{H} = (\mathcal{H}_i)_{i=1}^n$ on Ψ, where p_i, q_i are real-valued continuous functions in $[0, L_i]$, $p_i(x) \neq 0$ for each $x \in [0, L_i]$ ($i = 1, 2, \ldots, n$), and the fractional

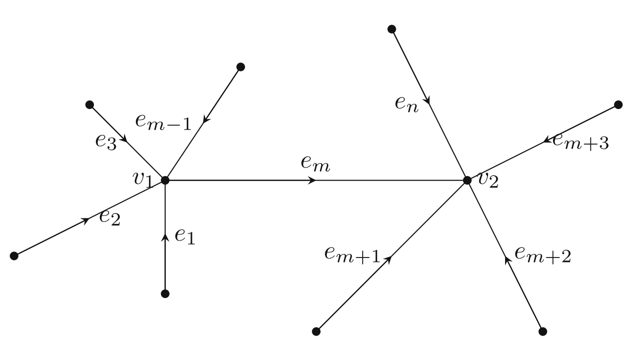

Figure 1: Compact metric graph Ψ.

orders α_i lie in $(0, 1)$. We impose a transmission condition on the common edge e_m and similar boundary conditions at other vertices as in the previous section.

We investigate the self-adjointness of this boundary value transmission problem with two approaches. First, we treat this as a transmission problem on the graph Ψ and define a particular inner product (depending on the transmission coefficients) in $L^2(\Psi)$, which makes the operator self-adjoint. In the second approach, we consider the transmission conditions as vertex conditions at a new vertex introduced at $x_{e_m} = c$ and thus transfer the transmission problem to a boundary value problem on a new graph. Then, we investigate the self-adjointness of this new quantum graph Hamiltonian with respect to the usual inner product (A3). We also define a particular inner product with the help of the one in the first approach, which makes the Hamiltonian self-adjoint for arbitrary transmission coefficients.

We impose the following boundary conditions:

$$(\cos\beta_i)Q(f_i, v_i)(0) + (\sin\beta_i)Q(f_i, u_i)(0) = 0, \quad i = 1, 2, \ldots, n, \tag{18}$$

where $u_i(x)$ and $v_i(x)$ are the real solutions of (10) corresponding to the parameter $\lambda = 0$ that satisfy (11) and $\beta_i \in \mathbb{R}$. Moreover, we impose the following boundary conditions:

$$I_{0^+}^{1-\alpha_1} f_1(L_1) = I_{0^+}^{1-\alpha_2} f_2(L_2) = \ldots = I_{0^+}^{1-\alpha_{m-1}} f_{m-1}(L_{m-1}), \tag{19}$$

$$\sum_{i=1}^{m-1} p_i(L_i) D_{0^+}^{\alpha_i} f_i(L_i) = 0, \tag{20}$$

$$I_{0^+}^{1-\alpha_m} f_m(L_m) = I_{0^+}^{1-\alpha_{m+1}} f_{m+1}(L_{m+1}) = \ldots = I_{0^+}^{1-\alpha_n} f_n(L_n), \tag{21}$$

$$\sum_{i=m}^{n} p_i(L_i) D_{0^+}^{\alpha_i} f_i(L_i) = 0. \tag{22}$$

In addition, we consider a transmission condition at an interior point c of the edge e_m (Figure 2) given by

$$I_{0^+}^{1-\alpha_m} f_m(c-) = \delta_{11} I_{0^+}^{1-\alpha_m} f_m(c+) + \delta_{12} p_m(c+) D_{0^+}^{\alpha_m} f_m(c+), \tag{23}$$

$$p_m(c-) D_{0^+}^{\alpha_m} f_m(c-) = \delta_{21} I_{0^+}^{1-\alpha_m} f_m(c+) + \delta_{22} p_m(c+) D_{0^+}^{\alpha_m} f_m(c+), \tag{24}$$

where δ_{11}, δ_{12}, δ_{21}, and δ_{22} are real numbers such that $\delta \coloneqq \delta_{11}\delta_{22} - \delta_{12}\delta_{21} > 0$.

We investigate the self-adjointness of the fractional Sturm-Liouville operator $\mathcal{H} = (\mathcal{H}_i)_{i=1}^n$ given by (8)–(9) acting on Ψ together with vertex conditions (18)–(22) and transmission conditions (23)–(24). Let us define the Hilbert space

$$H \coloneqq L^2(\Psi_1) \oplus L^2(\Psi_2) \oplus L^2[0, c) \oplus L^2(c, L_m]$$

with the inner product

$$\langle f, g \rangle_H \coloneqq \langle f_{|\Psi_1}, g_{|\Psi_1} \rangle_{L^2(\Psi_1)} + \delta \langle f_{|\Psi_2}, g_{|\Psi_2} \rangle_{L^2(\Psi_2)} + \int_0^c f_{m_1}(x)\,\overline{g_{m_1}}(x)\mathrm{d}x + \delta \int_c^{L_m} f_{m_2}(x)\,\overline{g_{m_2}}(x)\mathrm{d}x, \tag{25}$$

where $f_m, g_m \in L^2[0, c) \oplus L^2(c, L_m]$ are the mth components of $f = (f_i)_{i=1}^n$ and $g = (g_i)_{i=1}^n$ living on the edge e_m, respectively, and they are given by

$$f_m(x) = \begin{cases} f_{m_1}(x), & x \in [0, c) \\ f_{m_2}(x), & x \in (c, L_m] \end{cases}, \quad g_m(x) = \begin{cases} g_{m_1}(x), & x \in [0, c) \\ g_{m_2}(x), & x \in (c, L_m] \end{cases}. \tag{26}$$

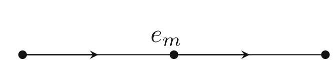

Figure 2: An interior point c on the edge e_m.

The domain $D_2(\mathcal{H})$ of the fractional Sturm-Liouville operator $\mathcal{H}$ consists of functions $f = (f_i)_{i=1}^n \in H$ such that

(i) $f_i, p_i D_{0^+}^{\alpha_i} f_i \in AC[0, L_i]$ for $i = 1, 2, \ldots, n$,
(ii) $\mathcal{H}f \in H$,
(iii) f satisfies the vertex conditions (18)–(22) and transmission conditions (23)–(24).

Theorem 3.1. *The fractional Sturm-Liouville operator $\mathcal{H} = (\mathcal{H}_i)_{i=1}^n$ acting on Ψ with domain $D_2(\mathcal{H})$ is self-adjoint in the Hilbert space $(H, \langle , \rangle_H)$.*

Proof. Let $f = (f_i)_{i=1}^n$, $g = (g_i)_{i=1}^n \in H$. As in the proof of Theorem 2.2, we obtain

$$\begin{aligned}\langle \mathcal{H}f, g\rangle_H = \langle f, \mathcal{H}g\rangle_H + \sum_{i=1}^{m-1} Q(f_i, \overline{g_i})(x)\Big|_{x=0}^{L_i} + \delta \sum_{i=m+1}^{n} Q(f_i, \overline{g_i})(x)\Big|_{x=0}^{L_i} \\ + Q(f_{m_1}, \overline{g_{m_1}})(x)|_{x=0}^{c} + \delta Q(f_{m_2}, \overline{g_{m_2}})(x)|_{x=c}^{L_m},\end{aligned} \tag{27}$$

where

$$p_m(x) = \begin{cases} p_{m_1}(x), & x \in [0, c], \\ p_{m_2}(x), & x \in [c, L_m]. \end{cases}$$

Suppose $f, g \in D_2(\mathcal{H})$. From (18), we have

$$\sum_{i=1}^{m-1} Q(f_i, \overline{g_i})(0) = 0, \quad \sum_{i=m+1}^{n} Q(f_i, \overline{g_i})(0) = 0, \quad Q(f_{m_1}, \overline{g_{m_1}})(0) = 0.$$

Moreover, the conditions (19)–(20) imply that

$$\sum_{i=1}^{m-1} Q(f_i, \overline{g_i})(L_i) = 0.$$

From (21)–(22), we obtain

$$Q(f_{m_2}, \overline{g_{m_2}})(L_m) + \sum_{i=m+1}^{n} Q(f_i, \overline{g_i})(L_i) = 0.$$

Moreover, the transmission conditions (23) and (24) yield that

$$Q(f_{m_1}, \overline{g_{m_1}})(c-) = \delta Q(f_{m_2}, \overline{g_{m_2}})(c+).$$

Consequently, from (27), we have $\langle \mathcal{H}f, g\rangle_H = \langle f, \mathcal{H}g\rangle_H$. It can be shown that the boundary terms in (27) vanish for every $f \in D_2(\mathcal{H})$ iff $g \in D_2(\mathcal{H})$, and hence, $\mathcal{H}$ is self-adjoint in H. □

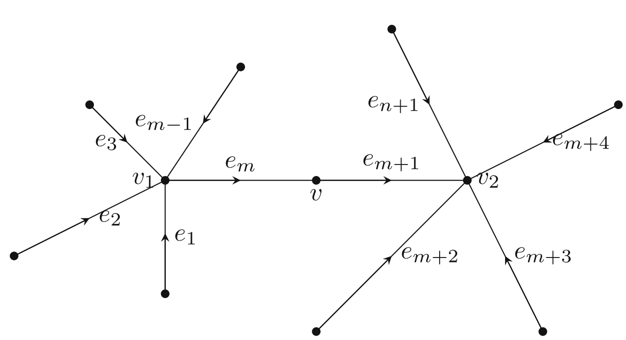

Figure 3: The metric graph Ψ' obtained by introducing a new vertex v to the metric graph Ψ at the point $x_{e_m} = c$ on the edge e_m of Ψ.

Let us denote by Ψ' the metric graph obtained by introducing a new vertex v to the metric graph Ψ at the point $x_{e_m} = c$ on the edge e_m. Now, the number of edges has been increased by one, and we label the edges of Ψ' as in Figure 3. In the new graph Ψ', we denote the lengths of the edges e_m and e_{m+1} by L'_m and L'_{m+1}, respectively. Obviously $L'_m = c$ and $L'_{m+1} = L_m - c$. In the new graph Ψ', the lengths of the edges e_i $(i = m + 2, m + 3, \ldots, n + 1)$ are L_{i-1} $(i = m + 2, m + 3, \ldots, n + 1)$ correspondingly and the lengths of the edges e_i $(i = 1, 2, \ldots, m - 1)$ remain the same as in Ψ. We assume the same vertex conditions as in Ψ at all vertices and consider the transmission conditions at $x_{e_m} = c$ as vertex conditions

$$I_{0^+}^{1-\alpha_m} f_m(L'_m) = \delta_{11} Q(f_{m+1}, v_{m+1})(0) - \delta_{12} Q(f_{m+1}, u_{m+1})(0), \tag{28}$$

$$p_m(L'_m) D_{0^+}^{\alpha_m} f_m(L'_m) = \delta_{21} Q(f_{m+1}, v_{m+1})(0) - \delta_{22} Q(f_{m+1}, u_{m+1})(0) \tag{29}$$

at the newly introduced vertex v, where δ_{11}, δ_{12}, δ_{21}, and δ_{22} are real numbers appearing in (23) and (24) such that $\delta = \delta_{11}\delta_{22} - \delta_{12}\delta_{21} > 0$, and $u_{m+1}(x)$, $v_{m+1}(x)$ are the solutions on the edge e_{m+1} satisfying the initial conditions (11).

Corollary 3.2. *If $\delta = 1$, then the fractional Sturm-Liouville operator $\mathcal{H}$ acting on Ψ' is self-adjoint in $L^2(\Psi')$.*

Proof. If $\delta = 1$, then it easily follows that H coincides with $L^2(\Psi')$ and the inner product $\langle , \rangle_H$ reduces to the usual inner product in $L^2(\Psi')$. The proof is completed using Theorem 3.1. □

By using the local coordinates in the new graph Ψ', we can also define a special inner product in $L^2(\Psi')$ with the help of the inner product $\langle , \rangle_H$ defined by (25). Namely, the local coordinate $x_{e_m} = c$ in Ψ corresponds to $x_{e_m} = L'_m$ on e_m and to $x_{e_{m+1}} = 0$ on e_{m+1} in Ψ'. Therefore, we can define a new special inner product in $L^2(\Psi')$ by

$$\langle f, g \rangle_1 := \langle f_{|\Psi_1}, g_{|\Psi_1} \rangle_{L^2(\Psi_1)} + \delta \langle f_{|\Psi_2}, g_{|\Psi_2} \rangle_{L^2(\Psi_2)} + \int_0^{L'_m} f_m(x) \overline{g_m}(x) \mathrm{d}x + \delta \int_0^{L'_{m+1}} f_{m+1}(x) \overline{g_{m+1}}(x) \mathrm{d}x$$

or alternatively

$$\langle f, g \rangle_1 := \langle f_{|\Psi'_1}, g_{|\Psi'_1} \rangle_{L^2(\Psi'_1)} + \delta \langle f_{|\Psi'_2}, g_{|\Psi'_2} \rangle_{L^2(\Psi'_2)}, \tag{30}$$

where Ψ'_1 is the star graph with central vertex v_1 and edge set $\{e_1, e_2, \ldots, e_m\}$ (Figure 4), and Ψ'_2 is the star graph with the central vertex v_2 and edge set $\{e_{m+1}, e_{m+2}, \ldots, e_{n+1}\}$ (Figure 5).

Corollary 3.3. *The fractional Sturm-Liouville operator $\mathcal{H}$ acting on Ψ' is self-adjoint in $L^2(\Psi')$ with respect to the special inner product defined by* (30).

Proof. The proof is a direct consequence of Theorem 3.1 and equations (25) and (30). □

Remark 3.4. Note that by taking the star graphs Ψ'_1 and Ψ'_2 as single intervals (i.e., $m = n = 1$ in Figures 4 and 5) or equivalently the star graphs Ψ_1 and Ψ_2 as single isolated vertices, we obtain a fractional boundary value

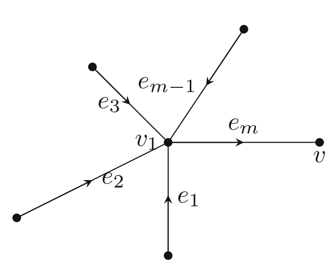

Figure 4: The star graph Ψ'_1.

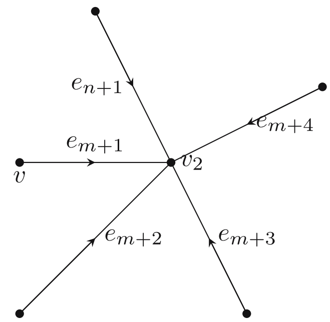

Figure 5: The star graph Ψ_2'.

transmission problem on an interval with one discontinuous point. Therefore, our problem on Ψ' or equivalently on Ψ is a generalization of the boundary value transmission problem on an interval to a much more complicated setting. Moreover, if $\alpha_i \to 1$, it easily follows that the fractional Sturm-Liouville operator $\mathcal{H}$ coincides with the usual second-order Sturm-Liouville operator and the vertex conditions (18)–(22) and the transmission conditions (23)–(24) reduce to their standard forms. Therefore, the problem under investigation also generalizes the boundary value transmission problem generated by ordinary Sturm-Liouville operator to its fractional counterpart.

4 Conclusions

Fractional calculus has various applications in real-world problems [10,13–16], and it continues to attract the attention of a significant number of mathematicians in recent years [4–12]. The theory of quantum graphs is also a developing field in the last two decades due to their wide-range applications in science, especially in physics [25,26,28,29]. In particular, they have been used to model dynamical systems happening on a network rather than a single interval [25]. In quantum graph studies, differential operators involving only integer-order derivatives have been considered in the literature. However, as many studies have demonstrated [10,13–16], Hamiltonians incorporating fractional-order derivatives on an interval can be more beneficial than standard Hamiltonians.

Motivated by the aforementioned ideas, we have introduced a fractional Sturm-Liouville operator

$$f = (f_i)_{i=1}^n \to ({}^cD_{L_i^-}^{\alpha_i}(p_i(x)D_{0^+}^{\alpha_i}f_i(x)) + f_i(x)q_i(x))_{i=1}^n \tag{31}$$

on a compact star graph where the orders α_i of the fractional derivatives lie in the interval $(0,1)$. We have determined the suitable vertex conditions that allow us to define a self-adjoint Hamiltonian. These vertex conditions are reduced to standard or Neumann-Kirchhoff conditions when $\alpha_i \to 1$. Therefore, in this study, we have provided the fractional counterpart of the well-known Schrödinger operator on metric graphs. This will enable scientists to model some physical phenomena that occur in a junction of intervals using fractional-order Hamiltonians.

Discontinuous boundary value problems with impulsive or transmission conditions have been investigated by several authors as they usually occur in physical problems with discontinuous conditions at an intermediate point of the considered interval [31–34]. Fractional counterparts of such problems have also been recently studied [37–40]. However, all these studies are concerned with boundary value transmission problems on an interval. As metric graphs are natural extensions of intervals, discontinuous conditions may also occur at an interior point of a metric graph's edge. These motivated us to consider a discontinuous boundary value problem incorporating the fractional Sturm-Liouville operator (31) on a compact metric graph for $\alpha_i \in (0,1)$. As a result, our study may lead some future research towards discontinuous boundary value problems on metric graphs rather than a single interval.

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Appendix

A.1 Basic definitions and properties of fractional integrals and derivatives

In this section, we present some auxiliary definitions and results from [8,42] about fractional derivatives and integrals.

Definition A1. Let $\mathrm{Re}(\alpha) > 0$ and f be an integrable function on $[a, b]$. The left-sided and right-sided Riemann-Liouville fractional integrals of order α are defined as follows:

$$(I_{a^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - s)^{\alpha - 1} f(s) \mathrm{d}s, \quad x > a,$$

$$(I_{b^-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (x - s)^{\alpha - 1} f(s) \mathrm{d}s, \quad x < b,$$

respectively.

Definition A2. Let $\mathrm{Re}(\alpha) > 0$, $m = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$ and $f \in AC^m[a, b]$. The left-sided and right-sided Riemann-Liouville fractional derivatives of order α are defined as follows:

$$(D_{a^+}^{\alpha} f)(x) = D^m (I_{a^+}^{m-\alpha} f)(x), \quad x > a,$$

$$(D_{b^-}^{\alpha} f)(x) = (-D)^m (I_{b^-}^{m-\alpha} f)(x), \quad x < b,$$

respectively. In particular, if $\alpha = n \in \mathbb{N}$, then

$$(D_{a^+}^{0} f)(x) = (D_{b^-}^{0} f)(x) = f(x),$$

$$(D_{a^+}^{n} f)(x) = D^n f(x), \quad n \geq 1,$$

$$(D_{b^-}^{n} f)(x) = (-D)^n f(x), \quad n \geq 1.$$

Definition A3. Let $\mathrm{Re}(\alpha) > 0$, $m = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$ and $f \in AC^m[a, b]$. The left-sided and right-sided Caputo fractional derivatives of order α of f are defined as follows:

$$({}^c D_{a^+}^{\alpha} f)(x) = (I_{a^+}^{m-\alpha} D^m f)(x), \quad x > a,$$

$$({}^c D_{b^-}^{\alpha} f)(x) = (I_{b^-}^{m-\alpha} (-D)^m f)(x), \quad x < b,$$

respectively. In particular, if $\alpha = n \in \mathbb{N}$, then

$$({}^c D_{a^+}^{0} f)(x) = ({}^c D_{b^-}^{0})(x) = f(x),$$

$$({}^c D_{a^+}^{n} f)(x) = D^n f(x), \quad n \geq 1,$$

$$({}^c D_{b^-}^{\alpha_i} f)(x) = (-D)^n f(x), \quad n \geq 1.$$

Lemma A4. [42] *Let $\alpha > 0$, $p, q \geq 1$ and $(1/p) + (1/q) \leq 1 + \alpha$ ($p \neq 1$, $q \neq 1$ when the equality holds). If $f \in L_p(a, b)$ and $g \in L_q(a, b)$, then*

$$\int_a^b f(x)(I_{a^+}^{\alpha} g)(x) \mathrm{d}x = \int_a^b g(x)(I_{b^-}^{\alpha} f)(x) \mathrm{d}x.$$

Lemma A5. [9] *The following integration by parts formulas hold*:

$$\int_a^b f(x)(D_{b^-}^{\alpha} g)(x)\mathrm{d}x = \int_a^b g(x)({}^cD_{a^+}^{\alpha} f)(x)\mathrm{d}x + \sum_{k=0}^{m-1} (-1)^{m-k} f^{(k)}(x) D^{m-k-1}(I_{b^-}^{m-\alpha} g)(x)\Big|_{x=a}^{b}, \tag{A1}$$

$$\int_a^b f(x)(D_{a^+}^{\alpha} g)(x)\mathrm{d}x = \int_a^b g(x)({}^cD_{b^-}^{\alpha} f)(x)\mathrm{d}x + \sum_{k=0}^{m-1} (-1)^{m-k} f^{(k)}(x) D^{m-k-1}(I_{a^+}^{m-\alpha} g)(x)\Big|_{x=a}^{b}. \tag{A2}$$

A.2 Basic definitions about quantum graphs

In this section, we summarize basic notions about quantum graphs. The definitions in this subsection are taken from [28].

Let $G = (V, E)$ be a graph with a non-empty vertex set V and edge set E. For simplicity, we assume V is finite and every vertex has a finite, positive degree. Loops and multiple edges are allowed. If we identify each edge $e \in E$ with an interval (finite or infinite) $[0, L_e]$ of the real line, we obtain a metric graph. Such an identification induces a local coordinate on each edge. Namely, if $L_e < \infty$, then the endpoints of the edge e have the local coordinate $x_e = 0$ or $x_e = L_e$ and all the other intermediate points have the local coordinate $x_e \in (0, L_e)$. If L_e is infinite, then e has only one endpoint and this point has the local coordinate $x_e = 0$, whereas all other points have a local coordinate $x_e > 0$. Moreover, a metric graph is a directed graph since each edge e can be endowed with a natural direction starting from $x_e = 0$.

If the metric graph $G = (V, E)$ has a finite number of edges with finite lengths, then $G = (V, E)$ is a compact metric graph. In this article, we are only interested in compact metric graphs. Obviously a function f on a metric graph is a vector of functions on the individual edges, i.e., $f = (f_e)_{e\in E}$, where $f_e : [0, L_e] \to \mathbb{C}$. Identifying each edge $e \in E$ with an interval $[0, L_e]$ enables us to define the Lebesgue measure and usual function spaces on a metric graph. Let us consider the Hilbert space $L^2(G)$ on the compact metric graph which is the direct sum of L^2 spaces on the edges, namely,

$$L^2(G) \coloneqq \bigoplus_{e\in E} L^2(0, L_e)$$

with the inner product

$$\langle f, g\rangle_{L^2(G)} \coloneqq \sum_{e\in E} \langle f_e, g_e\rangle_{L^2(0,L_e)} = \sum_{e\in E} \int_0^{L_e} f_e(x_e)\overline{g_e(x_e)}\mathrm{d}x_e. \tag{A3}$$

To turn a metric graph into a quantum graph, we need to consider a differential operator acting on the edges of the graph with appropriate boundary conditions imposed at the vertices (vertex conditions). In general, the differential operator is considered as the Laplace operator

$$f = (f_e)_{e\in E} \to -\frac{\mathrm{d}^2}{\mathrm{d}x^2} f = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2} f_e\right)_{e\in E} \tag{A4}$$

or the Schrödinger operator

$$f = (f_e)_{e\in E} \to -\frac{\mathrm{d}^2}{\mathrm{d}x^2} f + V(x) f = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2} f_e + V_e(x_e) f_e\right)_{e\in E} \tag{A5}$$

with a potential $V(x) = (V_e(x_e))_{e\in E}$.

Most frequently used vertex conditions are Neumann-Kirchhoff conditions given by

(i) $f = (f_e)_{e\in E}$ is continuous at v, i.e., $f_{e_1}(v) = f_{e_2}(v)$ whenever v is incident in edges e_1 and e_2,

(ii) $\sum_{e\in E_v} \frac{\mathrm{d}f_e}{\mathrm{d}x_e}(v) = 0$, where the derivatives are taken in outgoing directions (from the vertex into the edge).

The other types of vertex conditions include the vertex Dirichlet condition (i.e., $f = (f_e)_{e \in E}$ is continuous at v and $f(v) = 0$) and δ-type vertex conditions

(i) $f = (f_e)_{e \in E}$ is continuous at v,

(ii) $\sum_{e \in E_v} \frac{\mathrm{d}f_e}{\mathrm{d}x_e}(v) = \alpha_v f(v)$,

where α_v is a fixed number. Note that allowing $\alpha_v = 0$, we recover Neumann-Kirchhoff conditions. Further, letting $\alpha_v \to \infty$, we get the Dirichlet condition.

In general, we can write any vertex condition at a vertex v in the matrix form

$$A_v F(v) + B_v F'(v) = 0, \tag{A6}$$

where A_v, B_v are $d_v \times d_v$ matrices, and $F(v) = (f_e(v))^T_{e \in E_v}$ and $F'(v) = \left(\frac{\mathrm{d}f_e}{\mathrm{d}x_e}(v)\right)^T_{e \in E_v}$ are d_v dimensional vectors.

Now we have all the ingredients to form a quantum graph. A quantum graph is a collection of a metric graph $G = (V, E)$ and a differential operator Ξ (called Hamiltonian) acting on the edges $e \in E$ with vertex conditions imposed at every vertex $v \in V$. By the spectrum of a quantum graph, we mean the spectrum of its Hamiltonian. It has been proven [43] that the compact quantum graph $G = (V, E)$ with the differential operator (A4) and vertex conditions (A6) is self-adjoint if and only if for every vertex $v \in V$ the following conditions hold

(i) $\operatorname{rank}(A_v B_v) = d_v$,

(ii) $A_v B_v^* = B_v A_v^*$.