Research Article

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Ground state solutions and multiple positive solutions for nonhomogeneous Kirchhoff equation with Berestycki-Lions type conditions

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Abstract: This article is concerned with the following Kirchhoff equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(u) + h(x) \quad \text{in } \mathbb{R}^3,$$

where a and b are positive constants and $h \neq 0$. Under the Berestycki-Lions type conditions on g, we prove that the equation has at least two positive solutions by using variational methods. Furthermore, we obtain the existence of ground state solutions.

Keywords: nonhomogeneous Kirchhoff equation, variational methods, Berestycki-Lions type conditions, multiple positive solutions, ground state solutions

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1 Introduction and main results

In this article, we study the following Kirchhoff equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(u) + h(x) \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where a and b are positive constants, $g \in C(\mathbb{R}, \mathbb{R})$ and $h \in L^2(\mathbb{R}^3)$. In equation (1.1), if we replace $g(u) + h(x)$ and $\mathbb{R}^3$ by $g(x, u)$ and a bounded domain $\Omega \subset \mathbb{R}^3$, respectively, then it reduces to

$$-\left(a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x, u) \quad \text{in } \Omega. \tag{1.2}$$

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Equation (1.2) is essentially related to the stationary analogous of the Kirchhoff equation:

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x, u),$$

which was proposed by Kirchhoff as a generalization of the well-known D'Alembert wave equation

$$\varrho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{\lambda} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = g(x, u)$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, λ is the area of the cross section, E is the Young modulus of the material, ϱ is the mass density, and P_0 is the initial tension. For more mathematical and physical background on Kirchhoff type problems, we refer to Arosio and Panizzi [1], Chipot and Lovat [2], and the references therein.

In the last decade, many authors have studied the case, where $h = 0$, namely, the following classical Kirchhoff type problem

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(u) \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

where a and b are positive constants and $g \in C(\mathbb{R}, \mathbb{R})$. By means of the Fountain theorem, Jin and Wu [3] established the existence of infinitely many radial solutions for equation (1.3) with the nonlinearity $g(x, u) - u$. To obtain the boundedness of Palais-Smale sequence, it required that $g(x, u)$ satisfies the (AR) condition:

$$\mu G(x, u) := \mu\int_0^u g(x, s)\mathrm{d}s \leqslant ug(x, u) \quad \text{for some } \mu > 4 \text{ and all } (x, u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Azzollini [4] studied equation (1.3) with g satisfying the following conditions:

(g_1) $-\infty < \liminf_{t\to 0}\frac{g(t)}{t} \leqslant \limsup_{t\to 0}\frac{g(t)}{t} = -m < 0$,

(g_2) $-\infty < \limsup_{|t|\to\infty}\frac{g(t)}{|t|^4 t} \leqslant 0$,

(g_3) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^{\zeta} g(s)\mathrm{d}s > 0$,

and obtained a ground state solution by using minimizing arguments on a suitable natural constraint. In the literature, (g_1)–(g_3) are known as the Berestycki-Lions conditions introduced by Berestycki and Lions [5]. Moreover, Berestycki and Lions [5] showed that (g_1)–(g_3) were "almost" necessary for the existence of nontrivial solutions to equation (1.3) with $b = 0$. Lu [6] proved that (1.3) has infinitely many distinct radial solutions if g is odd and verifies (g_1), (g_2'), and (g_3),

(g_2') $\lim\limits_{|t|\to\infty}\frac{g(t)}{|t|^4 t} = 0$.

Recently, Chen et al. [7] discussed equation (1.3) with $g(u) = |u|^{p-2}u - u$, $p \in (2, 6)$ and established the existence of positive radial solutions by carrying out the constrained minimization on Nehari-Pohožaev manifold. For the related works of equation (1.3) including certain potentials, we refer readers to Chen and Tang [8], Figueiredo et al. [9], He et al. [10], Li and Ye [11], Li et al. [12], Liang et al. [13], Liu et al. [14], Tang and Chen [15], Ye and Tang [16], Zhang and Du [17], and the references therein.

For $h \neq 0$, only a few of results are obtained. The nonhomogeneous Kirchhoff equation with nonconstant coefficient of the form

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = g(x, u) + h(x) \quad \text{in } \mathbb{R}^3 \tag{1.4}$$

has been investigated by Chen and Li [18], and Cheng [19], where $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies the compactness assumption introduced by Bartsch and Wang [20]. By applying the Ekeland's variational principle and the mountain pass theorem [21], Chen and Li [18] proved two positive solutions, where g is superlinear at infinity and satisfies the (AR) condition. Subsequently, Cheng [19] weakened the assumptions on the nonlinear term, and then improved the results of Chen and Li [18]. Ding et al. [22] discussed equation (1.4) with $V(x) = 1$, $g(x, u) = a(x)f(u)$ and showed two positive solutions, where $f \in C(\mathbb{R}, \mathbb{R}_+)$ and f is asymptotically linear at infinity. Liu et al. [23] studied equation (1.4) with $V(x) = 1$, $g(x, u) = |u|^4u$ and obtained two positive solutions. For related works of equation (1.4) and similar nonhomogeneous problems, we refer to He et al. [24], Huang and Su [25], Huang et al. [26], and Lü [27]. Particularly, Zhang and Zhu [28] studied the Kirchhoff equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + u = |u|^{p-2}u + h(x) \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where a and b are positive constants, $p \in (2, 6)$, $h \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ being a nonnegative radial function and satisfying $(x, \nabla h) \in L^2(\mathbb{R}^3)$. By using the Ekeland's variational principle and the mountain pass theorem, they established the existence of two nontrivial solutions for $p \in (2, 6)$ with small L^2–norm $|h|_2$ of h. Note that for $p \in (2, 4]$, the function $|u|^{p-2}u$ neither satisfies the (AR) condition nor is 4-superlinear. In view of this, Zhang and Zhu [28] used Struwe's monotonicity trick to guarantee the boundedness of Palais-Smale sequence.

Inspired by the aforementioned works, especially by the results of Azzollini [4], Berestycki and Lions [5], and Zhang and Zhu [28], we are interested in finding the ground state solution and multiple positive solutions for equation (1.1) with g satisfying the Berestycki-Lions type conditions and $h \neq 0$. Furthermore, we make the following hypotheses:

(h_1) $h \in L^2(\mathbb{R}^3)$ is a nonzero radial function.

(h_2) $(x, \nabla h) \in L^{\frac{6}{5}}(\mathbb{R}^3)$, where the gradient ∇h is in the weak sense.

The main results of this article are as follows.

Theorem 1.1. *Assume that* (g_1), (g_2'), (g_3), (h_1), *and* (h_2) *hold. Then there exists* $\Lambda > 0$ *such that equation* (1.1) *admits two different nontrivial solutions and a ground state solution for* $|h|_2 < \Lambda$.

Corollary 1.2. *If* $h(x) \geqslant 0$ *in* $\mathbb{R}^3$ *and the assumptions of Theorem 1.1 hold, then there exists* $\tilde{\Lambda} > 0$ *such that equation* (1.1) *admits two different positive solutions and a positive ground state solution for* $|h|_2 < \tilde{\Lambda}$.

Remark 1.3.

(1) Comparing with the results of Zhang and Zhu [28], the novelty of our Theorem 1.1 is that we consider equation (1.1) for a much more general class of nonlinearities, including, for example, the nonlinearities

$$g(t) = -t + \frac{4t^3}{1+t^4}, \quad g(t) = -t + t\sin t, \quad \text{and} \quad g(t) = -t + \sum_{i=1}^{n} d_i\,|t|^{\delta_i - 2}t,$$

where $d_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, $d_1 > 0$ and $6 > \delta_1 > \delta_2 > \cdots > \delta_n > 2$. Another novelty of this article is that we first establish the existence of ground state solutions to (1.1) for $h \neq 0$.

(2) From assumption (g_3), it follows that $G(u) \coloneqq \int_0^u g(s)\mathrm{d}s > 0$ may not hold for all $u \in \mathbb{R}$. So the Struwe's monotonicity trick used in Zhang and Zhu [28] to obtain a bounded Palais-Smale sequence does not apply here. To overcome this difficulty, we employ a scaling technique introduced by Jeanjean [29] to guarantee the boundedness of Palais-Smale sequence.

This article is organized as follows. In Section 2, we recall some preliminaries and provide some lemmas. In Section 3, we give the proofs of Theorem 1.1 and Corollary 1.2.

2 Preliminaries

For $1 \leqslant p < \infty$, $L^p(\mathbb{R}^3)$ denotes the Lebesgue space with the usual norm

$$|u|_p = \left(\int_{\mathbb{R}^3} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

It follows from the classical Sobolev embedding theorems that $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ are continuous for all $p \in [2, 6]$. Thus for any $p \in [2, 6]$, there exists $S_p > 0$ such that

$$|u|_p \leqslant S_p \|u\|, \quad \forall u \in H^1(\mathbb{R}^3). \tag{2.1}$$

We will work on the space

$$H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : \; u(x) = u(|x|)\}.$$

It holds that the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is compact for any $p \in (2, 6)$ (Willem [30]). As a consequence, the functional $I : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} G(u)\mathrm{d}x - \int_{\mathbb{R}^3} h(x)u\mathrm{d}x$$

is well defined, and it is of class C^1 with derivative

$$\langle I'(u), v\rangle = \left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\int_{\mathbb{R}^3} \nabla u \nabla v \mathrm{d}x - \int_{\mathbb{R}^3} g(u)v\mathrm{d}x - \int_{\mathbb{R}^3} h(x)v\mathrm{d}x$$

for all $u, v \in H_r^1(\mathbb{R}^3)$. Moreover, if u is a critical point of I, then u is a weak solution of (1.1).

Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_D = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. Denoted by $S > 0$ the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$:

$$S \coloneqq \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}}}. \tag{2.2}$$

It is known that S can be achieved by a positive radial function. We will use C, C_i to denote various positive constants.

Now we give the following technical lemmas used to prove our main results.

Lemma 2.1. *Suppose that (g_1), (g_2'), (g_3), and (h_1) hold. Then there exist $\Lambda > 0$, $\rho > 0$, and $\alpha > 0$ such that for $|h|_2 < \Lambda$, there hold*

(i) *$I(u) \geqslant \alpha$ with $\|u\| = \rho$,*

(ii) *there exists a function $v \in H_r^1(\mathbb{R}^3)\setminus\{0\}$ such that $\|v\| > \rho$ and $I(v) < 0$.*

Proof. (i) It follows from (g_1) and (g_2') that there exist constants $L > 0$ and $C > 0$ such that

$$G(t) \leqslant -Lt^2 + C\,|t|^6 \quad \text{for all } \; t \in \mathbb{R}. \tag{2.3}$$

For $u \in H_r^1(\mathbb{R}^3)$, by (2.1), (2.3), and the Hölder inequality, we have

$$\begin{aligned} I(u) &\geqslant \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2 + L\int_{\mathbb{R}^3}u^2\mathrm{d}x - C\int_{\mathbb{R}^3}|u|^6\mathrm{d}x - |h|_2|u|_2 \\ &\geqslant \min\left\{\frac{a}{2}, L\right\}\|u\|^2 - CS_6^6\|u\|^6 - S_2\,|h|_2\,\|u\| \\ &= \|u\|\left(\min\left\{\frac{a}{2}, L\right\}\|u\| - CS_6^6\|u\|^5 - S_2\,|h|_2\right). \end{aligned} \tag{2.4}$$

Consider the function $f(t) = \min\left\{\frac{a}{2}, L\right\}t - CS_6^6 t^5$ for $t \geqslant 0$. Then, we conclude that for $\rho = \left(\frac{\min\{\frac{a}{2}, L\}}{5CS_6^6}\right)^{\frac{1}{4}} > 0$, it holds that

$$f(\rho) = \frac{4\left(\min\left\{\frac{a}{2}, L\right\}\right)^{\frac{5}{4}}}{5(5CS_6^6)^{\frac{1}{4}}} > 0.$$

Taking

$$\Lambda = \frac{f(\rho)}{2S_2} \quad \text{and} \quad \alpha = \frac{\rho f(\rho)}{2},$$

we deduce that if $|h|_2 < \Lambda$, then $I(u) \geqslant \alpha$ with $\|u\| = \rho$.

(ii) Borrowing the method in Berestycki and Lions [5], for $R > 1$, we define

$$w_R(x) = \begin{cases} \zeta, & |x| \leqslant R, \\ \zeta(R + 1 - |x|), & R \leqslant |x| \leqslant R + 1, \\ 0, & |x| \geqslant R + 1, \end{cases}$$

where ζ is given by the assumption (g_3). Thus, $w_R \in H_r^1(\mathbb{R}^3)$. Through a direct calculation, we conclude that

$$\begin{aligned} &\int_{\mathbb{R}^3}|\nabla w_R|^2\mathrm{d}x = \zeta^2\mathrm{meas}\{B_{R+1}(0) - B_R(0)\}, \\ &\int_{\mathbb{R}^3}G(w_R)\mathrm{d}x \geqslant G(\zeta)\mathrm{meas}\{B_R(0)\} - \mathrm{meas}\{B_{R+1}(0) - B_R(0)\}\left(\max_{s\in[0,\zeta]}|G(s)|\right), \\ &\int_{\mathbb{R}^3}|h(x)w_R|\mathrm{d}x \leqslant \Lambda\zeta(\mathrm{meas}\{B_{R+1}(0)\})^{\frac{1}{2}}, \end{aligned}$$

where meas$\{\cdot\}$ denotes Lebesgue measure, $B_r(y) \coloneqq \{x \in \mathbb{R}^3 : |x - y| < r\}$. Then there exist some $C_i > 0$ $(i = 1, 2, 3, 4)$ such that

$$\begin{cases} \int_{\mathbb{R}^3}|\nabla w_R|^2\mathrm{d}x = C_1R^2, \\ \int_{\mathbb{R}^3}G(w_R)\mathrm{d}x \geqslant C_2R^3 - C_3R^2, \\ \int_{\mathbb{R}^3}|h(x)w_R|\mathrm{d}x \leqslant C_4(R + 1)^{\frac{3}{2}}. \end{cases} \tag{2.5}$$

Defining $w_{R,\theta}(x) \coloneqq w_R(\frac{x}{\theta})$ for $\theta > 0$ and combining (2.5), we obtain

$$\begin{aligned} I(w_{R,\theta}) &= \frac{a\theta}{2}\int_{\mathbb{R}^3}|\nabla w_R|^2\mathrm{d}x + \frac{b\theta^2}{4}\left(\int_{\mathbb{R}^3}|\nabla w_R|^2\mathrm{d}x\right)^2 - \theta^3\int_{\mathbb{R}^3}G(w_R)\mathrm{d}x - \int_{\mathbb{R}^3}hw_{R,\theta}\mathrm{d}x \\ &\leqslant \frac{a\theta}{2}C_1R^2 + \frac{b\theta^2}{4}C_1^2R^4 - \theta^3(C_2R^3 - C_3R^2) + \theta^{\frac{3}{2}}C_4(R + 1)^{\frac{3}{2}}. \end{aligned} \tag{2.6}$$

Therefore, let $\theta = R^2$ sufficiently large, we easily obtain

$$I(w_{R,\theta}) \leqslant \frac{aC_1}{2}\theta^2 + \frac{bC_1^2}{4}\theta^4 - C_2\theta^{\frac{9}{2}} + C_3\theta^4 + C_4\theta^{\frac{3}{2}}\left(\theta^{\frac{1}{2}} + 1\right)^{\frac{3}{2}} < 0$$

and

$$\|w_{R,\theta}\|^2 \geqslant \int_{\mathbb{R}^3} |\nabla \omega_{R,\theta}|^2 \mathrm{d}x = C_1\theta^2 > \rho^2.$$

The conclusion (ii) follows by taking $v = w_{R,\theta}$. □

Lemma 2.2. *Suppose that* (g_1), (g_2'), *and* (h_1) *hold. Then any bounded sequence* $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ *satisfying* $I'(u_n) \to 0$ *has a strongly convergent subsequence.*

Proof. Since $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ is bounded, there exists $u \in H_r^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_r^1(\mathbb{R}^3), \\ u_n \to u & \text{in } L^s(\mathbb{R}^3),\ s \in (2,6), \\ u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{2.7}$$

By combining (2.7) and $I'(u_n) \to 0$, we obtain

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

We will end the proof by showing $u_n \to u$ in $H_r^1(\mathbb{R}^3)$. Motivated by Berestycki and Lions [5], define

$$g_1(t) = \begin{cases} (g(t) + mt)^+, & t \geqslant 0, \\ (g(t) + mt)^-, & t \leqslant 0, \end{cases} \tag{2.9}$$

$$g_2(t) = g_1(t) - g(t) - mt \quad \text{for all } t \in \mathbb{R}, \tag{2.10}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. Clearly, g_1 and g_2 satisfy

$$\lim_{t\to 0} \frac{g_1(t)}{t} = \lim_{|t|\to\infty} \frac{g_1(t)}{|t|^4 t} = 0, \tag{2.11}$$

$$g_2(t)t \geqslant 0 \quad \text{for all } t \in \mathbb{R}. \tag{2.12}$$

From (2.11), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g_1(t)| \leqslant \varepsilon(|t| + |t|^5) + C_\varepsilon |t|^3 \quad \text{for all } t \in \mathbb{R}. \tag{2.13}$$

Therefore, it follows from (2.7), (2.13), and the Hölder inequality that

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} (g_1(u_n) - g_1(u))(u_n - u)\mathrm{d}x\right| \\ &\leqslant \varepsilon \int_{\mathbb{R}^3} (|u_n| + |u| + |u_n|^5 + |u|^5)|u_n - u|\mathrm{d}x + C_\varepsilon \int_{\mathbb{R}^3} (|u_n|^3 + |u|^3)|u_n - u|\mathrm{d}x \\ &\leqslant \varepsilon(|u_n|_2 + |u|_2)|u_n - u|_2 + \varepsilon(|u_n|_6^5 + |u|_6^5)|u_n - u|_6 + C_\varepsilon\left(|u_n|_{\frac{9}{2}}^3 + |u|_{\frac{9}{2}}^3\right)|u_n - u|_3 \\ &\leqslant C\varepsilon + o(1). \end{aligned} \tag{2.14}$$

Hence, by the arbitrariness of ε, we obtain

$$\int_{\mathbb{R}^3} (g_1(u_n) - g_1(u))(u_n - u)\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

By (2.12) and Fatou's lemma, one deduces

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3} g_2(u_n)u_n\mathrm{d}x \geqslant \int_{\mathbb{R}^3} g_2(u)u\mathrm{d}x.$$

Furthermore, one obtains

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} g_2(u_n)u_n - g_2(u_n)u - g_2(u)(u_n-u)\mathrm{d}x \geqslant 0. \tag{2.16}$$

Then, we define the functional $k_u : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$k_u(v) = \int_{\mathbb{R}^3} \nabla u \nabla v \mathrm{d}x$$

for all $v \in H_r^1(\mathbb{R}^3)$. Since $|k_u(v)| \leqslant \int_{\mathbb{R}^3}|\nabla u\nabla v|\mathrm{d}x \leqslant \|u\|\|v\|$, we deduce that k_u is continuous on $H_r^1(\mathbb{R}^3)$. By using $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, we have as $n \to \infty$,

$$\int_{\mathbb{R}^3} \nabla u \nabla(u_n - u)\mathrm{d}x \to 0.$$

Thus, from the boundedness of $\{u_n\}$, one has

$$b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 - |\nabla u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3} \nabla u\nabla(u_n-u)\mathrm{d}x \to 0 \quad \text{as } n\to\infty. \tag{2.17}$$

In view of (2.10), one has

$$\begin{aligned}
&\langle I'(u_n) - I'(u), u_n - u\rangle\\
&\quad= \left(a + b\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\nabla u_n\nabla(u_n-u)\mathrm{d}x - \left(a + b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)\mathrm{d}x\\
&\qquad+ \int_{\mathbb{R}^3} m(u_n-u)^2\mathrm{d}x - \int_{\mathbb{R}^3}(g_1(u_n) - g_1(u))(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^3}(g_2(u_n)-g_2(u))(u_n-u)\mathrm{d}x\\
&\quad= \left(a + b\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}|\nabla(u_n-u)|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 - |\nabla u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)\mathrm{d}x\\
&\qquad+ \int_{\mathbb{R}^3} m(u_n-u)^2\mathrm{d}x - \int_{\mathbb{R}^3}(g_1(u_n) - g_1(u))(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^3}(g_2(u_n)-g_2(u))(u_n-u)\mathrm{d}x\\
&\quad\geqslant \min\{a, m\}\|u_n-u\|^2 + b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 - |\nabla u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)\mathrm{d}x\\
&\qquad- \int_{\mathbb{R}^3}(g_1(u_n)-g_1(u))(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^3} g_2(u_n)u_n - g_2(u_n)u - g_2(u)(u_n-u)\mathrm{d}x.
\end{aligned}$$

By (2.8) and (2.15)–(2.17), we conclude that $\|u_n - u\| \to 0$ as $n \to \infty$. □

By Lemma 2.1, we can define a mountain pass level of I as follows.

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)) > 0,$$

where $\Gamma \coloneqq \{\gamma \in C([0,1], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0, I(\gamma(1)) < 0\}$. We shall use the following Pohožaev type identity. The proof can be done similarly to that in Berestycki and Lions [5] and details are omitted here.

Lemma 2.3. *Suppose that* (g_1), (g_2'), (g_3), (h_1), *and* (h_2) *hold. Let* $u \in H_r^1(\mathbb{R}^3)$ *be a weak solution of equation* (1.1), *then the following Pohožaev type identity holds*

$$P(u) := \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - 3\int_{\mathbb{R}^3}G(u)dx - 3\int_{\mathbb{R}^3}hu dx - \int_{\mathbb{R}^3}(x, \nabla h)u dx = 0.$$

In the following, we employ a scaling technique introduced by Jeanjean [29] and Jeanjean and Tanaka [31] (see also Hirata et al. [32]) to construct a special Palais-Smale sequence that satisfies asymptotically the Pohožaev identity.

Lemma 2.4. *Suppose that* (g_1), (g_2'), (g_3), (h_1), *and* (h_2) *hold. There exists a bounded sequence* $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ *satisfying*

$$I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{and} \quad P(u_n) \to 0.$$

Proof. Following Jeanjean [29], we define the map $\Phi : \mathbb{R} \times H_r^1(\mathbb{R}^3) \to H_r^1(\mathbb{R}^3)$ for $\sigma \in \mathbb{R}$ and $v \in H_r^1(\mathbb{R}^3)$ by $\Phi(\sigma, v)(x) = v(e^{-\sigma}x)$. The functional $I \circ \Phi$ is computed as follows:

$$I(\Phi(\sigma, v)) = \frac{ae^{\sigma}}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx + \frac{be^{2\sigma}}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2 dx\right)^2 - e^{3\sigma}\int_{\mathbb{R}^3}G(v) + h(e^{\sigma}x)v dx.$$

It is standard to verify that $I \circ \Phi$ is continuously Fréchet-differential on $\mathbb{R} \times H_r^1(\mathbb{R}^3)$. Together with $I(\Phi(0, 0)) = 0$, we set the family of paths

$$\bar{\Gamma} = \{\bar{\gamma} \in C([0, 1], \mathbb{R} \times H_r^1(\mathbb{R}^3)) : \bar{\gamma}(0) = (0, 0), (I \circ \Phi)(\bar{\gamma}(1)) < 0\}.$$

As $\Gamma = \{\Phi \circ \bar{\gamma} : \bar{\gamma} \in \bar{\Gamma}\}$, the mountain pass levels of I and $I \circ \Phi$ coincide:

$$c = \inf_{\bar{\gamma}\in\bar{\Gamma}}\sup_{t\in[0,1]}(I \circ \Phi)(\bar{\gamma}(t)).$$

Let $\bar{\gamma} = (0, \gamma)$. For every $\varepsilon \in \left(0, \frac{c}{2}\right)$, there exists $\gamma \in \Gamma$ such that

$$\sup(I \circ \Phi)(0, \gamma) \leqslant c + \varepsilon.$$

Then, by Willem [30, Theorem 2.8], there exists $(\sigma, v) \in \mathbb{R} \times H_r^1(\mathbb{R}^3)$ such that
(a) $c - 2\varepsilon \leqslant (I \circ \Phi)(\sigma, v) \leqslant c + 2\varepsilon$,
(b) $\text{dist}\{(\sigma, v), (0, \gamma)\} \leqslant 2\sqrt{\varepsilon}$, where $\text{dist}\{(\sigma, v), (\tau, \vartheta)\} = (|\sigma - \tau|^2 + \|v - \vartheta\|^2)^{\frac{1}{2}}$,
(c) $\|(I \circ \Phi)'(\sigma, v)\| \leqslant 2\sqrt{\varepsilon}$.

Therefore, there exists a sequence $\{(\sigma_n, v_n)\} \subset \mathbb{R} \times H_r^1(\mathbb{R}^3)$ such that, as $n \to \infty$,

$$\sigma_n \to 0, \quad (I \circ \Phi)(\sigma_n, v_n) \to c \quad \text{and} \quad (I \circ \Phi)'(\sigma_n, v_n) \to 0.$$

For every $(\zeta, w) \in \mathbb{R} \times H_r^1(\mathbb{R}^3)$, we have

$$(I \circ \Phi)'(\sigma_n, v_n)[\zeta, w] = I'(\Phi(\sigma_n, v_n))[\Phi(\sigma_n, w)] + P(\Phi(\sigma_n, v_n))\zeta.$$

Taking $u_n = \Phi(\sigma_n, v_n)$, we obtain

$$I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{and} \quad P(u_n) \to 0. \tag{2.18}$$

Now, we need to check that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. By (2.18), for n large enough,

$$c + 1 \geqslant I(u_n) - \frac{1}{3}P(u_n) = \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\right)^2 + \frac{1}{3}\int_{\mathbb{R}^3}(x, \nabla h)u_n dx. \tag{2.19}$$

By (h_2), (2.1)–(2.2), and the Hölder inequality, one has

$$\left|\int_{\mathbb{R}^3}(x,\nabla h)u_n\mathrm{d}x\right| \leqslant \left(\int_{\mathbb{R}^3}|(x,\nabla h)|^{\frac{6}{5}}\mathrm{d}x\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x\right)^{\frac{1}{6}} \leqslant C\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^{\frac{1}{2}} \tag{2.20}$$

for all $x\in\mathbb{R}^3$. Then, by the Hölder and Sobolev inequalities, (2.19) and (2.20) imply that

$$a\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^2 \leqslant 3(c+1)-\int_{\mathbb{R}^3}(x,\nabla h)u_n\mathrm{d}x \leqslant 3(c+1)+C\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.21}$$

From this, $\{|\nabla u_n|_2\}$ is bounded. Furthermore, by (2.3) and $I(u_n)\to c$,

$$\begin{aligned}\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x &\leqslant \int_{\mathbb{R}^3}G(u_n)+hu_n\mathrm{d}x+c+o(1)\\ &\leqslant -L\int_{\mathbb{R}^3}u_n^2\mathrm{d}x+C\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x+|h|_2|u_n|_2+c+o(1)\\ &\leqslant -L\int_{\mathbb{R}^3}u_n^2\mathrm{d}x+CS^{-3}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^3+|h|_2|u_n|_2+c+o(1),\end{aligned} \tag{2.22}$$

which implies that $\{|u_n|_2\}$ is bounded. This completes the proof. □

3 Proof of main results

Proof of Theorem 1.1. The proof of this theorem is divided into three steps.
Step 1. There exists $u_1\in H_r^1(\mathbb{R}^3)$ such that $I(u_1)=c_0<0$ and $I'(u_1)=0$.

By (h_1), we can choose a function $\varphi\in H_r^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3}h(x)\varphi(x)\mathrm{d}x>0$. Indeed, there exists a radial sequence $\{\varphi_n\}\subset C_0^\infty(\mathbb{R}^3)$ such that $\varphi_n\to h$ in $L^2(\mathbb{R}^3)$ since $C_0^\infty(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$. By combining this and the Hölder inequality, we conclude as $n\to\infty$,

$$\int_{\mathbb{R}^3}h(x)\varphi_n(x)\mathrm{d}x\to\int_{\mathbb{R}^3}h^2(x)\mathrm{d}x>0.$$

Then we can choose $n_0\in\mathbb{N}$ large enough to make $\int_{\mathbb{R}^3}h(x)\varphi_{n_0}(x)\mathrm{d}x\neq0$ hold. It is clear that $\varphi_{n_0}(x)\in H_r^1(\mathbb{R}^3)$. Taking $\varphi(x)=\varphi_{n_0}(x)$ or $-\varphi_{n_0}(x)$, we obtain that $\int_{\mathbb{R}^3}h(x)\varphi(x)\mathrm{d}x>0$.

By (g_1) and (g_2'), for any $\delta>0$, there exists $C_\delta>0$ such that

$$|G(t)|\leqslant C_\delta\,|t|^2+\delta\,|t|^6\quad\text{for all }\ t\in\mathbb{R}. \tag{3.1}$$

Hence, for $t>0$ small enough, we have

$$I(t\varphi)\leqslant\frac{at^2}{2}\int_{\mathbb{R}^3}|\nabla\varphi|^2\mathrm{d}x+\frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla\varphi|^2\mathrm{d}x\right)^2+\int_{\mathbb{R}^3}C_\delta t^2\,|\varphi|^2+\delta t^6\,|\varphi|^6\mathrm{d}x-t\int_{\mathbb{R}^3}h\varphi\mathrm{d}x<0.$$

This shows that

$$c_0\coloneqq\inf_{u\in\bar{B}_\rho}I(u)<0,$$

where $\bar{B}_\rho=\{u\in H_r^1(\mathbb{R}^3):\|u\|\leqslant\rho\}$ and ρ was given by Lemma 2.1. Thus, by Lemma 2.1 and Ekeland's variational principle [33], there is a minimizing sequence $\{u_n\}\subset\bar{B}_\rho$ of c_0 such that

$$c_0\leqslant I(u_n)\leqslant c_0+\frac{1}{n}, \tag{3.2}$$

$$I(v)\geqslant I(u_n)-\frac{1}{n}\|v-u_n\|\quad\text{for all }v\in\bar{B}_\rho. \tag{3.3}$$

Now, we will prove that $\{u_n\}$ is a bounded $(\mathrm{PS})_{c_0}$ sequence of I. First, we claim that $\|u_n\| < \rho$ for all large $n \in \mathbb{N}$. Otherwise, we may assume that $\|u_n\| = \rho$, up to a subsequence. From Lemma 2.1, we deduce that if $|h|_2 < \Lambda$, then $I(u_n) \geqslant \alpha$ for $\|u_n\| = \rho$. Taking the limit as $n \to \infty$ and by using (3.2), we obtain $0 > c_0 \geqslant \alpha > 0$, which is a contradiction. In general, we suppose that $\|u_n\| < \rho$ for all $n \in \mathbb{N}$. Next, we show that $I'(u_n) \to 0$ in $[H_r^1(\mathbb{R}^3)]^*$. For any $\phi \in H_r^1(\mathbb{R}^3)$ with $\|\phi\| = 1$, we choose sufficiently small $\delta > 0$ such that $\|u_n + t\phi\| < \rho$ for all $|t| < \delta$. By using (3.3), we can write

$$\frac{I(u_n + t\phi) - I(u_n)}{t} \geqslant -\frac{1}{n}.$$

Taking the limit as $t \to 0$, we obtain $\langle I'(u_n), \phi\rangle \geqslant -\frac{1}{n}$. Similarly, by replacing ϕ with $-\phi$ in the aforementioned arguments, we obtain $\langle I'(u_n), \phi\rangle \leqslant \frac{1}{n}$. Then, for all $\phi \in H_r^1(\mathbb{R}^3)$ with $\|\phi\| = 1$, we conclude that $\langle I'(u_n), \phi\rangle \to 0$ as $n \to \infty$. This shows at once $\{u_n\}$ is a bounded $(PS)_{c_0}$ sequence of the functional I. Thus, by Lemma 2.2, there exists $u_1 \in H_r^1(\mathbb{R}^3)$ such that $I(u_1) = c_0 < 0$ and $I'(u_1) = 0$ for $|h|_2 < \Lambda$.

Step 2. There exists $u_2 \in H_r^1(\mathbb{R}^3)$ such that $I(u_2) = c > 0$ and $I'(u_2) = 0$.

It follows from Lemmas 2.1, 2.2, 2.4, and the mountain pass theorem.

Step 3. There exists a ground state solution $u_* \in H_r^1(\mathbb{R}^3)$ such that $I(u_*) = m_0 < 0$.

Define

$$\mathcal{S} = \{u \in H_r^1(\mathbb{R}^3) : I'(u) = 0\} \quad \text{and} \quad m_0 = \inf_{u \in \mathcal{S}} I(u).$$

From the aforementioned discussions, $\mathcal{S}$ is nonempty. By Lemma 2.3, for every $u \in \mathcal{S}$, there is $P(u) = 0$. Then for all $u \in \mathcal{S}$, we deduce from (2.20), the Hölder and Sobolev inequalities that

$$\begin{aligned} I(u) - \frac{1}{3}P(u) &= \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 + \frac{1}{3}\int_{\mathbb{R}^3} (x, \nabla h) u \mathrm{d}x \\ &\geqslant \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - C\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \geqslant -\frac{3C^2}{4a}, \end{aligned}$$

which and Step 1 imply $m_0 \in (-\infty, 0)$. Letting $\{u_n\} \subset \mathcal{S}$ be a minimizing sequence of m_0. Combining Lemma 2.3 with the proof of (2.21)–(2.22), we can prove that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. It follows from Lemma 2.2 that there exists $u_* \in H_r^1(\mathbb{R}^3)$ such that $u_n \to u_*$ in $H_r^1(\mathbb{R}^3)$. It is standard to prove that $I(u_*) = m_0 < 0$ and $I'(u_*) = 0$. Therefore, u_* is a ground state solution. Now, we complete the proof of Theorem 1.1. □

Proof of Corollary 1.2. For proving Corollary 1.2, we construct a new equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u = \tilde{g}(u) + h(x) \quad \text{in } \mathbb{R}^3, \tag{3.4}$$

where $\tilde{g} : \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$\tilde{g}(t) = \begin{cases} -mt, & t \leqslant 0, \\ g(t), & t \geqslant 0, \end{cases}$$

and define the energy functional $J : H_r^1(\mathbb{R}^3) \mapsto \mathbb{R}$ given by

$$J(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} \tilde{G}(u)\mathrm{d}x - \int_{\mathbb{R}^3} h(x) u \mathrm{d}x,$$

where $\tilde{G}(t) = \int_0^t \tilde{g}(s)\mathrm{d}s$. It is standard to prove that J is a well-defined C^1-functional. Then, under the assumptions of Theorem 1.1, there exists $\tilde{\Lambda} > 0$ such that equation (3.4) has two different nontrivial solutions $\tilde{u}_1$, $\tilde{u}_2$, and a ground state solution $\tilde{u}_*$ for $|h|_2 < \tilde{\Lambda}$. Further, letting $\tilde{u}_1^- = \min\{\tilde{u}_1, 0\}$ be a test function, one has

$$\langle J'(\tilde{u}_1), \tilde{u}_1^-\rangle = a\int_{\mathbb{R}^3} |\nabla\tilde{u}_1^-|^2 \mathrm{d}x + b\int_{\mathbb{R}^3} |\nabla\tilde{u}_1|^2 \mathrm{d}x \int_{\mathbb{R}^3} |\nabla\tilde{u}_1^-|^2 \mathrm{d}x + \int_{\mathbb{R}^3} m\,|\tilde{u}_1^-|^2 \mathrm{d}x - \int_{\mathbb{R}^3} h\tilde{u}_1^- \mathrm{d}x,$$

which implies that $\tilde{u}_1^- = 0$ since $h(x) \geqslant 0$ in $\mathbb{R}^3$. Then $\tilde{u}_1(x) \geqslant 0$ in $\mathbb{R}^3$. It follows from the definition of $\tilde{g}$ that $\tilde{u}_1$ is also the nonnegative solution of equation (1.1). By (g_1) and (g_2'), there exists a constant $\tilde{L} > 0$ such that

$$g(t) \geqslant -\tilde{L}(|t| + |t|^5) \quad \text{for all } t \in \mathbb{R}.$$

It is clear that $\tilde{u}_1$ solves the following equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + \tilde{L}(1 + u^4)u = g(u) + \tilde{L}(u + u^5) + h(x) \quad \text{in } \mathbb{R}^3.$$

From the regular estimates of elliptic equations, we may deduce that $\tilde{u}_1 \in L^\infty_{loc}(\mathbb{R}^3)$. Therefore, there exists $C(\Omega) > 0$ such that

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla\tilde{u}_1|^2 \mathrm{d}x\right)\Delta\tilde{u}_1 + C(\Omega)\tilde{u}_1 \geqslant 0$$

in any bounded domain Ω. Applying the strong maximum principle (see Gilbarg and Trudinger [34, Theorem 8.19]), we derive that $\tilde{u}_1(x) > 0$ in $\mathbb{R}^3$. Similarly, it can be proved that $\tilde{u}_2(x) > 0$ and $\tilde{u}_*(x) > 0$ in $\mathbb{R}^3$. □

Remark 3.1. Indeed, we have no way of proving whether the negative energy solution u_1 in Theorem 1.1 is equal to the ground state solution u_*. Therefore, determining the relationship between u_1 and u_* becomes a meaningful open problem.

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