Research Article

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Multiplicity of *k*-convex solutions for a singular *k*-Hessian system

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Abstract: In this article, we study the following nonlinear k-Hessian system with singular weights

$$\begin{cases} S_k^{\frac{1}{k}}(\sigma(D^2u_1)) = \lambda b(|x|)f(-u_1, -u_2), & \text{in } \Omega, \\ S_k^{\frac{1}{k}}(\sigma(D^2u_2)) = \lambda h(|x|)g(-u_1, -u_2), & \text{in } \Omega, \\ u_1 = u_2 = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, $1 \le k \le N$ is an integer, Ω stands for the open unit ball in $\mathbb{R}^N$, and $S_k(\sigma(D^2u))$ is the k-Hessian operator of u. By using the fixed point index theory, we prove the existence and nonexistence of negative k-convex radial solutions. Furthermore, we establish the multiplicity result of negative k-convex radial solutions based on *a priori* estimate achieved. More precisely, there exists a constant $\lambda^* > 0$ such that the system admits at least two negative k-convex radial solutions for $\lambda \in (0, \lambda^*)$.

Keywords: k-Hessian system, k-convex solution, multiplicity, upper and lower solution

MSC 2020: 35J60, 35J75, 35B09, 35A16

1 Introduction

In this article, we deal with a class of nonlinear singular k-Hessian system with the 0-Dirichlet boundary value condition

$$\begin{cases} S_k^{\frac{1}{k}}(\sigma(D^2u_1)) = \lambda b(|x|)f(-u_1, -u_2), & \text{in } \Omega, \\ S_k^{\frac{1}{k}}(\sigma(D^2u_2)) = \lambda h(|x|)g(-u_1, -u_2), & \text{in } \Omega, \\ u_1 = u_2 = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0, \Omega = \{x \in \mathbb{R}^N : |x| < 1\}, 1 \le k \le N$ is an integer, the nonlinearities f, g and the weights b, h satisfy the following conditions:

(A1): $f, g \in C([0, \infty) \times [0, \infty), (0, \infty))$.

(A2): $b, h \in C([0, 1), [0, \infty))$ are singular at 1, and not vanishing identically on any subinterval of $[0, 1)$, $\int_0^1 b^k(s)\mathrm{d}s < +\infty$ and $\int_0^1 h^k(s)\mathrm{d}s < +\infty$.

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(A3):

$$f_\infty = \lim_{s+t\to\infty} \frac{f(s,t)}{s+t} = \infty, \quad g_\infty = \lim_{s+t\to\infty} \frac{g(s,t)}{s+t} = \infty.$$

For $k = 1, 2, \ldots, N$ and $u \in C^2(\Omega)$, $S_k(\sigma(D^2u))$ is the k-Hessian operator of u, and given by

$$S_k(\sigma(D^2u)) = S_k(\sigma_1, \sigma_2, \ldots, \sigma_N) = \sum_{1\le j_1<\cdots<j_k\le N} \sigma_{j_1}\cdot\sigma_{j_2}\ldots\sigma_{j_k},$$

where $\sigma_1, \sigma_2, \ldots, \sigma_N$ are the eigenvalues of the Hessian matrix (D^2u) of second derivatives of u. The k-Hessian equation was initiated by Caffarelli et al. [1], and treated as well by Ivochkina [2] in 1985. A natural class of functions for the solutions to the k-Hessian equations is the k-convex function, which is discussed by Trudinger and Wang [3]. A function $u \in C^2(\Omega)$ is called k-convex if and only if $\sigma(D^2u(x)) \in \bar{\Gamma}_k$, where

$$\Gamma_k \coloneqq \{\sigma(D^2u) \in \mathbb{R}^N : S_l(\sigma(D^2u)) > 0, 1 \le l \le k\}.$$

When $k = 1$, the k-Hessian operator is the Laplace operator; while for $k = N$, it corresponds to the Monge-Ampère operator, namely,

Laplace operator	Monge-Ampère operator
$S_1(\sigma(D^2u(x))) = \sum_{i=1}^N \sigma_i = \Delta u,$	$S_N(\sigma(D^2u(x))) = \prod_{i=1}^N \sigma_i = \det D^2u$

For $2 \le k \le N$, it is a fully nonlinear partial differential operator [4]. In recent years, more attention has been paid to the study of the k-Hessian equations motivated by application to differential geometry, fluid dynamics, physics, and other applied subjects. In [5], it is indicated that Monge-Ampère problems have a connection with a reflector shape design or Weingarten curvature. It is noted in [6,7] that these equations are related to theoretical condensed matter physics and renormalization group and are used to describe some phenomena in statistical physics. There are many excellent results about the existence, uniqueness, and multiplicity of the solutions for the k-Hessian equations by means of different methods [8–30].

In 2016, by applying a Pohozaev-type identity and the monotone separation techniques, Wei [12] studied the uniqueness of negative radial solutions for the k-Hessian equation:

$$\begin{cases} S_k(\sigma(D^2u)) = f(-u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f \in C^2([0,\infty))$, $f(0) = 0$, $f(s) > 0$ on $(0,\infty)$ and satisfies $-sf'(-s) > kf(-s)$, or $-sf'(-s) < kf(-s)$ for $s < 0$.

In 2020, by applying the monotone iterative method, Wang et al. [17] showed the existence of entire positive bounded and blow-up radial solution for the following problem with the k-Hessian operator

$$\begin{cases} \mathcal{G}\left(S_k^{\frac{1}{k}}(\sigma(D^2u_1))\right)S_k^{\frac{1}{k}}(\sigma(D^2u_1)) = b(|x|)f(u_1,u_2), & x \in \mathbb{R}^N, \\ \mathcal{G}\left(S_k^{\frac{1}{k}}(\sigma(D^2u_2))\right)S_k^{\frac{1}{k}}(\sigma(D^2u_2)) = h(|x|)g(u_1,u_2), & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $\mathcal{G}$ is a nonlinear operator, $f, g \in C([0,\infty)\times[0,\infty))$ are positive and increasing for every variables.

In 2022, by using Guo-Krasnosel'skii fixed-point theorem [31,32], we [23] considered the k-Hessian problem

$$\begin{cases} S_k(\sigma(D^2u_1)) = \lambda_1 b(|x|)f(-u_1,-u_2), & \text{in } \Omega, \\ S_k(\sigma(D^2u_2)) = \lambda_2 h(|x|)g(-u_1,-u_2), & \text{in } \Omega, \\ u_1 = u_2 = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

under the assumption that f and g are k-sublinear or k-superlinear growth at 0 or ∞. The problem (1.4) has at least one radial solution for all $\lambda_i > 0$, and at least two radial solutions for the case $\lambda_i \ge \mu_1^k > 0$ or $\lambda_i \le \mu_2^k$, where $\lambda_i, \mu_i > 0$, $i = 1, 2$.

Recently, Zhang [24] used the index theory of fixed points for compact maps [33] to deal with the k_i-Hessian problem

$$\begin{cases}(-1)^{k_i}S_{k_i}^{\frac{1}{k_i}}(\sigma(D^2u_i)) = b_i(|x|)f(u_1, u_2, \ldots, u_m), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega,\end{cases} \tag{1.5}$$

where $k_i \in \{1, 2, \ldots, N\}$, $N < 2k_i$, $i \in \{1, 2, \ldots, m\}$, $f \in C([0, \infty)^m, [0, \infty))$. He showed that problem (1.5) has one, two, and many positive radial solutions under various hypotheses on the function f.

Inspired by the above works, by using the fixed point index theory [31,32], we present new results on the existence and nonexistence of negative k-convex radial solutions for the singular k-Hessian problem (1.1) in a ball Ω. Furthermore, by establishing a prior estimate and combining upper and lower solution method, the multiplicity of negative k-convex radial solutions for the system (1.1) is also obtained. Compared with the works [21–24], our results consider not only the existence and multiplicity of radial solutions but also the nonexistence of radial solutions, which complement and improve the previous research works.

2 Preliminaries

In this section, we first present a lemma of the property of radial functions with respect to $S_k(\sigma(D^2u))$, which will be used in the proof of the main results.

Lemma 2.1. [8] *Suppose that $v(r) \in C^2[0, R)$ and $v'(0) = 0$, then the function $u(x) = v(r)$ with $r = |x| < R$ belongs to $C^2(B_R)$, and*

$$\sigma(D^2u) = \begin{cases}\left(v''(r), \dfrac{v'(r)}{r}, \ldots, \dfrac{v'(r)}{r}\right), & r \in (0, R),\\ (v''(0), v''(0), \ldots, v''(0)), & r = 0,\end{cases}$$

$$S_k(\sigma(D^2u)) = \begin{cases}C_{N-1}^{k-1}v''(r)\left(\dfrac{v'(r)}{r}\right)^{k-1} + C_{N-1}^{k}\left(\dfrac{v'(r)}{r}\right)^{k}, & r \in (0, R),\\ C_N^k(v''(0))^k, & r = 0,\end{cases}$$

where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and $C_N^k = \frac{N!}{k!(N-k)!}$.

Lemma 2.2. [8] *Let $f(t)$ be a continuous function defined on $\mathbb{R}$ and satisfying that $f(t) > 0$ is monotone non-decreasing in $(0, +\infty)$ and $f(t) = 0$ in $(-\infty, 0]$. For any positive number a, assume $v(r) \in C^0[0, R) \cap C^1(0, R)$ is a solution of the Cauchy problem*

$$\begin{cases}v'(r) = \left(\dfrac{k}{C_{N-1}^{k-1}}r^{k-N}\displaystyle\int_0^r s^{N-1}f^k(v(s))\mathrm{d}s\right)^{\frac{1}{k}}, & r > 0,\\ v(0) = a.\end{cases}$$

Then $v(r) \in C^2[0, R)$, and it satisfies equation

$$C_{N-1}^{k-1}v''(r)\left(\frac{v'(r)}{r}\right)^{k-1} + C_{N-1}^{k}\left(\frac{v'(r)}{r}\right)^{k} = f^k(v(r)),$$

with $v'(0) = 0$ and

$$\sigma_r := \left(v''(r), \frac{v'(r)}{r}, \ldots, \frac{v'(r)}{r}\right) \in \Gamma_k,$$

for $0 \le r < R$.

By using the analogous proof of Lemma 2.2, we have the following Lemma.

Lemma 2.3. *Assume that* (A1) *and* (A2) *are satisfied and* $(v_1(r), v_2(r)) \in (C^0[0,1] \cap C^1(0,1])^2$ *is a solution of the Cauchy problem*

$$\begin{cases} v_1'(r) = -\left(\dfrac{k}{C_{N-1}^{k-1}} r^{k-N} \displaystyle\int_0^r s^{N-1}[\lambda b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}}, & 0 < r < 1, \\ v_2'(r) = -\left(\dfrac{k}{C_{N-1}^{k-1}} r^{k-N} \displaystyle\int_0^r s^{N-1}[\lambda h(s) g(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}}, & 0 < r < 1, \\ v_1(1) = 0, \quad v_2(1) = 0. \end{cases}$$

Then $(v_1(r), v_2(r)) \in C^2[0,1] \times C^2[0,1]$, *and it satisfies the system*

$$\begin{cases} C_{N-1}^{k-1}(-v_1)''(r)\left(\dfrac{-v_1'(r)}{r}\right)^{k-1} + C_{N-1}^{k}\left(\dfrac{-v_1'(r)}{r}\right)^{k} = [\lambda b(s) f(v_1(s), v_2(s))]^k, \\ C_{N-1}^{k-1}(-v_2)''(r)\left(\dfrac{-v_2'(r)}{r}\right)^{k-1} + C_{N-1}^{k}\left(\dfrac{-v_2'(r)}{r}\right)^{k} = [\lambda h(s) g(v_1(s), v_2(s))]^k, \end{cases}$$

with $v_i'(0) = 0$ *and*

$$\sigma_{i,r} := \left(-v_i''(r), \frac{-v_i'(r)}{r}, \ldots, \frac{-v_i'(r)}{r}\right) \in \overline{\Gamma}_k,$$

for $0 \le r < 1$, *where* $i = 1, 2$.

By the standard derivation, one can convert the singular problem (1.1) with the 0-Dirichlet boundary value condition $u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0$ to the following boundary value problem. An analogous proof process can also be found in [21,23].

Lemma 2.4. $(u_1(x), u_2(x)) = (-v_1(r), -v_2(r))$ *is a radial solution of the singular problem* (1.1) if and only if $(v_1(r), v_2(r))$ *is a solution of the following boundary value problem*

$$\begin{cases} \left\{\dfrac{r^{N-k}}{k}[(-v_1(r))']^k\right\}' = \dfrac{r^{N-1}}{C_{N-1}^{k-1}}[\lambda b(r) f(v_1(r), v_2(r))]^k, & 0 < r < 1, \\ \left\{\dfrac{r^{N-k}}{k}[(-v_2(r))']^k\right\}' = \dfrac{r^{N-1}}{C_{N-1}^{k-1}}[\lambda h(r) g(v_1(r), v_2(r))]^k, & 0 < r < 1, \\ v_1'(0) = 0, \quad v_2'(0) = 0, \quad v_1(1) = 0, \quad v_2(1) = 0. \end{cases} \tag{2.1}$$

Next, we consider the following equivalent integral system of (2.1).

$$\begin{cases} v_1(r) = \displaystyle\int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t, & r \in [0,1], \\ v_2(r) = \displaystyle\int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda h(s) g(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t, & r \in [0,1]. \end{cases} \tag{2.2}$$

In the following lemmas, we assume that E is a real Banach space and H is a cone in E.

Lemma 2.5. [31,32] *For $r > 0$, define $H_r = \{v \in H : \|v\| < r\}$. Assume that $T : \overline{H}_r \to H$ is a compact map such that $Tv \neq v$ for $v \in \partial H_r$.*
(1) If $\|Tv\| \leq \|v\|$ for $v \in \partial H_r$, *then* $i(T, H_r, H) = 1$.
(2) If $\|Tv\| \geq \|v\|$ for $v \in \partial H_r$, *then* $i(T, H_r, H) = 0$.

Lemma 2.6. [31] *Let $\Omega \subset E$ be a bounded open set in E, $0 \in \Omega$ and $T : H \cap \overline{\Omega} \to H$ be condensing. Assume that $Tv \neq \theta v$ for $v \in H \cap \partial\Omega$* and $\theta \geq 1$. *Then* $i(T, H \cap \Omega, H) = 1$.

To apply Lemmas 2.5 and 2.6 to the system (2.2), let $E = (C[0,1])^2$ be the Banach space with $\|(v_1, v_2)\| = \|v_1\|_0 + \|v_2\|_0$, where $\|v_i\|_0 = \max_{r\in[0,1]}|v_i(r)|$, $i = 1, 2$. Define H be the positive cone in E by

$$H = \left\{(v_1, v_2) : (v_1, v_2) \in E, v_1, v_2 \geq 0, \min_{r\in[\frac{1}{4},\frac{3}{4}]} (v_1(r) + v_2(r)) \geq \frac{1}{4}(\|v_1\|_0 + \|v_2\|_0)\right\}.$$

Note that H is a nonempty set by Remark 2.1 [23].

Define

$$T_1(v_1, v_2)(r) = \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t,$$

$$T_2(v_1, v_2)(r) = \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda h(s) g(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t,$$

and

$$T(v_1, v_2)(r) = (T_1(v_1, v_2)(r), T_2(v_1, v_2)(r)).$$

In our setting, (2.2) is equivalent to

$$T(v_1, v_2) = (v_1, v_2).$$

Lemma 2.7. *Assume that* (A1) *and* (A2) *are satisfied, then $T(H) \subset H$ and $T : E \to E$ is completely continuous.*

Proof. It is a standard procedure to prove that $T : E \to E$ is completely continuous, which is divided into three steps.

(i) The operator T is continuous in view of the continuity of b, h, f, g.

(ii) Let Ω_l be a bounded subset of H, i.e.,

$$\Omega_l \coloneqq \{(v_1, v_2) \in H : \|(v_1, v_2)\| \leq l\},$$

where $l > 0$ is a constant. Let

$$M_1 = \max_{\|(v_1,v_2)\|\leq l} f(v_1, v_2), \quad M_2 = \max_{\|(v_1,v_2)\|\leq l} g(v_1, v_2), \quad c_1 = \int_0^1 b^k(t)\mathrm{d}t, \quad c_2 = \int_0^1 h^k(t)\mathrm{d}t,$$

then for $(v_1, v_2) \in \Omega_l$, we have

$$\begin{aligned}
\|T_1(v_1, v_2)(r)\| &\leq \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&\leq \lambda \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \int_0^1 \left(t^{k-1} \int_0^1 [b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&\leq \lambda M_1 \left(\frac{k c_1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \int_0^1 t^{\frac{k-1}{k}} \mathrm{d}t \\
&= \lambda M_1 \left(\frac{k c_1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \frac{k}{2k-1}
\end{aligned}$$

and

$$\|T_2(v_1,v_2)(r)\| \le \lambda M_2\left(\frac{kc_2}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\frac{k}{2k-1},$$

which mean that $T(\Omega_l)$ is uniformly bounded;

(iii) for $(v_1,v_2)\in\Omega_l$, one has

$$\begin{aligned}|(T_1(v_1,v_2))'(t)| &= \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda b(s)f(v_1(s),v_2(s))]^k\mathrm{d}s\right)^{\frac{1}{k}}\\ &\le \left(\frac{k}{C_{N-1}^{k-1}}t^{k-1}\int_0^1[\lambda b(s)f(v_1(s),v_2(s))]^k\mathrm{d}s\right)^{\frac{1}{k}}\\ &\le \lambda M_1\left(\frac{kc_1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}t^{\frac{k-1}{k}} \le \lambda M_1\left(\frac{kc_1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\end{aligned}$$

and

$$|(T_2(v_1,v_2))'(t)| \le \lambda M_2\left(\frac{kc_2}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}.$$

By the mean-value formula, we can see that for $(v_1,v_2)\in\Omega_l$ and $t_1,t_2\in[0,1]$, there exists $\zeta\in[t_1,t_2]$ such that

$$|T_i(v_1,v_2)(t_1)-T_i(v_1,v_2)(t_2)| = |(T_i(v_1,v_2))'(\zeta)||t_1-t_2| \le \lambda M_i\left(\frac{kc_i}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}|t_1-t_2|,$$

where $i=1,2$. When $|t_1-t_2|\to 0$, the right-hand side of the above inequality tends to zero. This indicates that $T(\Omega_l)$ is equicontinuous.

By the Arzela-Ascoli theorem, we obtain that T is completely continuous.

Next, we prove that $T(H)\subset H$. Choosing $(v_1,v_2)\in H$, it follows from (A1) and (A2) that $T_i(v_1,v_2)(r)\ge 0$, $(T_i(v_1,v_2))'(r)\le 0$, $(T_i(v_1,v_2))''(r)\le 0$. So, for every $r\in[\frac{1}{4},\frac{3}{4}]$, we have

$$\frac{T_i(v_1,v_2)(1)-T_i(v_1,v_2)(0)}{1} \le \frac{T_i(v_1,v_2)(r)-T_i(v_1,v_2)(0)}{r},$$

i.e., $T_i(v_1,v_2)(r)-T_i(v_1,v_2)(0)\ge r(T_i(v_1,v_2)(1)-T_i(v_1,v_2)(0))$, and then

$$\min_{r\in[\frac{1}{4},\frac{3}{4}]} T_i(v_1,v_2)(r) \ge \frac{1}{4}\|T_i(v_1,v_2)\|_0, \qquad i=1,2.$$

Thus,

$$\begin{aligned}\min_{r\in[\frac{1}{4},\frac{3}{4}]}(T_1(v_1,v_2)(r)+T_2(v_1,v_2)(r)) &= \min_{r\in[\frac{1}{4},\frac{3}{4}]}T_1(v_1,v_2)(r) + \min_{r\in[\frac{1}{4},\frac{3}{4}]}T_2(v_1,v_2)(r)\\ &\ge \frac{1}{4}\|T_1(v_1,v_2)\|_0 + \frac{1}{4}\|T_2(v_1,v_2)\|_0\\ &= \frac{1}{4}(\|T_1(v_1,v_2)\|_0 + \|T_2(v_1,v_2)\|_0)\\ &= \frac{1}{4}\|(T_1(v_1,v_2),T_2(v_1,v_2))\|,\end{aligned}$$

which means that $T(H)\subset H$. The proof is completed. □

3 Existence and nonexistence

Denote

$$L_1 = \frac{k}{2k-N}\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \quad \text{and} \quad L_2 = \frac{k}{2k-N}\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(1-\left(\frac{3}{4}\right)^{\frac{2k-N}{k}}\right),$$

where $N < 2k$.

Theorem 3.1. *Assume that* (A1), (A2), *and* (A3) *are satisfied. Then,*
(1) *there exists a constant $\mu_1 > 0$ such that for all $\lambda < \mu_1$, the system* (1.1) *admits at least one negative k-convex radial solution*;
(2) *for λ is sufficiently large, system* (1.1) *admits no negative k-convex radial solution.*

Proof. (1) If $q > 0$, then it follows from (A1) and (A2) that

$$M_1(q) = L_1 \max_{(v_1,v_2)\in H, \|(v_1,v_2)\|=q}\left(\int_0^1 s^{N-1}[b(s)f(v_1(s),v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} > 0,$$

and

$$M_2(q) = L_1 \max_{(v_1,v_2)\in H, \|(v_1,v_2)\|=q}\left(\int_0^1 s^{N-1}[h(s)g(v_1(s),v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} > 0.$$

Let $M(q) = \max\{M_1(q), M_2(q)\}$. Choose a number $J_1 > 0$, let $\mu_1 = \frac{J_1}{2M(J_1)}$ and set

$$H_{J_1} = \{(v_1,v_2) : (v_1,v_2) \in H, \|(v_1,v_2)\| < J_1\}.$$

Then for $\lambda < \mu_1$ and $(v_1,v_2) \in \partial H_{J_1}$, we have

$$\begin{aligned}
T_1(v_1,v_2)(r) &= \int_r^1\left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda b(s)f(v_1(s),v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&\leq \int_0^1\left(\frac{k}{t^{N-k}}\int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda b(s)f(v_1(s),v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&< \mu_1\int_0^1\left(\frac{k}{t^{N-k}}\int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}}[b(s)f(v_1(s),v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&= \mu_1\frac{k}{2k-N}\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(\int_0^1 s^{N-1}[b(s)f(v_1(s),v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \\
&= \mu_1 L_1\left(\int_0^1 s^{N-1}[b(s)f(v_1(s),v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \\
&\leq \mu_1 M(J_1) \\
&= \frac{J_1}{2}.
\end{aligned}$$

In a similar way, $T_2(v_1,v_2)(r) < \frac{J_1}{2}$, which implies

$$\|T(v_1,v_2)\| = \|T_1(v_1,v_2)\| + \|T_2(v_1,v_2)\| < J_1 = \|(v_1,v_2)\|$$

for $(v_1,v_2) \in \partial H_{J_1}$. By (1) of Lemma 2.5, we have $i(T, H_{J_1}, H) = 1$.

Here, we shall address the case $f_\infty = \infty$, and the proof for the case $g_\infty = \infty$ is similar. From $f_\infty = \infty$, we then can choose $\widehat{J} > 0$ so that $f(v_1, v_2) \geq \delta(v_1 + v_2)$ for $v_1 + v_2 \geq \widehat{J}$, where $\delta > 0$ is chosen so that

$$\frac{\lambda\delta L_2}{4}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1}b^k(s)\mathrm{d}s\right)^{\frac{1}{k}} > 1.$$

Let $J_2 = \max\{2J_1, 4\widehat{J}\}$ and set

$$H_{J_2} = \{(v_1, v_2) : (v_1, v_2) \in H, \|(v_1, v_2)\| < J_2\}.$$

If $(v_1, v_2) \in \partial H_{J_2}$, then we have

$$\min_{r\in[\frac{1}{4},\frac{3}{4}]} (v_1(r) + v_2(r)) \geq \frac{1}{4}(\|v_1\|_0 + \|v_2\|_0) = \frac{1}{4}\|(v_1, v_2)\| \geq \widehat{J}$$

and

$$\begin{aligned}
T_1(v_1, v_2)(r) &= \int_r^1\left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda b(s)f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&\geq \int_{\frac{3}{4}}^1\left(\frac{k}{t^{N-k}}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda b(s)f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&\geq \int_{\frac{3}{4}}^1\left(\frac{k}{t^{N-k}}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}}[\lambda\delta b(s)(v_1(s) + v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&\geq \frac{\lambda\delta}{4}(\|v_1\|_0 + \|v_2\|_0)\int_{\frac{3}{4}}^1\left(\frac{k}{t^{N-k}}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}}b^k(s)\mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&= \frac{\lambda\delta}{4}\|(v_1, v_2)\|\frac{k}{2k-N}\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(1 - \left(\frac{3}{4}\right)^{\frac{2k-N}{k}}\right)\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1}b^k(s)\mathrm{d}s\right)^{\frac{1}{k}}\\
&= \frac{\lambda\delta}{4}\|(v_1, v_2)\|L_2\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1}b^k(s)\mathrm{d}s\right)^{\frac{1}{k}}\\
&= \|(v_1, v_2)\|\frac{\lambda\delta L_2}{4}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1}b^k(s)\mathrm{d}s\right)^{\frac{1}{k}}\\
&> \|(v_1, v_2)\|.
\end{aligned}$$

Therefore,

$$\|T(v_1, v_2)\| = \|T_1(v_1, v_2)\| + \|T_2(v_1, v_2)\| \geq T_1(v_1, v_2)(r) > \|(v_1, v_2)\|$$

for $(v_1, v_2) \in \partial H_{J_2}$. By (2) of Lemma 2.5, we have $i(T, H_{J_2}, H) = 0$.

Since $J_1 < J_2$, by the additivity of the fixed point index, we have $i(T, H_{J_2}\backslash\overline{H}_{J_1}, H) = -1$. Consequently, T has a fixed point in $H_{J_2}\backslash\overline{H}_{J_1}$, which is the positive solution of system (2.2).

(2) Since $f > 0$ and $f_\infty = \infty$ (the proof is similar if $g_\infty = \infty$), we have that the existence of a constant $\widetilde{J} > 0$ such that $f(v_1, v_2) \geq \widetilde{J}(v_1 + v_2)$ for $v_1, v_2 \geq 0$, where $\widetilde{J}$ is chosen so that

$$\frac{\lambda \widetilde{J} L_2}{4}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} b^k(s)\mathrm{d}s\right)^{\frac{1}{k}} > 1.$$

Let $(v_1, v_2) \in E$ be a positive solution of system (2.2). By Lemma 2.7, we have $(v_1, v_2) \in H$ and

$$\min_{r\in[\frac{1}{4},\frac{3}{4}]} (v_1(r) + v_2(r)) \geq \frac{1}{4}(\|v_1\|_0 + \|v_2\|_0).$$

For λ is large enough, we have

$$\begin{aligned}
v_1(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&\geq \int_{\frac{3}{4}}^1 \left(\frac{k}{t^{N-k}} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&\geq \int_{\frac{3}{4}}^1 \left(\frac{k}{t^{N-k}} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda \widetilde{J}\, b(s)(v_1(s) + v_2(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&\geq \frac{\lambda \widetilde{J}}{4}(\|v_1\|_0 + \|v_2\|_0) \int_{\frac{3}{4}}^1 \left(\frac{k}{t^{N-k}} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}} b^k(s)\mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t \\
&= \frac{\lambda \widetilde{J}}{4} \|(v_1, v_2)\| \frac{k}{2k-N} \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \left(1 - \left(\frac{3}{4}\right)^{\frac{2k-N}{k}}\right) \left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} b^k(s)\mathrm{d}s\right)^{\frac{1}{k}} \\
&= \frac{\lambda \widetilde{J}}{4} \|(v_1, v_2)\| L_2 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} b^k(s)\mathrm{d}s\right)^{\frac{1}{k}} \\
&= \|(v_1, v_2)\| \frac{\lambda \widetilde{J} L_2}{4} \left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} b^k(s)\mathrm{d}s\right)^{\frac{1}{k}} \\
&> \|(v_1, v_2)\|,
\end{aligned}$$

which is a contradiction. This completes the proof. □

4 Multiplicity result

Now let us list an assumption on f and g, which is to be used in Theorem 4.1.

(A4): $f(s_1, t_1) \leq f(s_2, t_2)$ and $g(s_1, t_1) \leq g(s_2, t_2)$ for $0 \leq s_1 \leq s_2$, $0 \leq t_1 \leq t_2$.

In order to obtain the multiplicity result, we first need the following auxiliary lemmas.

Lemma 4.1. *Assume that conditions* (A1), (A2), *and* (A3) *are satisfied, and λ belongs to a compact subset I of $(0,\infty)$. Then there exists a constant $\alpha_I > 0$ such that $\|(v_1, v_2)\| \le \alpha_I$ for all possible positive solutions (v_1, v_2) of the system* (2.2).

Proof. Assume that there exists a function sequence $\{(v_1^{(m)}, v_2^{(m)})\}_{m\ge1}$ of positive solutions of the system (2.2), with corresponding $\{\lambda_m\}_{m\ge1} \subset I$, such that

$$\lim_{m\to\infty} \|(v_1^{(m)}, v_2^{(m)})\| = \infty.$$

By Lemma 2.7, we have $(v_1^{(m)}, v_2^{(m)}) \in H$ and

$$\min_{r\in[\frac{1}{4},\frac{3}{4}]} (v_1^{(m)}(r) + v_2^{(m)}(r)) \ge \frac{1}{4}(\|v_1^{(m)}\|_0 + \|v_2^{(m)}\|_0).$$

As before, here, we only consider the case $f_\infty = \infty$, and the proof for the case $g_\infty = \infty$ is similar. From $f_\infty = \infty$, we may choose $\widehat{J} > 0$ so that $f(v_1, v_2) \ge \delta(v_1 + v_2)$ for $v_1 + v_2 \ge \widehat{J}$, where $\delta > 0$ is chosen so that

$$\frac{\lambda_m \delta L_2}{4}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} b^k(s) \mathrm{d}s\right)^{\frac{1}{k}} > 1.$$

Choosing m large enough so that $\frac{1}{4}(\|v_1^{(m)}\|_0 + \|v_2^{(m)}\|_0) \ge \widehat{J}$, we have

$$\begin{aligned}
\|v_1^{(m)}\|_0 \ge v_1^{(m)}(r)
&= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda_m b(s) f(v_1^{(m)}(s), v_2^{(m)}(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&\ge \int_{\frac{3}{4}}^1 \left(\frac{k}{t^{N-k}} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda_m b(s) f(v_1^{(m)}(s), v_2^{(m)}(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&\ge \int_{\frac{3}{4}}^1 \left(\frac{k}{t^{N-k}} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda_m \delta b(s)(v_1^{(m)}(s) + v_2^{(m)}(s))]^k \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&\ge \frac{\lambda_m \delta}{4}(\|v_1^{(m)}\|_0 + \|v_2^{(m)}\|_0) \int_{\frac{3}{4}}^1 \left(\frac{k}{t^{N-k}} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s^{N-1}}{C_{N-1}^{k-1}} b^k(s) \mathrm{d}s\right)^{\frac{1}{k}} \mathrm{d}t\\
&= \frac{\lambda_m \delta}{4}(\|v_1^{(m)}\|_0 + \|v_2^{(m)}\|_0) \frac{k}{2k-N}\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(1 - \left(\frac{3}{4}\right)^{\frac{2k-N}{k}}\right)\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} b^k(s) \mathrm{d}s\right)^{\frac{1}{k}}\\
&= (\|v_1^{(m)}\|_0 + \|v_2^{(m)}\|_0) \frac{\lambda_m \delta L_2}{4}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} b^k(s) \mathrm{d}s\right)^{\frac{1}{k}}\\
&> \|v_1^{(m)}\|_0 + \|v_2^{(m)}\|_0.
\end{aligned}$$

Obviously, this is contradictory. This completes the proof. □

For the convenience of writing, we will write $(v_1, v_2) \le (v_3, v_4)$ if $v_1(r) \le v_3(r)$ and $v_2(r) \le v_4(r)$ hold for all $r \in [0,1]$.

Definition 4.1. If the function pair $(\bar{v}_1, \bar{v}_2) \in C[0,1] \times C[0,1]$ satisfies

$$\begin{cases} \bar{v}_1(r) \geq \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(\bar{v}_1(s), \bar{v}_2(s))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t, \\ \bar{v}_2(r) \geq \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda h(s) g(\bar{v}_1(s), \bar{v}_2(s))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t, \end{cases}$$

we say $(\bar{v}_1, \bar{v}_2)$ is an upper solution of (2.2).

Definition 4.2. If the function pair $(\underline{v}_1, \underline{v}_2) \in C[0,1] \times C[0,1]$ satisfies

$$\begin{cases} \underline{v}_1(r) \leq \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(\underline{v}_1(s), \underline{v}_2(s))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t, \\ \underline{v}_2(r) \leq \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda h(s) g(\underline{v}_1(s), \underline{v}_2(s))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t, \end{cases}$$

we say $(\bar{v}_1, \bar{v}_2)$ is a lower solution of (2.2).

Lemma 4.2. *Assume that* (A1), (A2), *and* (A4) *are satisfied. Let* $(\bar{v}_1, \bar{v}_2)$ *is an upper solution of system* (2.2) *such that* $(0,0) \leq (\bar{v}_1, \bar{v}_2)$, *then there exists a positive solution* (v_1, v_2) *of the system* (2.2) *with* $(0,0) \leq (v_1, v_2) \leq (\bar{v}_1, \bar{v}_2)$.

Proof. On view of the monotonicity conditions (A4) on f and g, we first note that $(0,0)$ is a lower solution of the system (2.2), and it is clear that the result can be proved by the usual monotonic iterative scheme. This completes the proof. □

Denote

$$\Gamma = \{\lambda > 0 : \ (2.2) \text{ has a positive solution}\}$$

and $\lambda^* = \sup\Gamma$. By Theorem 3.1, Γ is nonempty and bounded, and thus, $\lambda^* \in (0,\infty)$. We claim that $\lambda^* \in \Gamma$. In face, let $\lambda_m \to \lambda^*$, where $\lambda_m \in \Gamma$. Because $\{\lambda_m\}_{m\geq 1}$ are bounded, by Lemma 4.1, one see that the corresponding positive solutions $(v_1^{(m)}, v_2^{(m)})$ are bounded. It follows from the compactness of the integral operators T_1 and T_2 that $\lambda^* \in \Gamma$. Let (v_1^*, v_2^*) be a positive solution of system (2.2) corresponding to λ^*.

Lemma 4.3. *Assume that* (A1), (A2), *and* (A4) *are satisfied. Let* $0 < \lambda < \lambda^*$. *Then there exists* $\varepsilon^* > 0$ *such that for* $0 < \varepsilon \leq \varepsilon^*$, $(v_1^* + \varepsilon, v_2^* + \varepsilon)$ *is an upper solution of system* (2.2).

Proof. It follows from (A1), (A4), and $(v_1^*, v_2^*) \geq (0,0)$ that there exists a constant $c > 0$ so that $f(v_1^*(r), v_2^*(r)) \geq c > 0$ and $g(v_1^*(r), v_2^*(r)) \geq c$ for all $r \in [0,1]$. By uniform continuity, there exists $\varepsilon^* > 0$ such that

$$|f(v_1^*(r) + \varepsilon, v_2^*(r) + \varepsilon) - f(v_1^*(r), v_2^*(r))| < \frac{c(\lambda^* - \lambda)}{\lambda},$$

and

$$|g(v_1^*(r) + \varepsilon, v_2^*(r) + \varepsilon) - g(v_1^*(r), v_2^*(r))| < \frac{c(\lambda^* - \lambda)}{\lambda},$$

for all $t \in [0,1]$ and $0 < \varepsilon \leq \varepsilon^*$. Further, one has

$$v_1^*(r) + \varepsilon > \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda^* b(s) f(v_1^*(s), v_2^*(s))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t$$

$$
\begin{aligned}
&= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1^*(s)+\varepsilon, v_2^*(s)+\varepsilon)]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t \\
&\quad - \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s)(f(v_1^*(s)+\varepsilon, v_2^*(s)+\varepsilon) - f(v_1^*(s), v_2^*(s)))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t \\
&\quad + \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [(\lambda^* - \lambda) b(s) f(v_1^*(s), v_2^*(s))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t, \\
&\geq \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1^*(s)+\varepsilon, v_2^*(s)+\varepsilon)]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t.
\end{aligned}
$$

Similarly,

$$
v_2^*(r) + \varepsilon > \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda h(s) g(v_1^*(s)+\varepsilon, v_2^*(s)+\varepsilon)]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t.
$$

This completes the proof. □

Theorem 4.1. *Assume that* (A1), (A2), (A3), *and* (A4) *are satisfied.*

Then there exists a constant $\lambda^* > 0$ *such that the system* (1.1) *admits at least two negative k-convex radial solutions for* $\lambda \in (0, \lambda^*)$, *at least one negative k-convex radial solution for* $\lambda = \lambda^*$ *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Proof. Let $0 < \lambda < \lambda^*$. Because (v_1^*, v_2^*) is an upper solution of system (2.2), by Lemma 4.2, we see that there is a positive solution $(\hat{v}_1, \hat{v}_2)$ of the system (2.2) with $(0, 0) \leq (\hat{v}_1, \hat{v}_2) \leq (v_1^*, v_2^*)$. Thus, there exists a positive solution for $0 < \lambda \leq \lambda^*$, and a positive solution does not exist for $\lambda > \lambda^*$. Next, we find another positive solution of the system (2.2) for $0 < \lambda < \lambda^*$. Set

$$
\Omega_1 = \{(v_1, v_2) : (v_1, v_2) \in E, -\varepsilon < v_i(r) < v_i^*(r) + \varepsilon, r \in [0, 1], i = 1, 2\},
$$

where $\varepsilon > 0$. It is clear that Ω_1 is bounded and open in E and $(0, 0) \in \Omega_1$. Because T is completely continuous, $T : H \cap \overline{\Omega}_1 \to H$ is condensing. Furthermore, $(\hat{v}_1, \hat{v}_2) \in \Omega_1$ for $0 < \lambda \leq \lambda^*$.

Let $(v_1, v_2) \in H \cap \partial\Omega_1$. Then there exists $r_0 \in [0, 1]$ such that $v_i(r_0) = v_i^*(r_0) + \varepsilon$ $(i = 1, 2)$. Here, assume that $v_1(r_0) = v_1^*(r_0) + \varepsilon$, by (A_4), we obtain for $\theta \geq 1$

$$
\begin{aligned}
T_1(v_1, v_2)(r_0) &= \int_{r_0}^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1(s), v_2(s))]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t \\
&\leq \int_{r_0}^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\lambda b(s) f(v_1^*(s)+\varepsilon, v_2^*(s)+\varepsilon)]^k \mathrm{d}s \right)^{\frac{1}{k}} \mathrm{d}t \\
&< v_1^*(r_0) + \varepsilon \\
&= v_1(r_0) \\
&\leq \theta v_1(r_0).
\end{aligned}
$$

Similarly, $T_1(v_1, v_2)(r_0) < \theta v_2(r_0)$. Thus, for $(v_1, v_2) \in H \cap \partial\Omega_1$ and $\theta \geq 1$, $T(v_1, v_2) \neq \theta(v_1, v_2)$. By Lemma 2.6, one has

$$
i(T, H \cap \Omega_1, H) = 1.
$$

Let $J = \max\{a_I + 1, J_2, \|(v_1^* + \varepsilon, v_2^* + \varepsilon)\|\}$, where a_I and J_2 have been defined in Lemma 4.1 and Theorem 3.1, respectively. Set

$$H_J = \{(v_1, v_2) : (v_1, v_2) \in H, \|(v_1, v_2)\| < J\}.$$

From Lemma 4.1, we know that $T(v_1, v_2) \neq (v_1, v_2)$ for $(v_1, v_2) \in \partial H_J$. If $(v_1, v_2) \in \partial H_J$, then, by the proof of Theorem 3.1, we can obtain that $\|T(v_1, v_2)\| \geq \|(v_1, v_2)\|$. Consequently, by Lemma 2.5, one has

$$i(T, H_J, H) = 0.$$

It follows from the additivity of the fixed point index that

$$i(T, H_J \backslash \overline{H \cap \Omega_1}, H) = -1.$$

Thus, T has another fixed point in $H_J \backslash \overline{H \cap \Omega_1}$, which establishes that the system (2.2) admits at least two positive solutions for $\lambda \in (0, \lambda^*)$. This completes the proof. □

Remark 4.1. What we need to note is that when $f, g \in C([0, \infty) \times [0, \infty))$ satisfy $f(0, 0) = g(0, 0) = 0$, $f(s, t) > 0$, $g(s, t) > 0$ for $s, t \in (0, \infty)$, that is weaker than the condition (A1), Theorems 3.1 and 4.1 are still valid.

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