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Research Article

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A note on λ-analogue of Lah numbers and λ-analogue of r-Lah numbers

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Abstract: In this study, we introduce the λ-analogue of Lah numbers and λ-analogue of r-Lah numbers in the view of degenerate version, respectively. We investigate their properties including recurrence relation and several identities of λ-analogue of Lah numbers arising from degenerate differential operators. Using these new identities, we study the normal ordering of degenerate integral power of the number operator in terms of boson operators, which is represented by means of λ-analogue of Lah numbers and λ-analogue of r-Lah numbers, respectively.

Keywords: Lah numbers, degenerate Stirling numbers of the second kind, degenerate differential operator, normal ordering, number operator

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1 Introduction

The degenerate Stirling numbers of the first kind (5) and the second kind (6) have been studied by some scholars, starting with Carlitz [1]. In detail, when $\lambda \to 0$ in (5) and (6), we obtain the Stirling numbers. However, we cannot consider degenerate versions of Lah numbers from Lah numbers (15). From this point of view, we consider the λ-analogue r-Lah numbers (27) that replaces the rising factorials and the falling factorials with the generalized rising factorials and the generalized falling factorials on both sides at the same time. When $\lambda \to 1$ in (27), we obtain the Lah numbers. The outline of this article is as follows. In Section 1, we recall the Stirling numbers of both kinds, the degenerate Stirling numbers of both kinds and the unsigned Stirling numbers of the first kind. We remind the reader of the normal ordering in terms of boson operators and its analogue version, namely the normal ordering of an analogue integral power of the number operators. In Section 2, we consider the λ-analogue of Lah numbers in the view of degenerate version and derive several identities including these numbers. From these new identities, we investigate the normal ordering of a degenerate integral power of the number operator in terms of boson operators, which is represented by means of analogue of Lah numbers and the degenerate Stirling numbers of the second kind. In Section 3, we introduce the λ-analogue of r-Lah numbers in the view of degenerate version and explore some interesting identities including the normal ordering of analogue integral power of the number operator in terms of boson operators, which is represented by means of analogue of r-Lah numbers and the degenerate Stirling numbers of the second kind.

First, we introduce several definitions and properties needed in this article.

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For any $\lambda \in \mathbb{R}$, the degenerate exponential function $e_\lambda^x(t)$ is given by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \quad (\text{see [1–7]}). \tag{1}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\dots(x-(n-1)\lambda)$, $(n \geq 1)$.

When $\lambda \to 0$, we note that $(x)_0 = 1$, $(x)_n = x(x-1)(x-2)\dots(x-n+1)$, and $e_\lambda^x(t) = e^{xt}$.

Kim and Kim [2] introduced the degenerate logarithm function $\log_\lambda(1+t)$, which is the inverse of the degenerate exponential function $e_\lambda(t)$, as follows:

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n,1/\lambda} \frac{t^n}{n!} = \frac{1}{\lambda} \sum_{n=1}^{\infty} (\lambda)_n \frac{t^n}{n!} = \frac{1}{\lambda}((1+t)^\lambda - 1) \quad (\text{see [2]}). \tag{2}$$

Here, $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

When $\lambda \to 0$, we have $\log_\lambda(t) = \log t$.

For $n \geq 0$, the Stirling numbers of the first and second kind are defined by, respectively,

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, \quad \text{and} \quad \frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!} \quad (\text{see [1, 8–10]}) \tag{3}$$

and

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad \text{and} \quad \frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} \quad (\text{see [1, 8–12]}). \tag{4}$$

The degenerate Stirling numbers of the second kind are given by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0) \ (\text{see [2,3]}). \tag{5}$$

As an inversion formula of the degenerate Stirling numbers of the second kind, the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \quad (n \geq 0) \ (\text{see [3,7]}). \tag{6}$$

From (5) and (6), it is well known that

$$\frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0) \ (\text{see [2,3,6,7]}) \tag{7}$$

and

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0), \ (\text{see [3,7]}). \tag{8}$$

When $\lambda \to 0$, we note that $S_{1,\lambda}(n,k) = S_1(n,k)$ and $S_{2,\lambda}(n,k) = S_2(n,k)$.

The partial degenerate Bell polynomials were introduced by

$$\mathrm{bel}_{n,\lambda}(x) = \sum_{j=1}^{n} S_{2,\lambda}(n,j) x^j, \quad (n \geq 0) \ (\text{see [6,7]}). \tag{9}$$

When $x = 1$, we denote $\mathrm{bel}_{n,\lambda}(1)$ simply by $\mathrm{bel}_{n,\lambda}$.

From (9), we obtain the generating function of $\mathrm{bel}_{n,\lambda}(x)$

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{bel}_{n,\lambda}(x) \frac{t^n}{n!} \quad (\text{see [6,7,12]}). \tag{10}$$

The rising factorial sequences and the generalized rising factorial sequences are given by

$$\langle x \rangle_0 = 1, \quad \langle x \rangle_n = x(x+1)\dots(x+n-1), \quad (n \geq 1) \tag{11}$$

and

$$\langle x\rangle_{0,\lambda}=1,\quad \langle x\rangle_{n,\lambda}=x(x+\lambda)\ldots(x+(n-1)\lambda),\quad (n\geq 1)\ (\text{see [3]}), \tag{12}$$

respectively.

It is well known that

$$\frac{1}{(1-t)^x}=\sum_{n=0}^{\infty}\langle x\rangle_n\frac{t^n}{n!}\quad (\text{see [9]}). \tag{13}$$

As a generalization of Stirling numbers of the second kind, the r-Stirling numbers $\left\{ {n+r \atop k+r} \right\}_r$ of the second kind count the partitions of $1, 2, \ldots, n+r$ into $k+r$ nonempty subsets such that $1, 2, \ldots, r$ are contained in distinct blocks ($n \geq k \geq 0, r \geq 0$).

When $r = 0$, these numbers are the ordinary Stirling numbers of the second kind.

The degenerate r-Stirling numbers of the second kind $\left\{ {n+r \atop k+r} \right\}_{r,\lambda}$ are given by

$$(x+r)_{n,\lambda}=\sum_{k=0}^{n}\left\{ {n+r \atop k+r} \right\}_{r,\lambda}(x)_k,\quad (n,k\geq 0)\ (\text{see [5]}). \tag{14}$$

The Lah numbers are defined by $\langle x\rangle_n$ and $(x)_n$, $(n \geq 0)$, to be

$$\langle x\rangle_n=\sum_{k=0}^{n}L(n,k)(x)_k,\quad (n\geq 0)\ (\text{see [4,9,10]}), \tag{15}$$

or

$$(x)_n=\sum_{k=0}^{n}(-1)^{n-k}L(n,k)\langle x\rangle_k\quad (\text{see [9,10]}). \tag{16}$$

From (15), we note that

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},\quad (k\geq 0)\ (\text{see [4,9]}) \tag{17}$$

and

$$L(n,k)=\frac{n!}{k!}\binom{n-1}{k-1},\quad (n,k\geq 0)\ (\text{see [4,9,10]}). \tag{18}$$

Recently, Kim and Kim [3] introduced the degenerate differential operator given by

$$\left(x\frac{d}{dx}\right)_{k,\lambda}=\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-\lambda\right)\ldots\left(x\frac{d}{dx}-(k-1)\lambda\right). \tag{19}$$

From (19), we note that

$$\left(x\frac{d}{dx}\right)_{k,\lambda}x^n=(n)_{k,\lambda}x^n. \tag{20}$$

For a formal power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $k \geq 0$, the degenerate differential equation is given by

$$\left(x\frac{d}{dx}\right)_{k,\lambda}f(x)=\sum_{j=0}^{k}S_{2,\lambda}(k,j)x^j\left(\frac{d}{dx}\right)^jf(x)\quad (\text{see [3]}). \tag{21}$$

From (21), we have

$$\left(x\frac{d}{dx}\right)_{k,\lambda}=\sum_{j=0}^{k}S_{2,\lambda}(k,j)x^j\left(\frac{d}{dx}\right)^j\quad (\text{see [3]}). \tag{22}$$

Let a and a^+ be the boson annihilation and creation operators satisfying the commutation relation

$$[a, a^+] = aa^+ - a^+a = 1 \quad (\text{see [5,13–16]}). \tag{23}$$

Using the degenerate differential operator (22), the normal ordering of the degenerate kth power of the number operator a^+a, namely $(a^+a)_{k,\lambda}$, in terms of boson operators a and a^+ can be written in the form

$$(a^+a)_{k,\lambda} = \sum_{l=0}^{k} S_{2,\lambda}(k, l)(a^+)^l a^l \quad (\text{see [5]}), \tag{24}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x - \lambda)\ldots(x - (n - 1)\lambda)$, $(n \geq 1)$.

By inversion, from (24), we obtain

$$(a^+)^k a^k = \sum_{l=0}^{k} S_{1,\lambda}(k, l)(a^+a)_{l,\lambda} \quad (\text{see [5]}). \tag{25}$$

The number states $|m\rangle$, $m = 1, 2, \ldots$, are defined as

$$a|m\rangle = \sqrt{m}\,|m - 1\rangle, \quad a^+|m\rangle = \sqrt{m + 1}\,|m + 1\rangle \quad (\text{see [13,15,17]}). \tag{26}$$

By (26), we obtain $a^+a|m\rangle = m|m\rangle$ (see [7,13,15,17]). The coherent states $|z\rangle$, where z is a complex number, satisfy $a|z\rangle = z|z\rangle$, $z|z\rangle = 1$. To show a connection to coherent states, we recall that the harmonic oscillator has Hamiltonian $\hat{n} = a^+a$ (neglecting the zero point energy) and the usual eigenstates $|n\rangle$ (for $n \in \mathbb{N}$) satisfying $\hat{n}|n\rangle = n|n\rangle$ and $\langle m|n\rangle = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker's symbol.

2 λ-analogue of Lah numbers

First, we consider the analogue of Lah numbers in the view of degenerate version and investigate their properties associated with special numbers. In addition, we derive some new identities arising from degenerate differential operators.

In the view of (15) and (16), we consider the analogue of Lah numbers, which are defined by the degenerate rising factorial and falling factorial sequences as follows:

$$\langle x\rangle_{n,\lambda} = \sum_{k=0}^{n} L_\lambda(n, k)(x)_{k,\lambda} \tag{27}$$

and

$$(x)_{k,\lambda} = \sum_{k=0}^{n} (-1)^{n-k} L_\lambda(n, k)\langle x\rangle_{k,\lambda} \quad (n \geq 0). \tag{28}$$

When $\lambda \to 1$, we have $L_\lambda(n, k) = L(n, k)$.

Theorem 1. *For $n, k \geq 0$, we have the explicit formula and generating function of λ-analogue of Lah numbers as follows*:

$$L_\lambda(n, k) = \lambda^{n-k}\frac{n!}{k!}\binom{n - 1}{k - 1}$$

and

$$\frac{1}{k!}\left(\frac{t}{1 - \lambda t}\right)^k = \sum_{n=k}^{\infty} L_\lambda(n, k)\frac{t^n}{n!} \quad (k \geq 0).$$

Proof. From (27), we observe that

$$\left(\frac{1}{1-\lambda t}\right)^{\frac{x}{\lambda}} = e_{-\lambda}^{x}(t) = \sum_{n=0}^{\infty} \langle x\rangle_{n,\lambda}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} L_\lambda(n,k)(x)_{k,\lambda}\right)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} L_\lambda(n,k)\frac{t^n}{n!}\right)(x)_{k,\lambda}. \tag{29}$$

On the other hand, we easily obtain

$$(1-\lambda t)^{-\frac{x}{\lambda}} = \left(\frac{\lambda t}{1-\lambda t}+1\right)^{\frac{x}{\lambda}} = \sum_{k=0}^{\infty}\left(\frac{t}{1-\lambda t}\right)^k \frac{1}{k!}(x)_{k,\lambda}. \tag{30}$$

By (29) and (30), we obtain

$$\frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^k = \sum_{n=k}^{\infty} L_\lambda(n,k)\frac{t^n}{n!} \quad (k \geq 0). \tag{31}$$

From (31), we note that

$$\sum_{n=k}^{\infty} L_\lambda(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^k = \frac{t^k}{k!}\sum_{n=0}^{\infty}\binom{n+k-1}{n}\lambda^n t^n = \sum_{n=k}^{\infty}\lambda^{n-k}\binom{n-1}{k-1}\frac{n!}{k!}\frac{t^n}{n!}. \tag{32}$$

By comparing the coefficients on both sides of (32), we obtain the the explicit formula of $L_\lambda(n,k)$. □

Theorem 2. *For $n, k \geq 0$, we have*

$$L_\lambda(n,k) = \lambda^{n-k}\sum_{l=k}^{n}(-1)^{n-l}S_{2,\lambda}(l,k)S_{1,-\lambda}(n,l),$$

where $S_{2,\lambda}(n,k)$ are the degenerate Stirling numbers of the second kind.

Proof. By substituting t by $\log_\lambda(\frac{1}{1-\lambda t})$ into (7), we obtain

$$\begin{aligned}\frac{1}{k!}\left(\frac{1}{1-\lambda t}-1\right)^k &= \sum_{l=k}^{\infty} S_{2,\lambda}(l,k)\frac{1}{l!}\left(\log_\lambda\left(\frac{1}{1-\lambda t}\right)\right)^l \\ &= \sum_{l=k}^{\infty} S_{2,\lambda}(l,k)\frac{1}{l!}(-\log_{-\lambda}(1-\lambda t))^l \\ &= \sum_{l=k}^{\infty}(-1)^l S_{2,\lambda}(l,k)\sum_{n=l}^{\infty}S_{1,-\lambda}(n,l)\frac{(-\lambda t)^n}{n!} \\ &= \sum_{n=k}^{\infty}\lambda^n\left(\sum_{l=k}^{n}(-1)^{n-l}S_{2,\lambda}(l,k)S_{1,-\lambda}(n,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{33}$$

On the other hand, by (31), we obtain

$$\frac{1}{k!}\left(\frac{1}{1-\lambda t}-1\right)^k = \sum_{n=k}^{\infty}\lambda^k L_\lambda(n,k)\frac{t^n}{n!}. \tag{34}$$

By (33) and (34), we obtain the desired identity. □

Next, we can obtain the inverse formula of Theorem 2.

Theorem 3. *For $n, k \geq 0$, we have*

$$S_{2,-\lambda}(n,k) = \sum_{l=k}^{n}\lambda^{k-l}(-1)^{n-l}L_\lambda(l,k)S_{2,\lambda}(n,l),$$

where $S_{2,\lambda}(n,k)$ are the degenerate Stirling numbers of the second kind.

Proof. By substituting t by $t = \frac{1}{\lambda}(1 - e_\lambda(-t))$ into (31), we have

$$\begin{aligned}\frac{1}{k!}\left(\frac{1}{e_\lambda(-t)} - 1\right)^k &= \sum_{l=k}^{\infty} \lambda^k L_\lambda(l,k) \frac{1}{\lambda^l}\frac{1}{l!}(1 - e_\lambda(-t))^l \\ &= \sum_{l=k}^{\infty} \lambda^{k-l} L_\lambda(l,k)(-1)^l \sum_{n=k}^{\infty} S_{2,\lambda}(l,k)\frac{(-t)^n}{n!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n} \lambda^{k-l}(-1)^{n-l} L_\lambda(l,k) S_{2,\lambda}(n,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{35}$$

On the other hand, by (7), we set

$$\frac{1}{k!}\left(\frac{1}{e_\lambda(-t)} - 1\right)^k = \frac{1}{k!}(e_{-\lambda}(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,-\lambda}(n,k)\frac{t^n}{n!}. \tag{36}$$

Therefore, by (35) and (36), we obtain the desired result. □

Now, for any real number λ and any nonnegative integer k, we consider the degenerate rising differential operator, which is given by $\left\langle x\frac{d}{dx}\right\rangle_{k,\lambda}$:

$$\left\langle x\frac{d}{dx}\right\rangle_{k,\lambda} = \left(x\frac{d}{dx}\right)\left(x\frac{d}{dx} + \lambda\right)\left(x\frac{d}{dx} + 2\lambda\right)\cdots\left(x\frac{d}{dx} + (k-1)\lambda\right). \tag{37}$$

From (36), we note that

$$\left\langle x\frac{d}{dx}\right\rangle_{k,\lambda} x^n = \langle n\rangle_{k,\lambda} x^n \quad (n \ge 1) \tag{38}$$

and

$$\left\langle x\frac{d}{dx}\right\rangle_{k,\lambda} (1+x)^n = \sum_{l=0}^{n}\binom{n}{l}\left\langle x\frac{d}{dx}\right\rangle_{k,\lambda} x^l = \sum_{l=0}^{n}\binom{n}{l}\langle l\rangle_{k,\lambda} x^l. \tag{39}$$

Theorem 4. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be formal power series. For $k \ge 0$, we have*

$$\left\langle x\frac{d}{dx}\right\rangle_{k,\lambda} f(x) = \sum_{l=0}^{k} L_\lambda(k,l)\left(x\frac{d}{dx}\right)_{l,\lambda} f(x).$$

Proof. For the formal power series $f(x)$, by applying (27) and (38), we obtain

$$\begin{aligned}\left\langle x\frac{d}{dx}\right\rangle_{k,\lambda} f(x) &= \sum_{n=0}^{\infty} a_n \langle n\rangle_{k,\lambda} x^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{l=0}^{k} L_\lambda(k,l)(n)_{l,\lambda} x^n \\ &= \sum_{l=0}^{k} L_\lambda(k,l)\left(x\frac{d}{dx}\right)_{l,\lambda}\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{l=0}^{k} L_\lambda(k,l)\left(x\frac{d}{dx}\right)_{l,\lambda} f(x).\end{aligned} \tag{40}$$

By comparing the coefficients n on both sides of (40), we obtain the desired identity. □

Corollary 5. *For* $k \geq 0$, *we have*

$$\left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{k,\lambda} = \sum_{j=0}^{k} x^j \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \sum_{l=j}^{k} L_\lambda(k,l) S_{2,\lambda}(l,j),$$

where $S_{2,\lambda}(n,k)$ *are the degenerate Stirling numbers of the second kind.*

Proof. Since $(x\frac{\mathrm{d}}{\mathrm{d}x})_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k) x^k (\frac{\mathrm{d}}{\mathrm{d}x})^k$, from Theorem 4, we observe that

$$\left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{k,\lambda} = \sum_{l=0}^{k} L_\lambda(k,l)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)_{l,\lambda} = \sum_{l=0}^{k} L_\lambda(k,l) \sum_{j=0}^{l} S_{2,\lambda}(l,j) x^j \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j = \sum_{j=0}^{k} x^j \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \sum_{l=j}^{k} L_\lambda(k,l) S_{2,\lambda}(l,j). \tag{41}$$

By (41), we obtain the desired identity. □

The next corollary is the inverse formula of Corollary 5.

Theorem 6. *For* $k \geq 0$, *we have*

$$x^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k = \sum_{j=0}^{k} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{j,\lambda} \sum_{l=j}^{k} (-1)^{l-j} S_{1,\lambda}(l,k) L_\lambda(l,j),$$

where $S_{1,\lambda}(n,k)$ *are the degenerate Stirling numbers of the first kind.*

Proof. It is known that

$$x^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n = \sum_{k=0}^{n} S_{1,\lambda}(n,k) \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)_{k,\lambda} \quad \text{(see [16])}. \tag{42}$$

Let $f(x)$ be the formal power series with $f(x) = \sum_{n=0}^{\infty} a_n x^n$. From (20), (28), and (38), we note that

$$\begin{aligned}
\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)_{k,\lambda} f(x) &= \sum_{n=0}^{\infty} a_n (n)_{k,\lambda} x^n = \sum_{n=0}^{\infty} a_n \sum_{j=0}^{k} L_\lambda(k,j)(-1)^{k-j} \langle n\rangle_{j,\lambda} x^n \\
&= \sum_{j=0}^{k} L_\lambda(k,j)(-1)^{k-j} \sum_{n=0}^{\infty} a_n \langle n\rangle_{j,\lambda} x^n \\
&= \sum_{j=0}^{k} L_\lambda(k,j)(-1)^{k-j} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{j,\lambda} \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{j=0}^{k} L_\lambda(k,j)(-1)^{k-j} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{j,\lambda} f(x).
\end{aligned} \tag{43}$$

From (43), we have

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)_{k,\lambda} = \sum_{j=0}^{k} L_\lambda(k,j)(-1)^{k-j} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{j,\lambda}, \quad (k \geq 0). \tag{44}$$

By (42) and (44), we obtain

$$\begin{aligned}
x^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n &= \sum_{k=0}^{n} S_{1,\lambda}(n,k) \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)_{k,\lambda} \\
&= \sum_{k=0}^{n} S_{1,\lambda}(n,k) \sum_{j=0}^{k} L_\lambda(k,j)(-1)^{k-j} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{j,\lambda} \\
&= \sum_{j=0}^{n} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle_{j,\lambda} \sum_{k=j}^{n} (-1)^{k-j} S_{1,\lambda}(n,k) L_\lambda(k,j).
\end{aligned} \tag{45}$$

Therefore, we obtain the desired result. □

The number states $|m\rangle$, that satisfy $a^{\dagger}a|m\rangle = m|m\rangle$, $\langle m|m\rangle = 1$ and the coherent states $|r\rangle$ that satisfy $a|r\rangle = r|r\rangle$, $\langle r|r\rangle = 1$ are the two most important sets of states within the boson-operator Fock space. They are related by the well-known expression

$$|z\rangle = e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle \quad (\text{see [5,15,18]}). \tag{46}$$

For $x, y \in \mathbb{C}$, we have

$$\begin{aligned}\langle x|y\rangle &= e^{-\frac{|x|^2}{2}}\sum_{m=0}^{\infty}\frac{(\overline{x})^m}{\sqrt{m!}}e^{-\frac{|y|^2}{2}}\sum_{n=0}^{\infty}\frac{y^n}{\sqrt{n!}}\langle m|n\rangle \\ &= e^{-\frac{|x|^2}{2}-\frac{|y|^2}{2}}\sum_{n=0}^{\infty}\frac{(\overline{x}y)^n}{n!} = e^{-\frac{1}{2}(|x|^2+|y|^2)+\overline{x}y} \quad (\text{see [5,15,18]}).\end{aligned} \tag{47}$$

By letting $a = \frac{d}{dx}$ and $a^{\dagger} = x$ (the operator of multiplication by x) and Corollary 5, we rewrite

$$\langle a^{\dagger}a\rangle_{k,\lambda} = \sum_{j=0}^{k}(a^{\dagger})^j(a)^j\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j) \quad (k \in \mathbb{N}). \tag{48}$$

Theorem 7. *For $n \geq 0$, we have*

$$\langle m\rangle_{k,\lambda} = \sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)(m)_j \quad (k \geq 1)$$

and

$$\sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)|z|^{2j} = \mathrm{bel}_{k,-\lambda}(|z|^2),$$

where $S_{2,\lambda}(n,k)$ are the degenerate Stirling numbers of the second kind and $\mathrm{bel}_{n,\lambda}(x)$ are the partial degenerate Bell polynomials.

Proof. From (12), (26), and (48), we note that

$$\langle a^{\dagger}a\rangle_{k,\lambda}|m\rangle = (a^{\dagger}a)(a^{\dagger}a+\lambda)\ldots(a^{\dagger}a+(k-1)\lambda)|m\rangle = \langle m\rangle_{k,\lambda}|m\rangle \tag{49}$$

and

$$\langle a^{\dagger}a\rangle_{k,\lambda}|m\rangle = \sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)(a^{\dagger})^j(a)^j|m\rangle = \sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)(m)_j|m\rangle. \tag{50}$$

Thus, by (49) and (50), we obtain

$$\langle m\rangle_{k,\lambda} = \sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)(m)_j \quad (k \geq 1). \tag{51}$$

Furthermore, from (48), we observe that

$$\begin{aligned}\langle z|\langle a^{\dagger}a\rangle_{k,\lambda}|z\rangle &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)\langle z|(a^{\dagger})^j a^j|z\rangle \\ &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)(\overline{z})^j z^j\langle z|z\rangle \\ &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{\lambda}(k,l)S_{2,\lambda}(l,j)|z|^{2j}.\end{aligned} \tag{52}$$

For any $k \in \mathbb{N}$, it is easy to see that

$$a^\dagger a (a^\dagger a)_{k,\lambda} = a^\dagger (a a^\dagger)_{k,\lambda} a$$

and

$$a^\dagger a e_{-\lambda}^{a^\dagger a+\lambda}(t) = e_{-\lambda}^{a^\dagger a+\lambda}(t) a^\dagger a = a^\dagger e_{-\lambda}^{aa^\dagger+\lambda}(t) a = a^\dagger e_{-\lambda}^{a^\dagger a+1+\lambda}(t) a. \tag{53}$$

Let $f(t) = \langle z | e_{-\lambda}^{a^\dagger a}(t) | z\rangle$. Then, by (52) and (53), we obtain

$$f(t) = \langle z | e_{-\lambda}^{a^\dagger a}(t) | z\rangle = \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle z | \langle a^\dagger a\rangle_{k,\lambda} | z\rangle = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=j}^{k} L_\lambda(k,l) S_{2,\lambda}(l,j) |z|^{2j} \frac{t^k}{k!}. \tag{54}$$

From (26), we note that

$$\begin{aligned} \frac{\partial f(t)}{\partial t} &= \frac{\partial}{\partial t} \langle z | e_{-\lambda}^{a^\dagger a}(t) | z\rangle = \langle z | a^\dagger a (e_{-\lambda}^{a^\dagger a+\lambda}(t)) | z\rangle \\ &= \langle z | a^\dagger (e_{-\lambda}^{a^\dagger a+\lambda+1}(t)) a | z\rangle \\ &= e_{-\lambda}^{\lambda+1}(t) \bar{z} z \langle z | e_{-\lambda}^{a^\dagger a}(t) | z\rangle = e_{-\lambda}^{\lambda+1}(t) |z|^2 f(t). \end{aligned} \tag{55}$$

By (55), we observe that

$$\frac{\partial f(t)}{\partial t} = e_{-\lambda}^{\lambda+1}(t) |z|^2 f(t) \quad \Leftrightarrow \quad \frac{f'(t)}{f(t)} = e_{-\lambda}^{\lambda+1}(t)|z|^2, \quad \left(f'(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)\right). \tag{56}$$

Assume that $f(0) = 1$, for the initial value. Then, by (56), we obtain

$$\log f(t) = \int_0^t \frac{f'(t)}{f(t)} \mathrm{d}t = \int_0^t e_{-\lambda}^{\lambda+1}(t) |z|^2 \mathrm{d}t = (e_{-\lambda}(t) - 1)|z|^2. \tag{57}$$

From (9) and (57), it can be rewritten as

$$\begin{aligned} f(t) &= e^{|z|^2(e_{-\lambda}(t)-1)} \\ &= \sum_{j=0}^{\infty} |z|^{2j} \frac{1}{j!} (e_{-\lambda}(t) - 1)^j = \sum_{j=0}^{\infty} |z|^{2j} \sum_{n=j}^{\infty} S_{2,-\lambda}(n,j) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} |z|^{2j} S_{2,-\lambda}(n,j) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathrm{bel}_{n,-\lambda}(|z|^2) \frac{t^n}{n!}. \end{aligned} \tag{58}$$

Combining (54) with (58), we obtain

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{l=j}^{n} L_\lambda(n,l) S_{2,\lambda}(l,j) |z|^{2j} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathrm{bel}_{n,-\lambda}(|z|^2) \frac{t^n}{n!}. \tag{59}$$

By comparing with the coefficients of both sides of (59), we have

$$\mathrm{bel}_{n,-\lambda}(|z|^2) = \sum_{j=0}^{n} \sum_{l=j}^{n} L_\lambda(n,l) S_{2,\lambda}(l,j) |z|^{2j}. \qquad \square$$

Theorem 8. *For* $n \geq 0$, *we have*

$$\mathrm{bel}_{n+1,-\lambda}(|z|^2) = |z|^2 \left(\sum_{l=0}^{n} \binom{n}{l} (\lambda+1)_{n-l,\lambda} \mathrm{bel}_{l,-\lambda}(|z|^2) \right),$$

where $\mathrm{bel}_{n,\lambda}(x)$ *are the partial degenerate Bell polynomials.*

Proof. From (58), we have

$$f(t) = e^{|z|^2(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{bel}_{n,-\lambda}(|z|^2)\frac{t^n}{n!}. \tag{60}$$

Differentiating (58) with respect to t, the right-hand side of (60) is

$$\frac{\partial f(t)}{\partial t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}\mathrm{bel}_{n,-\lambda}(|z|^2) = \sum_{n=0}^{\infty} \mathrm{bel}_{n+1,-\lambda}(|z|^2)\frac{t^n}{n!}. \tag{61}$$

By (55) and (58), the left-hand side of (60) is

$$\begin{aligned}\frac{\partial f(t)}{\partial t} &= e_{-\lambda}^{\lambda+1}(t)|z|^2 f(t) = e_{-\lambda}^{\lambda+1}(t)|z|^2 \sum_{l=0}^{\infty} \mathrm{bel}_{l,-\lambda}(|z|^2)\frac{t^l}{l!} \\ &= |z|^2 \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(\lambda+1)_{n-l,\lambda}\mathrm{bel}_{l,-\lambda}(|z|^2)\right)\frac{t^n}{n!}.\end{aligned} \tag{62}$$

Comparing the coefficients of (61) and (62), we obtain the desired identity. □

In particular, for $|z| = 1$, we have

$$\mathrm{bel}_{n+1,-\lambda} = \sum_{l=0}^{n}\binom{n}{l}(\lambda+1)_{n-l,\lambda}\mathrm{bel}_{l,-\lambda}. \tag{63}$$

By (47) and (49), we have

$$\langle z|\langle a^\dagger a\rangle_{k,\lambda}|z\rangle = e^{-\frac{|z|^2}{2}}e^{-\frac{|z|^2}{2}} \sum_{m,n=0}^{\infty} \frac{\bar{z}^m z^n}{\sqrt{m!}\sqrt{n!}}\langle n\rangle_{k,\lambda}\langle m|n\rangle = e^{-|z|^2}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\langle n\rangle_{k,\lambda}. \tag{64}$$

Thus, by (58), (60), and (64), we obtain

$$f(t) = \sum_{n=0}^{\infty} \mathrm{bel}_{n,-\lambda}(|z|^2)\frac{t^n}{n!} = e^{-|z|^2}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\langle n\rangle_{k,\lambda} \quad (k \in \mathbb{N}). \tag{65}$$

3 λ-analogue of r-Lah numbers

In this section, we consider the analogue of r-Lah numbers in the view of degenerate version and investigate their properties associated with special numbers as in Section 2. Moreover, we also derive some combinatorial identities arising from degenerate differential operators.

For $n,\ r \geq 0$, the r-Lah number is defined by

$$\langle x + 2r\rangle_n = \sum_{k=0}^{n} L_r(n,k)(x)_k \quad (\text{see [4]}). \tag{66}$$

In the view of (28), we consider the analogue of r-Lah numbers, which are given by

$$\langle x + 2r\lambda\rangle_{n,\lambda} = \sum_{k=0}^{n} L_{r,\lambda}(n,k)(x)_{k,\lambda} \quad (n \geq 0). \tag{67}$$

Theorem 9. *For $n, k \geq 0$, we have the explicit formula and generating function of analogue of r-Lah numbers as follows*:

$$L_{r,\lambda}(n,k) = \lambda^{n-k}\binom{n+2r-1}{k+2r-1}\frac{n!}{k!} \quad (n,r,k \geq 0) \tag{68}$$

and

$$\sum_{n=k}^{\infty} L_{r,\lambda}(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\frac{1}{1-\lambda t}\right)^{2r}\left(\frac{t}{1-\lambda t}\right)^{k} \quad (k \geq 0). \tag{69}$$

Proof. From (13), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\langle x+2r\lambda\rangle_{n,\lambda}\frac{t^n}{n!} &= \left(\frac{1}{1-\lambda t}\right)^{\frac{x+2r\lambda}{\lambda}} \\ &= e_{-\lambda}^{2r\lambda}(t)\cdot e_{-\lambda}^{x}(t) \\ &= \left(\frac{\lambda t}{1-\lambda t}+1\right)^{\frac{x}{\lambda}}\left(\frac{1}{1-\lambda t}\right)^{2r} \\ &= \sum_{k=0}^{\infty}\left(\frac{t}{1-\lambda t}\right)^{k}\left(\frac{1}{1-\lambda t}\right)^{2r}\frac{1}{k!}(x)_{k,\lambda}.\end{aligned} \tag{70}$$

On the other hand, by (67), we obtain

$$\sum_{n=0}^{\infty}\langle x+2r\lambda\rangle_{n,\lambda}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}L_{r,\lambda}(n,k)(x)_{k,\lambda}\right)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}L_{r,\lambda}(n,k)\frac{t^n}{n!}\right)(x)_{k,\lambda}. \tag{71}$$

Thus, by (70) and (71), we obtain

$$\frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^{k}\left(\frac{1}{1-\lambda t}\right)^{2r} = \sum_{n=k}^{\infty}L_{r,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0). \tag{72}$$

From (72), we obtain

$$\sum_{n=k}^{\infty}L_{r,\lambda}(n,k)\frac{t^n}{n!} = \frac{t^k}{k!}\left(\frac{1}{1-\lambda t}\right)^{k+2r} = \frac{t^k}{k!}\sum_{n=0}^{\infty}\binom{n+k+2r-1}{n}t^n\lambda^n = \sum_{n=k}^{\infty}\lambda^{n-k}\binom{n+2r-1}{k+2r-1}\frac{n!}{k!}\frac{t^n}{n!}. \tag{73}$$

By comparing the coefficients on both sides of (73), we have the desired result. □

By (1) and Theorem 1, we easily obtain

$$\begin{aligned}\sum_{n=k}^{\infty}L_{r,\lambda}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^{k}e_{-\lambda}^{2r}(t) \\ &= \left(\sum_{l=k}^{\infty}L_{\lambda}(l,k)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{(2r)_{m,-\lambda}}{m!}t^m\right) \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{n}{l}L_{\lambda}(l,k)\langle 2r\rangle_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{74}$$

By comparing the coefficients on both sides of (74), we obtain

$$L_{r,\lambda}(n,k) = \sum_{l=k}^{n}\binom{n}{l}L_{\lambda}(l,k)\langle 2r\rangle_{n-l,\lambda} \quad (n,k \geq 0). \tag{75}$$

Theorem 10. *For* $n, k, r \geq 0$, *we have*

$$\begin{Bmatrix} n+2r \\ k+2r \end{Bmatrix}_{2r,\lambda} = \sum_{l=k}^{n}L_{r,\lambda}(l,k)(-1)^{n-l}S_{2,-\lambda}(n,l)\lambda^{k-l},$$

where $\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda}$ *is the degenerate r-Stirling number of the second kind.*

Proof. By substituting t by $\frac{1}{\lambda}(1-e_{-\lambda}(t))$ into (69), the left-hand side of (69) is

$$\begin{aligned}\sum_{l=k}^{\infty}L_{r,\lambda}(l,k)\frac{1}{l!}\frac{(1-e_{-\lambda}(t))^{l}}{\lambda^{l}}&=\sum_{l=k}^{\infty}L_{r,\lambda}(l,k)(-1)^{l}\lambda^{-l}\sum_{n=l}^{\infty}S_{2,-\lambda}(n,l)\frac{t^{n}}{n!}\\&=\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}L_{r,\lambda}(l,k)S_{2,-\lambda}(n,l)(-1)^{l}\lambda^{-l}\right)\frac{t^{n}}{n!}.\end{aligned}\tag{76}$$

The right-hand side of (69) is

$$\frac{1}{k!}e_{-\lambda}^{-2r}(t)(e_{-\lambda}^{-1}(t)-1)^{k}\lambda^{-k}=\frac{1}{k!}e_{\lambda}^{2r}(-t)(e_{\lambda}(-t)-1)^{k}\lambda^{-k}=\sum_{n=k}^{\infty}\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}\lambda^{-k}(-1)^{n}\frac{t^{n}}{n!}.\tag{77}$$

Therefore, by (76) and (77), we obtain the desired identity. □

Theorem 11. *For $n,k,r\geq 0$, we have*

$$L_{r,\lambda}(n,k)=\lambda^{n-k}\sum_{l=k}^{n}(-1)^{n-l}\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}S_{1,-\lambda}(n,l).$$

Proof. From (14), the generating function of $\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}$ is given by

$$\frac{1}{k!}e_{\lambda}^{2r}(t)(e_{\lambda}(t)-1)^{k}=\sum_{n=k}^{\infty}\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}\frac{t^{n}}{n!}.\tag{78}$$

Let us take $t=\log_{\lambda}(\frac{1}{1-\lambda t})$ in (78). Then, from (8), we have

$$\begin{aligned}\frac{1}{k!}\left(\frac{1}{1-\lambda t}\right)^{2r}\left(\frac{\lambda t}{1-\lambda t}\right)^{k}&=\sum_{l=k}^{\infty}\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}\frac{1}{l!}\left(\log_{\lambda}\frac{1}{1-\lambda t}\right)^{l}=\sum_{l=k}^{\infty}\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}\frac{1}{l!}(-\log_{\lambda}(1-\lambda t))^{l}\\&=\sum_{l=k}^{\infty}(-1)^{l}\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}\sum_{n=l}^{\infty}S_{1,\lambda}(n,l)(-\lambda)^{n}\frac{t^{n}}{n!}\\&=\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}(-1)^{n-l}\begin{Bmatrix}n+2r\\k+2r\end{Bmatrix}_{2r,\lambda}S_{1,\lambda}(n,l)\right)\lambda^{n}\frac{t^{n}}{n!}.\end{aligned}\tag{79}$$

On the other hand, by (69), we obtain

$$\frac{1}{k!}\left(\frac{1}{1-\lambda t}\right)^{2r}\left(\frac{\lambda t}{1-\lambda t}\right)^{k}=\sum_{n=k}^{\infty}L_{r,\lambda}(n,k)\lambda^{k}\frac{t^{n}}{n!}.\tag{80}$$

Therefore, by (79) and (80), we obtain the desired result. □

Theorem 12. *For $n,k\in\mathbb{N}$ with $n\geq k$ and $r\in\mathbb{Z}$ with $r\geq 0$, we have a recurrence relation*

$$L_{r,\lambda}(n+1,k)=L_{r,\lambda}(n,k-1)+((n+k)\lambda+2r\lambda)L_{r,\lambda}(n,k).$$

Proof. From (67), we note that

$$\begin{aligned}\sum_{k=0}^{n+1}L_{r,\lambda}(n+1,k)(x)_{k,\lambda}=\langle x+2r\lambda\rangle_{n+1,\lambda}&=\langle x+2r\lambda\rangle_{n,\lambda}(x+2r\lambda+n\lambda)\\&=\sum_{k=0}^{n}L_{r,\lambda}(n,k)(x)_{k,\lambda}(x-k\lambda+k\lambda+n\lambda+2r\lambda)\\&=\sum_{k=0}^{n+1}\{L_{r,\lambda}(n,k-1)+((n+k)\lambda+2r\lambda)L_{r,\lambda}(n,k)\}(x)_{k,\lambda}.\end{aligned}\tag{81}$$

By comparing the coefficients on both sides of (81), we have the desired recurrence relation. □

To obtain next theorem, we observe that

$$\left\langle x\frac{\mathrm{d}}{\mathrm{d}x} + 2r\lambda \right\rangle_{k,\lambda} x^n = \langle n + 2r\lambda \rangle_{k,\lambda} \quad (n \geq 1). \tag{82}$$

Theorem 13. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *be the formal power series. For* $k \geq 0$, *we have*

$$\left\langle x\frac{\mathrm{d}}{\mathrm{d}x} + 2r\lambda \right\rangle_{k,\lambda} f(x) = \sum_{l=0}^{k} L_{r,\lambda}(k,l) \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)_{l,\lambda} f(x).$$

Proof. For the formal power series $f(x)$, by applying (27) and (82), we obtain

$$\begin{aligned} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x} + 2r\lambda \right\rangle_{k,\lambda} f(x) &= \sum_{n=0}^{\infty} a_n \langle n + 2r\lambda \rangle_{k,\lambda} x^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{l=0}^{k} L_{r,\lambda}(k,l)(n)_{l,\lambda} x^n \\ &= \sum_{l=0}^{k} L_{r,\lambda}(k,l) \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)_{l,\lambda} \sum_{n=0}^{\infty} a_n x^n = \sum_{l=0}^{k} L_{r,\lambda}(k,l) \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)_{l,\lambda} f(x). \end{aligned} \tag{83}$$

By comparing the coefficients n on both sides of (83), we obtain the desired identity. □

Corollary 14. *For* $k \geq 0$, *we have*

$$\left\langle x\frac{\mathrm{d}}{\mathrm{d}x} + 2r\lambda \right\rangle_{k,\lambda} = \sum_{j=0}^{k} x^j \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^j \sum_{l=j}^{k} L_{r,\lambda}(k,l) S_{2,\lambda}(l,j).$$

Proof. Since $(x\frac{\mathrm{d}}{\mathrm{d}x})_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k) x^k (\frac{\mathrm{d}}{\mathrm{d}x})^k$, from Theorem 13, we observe that

$$\begin{aligned} \left\langle x\frac{\mathrm{d}}{\mathrm{d}x} + 2r\lambda \right\rangle_{k,\lambda} &= \sum_{l=0}^{k} L_{r,\lambda}(k,l) \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)_{l,\lambda} \\ &= \sum_{l=0}^{k} L_{r,\lambda}(k,l) \sum_{j=0}^{l} S_{2,\lambda}(l,j) x^j \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^j = \sum_{j=0}^{k} x^j \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^j \sum_{l=j}^{k} L_{r,\lambda}(k,l) S_{2,\lambda}(l,j). \end{aligned} \tag{84}$$

By (84), we obtain the desired identity. □

From Corollary 14, it can rewritten as

$$\langle a^\dagger a + 2r\lambda \rangle_{k,\lambda} = \sum_{j=0}^{k} (a^\dagger)^j (a)^j \sum_{l=j}^{k} L_{r,\lambda}(k,l) S_{2,\lambda}(l,j), \quad (k \in \mathbb{N}).$$

The λ-analogue of r-Lah numbers will be studied in a similar way to λ-analogue of Lah numbers.

Theorem 15. *For* $n \geq 0$, *we have*

$$\left(\frac{1}{1 - \lambda t} \right)^{2r} \mathrm{bel}_{n,-\lambda}(|z|^2) \frac{t^n}{n!} = \sum_{j=0}^{n} \sum_{l=j}^{n} L_{r,\lambda}(n,l) S_{2,\lambda}(l,j) |z|^{2j},$$

where $S_{2,\lambda}(n,k)$ *are the degenerate Stirling numbers of the second kind and* $\mathrm{bel}_{n,\lambda}(x)$ *are the partial degenerate Bell polynomials.*

Proof. In the same way in Theorem 7, we observe that

$$\langle a^{\dagger}a + 2r\lambda\rangle_{k,\lambda}|m\rangle = (a^{\dagger}a + 2r\lambda)(a^{\dagger}a + 2r\lambda + \lambda)\ldots(a^{\dagger}a + 2r\lambda) + (k-1)\lambda|m\rangle = \langle m + 2r\lambda\rangle_{k,\lambda}|m\rangle \tag{85}$$

and

$$\begin{aligned}\langle a^{\dagger}a + 2r\lambda\rangle_{k,\lambda}|m\rangle &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{r,\lambda}(k,l)S_{2,\lambda}(l,j)(a^{\dagger})^{j}(a)^{j}|m\rangle \\ &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{r,\lambda}(k,l)S_{2,\lambda}(l,j)(m)_{j}|m\rangle.\end{aligned} \tag{86}$$

Thus, by (85) and (86), we obtain

$$\langle m + 2r\lambda\rangle_{k,\lambda} = \sum_{j=0}^{k}\sum_{l=j}^{k}L_{r,\lambda}(k,l)S_{2,\lambda}(l,j)(m)_{j}, \quad (k \geq 1). \tag{87}$$

From (87), we observe that

$$\begin{aligned}\langle z|\langle a^{\dagger}a + 2r\lambda\rangle_{k,\lambda}|z\rangle &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{r,\lambda}(k,l)S_{2,\lambda}(l,j)\langle z|(a^{\dagger})^{j}a^{j}|z\rangle \\ &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{r,\lambda}(k,l)S_{2,\lambda}(l,j)(\bar{z})^{j}z^{j}\langle z|z\rangle \\ &= \sum_{j=0}^{k}\sum_{l=j}^{k}L_{r,\lambda}(k,l)S_{2,\lambda}(l,j)|z|^{2j}.\end{aligned} \tag{88}$$

Let $f(t) = \langle z|e_{-\lambda}^{a^{\dagger}a+2r\lambda}(t)|z\rangle$. Then, by (88), we obtain

$$f(t) = \langle z|e_{-\lambda}^{a^{\dagger}a+2r\lambda}(t)|z\rangle = \sum_{k=0}^{\infty}\frac{t^{k}}{k!}\langle z|\langle a^{\dagger}a + 2r\lambda\rangle_{k,\lambda}|z\rangle = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=j}^{k}L_{r,\lambda}(k,l)S_{2,\lambda}(l,j)|z|^{2j}\frac{t^{k}}{k!}. \tag{89}$$

From (53), we note that

$$\begin{aligned}\frac{\partial f(t)}{\partial t} &= \frac{\partial}{\partial t}\langle z|e_{-\lambda}^{a^{\dagger}a+2r\lambda}(t)|z\rangle = \langle z|(a^{\dagger}a + 2r\lambda)e_{-\lambda}^{a^{\dagger}a+2r\lambda+\lambda}(t)|z\rangle \\ &= \langle z|(a^{\dagger}a)e_{-\lambda}^{a^{\dagger}a+2r\lambda+\lambda}(t)|z\rangle + 2r\lambda\langle z|e_{-\lambda}^{a^{\dagger}a+2r\lambda+\lambda}(t)|z\rangle \\ &= \langle z|a^{\dagger}(e_{-\lambda}^{a^{\dagger}a+2r\lambda+\lambda+1}(t))a|z\rangle + 2r\lambda e_{-\lambda}^{\lambda}(t)\langle z|e_{-\lambda}^{a^{\dagger}a+2r\lambda}(t)|z\rangle \\ &= e_{-\lambda}^{\lambda+1}(t)\bar{z}z\langle z|e_{-\lambda}^{a^{\dagger}a}(t)|z\rangle + 2r\lambda e_{-\lambda}^{\lambda}(t)\langle z|e_{-\lambda}^{a^{\dagger}a+2r\lambda}(t)|z\rangle \\ &= \{e_{-\lambda}^{\lambda+1}(t)|z|^{2} + 2r\lambda e_{-\lambda}^{\lambda}(t)\}f(t).\end{aligned} \tag{90}$$

By (90), we observe that

$$\frac{\partial f(t)}{\partial t} = e_{-\lambda}^{\lambda+1}(t)|z|^{2}f(t) \quad \Leftrightarrow \quad \frac{f'(t)}{f(t)} = e_{-\lambda}^{\lambda+1}(t)|z|^{2} + 2r\lambda e_{-\lambda}^{\lambda}(t), \tag{91}$$

where $f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}f(t)$.

Assume that $f(0) = 1$, for the initial value. Then, by (91), we obtain

$$\log f(t) = \int_{0}^{t}\frac{f'(t)}{f(t)}\mathrm{d}t = \int_{0}^{t}\{e_{-\lambda}^{\lambda+1}(t)|z|^{2} + 2r\lambda e_{-\lambda}^{\lambda}(t)\}\mathrm{d}t = (e_{-\lambda}(t) - 1)|z|^{2} - 2r\log(1 - \lambda t). \tag{92}$$

From (9) and (92), it can be rewritten as

$$f(t) = e^{|z|^{2}(e_{-\lambda}(t)-1)}e^{-2r\log(1-\lambda t)} = \left(\frac{1}{1-\lambda t}\right)^{2r}\sum_{j=0}^{\infty}|z|^{2j}\frac{1}{j!}(e_{-\lambda}(t) - 1)^{j}$$

$$
\begin{aligned}
&= \left(\frac{1}{1-\lambda t}\right)^{2r} \sum_{j=0}^{\infty} |z|^{2j} \sum_{n=j}^{\infty} S_{2,-\lambda}(n,j)\frac{t^n}{n!} \\
&= \left(\frac{1}{1-\lambda t}\right)^{2r} \sum_{n=0}^{\infty} \sum_{j=0}^{n} S_{2,-\lambda}(n,j)\frac{t^n}{n!}\, |z|^{2j} \\
&= \left(\frac{1}{1-\lambda t}\right)^{2r} \sum_{n=0}^{\infty} \mathrm{bel}_{n,-\lambda}(|z|^2)\frac{t^n}{n!}.
\end{aligned} \tag{93}
$$

Combining (89) with (93), we obtain

$$
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{l=j}^{n} L_{r,\lambda}(n,l) S_{2,\lambda}(l,j)|z|^{2j} \frac{t^n}{n!} = \left(\frac{1}{1-\lambda t}\right)^{2r} \sum_{n=0}^{\infty} \mathrm{bel}_{n,-\lambda}(|z|^2)\frac{t^n}{n!}. \tag{94}
$$

By comparing with the coefficients of both sides of (94), we have the desired identity. □

Open problem. What are the interesting properties of degenerate versions and analogue versions of r ($r \in \mathbb{N}$)-Whitney Lah numbers, respectively?

4 Conclusion

In this article, we introduced the λ-analogue of Lah numbers and λ-analogue of r-Lah numbers as degenerate version, respectively. We also considered the degenerate rising differential operators (37) from degenerate differential operators defined by Kim and Kim [3]. We derived combinatorial identities using the analogue rising differential operators in Theorems 4, 6, 13 and Corollaries 5 and 14. From these identities, we obtained interesting identities, which the partial degenerate (r-)Bell polynomials are represented by means of λ-analogue of (r-)Lah numbers and the degenerate Stirling numbers of the second kind by identifying formally $a = \frac{\mathrm{d}}{\mathrm{d}x}$ and $a^\dagger = x$, which satisfy $[\frac{\mathrm{d}}{\mathrm{d}x}, x] = 1$.

We propose an open problem earlier for researchers in this field and continue to study effective ways to find properties of combinatorial numbers using the boson operators.

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