Research Article

Roqia Abdullah Jeli*

Approximations to precisely localized supports of solutions for non-linear parabolic *p*-Laplacian problems

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Abstract: The shrinking of support in non-linear parabolic p-Laplacian equations with a positive initial condition u_0 that decayed as $|x| \to \infty$ was explored in the Cauchy problem. Proofs were provided for establishing exact local estimates for the boundary of the support of the solutions.

Keywords: local solution estimations, singular support behavior, support reduction, parabolic singularities, non-linear dynamics

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1 Introduction

Consider the following non-linear parabolic equation:

$$\mathcal{L}u \equiv u_t - (\ell(u)|u_x|^{p-2}u_x)_x + H(u) = 0, \quad (x,t) \in \mathbb{R} \times [0,\infty), \tag{1}$$

where $p \in (1,2)$, the function $u = u(x,t)$, the coefficients H and ℓ are in the space $C[0,+\infty) \cap C^1(0,+\infty)$, and they satisfy the conditions $H(0) = 0, \ell(0) \geq 0, H(u) > 0$, and $\ell(u) > 0$ for $u > 0$. This equation applies in the theory of non-Newtonian fluids [1,2]. The equation

$$\mathcal{L}_1 u \equiv u_t - (u^\nu \, |u_x|^{p-2}u_x)_x + u^\alpha, \tag{2}$$

where $\nu \geq 0$ and $\alpha > 0$ are chosen randomly, serves as an example of equation (1) that frequently occurs. In this article, the Cauchy problems given by equations (1) and (2) are considered under the assumption

$$u(x,0) = u_0(x) \geq 0, \quad u_0 \in C(\mathbb{R}), \quad \sup u_0 < \infty. \tag{3}$$

We define

$$\psi(u) = \int_0^u \ell(\zeta)\mathrm{d}\zeta, \quad u \geq 0.$$

Due to the monotonicity of the function $\psi(u)$, an inverse function $\psi^{-1}(\mu)$ exists for $0 \leq \mu < \psi(\infty)$. This study employs the Sobolev space $W^{1,p}(\mathbb{R})$. It is a space of functions with a norm that combines the L^p-norms of the function and its first derivative. Specifically, the norm $||u||_{W^{1,p}}$ of a function $u \in W^{1,p}(\mathbb{R})$ is defined as

$$||u||_{W^{1,p}(\mathbb{R})} = \left(\int_{\mathbb{R}} |u(x)|^p \mathrm{d}x + \int_{\mathbb{R}} |u_x(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

*** Corresponding author: Roqia Abdullah Jeli,** Department of Mathematics, Faculty of Science, Jazan University, P.O. Box 114, Jazan 45142, Saudi Arabia, e-mail: rjeli@jazanu.edu.sa

The norm $||\cdot||_{W^{1,p}}$ ensures that $W^{1,p}(\mathbb{R})$ is complete, making it a Banach space in the context that every Cauchy sequence converges within the given space. This property is crucial in the mathematical analysis conducted in this study.

This work's primary innovation is the development of precise local estimates for the boundary of support of solutions to certain non-linear parabolic equations. Compared to previous studies that provide mostly qualitative characteristics or, at best, approximate behaviors, the present study presents an analytical solution for the phenomena of interest, from which exact asymptotic formulas for the dynamics of the support of the solution can be obtained. This significant development allows for a clearer and more comprehensive perspective on the dynamics at work. Moreover, the procedure described in this work is generalized for a wider family of p-Laplacian equations. With respect to other initial conditions and non-linear elements in physical and mathematical problems, the solutions generated here can be extended to other related problems. This broad approach maximizes the generalizability of the findings in various situations. A second practical contribution of this work is the formal analysis and description of the instantaneous localization (IL) phenomenon. The research reveals precise criteria that cause the solution's support to instantly shrink to a finite location. The existing knowledge of IL provides insights into how solutions behave under particular conditions, particularly emphasizing the balance between non-linear elements and initial conditions.

The investigation of non-linear parabolic equations, especially those involving the p-Laplacian operator, is highly complex. The inherent non-linearity and the potential encounters with singularities hinder the theoretical analysis and prevent accurate estimations. The non-linearity of the problem depends on the gradient of the solution, introducing new levels of complexity that must be addressed consistently throughout the analysis. Additionally, the complexity increases when the model under consideration involves general initial conditions that tend to decay as $|x| \to \infty$. These initial conditions could significantly alter the evolution of the support and might require the use of advanced mathematical analysis to achieve accurate local estimates. To maintain the validity of the results, the approach must navigate challenging lemmas and comparison principles, requiring proper mathematical rationale.

There is a list of applied problems that can be described by the non-linear parabolic p-Laplacian equation. In its application, the equation is used for non-Newtonian fluids, which include some polymers, toothpaste, and blood, where viscosity is limited by the shear rate in a non-linear manner. Understanding how these compounds behave under different conditions is essential for manufacturing processes and medical applications. This equation also models fluid flow through porous media, such as water in soil or oil in reservoirs. In hydrology and petroleum engineering, non-linear diffusion effects are extremely important. The p-Laplacian equation is also applied in image processing algorithms to maintain edges while reducing noise, which is crucial for accurate diagnosis in medical imaging. Additionally, it appears in many population dynamic models, representing the movement of individuals in the space of a habitat, assisting ecologists in assessing the patterns of species distribution.

If $p = 2$, then the Cauchy problems for equations (1) and (2) with condition (3) simplify to the following problems for non-linear parabolic equations:

$$\mathcal{L}u \equiv u_t - (\ell(u)u_x)_x + H(u) = 0 \tag{4}$$

and

$$\mathcal{L}_1 u \equiv u_t - (u^{\nu}u_x)_x + u^{\alpha}, \quad \text{for } \nu \geq 0, 0 < \alpha < 1, \tag{5}$$

respectively, where $x \in \mathbb{R}$ and $t \geq 0$, under the same condition (3). The boundary of support for solutions to these problems can be locally estimated [3]. The concept of localization was first introduced in [4] for problem (5) under condition (3) with $\alpha = 1$ and $\nu > 0$. In [5], it is proven that complete localization of perturbations exists in problem (4) under condition (3), if

$$\int_0^1 \left[\int_0^{\zeta} H(\psi^{-1}(\mu))\mathrm{d}\mu\right]^{-\frac{1}{2}} \mathrm{d}\zeta < \infty. \tag{6}$$

In [6], the phenomenon of shrinking support for equation (4) is discovered under the assumptions

$$\int_0^1 \frac{\mathrm{d}s}{H(s)} < +\infty \quad \text{and} \quad \int_0^1 [s \cdot H(\psi^{-1}(s))]^{-\frac{1}{2}} \mathrm{d}s < +\infty. \tag{7}$$

Later, [7] investigates the phenomenon of instantaneous support shrinking in equation (1) under the assumption that

$$\int_0^1 [s \cdot H(\psi^{-1}(s))]^{-\frac{1}{p}} \mathrm{d}s < +\infty. \tag{8}$$

This assumption meets the criteria for localization in problem (1) under condition (3), and is also a necessary requirement. This assumption is also valid for equation (2) if $0 < \alpha < 1$.

To guarantee the results with considerable evidence, it is necessary to close the gap between super-solutions (sub-solutions) and classical solutions. Super-solutions and sub-solutions provide upper and lower approximations of classical solutions, respectively. The presented solutions must be uniform to provide realistic estimates and ensure their applicability.

Super-solutions and sub-solutions are often established using weak formulations of the p-Laplacian equation. With the help of Sobolev space embeddings and fixed-point theorems, it is possible to show the existence of functions that satisfy the integral forms of the inequalities involved in the definitions of the super- and sub-solutions. Super- and sub-solutions are then related to classical solutions using comparison techniques. These principles state that if a super-solution is greater than or equal to a sub-solution at the start, this ordering is maintained for all subsequent times. This enables the generation of a sandwiching effect, where the classical solution is contained between the super-solution and the sub-solution.

Regularity results are critical for ensuring that the weak solutions (super- and sub-solutions) are smooth enough to be classified as classical solutions. This involves establishing results that corroborate the fact that the solutions are differentiable and satisfy the p-Laplacian equation in the classical sense.

Definition 1.1. [8] If every $u(x, t)$ in the set $\{(x, t) : x \in \mathbb{R}_+, 0 < t < T\}$, with some $T > 0$, satisfies $u(0, 0) > 0$ for equation (1) and allows for the spread of perturbations at an infinite speed, then there is a value τ within the interval $\in (0, T)$ such that $u(x, t) > 0$ for all $x > 0$ and $0 < t < \tau$.

One notes that the above-mentioned propagation property is considered for the forward x-direction. Similarly, the propagation property is described for the backward x-direction. Let Ω be a closed subdomain of $\mathbb{R}^2_+ = \{(x, t) : x \in \mathbb{R}, 0 < t < \infty\}$. A non-negative function $u(x, t)$ is said to be a super- (or sub-) solution of equation (1) in Ω if:

- $u \in C(\overline{\Omega})$ and $u_x \in L^p_{\mathrm{loc}}(\Omega)$.
- For any t_0, t_1, x_0, and x_1 that meet the conditions $t_0 < t_1$, $x_0 < x_1$, and $\Sigma = [t_0, t_1] \times [x_0, x_1] \subset \Omega$, the integral inequality

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} (u\varphi_t - \ell(u)|u_x|^{p-2}u_x\varphi_x - H(u)\varphi)\mathrm{d}x\mathrm{d}t - \int_{x_0}^{x_1} u\varphi \mathrm{d}x \Bigg|_{t=t_0}^{t=t_1} \geq (\leq)\ 0 \tag{9}$$

  holds if φ is a non-negative function belonging to $C^{1,2}_{t,x}(\Sigma)$ and vanishes when $x = x_i$ for $t_0 \leq t \leq t_1$, where $i = 0, 1$.

A super- (or sub-) solution of problem (1) under condition (3) is defined as a super- (or sub-) solution of equation (1) in $\mathbb{R}^2_+$ that fulfills the initial condition (3). A solution to equation (1) is described as a function $u(x, t)$ that satisfies the same conditions as a super- (or sub-) solution but additionally satisfies the integral identity

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} (u\varphi_t - \ell(u)|u_x|^{p-2}u_x\varphi_x - H(u)\varphi)\mathrm{d}x\mathrm{d}t - \int_{x_0}^{x_1} u\varphi \mathrm{d}x \Bigg|_{t=t_0}^{t=t_1} = 0 \tag{10}$$

for any t_0, t_1, x_0, and x_1 satisfy $t_0 < t_1$ and $x_0 < x_1$, and any non-negative function $\varphi \in C^{1,2}_{t,x}([t_0, t_1] \times [x_0, x_1])$ that vanishes when $x = x_i$ for $t_0 \le t \le t_1$, where $i = 0, 1$. A solution to problem (1) under condition (3) is a function $u(x, t)$ that is a super- (or sub-) solution and satisfies the initial condition (3).

The works [8–14] provide existence, uniqueness, comparison theorems, and regularity results for solutions related to problem (1) under condition (3). To establish precise local estimates for the support boundary of solutions to the same problem, the following lemmas are useful:

Lemma 1. (see [14]) *Consider a non-negative and continuous function $v(x, t)$ in $\mathbb{R}_+ \times [0, T]$, where $T > 0$ is a positive value, and assume v belongs to the class $C^{1,2}_{t,x}$ in the domain $\mathbb{R}_+ \times [0, T]$ and satisfies the inequality $Lv(x, t) \ge 0$ (or $Lv(x, t) \le 0$) outside a finite number of curves $x = \xi(t)$. Additionally, assume that the function $\ell(u)|g_x|^{p-2}g_x$ is continuous for $x = \xi(t)$. Then, $v(x, t)$ is a super- (or sub-) solution of equation* (1).

Lemma 2. (Comparison principle) [8]. *If $u(x, t)$ is a super-solution and $v(x, t)$ is a sub-solution of equation* (1) *in $G = \{(x, t) : c \le x\langle+\infty), 0 \le t \le T\}$, where $c \in \mathbb{R}$, and if $u(x, t) \ge v(x, t)$ on $\overline{G}\backslash G$, then $u(x, t) \ge v(x, t)$ on $\overline{G}$.*

In connection with equation (1), the shrinking of support is identified as one of the most remarkable phenomena. Let us introduce the functions

$$\zeta^+(t) = \sup\{x \in \mathbb{R} : u(x, t) > 0\}$$

and

$$\zeta^-(t) = \inf\{x \in \mathbb{R} : u(x, t) > 0\}.$$

Definition 1.2. (see [3]) Suppose $u_0(x)$ is a positive function for $x \in \mathbb{R}$ and

$$\lim_{|x|\to+\infty} u_0(x) = 0. \tag{11}$$

If there exists a positive value δ such that, for all t in the interval $(0, \delta]$, the solution to problem (1) under condition (3) exhibits instantaneous localization (IL), then this occurs if and only if $\zeta^+(t)$ converges to a finite value and $\zeta^-(t)$ diverges to negative infinity.

The primary objective of this article is to establish a precise asymptotic formula for the support boundary of solutions in problem (1) under condition (3), while always assuming that $u_0(x)$ is positive, meets condition (11), and is not constrained by the rate of decrease described below. This goal is achieved by applying the methodology developed in [3]. Finally, the asymptotic behavior of the solutions is analyzed to obtain exact local approximations of the support. The rates of decay and the speed of propagation provide significant insight into the evolution of the support over time.

The structure of this article is as follows: Section 1 discusses the non-linear parabolic p-Laplacian equation, the initial conditions, and key concepts such as super-solutions and sub-solutions. Section 2 presents the main findings, including theorems and corollaries that provide precise estimates for the support of solutions. Sections 3 and 4 contain detailed proofs of the main results, utilizing technical lemmas and comparison principles to substantiate the findings. Finally, the conclusion summarizes the most significant results and explores potential applications.

2 The main results

The results for the estimates on the support of the solution to problem (2) under condition (3) are given as follows:

Theorem 3. *Suppose $0 < \alpha < 1$ and $\nu \ge 1 - \alpha(p - 1)$, and let u_0 be a positive even function satisfying* (11). *Also, assume that there is a value $\hat{x} > 0$ in such a way that u_0 is twice continuously differentiable on $(\hat{x}, +\infty)$, and $u_0'(x) < 0$ for $x \ge \hat{x}$, and that $(u_0^\kappa)''$ has the order of growth of $O(1)$ as x approaches infinity, where $\kappa = 1 - \alpha$.*

Under these conditions, the solution to problem (1) *under condition* (3) *satisfies IL, and for any* $\varepsilon > 0$ *that is arbitrarily small, a positive value* δ *exists such that the functions characterizing the boundaries of the solution's support meet the estimate*:

$$u_0^{-1}\left((1-\varepsilon)^{-1}((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}\right) \le |\zeta^{\pm}(t)| \le u_0^{-1}\left((1+\varepsilon)^{-1}((1-\alpha-\varepsilon)t)^{\frac{1}{1-\alpha}}\right), \quad t \in (0,\delta]. \tag{12}$$

In this context, u_0^{-1} *represents the inverse function of* u_0 *on the domain* $x \ge \hat{x}$.

Corollary 4. *Assuming that the conditions outlined in Theorem* 3 *are met, the following exact asymptotic behavior is established*:

$$\zeta^{\pm}(t) \sim \pm\, u_0^{-1}\left[((1-\alpha)t)^{\frac{1}{1-\alpha}}\right], \quad t \to 0. \tag{13}$$

Corollary 5. *Assuming the validity of the assumptions in Theorem* 3, *for every* $\varepsilon > 0$ *that is sufficiently small, there exist* $x_\varepsilon > 0$ *and* $\delta > 0$ *such that problem* (2) *under condition* (3) *has a solution satisfying the following estimate*:

$$\{((1-\varepsilon)u_0(x))^{1-\alpha} - (1-\alpha+\varepsilon)t\}_+^{\frac{1}{1-\alpha}} \le u(x,t) \le \{((1+\varepsilon)u_0(x))^{1-\alpha} - (1-\alpha-\varepsilon)t\}_+^{\frac{1}{1-\alpha}}, \tag{14}$$
$(x,t) \in (|x| \ge x_\varepsilon) \times [0,\delta]$.

Theorem 6. *Let* $0 < \alpha < 1$ *and* $0 \le \nu < 1 - \alpha(p-1)$. *Assume that the function* u_0 *meets all the requirements of Theorem* 3 *for* $\kappa = \frac{\nu+p-1-\alpha}{p}$. *If* $\nu = 0$, *additionally assume that*

$$(u_0^{1-\alpha})'' = o(1), \quad x \to +\infty.$$

For the solution to problem (2) *under condition* (3), *IL holds. Additionally, for each positive* ε *that is sufficiently small and* $\rho > 0$, *there is a* δ *that depends on* ε *and* ρ, *such that the functions characterizing the boundaries of the solution's support satisfy the estimate*

$$u_0^{-1}\left(\frac{1}{1-\varepsilon}((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}\right) \le |\zeta^{\pm}(t)| \le u_0^{-1}(\rho t^{\frac{p}{\nu+p-1-\alpha}}), \quad 0 < t \le \delta. \tag{15}$$

Corollary 7. *In accordance with the conditions stated in Theorem* 6, *for each positive* ε *that is sufficiently small, there exist* $x_\varepsilon > 0$ *and* $\delta > 0$ *such that the solution of problem* (2) *under condition* (3) *satisfies the following local estimate*:

$$\{[(1-\varepsilon)u_0(x)]^{1-\alpha} - (1-\alpha+\varepsilon)t\}_+^{\frac{1}{1-\alpha}} \le u(x,t) \le \left\{[(1+\varepsilon)u_0(x)]^{\frac{\nu+p-1-\alpha}{p}} - \rho t\right\}_+^{\frac{p}{\nu+p-1-\alpha}}, \tag{16}$$

for $(x,t) \in (|x| \ge x_\varepsilon) \times [0,\delta]$.

The results for the estimates on the support of the solution of problem (1) under condition (3) are given as follows:

Theorem 8. *Consider the case where both conditions*

$$\int_0^1 \frac{\mathrm{d}\zeta}{H(\zeta)} < +\infty; \quad (\ell(x)H(x))' \ge 0, \quad x > 0 \tag{17}$$

and

$$d \equiv \lim_{x\to 0} \frac{[\ell(x)H(x)]'}{H(x)} < +\infty, \quad \ell(0) = 0 \tag{18}$$

are true. Assume u_0 *is an even positive function satisfying* (11). *Additionally, suppose that there exists a positive value* $\hat{x}$ *such that the initial function* u_0 *belongs to the class* $C^2(\hat{x}, +\infty)$, $u_0'(x) < 0$ *for* $x \geq \hat{x}$, *and*

$$\frac{\mathrm{d}^2\Phi(u_0(x))}{\mathrm{d}x^2} = O(1) \quad \text{as } x \to +\infty,$$

where Φ^{-1} *represents the inverse function of* Φ. *Then, for the solution to problem* (1) *under condition* (3), *IL holds. Moreover, for any* $\varepsilon > 0$ *that is sufficiently small, there is a positive value* δ *such that the functions that characterize the boundaries of the solution's support satisfy the estimate*

$$u_0^{-1}\left(\frac{1}{1-\varepsilon}\Phi^{-1}((1+\varepsilon))t\right) \leq |\zeta^{\pm}(t)| \leq u_0^{-1}\left(\frac{1}{1+\varepsilon}\Phi^{-1}((1-\varepsilon))t\right), \quad t \in (0, \delta]. \tag{19}$$

Corollary 9. *The precise asymptotic formula*

$$\zeta^{\pm}(t) \sim \pm\, u_0^{-1}(\Phi^{-1}(t)), \quad t \to 0 \tag{20}$$

holds under the conditions of the theorem.

Corollary 9 can be established using the same approach as Corollary 4.

Corollary 10. *If the requirements of Theorem* 8 *are met, then for every* $\varepsilon > 0$ *that is sufficiently small,* $x_\varepsilon > 0$ *and* $\delta > 0$ *exist such that the solution to problem* (1) *under condition* (3) *meets the estimate*:

$$\Phi^{-1}([\Phi((1-\varepsilon)u_0(x)) - (1+\varepsilon)t]_+) \leq u(x,t) \leq \Phi^{-1}([\Phi((1+\varepsilon)u_0(x)) - (1-\varepsilon)t]_+), \tag{21}$$

for $(x, t) \in (|x| \geq r_\varepsilon) \times [0, \delta]$.

Corollary 10 can be directly deduced from the proof of Theorem 8.

Theorem 11. *If we assume that conditions* (17), (8), *and*

$$\lim_{x\to 0}[\ell(x)H(x)]' = +\infty \tag{22}$$

are satisfied, along with a function $u_0(x)$ *that meets the requirements of Theorem* 8, *and*

$$\frac{\mathrm{d}^2\Phi(\psi(u_0(x)))}{\mathrm{d}x^2} = O(1) \quad \text{as } x \to +\infty.$$

If $\ell(0) > 0$, *we also assume that*

$$\frac{\mathrm{d}^2F(u_0(x))}{\mathrm{d}x^2} = o(1) \quad \text{as } x \to +\infty. \tag{23}$$

Then, for the solution to problem (1) *under condition* (3), *IL holds. Additionally, for any* $\varepsilon > 0$ *that is sufficiently small, there is a positive value* δ *such that the functions that characterize the boundaries of the solution's support satisfy the estimate*

$$u_0^{-1}\left[\frac{1}{1-\varepsilon}\Phi^{-1}((1+\varepsilon)t)\right] \leq |\zeta^{\pm}(t)| \leq u_0^{-1}\left[\frac{1}{1+\varepsilon}\psi^{-1}\left(\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}} t\right)\right)\right], \quad t \in (0, \delta], \tag{24}$$

where Ψ^{-1} *represents the inverse function of* Ψ.

Corollary 12. *In accordance with the conditions stated in Theorem* 11, *for each positive* ε *that is sufficiently small, there exist* $x_\varepsilon > 0$ *and* $\delta > 0$ *such that the solution to problem* (1) *under condition* (3) *satisfies the estimate*:

$$\Phi^{-1}([\Phi((1-\varepsilon)u_0(x)) - (1+\varepsilon)t]_+) \leq u(x,t) \leq \psi^{-1}\left(\Psi^{-1}\left[\Psi((1+\varepsilon)u_0(x)) - \left(\frac{p}{p-1}\right)^{\frac{1}{p}} t\right]_+\right), \tag{25}$$

for $(x, t) \in (|x| \geq x_\varepsilon) \times [0, \delta]$.

Note that it is not crucial to assume the initial function is even; we only assumed it to simplify the notation. For example, if we suppose that $\hat{x} > 0$ exists such that $u_0'(x) < 0$ for $x \geq \hat{x}$, then estimates (12), (13), (14), (17), (18), and (20) hold for $|\zeta^+(t)|$ (we can also write $\zeta^+(t)$ instead of $|\zeta^+(t)|$). The same estimates for $|\zeta^+(t)|$ can be proven using a completely analogous method if there is $\hat{x} < 0$ such that $u_0'(x) < 0$ for $x \leq \hat{x}$. In this case, the function $u_0^{-1}(x)$ represents the inverse of $u_0(x)$ for $x \leq \hat{x}$ (we can also write $-\zeta^-(t)$ instead of $|\zeta^-(t)|$).

3 Proof of the results of problem (2) under condition (3)

Proof of Theorem 3. Introduce the following function that exhibits the characteristics of a traveling wave:

$$h(x,t) = g(\eta), \quad \eta = A - t - \int_0^x \gamma(\zeta)\mathrm{d}\zeta, \tag{26}$$

where $A > 0$ is a constant, γ is a continuously differentiable function on $[0,+\infty)$ satisfying $|\gamma'(x)| \leq M$ and $0 < \gamma(x) \leq \mu$, and γ satisfies the following condition:

$$\int_0^\infty \gamma(x)\mathrm{d}x = A. \tag{27}$$

The function $h(x,t)$ represents the traveling wave solution, where $g(\eta)$ is a continuously differentiable function on $[0,+\infty)$. The function γ meets the necessary growth and decay conditions specified above. For any small positive value ε, the function $g(\eta)$ is defined as $g(\eta) = [(1-\alpha-\varepsilon)\eta]_+^{\frac{1}{1-\alpha}}$, where $[(x)]_+$ denotes the function that returns x if $x \geq 0$ and 0 otherwise. The function $h(x,t)$ satisfies the following conditions:

$$h(x,0) = \left[(1-\alpha-\varepsilon)\left(A - \int_0^x \gamma(\zeta)\mathrm{d}\zeta\right)\right]_+^{\frac{1}{1-\alpha}} \quad \text{for } x \in \mathbb{R}_+$$

and

$$h(0,t) = ((1-\alpha-\varepsilon)[A-t]_+)^{\frac{1}{1-\alpha}} \quad \text{for } t \geq 0.$$

Moreover, $h(x,t)$ satisfies $h(x,0) > 0$ for $x \geq 0$, and $h(x,0)$ approaches zero as x goes to positive infinity. Furthermore, for any $t \in (0,A]$,

$$h(x,t) = 0, \quad x \geq \mu_0(A-t),$$

where $x = \mu_0(w)$ is the inverse function of $w = \int_0^x \gamma(\zeta)\mathrm{d}\zeta$. It is asserted that the function $h(x,t)$ serves as a super-solution for equation (2) within the region $(x,t) \in [0,+\infty) \times [0,t_1]$, where t_1 is a positive constant. Then, we obtain

$$\begin{aligned}
\mathcal{L}_1 h &\equiv h_t - (h^\nu |h_x|^{p-2}h_x)_x + h^\alpha = -g' - \gamma^p(x)(g^\nu |g'|^{p-2}g')' + (p-1)\gamma^{p-2}(x)\gamma'(x)g^\nu |g'|^{p-2}g' + g^\alpha \\
&\geq -g' - \mu^p(g^\nu |g'|^{p-2}g')' - (p-1)\mu^{p-2}Mg^\nu |g'|^{p-2}g' + g^\alpha \\
&= -\left(\frac{1-\alpha-\varepsilon}{1-\alpha}\right)((1-\alpha-\varepsilon)\eta)^{\frac{\alpha}{1-\alpha}} - \mu^p(\alpha(p-1)+\nu)\left(\frac{1-\alpha-\varepsilon}{1-\alpha}\right)^p((1-\alpha-\varepsilon)\eta)^{\frac{\alpha p+\nu-1}{1-\alpha}} \\
&\quad - (p-1)\mu^{p-2}M\left(\frac{1-\alpha-\varepsilon}{1-\alpha}\right)^{p-1}((1-\alpha-\varepsilon)\eta)^{\frac{\alpha(p-1)+\nu}{1-\alpha}} + ((1-\alpha-\varepsilon)\eta)^{\frac{\alpha}{1-\alpha}} \\
&= ((1-\alpha-\varepsilon)\eta)^{\frac{\alpha}{1-\alpha}}\left[\frac{\varepsilon}{1-\alpha} - \mu^p(\alpha(p-1)+\nu)\left(\frac{1-\alpha-\varepsilon}{1-\alpha}\right)^p((1-\alpha-\varepsilon)\eta)^{\frac{\alpha(p-1)+\nu-1}{1-\alpha}}\right. \\
&\quad \left. - (p-1)\mu^{p-2}M\left(\frac{1-\alpha-\varepsilon}{1-\alpha}\right)^{p-1}((1-\alpha-\varepsilon)\eta)^{\frac{\alpha(p-2)+\nu}{1-\alpha}}\right].
\end{aligned} \tag{28}$$

Equation (28) imples that if $\nu > 1 - \alpha(p-1)$, $0 < \alpha < 1$, then there exists a positive value of η_0 such that

$$\mathcal{L}_1 h > 0, \quad \eta \in (0, \eta_0]. \tag{29}$$

Furthermore, if $\nu = 1 - \alpha(p-1)$, $0 < \alpha < 1$, then in the case

$$0 < \mu < \left(\frac{\varepsilon}{1-\alpha}\right)^{\frac{1}{p}}, \tag{30}$$

condition (29) also holds. Thus, if ν is greater than or equal to $1 - \alpha(p-1)$ where $0 < \alpha < 1$, then for any A in the interval $(0, \eta_0]$, the function h acts as a super-solution of equation (2) for all x in the positive real numbers and for all t in the interval $[0, t_1]$, where $t_1 \in (0, A]$. We choose a value x_ε that is greater than or equal to $\hat{x}$ and define

$$\gamma_\varepsilon(x) = -\frac{1}{1-\alpha-\varepsilon}([(1+\varepsilon)u_0(x)]^{1-\alpha})'$$

for $x_\varepsilon \le x < +\infty$. We also define A_ε as

$$A_\varepsilon = \frac{1}{1-\alpha-\varepsilon}[(1+\varepsilon)u_0(x_\varepsilon)]^{1-\alpha}.$$

Checking that

$$\int_{x_\varepsilon}^{+\infty} \gamma_\varepsilon(x)\mathrm{d}x = A_\varepsilon \tag{31}$$

is a straightforward task. As the function γ_ε' is bounded for $x \in [x_\varepsilon, +\infty)$ by hypothesis, it follows from (31) that

$$\lim_{x\to+\infty} \gamma_\varepsilon(x) = 0. \tag{32}$$

According to this limit, there exists x_1 such that $0 < \gamma_\varepsilon(x) \le \mu$ for $x \ge x_1$, where $\mu > 0$ satisfies (30). For $\varepsilon > 0$, we suppose that $x_\varepsilon \ge x_1$. We examine the function

$$h_\varepsilon(x,t) = \left[(1-\alpha-\varepsilon)\left(A_\varepsilon - t - \int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right)\right]^{\frac{1}{1-\alpha}}.$$

Using a similar approach as described previously, we can prove the existence of a value $t_1 \in (0, T)$ such that for (x,t) belonging to the set $[x_\varepsilon, \infty) \times [0, t_1]$, $h_\varepsilon(x,t)$ acts as a super-solution of equation (2). Equation (31) implies that for x greater than or equal to x_ε, the value of $h_\varepsilon(x, 0)$ is positive. Furthermore,

$$\begin{aligned} h_\varepsilon(x,0) &= \left[(1-\alpha-\varepsilon)\left(A_\varepsilon - \int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right)\right]^{\frac{1}{1-\alpha}} \\ &= \left[(1-\alpha-\varepsilon)\int_{x_\varepsilon}^{+\infty}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right]^{\frac{1}{1-\alpha}} \\ &= (1+\varepsilon)u_0(x) > u_0(x), \quad x_\varepsilon \le x < +\infty, \\ h_\varepsilon(x_\varepsilon, t) &= [(1-\alpha-\varepsilon)(A_\varepsilon - t)_+]^{\frac{1}{1-\alpha}}, \quad 0 \le t \le t_1. \end{aligned} \tag{33a}$$

Since

$$h_\varepsilon(x_\varepsilon, 0) = ((1-\alpha-\varepsilon)A_\varepsilon)^{\frac{1}{1-\alpha}} = (1+\varepsilon)u_0(x_\varepsilon) > u_0(x_\varepsilon),$$

there exists $\delta_1 > 0$ such that

$$h_\varepsilon(x_\varepsilon, t) \ge u(x_\varepsilon, t), \quad t \in [0, \delta_1]. \tag{33b}$$

Let us analyze the given argument step by step. First, we set δ_2 as

$$\delta_2 = \min\left\{\delta_1, t_1, A_\varepsilon, [(1+\varepsilon)u_0(\hat{x})]^{1-\alpha}\frac{1}{1-\alpha-\varepsilon}\right\}.$$

This choice of δ_2 guarantees that it satisfies all the conditions mentioned in Lemma 2. Next, we have the inequality

$$0 \le u(x,t) \le h_\varepsilon(x,t), \quad (x,t) \in [x_\varepsilon, +\infty) \times [0, \delta_2]. \tag{34}$$

This result is based on equation (33) and Lemma 2. It states that the solution $u(x,t)$ is bounded between 0 and $h_\varepsilon(x,t)$ for $(x,t) \in [x_\varepsilon, +\infty) \times [0,\delta_2]$. Then, we can conclude that

$$u(x,t) = 0, \quad x \ge \mu_\varepsilon(A_\varepsilon - t), 0 < t \le \delta_2, \tag{35}$$

where $x = \mu_\varepsilon(w)$ is the inverse function of $w = \int_{x_\varepsilon}^{x}\gamma(\zeta)\mathrm{d}\zeta$. Moreover, we have the expression

$$\mu_\varepsilon(A_\varepsilon - t) = u_0^{-1}\left[\frac{1}{1+\varepsilon}((1-\alpha-\varepsilon)t)^{\frac{1}{1-\alpha}}\right].$$

This is obtained from the previous equation and the given expression for $\int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta$. Finally, based on even symmetry arguments, it is possible to demonstrate

$$u(x,t) = 0 \quad \text{for } x \ge u_0^{-1}\left[\frac{1}{1+\varepsilon}((1-\alpha-\varepsilon)t)^{\frac{1}{1-\alpha}}\right], \quad t \in (0, \delta_2]$$

and

$$u(x,t) = 0 \quad \text{for } x \le -u_0^{-1}\left[\frac{1}{1+\varepsilon}((1-\alpha-\varepsilon)t)^{\frac{1}{1-\alpha}}\right], \quad t \in (0, \delta_2].$$

These relations are obtained from the previous equation using symmetry considerations. This allows us to prove these relations based on even solutions with respect to x for every $t > 0$ of problem (2) under condition (3) with an even initial function. Therefore, the argument is valid and proves the right-hand side estimate in (12).

To prove the left-hand side estimate in (12), we consider the function h defined in (26), where A and γ are as in (26) and (27). Next, we define the function g as

$$g(\eta) = [(1-\alpha+\varepsilon)\eta]_+^{\frac{1}{1-\alpha}}.$$

This function is constructed such that $g(\eta)$ acts as an upper bound for the non-negative values of the function $h(x,t)$. It is necessary to confirm that $h(x,t)$ meets the requirements of being a sub-solution for the equation (2) for $(x,t) \in [0,\infty) \times [0,t_1]$ with $t_1 > 0$. To do this, we compute the operator $\mathcal{L}_1 h$. We have

$$\begin{aligned}
\mathcal{L}_1 h &= -g' - \gamma^p(x)(g^\nu\,|g'|^{p-2}g')' + (p-1)\gamma^{p-2}(x)\gamma'(x)g^\nu\,|g'|^{p-2}g' + g^\alpha \\
&\le -g' - (p-1)\mu^{p-2}Mg^\nu\,|g'|^{p-2}g' + g^\alpha \\
&= -\frac{1-\alpha+\varepsilon}{1-\alpha}((1-\alpha+\varepsilon)\eta)^{\frac{\alpha}{1-\alpha}} - (p-1)\mu^{p-2}M\left(\frac{1-\alpha+\varepsilon}{1-\alpha}\right)^{p-1}((1-\alpha+\varepsilon)\eta)^{\frac{\alpha(p-1)+\nu}{1-\alpha}} + [(1-\alpha+\varepsilon)\eta]^{\frac{\alpha}{1-\alpha}} \\
&= [(1-\alpha+\varepsilon)\eta]^{\frac{\alpha}{1-\alpha}}\left\{-\frac{\varepsilon}{1-\alpha} - (p-1)\mu^{p-2}M\left(\frac{1-\alpha+\varepsilon}{1-\alpha}\right)^{p-1}((1-\alpha+\varepsilon)\eta)^{\frac{\alpha(p-2)+\nu}{1-\alpha}}\right\}.
\end{aligned} \tag{36}$$

From equation (36), we know that there exists a positive value of η_1 such that

$$\mathcal{L}_1 h < 0, \quad \eta \in (0, \eta_1].$$

Therefore, for every $A \in (0,\eta_1]$, the function h acts as a sub-solution of equation (2) for $x \in \mathbb{R}_+$ and $t \in [0,t_1]$ with $t_1 \in (0,A]$. To derive equations (31) and (32), we follow a similar approach as in the previous proof. We select $x_\varepsilon \ge \hat{x}$ and define the function $\gamma_\varepsilon(x)$ as follows:

$$\gamma_\varepsilon(x) = -\frac{1}{1-\alpha+\varepsilon}([(1-\varepsilon)u_0(x)]^{1-\alpha})', \quad x_\varepsilon \le x < +\infty,$$

$$A_\varepsilon = \frac{1}{1-\alpha+\varepsilon}[(1-\varepsilon)u_0(x_\varepsilon)]^{1-\alpha}.$$

By making these choices, we can proceed with the proof for equations (31) and (32) in the same manner. We introduce the function $h_\varepsilon(x,t)$ defined as

$$h_\varepsilon(x,t) = \left[(1-\alpha+\varepsilon)\left(A_\varepsilon - t - \int_{x_\varepsilon}^{x}\gamma(\zeta)\mathrm{d}\zeta\right)_+\right]^{\frac{1}{1-\alpha}}.$$

Using a similar argument as before, we show that h_ε serves as a sub-solution for equation (2) within the region $(x,t) \in [x_\varepsilon, +\infty) \times [0, t_1]$ for some $t_1 > 0$. From equation (32), it follows that $h_\varepsilon(x,0) > 0$ for $x \geq x_\varepsilon$. Additionally,

$$\begin{aligned} h_\varepsilon(x,0) &= (1-\varepsilon)u_0(x) < u_0(x), \quad x_\varepsilon \leq x < +\infty, \\ h_\varepsilon(x_\varepsilon,t) &= [(1-\alpha+\varepsilon)(A_\varepsilon - t)_+]^{\frac{1}{1-\alpha}}, \quad 0 \leq t \leq t_1. \end{aligned} \tag{37a}$$

Given that

$$h_\varepsilon(x_\varepsilon,0) = [(1-\alpha+\varepsilon)A_\varepsilon]^{\frac{1}{1-\alpha}} = (1-\varepsilon)u_0(x_\varepsilon) < u_0(x_\varepsilon),$$

there exists a positive value δ_3 such that

$$h_\varepsilon(x_\varepsilon,t) \leq u(x_\varepsilon,t), \quad 0 \leq t \leq \delta_3. \tag{37b}$$

If $\delta_3 < \min\left\{A_\varepsilon, \frac{[(1-\varepsilon)u_0(\hat{x})]^{1-\alpha}}{1-\alpha+\varepsilon}\right\}$, then applying Lemma 2 to the above, we obtain

$$h_\varepsilon(x,t) \leq u(x,t), \quad (x,t) \in [x_\varepsilon, +\infty) \times [0,\delta_3]. \tag{38a}$$

Based on the properties of $h_\varepsilon > 0$, it follows that

$$u(x,t) > 0 \quad \text{for } (x,t) \in [x_\varepsilon, \mu_\varepsilon(A_\varepsilon - t)) \times (0,\delta_3], \tag{38b}$$

where $x = \mu_\varepsilon(w)$ represents the inverse function of $w = \int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta$. Clearly,

$$\mu_\varepsilon(A_\varepsilon - t) = u_0^{-1}\left[\frac{1}{1-\varepsilon}((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}\right].$$

If necessary, we can choose a sufficiently small $\delta_3 > 0$. Thus, from the above, we can conclude that

$$u(x,t) > 0 \quad \text{for } 0 \leq x \leq u_0^{-1}\left[\frac{1}{1-\varepsilon}((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}\right], \quad t \in [0,\delta_3]. \tag{38c}$$

By following the same method, we can prove that

$$u(x,t) > 0 \quad \text{for } -u_0^{-1}\left[\frac{1}{1-\varepsilon}((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}\right] \leq x \leq 0, \quad t \in [0,\delta_3].$$

Consequently, inequality (12) holds with $\delta = \min(\delta_2, \delta_3) > 0$. □

Remark 1. It is important to note that for $t > 0$, there is no connected support for the solution $u(x,t)$. Thus, the points $x = \zeta^\pm(t)$ do not form the complete outer limit of the support of the solution $u(x,t)$ with respect to x for a fixed value $t > 0$. In addition to $x \leq \zeta^-(t)$ or $x \geq \zeta^+(t)$, it is possible for $x_1(t) \leq x \leq x_2(t)$ to also cause $u(x,t) = 0$, where $\zeta^-(t) < x_1(t) < x_2(t) < \zeta^+(t)$. However, as we demonstrate in Theorem 3, for every small value of $\varepsilon > 0$, there exists a positive value δ for which the intervals defined by estimates (36) fully capture the entire boundary of the solution's support. By selecting δ as the smaller value between δ_2 and δ_3, it can be deduced from equation (35) that

$$u(x,t) = 0 \quad \text{for } |x| \geq u_0^{-1}\left[\frac{1}{1+\varepsilon}((1-\alpha-\varepsilon)t)^{\frac{1}{1-\alpha}}\right], \quad t \in (0,\delta].$$

Furthermore, from (38a), we conclude that

$$u(x,t)>0 \quad \text{for } |x|\le u_0^{-1}\left[\frac{1}{1-\varepsilon}((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}\right], \quad t\in(0,\delta].$$

This observation applies to all subsequent theorems (corollaries) pertaining to local estimates for the support of the solutions to equations (1) and (2) [3].

Proof of Corollary 4. Given that the initial function u_0 is strictly monotonically decreasing as $x\to+\infty$, we deduce from inequality (12) that

$$\frac{((1-\alpha-\varepsilon)t)^{\frac{1}{1-\alpha}}}{1+\varepsilon}\le u_0(\zeta^+(t))\le\frac{((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}}{1-\varepsilon}, \quad t\in(0,\delta].$$

Simplifying further, we obtain

$$\frac{((1-\alpha-\varepsilon)t)^{\frac{1}{1-\alpha}}}{(1+\varepsilon)((1-\alpha)t)^{\frac{1}{1-\alpha}}}\le\frac{u_0(\zeta^+(t))}{((1-\alpha)t)^{\frac{1}{1-\alpha}}}\le\frac{((1-\alpha+\varepsilon)t)^{\frac{1}{1-\alpha}}}{(1-\varepsilon)((1-\alpha)t)^{\frac{1}{1-\alpha}}}, \quad t\in(0,\delta].$$

In the limit, as t tends to zero, it is observed that

$$\frac{1}{1+\varepsilon}\left(\frac{1-\alpha-\varepsilon}{1-\alpha}\right)^{\frac{1}{1-\alpha}}\le\lim_{t\to0}\frac{u_0(\zeta^+(t))}{((1-\alpha)t)^{\frac{1}{1-\alpha}}}\le\frac{1}{1-\varepsilon}\left(\frac{1-\alpha+\varepsilon}{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$

Since $\varepsilon>0$ is arbitrarily chosen, we conclude that

$$\lim_{t\to0}\frac{u_0(\zeta^+(t))}{((1-\alpha)t)^{\frac{1}{1-\alpha}}}=1,$$

or equivalently,

$$u_0(\zeta^+(t))\sim((1-\alpha)t)^{\frac{1}{1-\alpha}}, \quad t\to0.$$

Similarly, by following a similar argument, we demonstrate that the asymptotic formula (13) holds for $\zeta^-(t)$ as well. □

Proof of Corollary 5. Using the estimate (34), we find that

$$0\le u(x,t)\le h_\varepsilon(x,t)=\left[(1-\alpha-\varepsilon)\left(A_\varepsilon-t-\int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right)\right]^{\frac{1}{1-\alpha}}, \quad (x,t)\in[x_\varepsilon,+\infty)\times[0,\delta_2].$$

Given that

$$\int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta=A_\varepsilon-\frac{1}{1-\alpha-\varepsilon}[(1+\varepsilon)u_0(x)]^{1-\alpha},$$

we deduce that

$$0\le u(x,t)\le\{[(1+\varepsilon)u_0(x)]^{1-\alpha}-(1-\alpha-\varepsilon)t\}_+^{\frac{1}{1-\alpha}}, \quad (x,t)\in[x_\varepsilon,+\infty)\times[0,\delta_2].$$

Furthermore, utilizing estimate (38), we have

$$\left[(1-\alpha+\varepsilon)\left(A_\varepsilon-t-\int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right)\right]^{\frac{1}{1-\alpha}}=h_\varepsilon(x,t)\le u(x,t), \quad (x,t)\in[x_\varepsilon,+\infty)\times[0,\delta_3].$$

By considering

$$\int_{x_\varepsilon}^{x} \gamma_\varepsilon(\zeta)\mathrm{d}\zeta = A_\varepsilon - \frac{1}{1-\alpha+\varepsilon}((1-\varepsilon)u_0(x))^{1-\alpha},$$

it is determined that

$$\{((1-\varepsilon)u_0(x))^{1-\alpha} - (1-\alpha+\varepsilon)t\}_+^{\frac{1}{1-\alpha}} \le u(x,t), \quad (x,t) \in [x_\varepsilon, +\infty) \times [0, \delta_3].$$

By setting $\delta = \min(\delta_2, \delta_3)$, estimate (14) is satisfied. □

There are several examples that can be considered. For instance, consider $0 < \alpha < 1$ and $\nu \ge 1 - \alpha(p-1)$. Suppose that u represents a solution to problem (2) under condition (3), and assume that there exists a positive number $\hat{x}$ such that $u_0(x) = \psi(|x|)$ for $|x| \ge \hat{x}$. In such cases, it is possible to provide exact asymptotic formulas for $\zeta^{\pm}(t)$ as $t \to 0$. The specific formulas for $\zeta^{\pm}(t)$ can be found in Table 1 of the article referenced [3].

Proof of Theorem 6. As we observe, the expression on the left-hand side of (15) in Theorem 3 is established. It is evident that in the demonstration of the left-hand side of (12) (see (36)), only the condition $\nu > 0$ is utilized. Hence, it is still necessary to investigate the case where $\nu = 0$. In this case, the second term in the curly brackets on the right-hand side of (36) does not disappear for $\eta = 0$. However, according to the theorem's assumptions, for any given small value $\varepsilon > 0$, we can pick x_ε to be sufficiently large such that

$$M = \sup_{x \ge x_\varepsilon} |\gamma'_\varepsilon(x)| < \frac{\varepsilon}{1-\alpha+\varepsilon}.$$

Consequently, from (36), we deduce the existence of $\eta_1 > 0$ such that $\mathcal{L}_1 h < 0$ for $\eta \in [0, \eta_1]$. Evidently, it is enough to prove the right-hand side of inequality (15). The proof is constructed in the same manner as the one in Theorem 3. Consider the function h defined in (26), where A and γ are given by (26) and (27), respectively. Define the function $g(\eta)$ as follows:

$$g(\eta) = [\rho \cdot \eta]_+^{\frac{p}{\nu+p-1-\alpha}}, \quad \rho > 0.$$

We can verify that there exists a positive number t_1 such that h serves as a super-solution to equation (2) for $(x,t) \in [0, +\infty) \times [0, t_1]$. By computing $\mathcal{L}_1 h$, we find

$$\begin{aligned}
\mathcal{L}_1 h &= -g' - \gamma^p(x)(g^\nu\, |g'|^{p-2}g')' + (p-1)\gamma^{p-2}(x)\gamma'(x)g^\nu\, |g'|^{p-2}g' + g^\alpha \\
&\ge -g' - \mu^p(g^\nu\, |g'|^{p-2}g')' - M(p-1)\mu^{p-2}g^\nu\, |g'|^{p-2}g' + g^\alpha \\
&= -\frac{p\rho}{\nu+p-1-\alpha}[\rho\cdot\eta]^{\frac{-\nu+1+\alpha}{\nu+p-1-\alpha}} - \mu^p\rho^p\left(\frac{p}{\nu+p-1-\alpha}\right)^{p-1}\left(\frac{\nu+p-1+\alpha(p-1)}{\nu+p-1-\alpha}\right)[\rho\cdot\eta]^{\frac{p\alpha}{\nu+p-1-\alpha}} \\
&\quad - M(p-1)\mu^{p-2}\left(\frac{p\rho}{\nu+p-1-\alpha}\right)^{p-1}[\rho\cdot\eta]^{\frac{(-\nu+1+\alpha)(p-1)+p\nu}{\nu+p-1-\alpha}} + [\rho\cdot\eta]^{\frac{p\alpha}{\nu+p-1-\alpha}} \\
&= [\rho\cdot\eta]^{\frac{p\alpha}{\nu+p-1-\alpha}}\left\{1 - \frac{p\rho}{\nu+p-1-\alpha}[\rho\cdot\eta]^{\frac{1-\alpha(p-1)-\nu}{\nu+p-1-\alpha}} - \mu^p\rho^p\left(\frac{p}{\nu+p-1-\alpha}\right)^{p-1}\left(\frac{\nu+p-1+\alpha(p-1)}{\nu+p-1-\alpha}\right)\right. \\
&\quad \left. - M(p-1)\mu^{p-2}\rho^p\left(\frac{p}{\nu+p-1-\alpha}\right)^{p-1}\eta\right\}.
\end{aligned} \tag{39}$$

Assume that

$$0 < \mu^p < \frac{(\nu+p-1-\alpha)^p}{\rho^p p^{p-1}(\nu+p-1+\alpha(p-1))}. \tag{40}$$

Table 1: Adapted from [3]

$\psi(x)$	$\zeta^{\pm}(t) \sim \ldots,\ t \to 0$
$x^{-\kappa},\ \kappa > 0$	$\pm[(1-\alpha)t]^{-\frac{1}{\kappa(1-\alpha)}}$
$\exp(-x)$	$\mp\frac{\ln((1-\alpha)t)}{1-\alpha}$
$\exp(-x^2)$	$\pm\left[\frac{-\ln((1-\alpha)t)}{1-\alpha}\right]^{\frac{1}{2}}$
$(\ln x)^{-\mu},\ \mu > 0$	$\pm\exp\left([(1-\alpha)t]^{-\frac{1}{\mu(1-\alpha)}}\right)$
$(1+x^{\lambda})^{-1},\ \lambda > 0$	$\pm\left[((1-\alpha)t)^{-\frac{1}{1-\alpha}} - 1\right]^{\frac{1}{\lambda}}$

From (39), it follows that for some positive number η_0, we obtain

$$\mathcal{L}_1 h > 0, \quad \eta \in (0, \eta_0]. \tag{41}$$

Therefore, for all $A \in (0, \eta_0]$, the function h acts as a super-solution of equation (2) for $x \in \mathbb{R}_+$ and $t \in [0, t_1]$, with $t_1 \in (0, A]$.

Similar to the proof of Theorem 3, we select $x_\varepsilon \geq \hat{x}$ and define

$$\gamma_\varepsilon(x) = -\frac{1}{\rho}\left[((1+\varepsilon)u_0(x))^{\frac{\nu+p-1-\alpha}{p}}\right]', \quad x_\varepsilon \leq x < +\infty,$$
$$A_\varepsilon = \frac{1}{\rho}((1+\varepsilon)u_0(x_\varepsilon))^{\frac{\nu+p-1-\alpha}{p}}.$$

For sufficiently large values of x_ε, we have $A_\varepsilon \in (0, \eta_0]$. It is verified that

$$\int_{x_\varepsilon}^{+\infty} \gamma_\varepsilon(\zeta)\mathrm{d}\zeta = A_\varepsilon. \tag{42}$$

According to the theorem's hypothesis, the function $\gamma_\varepsilon'(x)$ is bounded for $x \geq x_\varepsilon$. Thus, from (42), we obtain

$$\lim_{x\to+\infty} \gamma_\varepsilon(x) = 0. \tag{43}$$

Therefore, there exists x_1 such that $0 < \gamma_\varepsilon(x) \leq \mu$ for $x \geq x_1$, where μ satisfies condition (40). We assume that $x_\varepsilon \geq x_1$.

Consider the function

$$h_\varepsilon(x,t) = \left[\rho\left(A_\varepsilon - t - \int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right)\right]^{\frac{p}{\nu+p-1-\alpha}}.$$

Similarly to the previous proof, we prove the existence of a positive t_1 such that h_ε serves as a super-solution to equation (2) within the region $(x,t) \in [x_\varepsilon, +\infty) \times [0, t_1]$. From (42), it follows that $h_\varepsilon(x, 0) > 0$ for $x \geq x_\varepsilon$. Furthermore, we obtain the following:

$$\begin{aligned} h_\varepsilon(x,0) &= \left[\rho\left(A_\varepsilon - \int_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right)\right]^{\frac{p}{\nu+p-1-\alpha}} = \left[\rho\int_{x}^{+\infty}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right]^{\frac{p}{\nu+p-1-\alpha}} \\ &= (1+\varepsilon)u_0(x) > u_0(x), \quad x_\varepsilon \leq x < +\infty, \\ h_\varepsilon(x_\varepsilon, t) &= [\rho(A_\varepsilon - t)]^{\frac{p}{\nu+p-1-\alpha}}, \quad 0 \leq t \leq t_1. \end{aligned} \tag{44a}$$

Since

$$h_\varepsilon(x_\varepsilon, 0) = (\rho A_\varepsilon)^{\frac{p}{\nu+p-1-\alpha}} = (1+\varepsilon)u_0(x_\varepsilon) > u_0(x_\varepsilon),$$

there exists a positive value δ_1 such that

$$h_\varepsilon(x_\varepsilon, t) \geq u(x_\varepsilon, t), \quad t \in (0, \delta_1]. \tag{44b}$$

We deduce from (44) using Lemma 2 that

$$\begin{gathered} 0 \leq u(x,t) \leq h_\varepsilon(x,t), \quad (x,t) \in [x_\varepsilon, +\infty) \times [0, \delta], \\ \delta = \min\left\{\delta_1, t_1, A_\varepsilon, \frac{[(1+\varepsilon)u_0(\hat{x})]^{\frac{\nu+p-1-\alpha}{p}}}{\rho}\right\}. \end{gathered} \tag{45a}$$

Consequently, we have

$$u(x,t) = 0 \quad \text{for } x \geq \mu_\varepsilon(A_\varepsilon - t), 0 < t \leq \delta, \tag{45b}$$

where $x = \mu_\varepsilon(w)$ is the inverse function of $w = \int_{x_\varepsilon}^{x} \gamma_\varepsilon(\zeta)\mathrm{d}\zeta$. Since

$$\int\limits_{x_\varepsilon}^{x} \gamma_\varepsilon(\zeta)\mathrm{d}\zeta = A_\varepsilon - \frac{1}{\rho}[(1+\varepsilon)u_0(x)]^{\frac{\nu+p-1-\alpha}{p}},$$

it implies that

$$\mu_\varepsilon(A_\varepsilon - t) = u_0^{-1}\left[\frac{1}{1+\varepsilon}(\rho t)^{\frac{p}{\nu+p-1-\alpha}}\right].$$

Accordingly, from (45), we obtain that

$$u(x,t) = 0 \quad \text{for } x \geq u_0^{-1}\left[\frac{1}{1+\varepsilon}(\rho t)^{\frac{p}{\nu+p-1-\alpha}}\right], t \in (0, \delta].$$

In a similar manner, it is demonstrated that

$$u(x,t) = 0 \quad \text{for } x \leq -u_0^{-1}\left[\frac{1}{1+\varepsilon}(\rho t)^{\frac{p}{\nu+p-1-\alpha}}\right], t \in (0, \delta].$$

Hence, the right-hand side of inequality (15) is proved.

Due to the arbitrary choice of ρ, we conclude that the right-hand side of (15) holds. The dependence of the length of the time interval δ on ρ is evident. □

Corollary 7 is easily derived from the previously established formulas (38) and (45).

Remark 2. It is important to note that if the initial function $u_0(x)$ satisfies condition (11) but does not meet the other conditions of Theorem 3 (Theorem 6), we have the flexibility to choose functions $u_1(x)$ and $u_2(x)$ such that they satisfy all the conditions of Theorem 3 (Theorem 6) and satisfy the inequality $u_1(x) \leq u_0(x) \leq u_2(x)$. By applying Theorem 3 (Theorem 6), along with the comparison principle of Lemma 2, we establish estimates (12)–(14) with the functions u_1^{-1} and u_2^{-1} appearing in the left-hand and right-hand inequalities, respectively [3].

Remark 3. We verify that each of the initial functions listed in Table 1 of [3] satisfies the conditions of Theorem 6. Therefore, the corresponding asymptotic estimates are derived for these initial functions [3].

4 Proof of the results of problem (1) under condition (3)

Proof of Theorem 8. The proof is constructed in the same manner as in Theorem 3. Consider the function h as defined in equation (26), where $A > 0$, $\gamma(x) \in C^1(\mathbb{R}_+)$ with $0 < \gamma(x) \le \mu$ and $|\gamma'(x)| \le M$, and equation (27) holds. We choose an arbitrarily small value $\varepsilon > 0$ and define the function $g(\eta)$ as follows:

$$g(\eta) = \Phi^{-1}([(1-\varepsilon)\eta]_+), \quad -\infty < \eta \le \eta_1 \equiv \frac{\Phi(+\infty)}{1-\varepsilon} \le +\infty.$$

Suppose that $A \in (0, \eta_1]$. The function h satisfies the following conditions:

$$\begin{aligned} h(x,0) &= \Phi^{-1}\left[(1-\varepsilon)(A - \int_0^x \sigma(\zeta)\mathrm{d}\zeta)\right], \quad x \in \mathbb{R}_+,\\ h(0,t) &= \Phi^{-1}((1-\varepsilon)[A-t]_+), \quad t \ge 0. \end{aligned}$$

We have $h(x,0) > 0$ for $x > 0$ and $h(x,0)\downarrow 0$ as $x \to +\infty$ due to condition (27). Next, we demonstrate the existence of a positive t_1 such that h serves as a super-solution to equation (1) within the region $(x,t) \in [0,+\infty) \times [0,t_1]$. To do this, we compute the operator $\mathcal{L}h$. We obtain

$$\begin{aligned} \mathcal{L}h &= -g' + (p-1)\gamma^{p-2}(x)\gamma'(x)\ell(g)|g'|^{p-2}g' - \gamma^p(x)(\ell(g)|g'|^{p-2}g')' + H(g)\\ &\ge -g' - M(p-1)\mu^{p-2}\ell(g)|g'|^{p-2}g' - \mu^p(\ell(g)|g'|^{p-2}g')' + H(g)\\ &= -(1-\varepsilon)H(g) - M(1-\varepsilon)^{p-1}(p-1)\mu^{p-2}\ell(g)H(g)^{p-1} - \mu^p(1-\varepsilon)^{p-1}(\ell(g)H(g)^{p-1})' \qquad (46)\\ &= H(g)\left[\varepsilon - M(1-\varepsilon)^{p-1}(p-1)\mu^{p-2}\ell(g)H(g)^{p-2} - \mu^p(1-\varepsilon)^{p-1}\frac{(\ell(g)H(g)^{p-1})'}{H(g)}\right]. \end{aligned}$$

Suppose that

$$0 < \mu^p < \frac{\varepsilon}{d}, \qquad (47)$$

which holds by condition (18). Then, from (46), it follows that there exists $\eta_0 \in (0,\eta_1)$ such that $\mathcal{L}h > 0$ for $\eta \in (0,\eta_0]$. Hence, for every $A \in (0,\eta_0]$, the function h acts as a super-solution to equation (1) for $x \in \mathbb{R}_+$ and $t \in [0,t_1]$, where $t_1 \in (0,A]$.

Similar to the proof of Theorem 3, we choose $x_\varepsilon \ge \hat{x}$ and define

$$\begin{aligned} \gamma_\varepsilon(x) &= -\frac{1}{1-\varepsilon}[\Phi((1+\varepsilon)u_0(x))]', \quad x_\varepsilon \le x < +\infty,\\ A_\varepsilon &= \frac{1}{1-\varepsilon}\Phi((1+\varepsilon)u_0(x_\varepsilon)). \end{aligned}$$

We carefully select a value for $x_\varepsilon \ge \hat{x}$ to ensure that the argument of Φ falls within the range $(0,\eta_1)$ and that A_ε lies within the interval $(0,\eta_0]$. It is straightforward to verify whether condition (31) is satisfied. Since the function $\gamma'_\varepsilon(x)$ $(x_\varepsilon \le x < \infty)$ is bounded, equation (31) implies (32). By considering (32), we choose x_ε such that $0 < \gamma_\varepsilon(x) \le \mu < (\varepsilon/d)^{1/p}$ for $x \ge x_\varepsilon$.

We now introduce the function $h_\varepsilon(x,t)$, which is defined as follows:

$$h_\varepsilon(x,t) = \Phi^{-1}\left[(1-\varepsilon)\left(A_\varepsilon - t - \int_{x_\varepsilon}^x \sigma_\varepsilon(\zeta)\mathrm{d}\zeta\right)_+\right].$$

Using a similar approach as described previously, we show that there exists a positive t_1 such that h_ε is a super-solution of equation (1) for $(x,t) \in [x_\varepsilon,\infty) \times [0,t_1]$. Moreover, we also have

$$h_\varepsilon(x,0) = \Phi^{-1}\left[(1-\varepsilon)\int_x^\infty \gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right] = (1+\varepsilon)u_0 > u_0 \quad \text{for } x_\varepsilon \le x < \infty$$

and

$$h_\varepsilon(x_\varepsilon, t) = \Phi^{-1}((1-\varepsilon)[A_\varepsilon - t]_+) \quad \text{for } 0 \le t \le t_1.$$

Since

$$h_\varepsilon(x_\varepsilon, 0) = \Phi^{-1}((1-\varepsilon)A_\varepsilon) = (1+\varepsilon)u_0(x_\varepsilon) > u_0(x_\varepsilon),$$

there exists a positive value δ_1 such that

$$0 \le u(x_\varepsilon, t) \le h_\varepsilon(x_\varepsilon, t) \quad \text{for } 0 \le t \le \delta_1.$$

As a result, we obtain inequality (34) with $\delta = \min\left\{\delta_1, t_1, A_\varepsilon, \frac{\Phi((1+\varepsilon)u_0(\hat{x}))}{1-\varepsilon}\right\}$. Using the equation

$$\int_{x_\varepsilon}^{x} \gamma_\varepsilon(\zeta)\mathrm{d}\zeta = A_\varepsilon - \frac{1}{1-\varepsilon}\Phi((1+\varepsilon)u_0(x)),$$

we deduce that

$$\mu_\varepsilon(A_\varepsilon - t) = u_0^{-1}\left(\frac{1}{1+\varepsilon}\Phi^{-1}((1-\varepsilon)t)\right).$$

From equation (35), it follows that

$$u(x,t) = 0 \quad \text{for } x \ge u_0^{-1}\left(\frac{1}{1+\varepsilon}\Phi^{-1}((1-\varepsilon)t)\right), \quad t \in (0, \delta_2].$$

Similarly, we demonstrated that

$$u(x,t) = 0 \quad \text{for } x \le -u_0^{-1}\left(\frac{1}{1+\varepsilon}\Phi^{-1}((1-\varepsilon)t)\right), \quad t \in (0, \delta_2].$$

Therefore, the right-hand side of equation (19) is established.

To prove the left-hand side of equation (19), we consider the function h defined in equation (26), where $A > 0$, $\gamma \in C^1(\mathbb{R}_+)$, $0 < \gamma(x) \le \mu$, $|\gamma'(x)| \le M$, and equation (27) holds. We then define the function g as follows:

$$g(\eta) = \Phi^{-1}([(1+\varepsilon)\eta]_+), \quad -\infty < \eta \le \eta_2 = \frac{\Phi(+\infty)}{1+\varepsilon} \le +\infty.$$

Now, we assert that for $(x,t) \in [0,\infty) \times [0, t_1]$, there exists a $t_1 > 0$ such that h is a sub-solution of equation (1). By differentiating, we find that

$$\mathcal{L}h = -g' + (p-1)\gamma^{p-2}(x)\gamma'(x)\ell(g)|g'|^{p-2}g' - \gamma^p(x)(\ell(g)|g'|^{p-2}g')' + H(g).$$

Simplifying the expression, we obtain

$$\begin{aligned}\mathcal{L}h &\le -(1+\varepsilon)H(g) - M(p-1)\mu^{p-2}(1+\varepsilon)^{p-1}\ell(g)H(g)^{p-1} + H(g)\\ &= H(g)[-\varepsilon - M(p-1)\mu^{p-2}(1+\varepsilon)^{p-1}\ell(g)H(g)^{p-2}].\end{aligned} \tag{48}$$

Therefore, there exists $\eta_0 \in (0, \eta_2)$ such that $\mathcal{L}h \le 0$ for $\eta \in [0, \eta_0]$. Hence, for every $A \in (0, \eta_0]$, the function h acts as a sub-solution of equation (1) within the region $x \in \mathbb{R}_+$ and $t \in [0, t_1]$, where $t_1 \in (0, A]$.

Following a similar approach as in the proof of Theorem 3, we select a value $x_\varepsilon \ge \hat{x}$ and define the functions

$$\begin{aligned}\gamma_\varepsilon(x) &= -\frac{1}{1+\varepsilon}[\Phi((1-\varepsilon)u_0(x))]', \quad x_\varepsilon \le x < +\infty,\\ A_\varepsilon &= \frac{1}{1+\varepsilon}\Phi((1-\varepsilon)u_0(x_\varepsilon)).\end{aligned}$$

We choose $x_\varepsilon \geq \hat{x}$ to be large enough so that the argument of Φ falls within the range $(0, \eta_2)$ and A_ε is in the range $(0, \eta_0]$. Now, consider the function

$$h_\varepsilon(x,t) = \Phi^{-1}\left((1+\varepsilon)\left[A_\varepsilon - t - \int_{x_\varepsilon}^{x} \gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right]_+\right).$$

Similar to what is mentioned earlier, we show that there exists a positive t_1 where h_ε acts as a sub-solution for equation (1) within the region $(x,t) \in [x_\varepsilon, +\infty) \times [0, t_1]$. Moreover, we also have

$$h_\varepsilon(x,0) = \Phi^{-1}\left((1+\varepsilon)\int_{x}^{\infty} \gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right) = (1-\varepsilon)u_0 < u_0 \quad \text{for } x_\varepsilon \leq x < +\infty,$$
$$h_\varepsilon(x_\varepsilon, t) = \Phi^{-1}((1+\varepsilon)[A_\varepsilon - t]_+) \quad \text{for } 0 \leq t \leq t_1.$$

Furthermore, since

$$h_\varepsilon(x_\varepsilon, 0) = \Phi^{-1}((1+\varepsilon)A_\varepsilon) = (1-\varepsilon)u_0(x_\varepsilon) < u_0(x_\varepsilon), \tag{49}$$

there exists a positive value δ_3 such that

$$u(x_\varepsilon, t) \geq h_\varepsilon(x_\varepsilon, t), \quad 0 \leq t \leq \delta_3. \tag{50}$$

Let us set $\delta_4 = \min\left\{\delta_3, t_1, A_\varepsilon, \frac{\Phi((1-\varepsilon)u_0(\hat{x}))}{1+\varepsilon}\right\}$. Lemma 2 yields

$$u(x,t) \geq h_\varepsilon(x,t), \quad (x,t) \in [x_\varepsilon, +\infty) \times [0, \delta_4]. \tag{51}$$

Therefore, by using the properties of h_ε in a similar manner as in the argument of Theorem 3, we establish the left-hand inequality and consequently the full estimate (19) for $\delta = \min(\delta_2, \delta_4)$. □

The examples presented in Table 1 remain applicable to problem (1) under condition (3), where $H(u) = u^\alpha$ with $0 < \alpha < 1$ and the function $\ell \in C[0, +\infty) \cap C^1(0, +\infty)$ satisfies the conditions $\ell(0) = 0$ and $\ell'(0) \neq +\infty$.

Proof of Theorem 11. We should note that Theorem 8 has previously proven the inequality on the left-hand side of (24). This proof only relies on the condition $\ell(0) = 0$ (as shown in equation (48)). Therefore, we now consider the case where $\ell(0) > 0$. When $\eta = 0$, the second term within the square brackets on the right-hand side of (48) does not disappear. However, condition (23) implies that we can select a sufficiently large value $x_\varepsilon > 0$ such that

$$M = \sup_{x \geq x_\varepsilon} |\gamma_\varepsilon(x)| < \frac{\varepsilon}{(1+\varepsilon)\ell(0)}.$$

Equation (48) once again implies that there exists a positive value of η_0 for which $\mathcal{L}h < 0$ for $\eta \in (0, \eta_0]$.

To prove the inequality on the right-hand side of (24), we focus on proving this side. The proof follows a similar structure to the previous theorems. We consider the function h as defined in (26), where $A > 0$, $\gamma \in C^1(\mathbb{R}^1_+)$, $0 < \gamma(x) \leq \frac{1}{2}$, $|\gamma'(x)| \leq M$, and (27) are satisfied. We define the function $g(\eta)$ using the formula

$$g(\eta) = \psi^{-1}\left[\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}} [\eta]_+\right)\right].$$

The function g meets the conditions

$$(\ell(g)|g'|^{p-2}g')' = H(g), \quad 0 < \eta < \eta_1 \equiv \left(\frac{p}{p-1}\right)^{-\frac{1}{p}} \Psi(\psi(\infty)), \quad g(0) = 0; \quad g(\eta_1) = \infty.$$

Let $A \in (0, \eta_1)$. The function h satisfies the relations

$$h(x,0) = \psi^{-1}\left[\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(A - \int_0^x \gamma(\zeta)\mathrm{d}\zeta\right)\right)\right], \quad x \in \mathbb{R}+,$$

$$h(0,t) = \psi^{-1}\left[\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}}[A-t]_+\right)\right], \quad t \geq 0.$$

Using (27), we see that $h(x,0) > 0$ for $x \geq 0$ and $h(x,0)\downarrow 0$ as $x \to +\infty$. We now verify the existence of $t_1 > 0$ such that h serves as a super-solution of (1) within the region $(x,t) \in [0,+\infty) \times [0,t_1]$. By calculating $\mathcal{L}h$, we have

$$\begin{aligned}
\mathcal{L}h &= -g' + (p-1)\gamma^{p-2}(x)\gamma'(x)\ \ell(g)|g'|^{p-2}g' - \gamma^p(x)(\ell(g)|g'|^{p-2}g')' + H(g)\\
&\geq -g' - (p-1)\gamma^{p-2}(x)M\ \ell(g)|g'|^{p-2}g' + (1-\gamma^p(x))H(g)\\
&\geq -g' - \left(\frac{p-1}{2^{p-2}}\right)M\ \ell(g)|g'|^{p-2}g' + \left(\frac{2^p-1}{2^p}\right)H(g)\\
&= -\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\frac{\left[\int_0^{\psi(g)} H(\psi^{-1}(\mu))\mathrm{d}\mu\right]^{\frac{1}{p}}}{\ell(g)} - \left(\frac{p-1}{2^{p-2}}\right)\left(\frac{p}{p-1}\right)^{\frac{1}{p}}M\left[\int_0^{\psi(g)} H(\psi^{-1}(\mu))\mathrm{d}\mu\right]^{\frac{p-1}{p}} + \left(\frac{2^p-1}{2^p}\right)H(g) \qquad (52)\\
&= H(g)\left\{\frac{2^p-1}{2^p} - \left(\frac{p}{p-1}\right)^{\frac{1}{p}}\frac{\left[\int_0^{\psi(g)} H(\psi^{-1}(\mu))\mathrm{d}\mu\right]^{\frac{1}{p}}}{H(g)\ell(g)} - \left(\frac{p-1}{2^{p-2}}\right)\left(\frac{p}{p-1}\right)^{\frac{1}{p}}M\frac{\left[\int_0^{\psi(g)} H(\psi^{-1}(\mu))\mathrm{d}\mu\right]^{\frac{p-1}{p}}}{H(g)}\right\}.
\end{aligned}$$

By using (22) and (17), we find that

$$\begin{aligned}
\lim_{g\to 0}\frac{\left[\int_0^{\psi(g)} H(\psi^{-1}(\mu))\mathrm{d}\mu\right]^{\frac{1}{p}}}{H(g)\ell(g)} &= \left(\lim_{g\to 0}\frac{\int_0^g H(x)\ell(x)\mathrm{d}x}{(H(g)\ell(g))^p}\right)^{\frac{1}{p}} = \left(\lim_{g\to 0}\frac{H(g)\ell(g)}{p(H(g)\ell(g))^{p-1}(H(g)\ell(g))'}\right)^{\frac{1}{p}}\\
&= \left(\lim_{g\to 0}\frac{1}{p(H(g)\ell(g))^{p-2}(H(g)\ell(g))'}\right)^{\frac{1}{p}} = 0\\
\lim_{g\to 0}\frac{\left[\int_0^{\psi(g)} H(\psi^{-1}(\mu))\mathrm{d}\mu\right]^{\frac{p-1}{p}}}{H(g)} &= \left(\lim_{g\to 0}\frac{\int_0^g H(x)\ell(x)\mathrm{d}x}{(H(g))^{\frac{p}{p-1}}}\right)^{\frac{p-1}{p}}\\
&= \left(\lim_{g\to 0}\frac{(p-1)H(g)\ell(g)}{p(H(g))^{\frac{1}{p-1}}H'(g)}\right)^{\frac{p-1}{p}} = \left(\lim_{g\to 0}\frac{(p-1)\ell(g)}{p(H(g))^{\frac{2-p}{p-1}}H'(g)}\right)^{\frac{p-1}{p}} = 0.
\end{aligned}$$

From (52), we conclude that there exists $\eta_0 \in (0,\eta_1)$ such that $\mathcal{L}h > 0$ for $\eta \in (0,\eta_0]$. In other words, for any value of A in the interval $(0,\eta_0]$, the function h acts as a super-solution to equation (1) within the domain $x \in \mathbb{R}_+$ and time range $t \in [0,t_1]$, with t_1 chosen from the interval $(0,A]$.

In a manner similar to the proof of Theorem 3, we choose $x_\varepsilon \geq \hat{x}$ and define

$$\gamma_\varepsilon(x) = -\left(\frac{p}{p-1}\right)^{-\frac{1}{p}}(\Psi(\psi((1+\varepsilon)u_0(x))))', \quad x_\varepsilon \leq x < +\infty,$$

$$A_\varepsilon = -\left(\frac{p}{p-1}\right)^{-\frac{1}{p}}\Psi(\psi((1+\varepsilon)u_0(x))).$$

We ensure that x_ε is sufficiently large to guarantee that the argument of the function $\Psi(\psi(\cdot))$ falls within the range of $(0, \eta_1)$ and that A_ε lies within the range of $(0, \eta_0]$. It is straightforward to check that conditions (31) and (32) are satisfied. Specifically, we choose $x_\varepsilon > 0$ such that $0 < \gamma_\varepsilon(x) \le \frac{1}{2\varepsilon}$ for $x \ge x_\varepsilon$. Next, we specify

$$h_\varepsilon(x,t) = \psi^{-1}\left(\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left[A_\varepsilon - t - \int\limits_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta\right]_+\right)\right).$$

Similar to before, we show the existence of $t_1 > 0$ such that h_ε fulfills the role of a super-solution of (1) within the domain of $(x,t) \in [x_\varepsilon, +\infty) \times [0, t_1]$. Moreover, we also have

$$\begin{aligned} &h_\varepsilon(x,0) = (1+\varepsilon)u_0(x) > u_0(x), \quad x_\varepsilon \le x < +\infty, \\ &h_\varepsilon(x_\varepsilon, t) = \psi^{-1}\left[\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}}[A_\varepsilon - t]_+\right)\right], \quad 0 \le t \le t_1. \end{aligned}$$

Since

$$h_\varepsilon(x_\varepsilon, 0) = \psi^{-1}\left[\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}}A_\varepsilon\right)\right] = (1+\varepsilon)u_0(x_\varepsilon) > u_0(x_\varepsilon),$$

there exists a positive value δ_1 such that

$$0 \le u(x_\varepsilon, t) \le h_\varepsilon(x_\varepsilon, t), \quad 0 \le t \le \delta_1.$$

Therefore, conditions (34) and (35) are satisfied, where $\delta_2 = \min\left\{\delta_1, t_1, A_\varepsilon, \left(\frac{p}{p-1}\right)^{-\frac{1}{p}}\Psi(\psi((1+\varepsilon)u_0(\hat{x})))\right\}$. It follows from

$$\int\limits_{x_\varepsilon}^{x}\gamma_\varepsilon(\zeta)\mathrm{d}\zeta = A_\varepsilon - \left(\frac{p}{p-1}\right)^{-\frac{1}{p}}\Psi(\psi((1+\varepsilon)u_0(x)))$$

that

$$\mu_\varepsilon(A_\varepsilon - t) = u_0^{-1}\left[\frac{1}{1+\varepsilon}\psi^{-1}\left(\Psi^{-1}\left(\left(\frac{p}{p-1}\right)^{\frac{1}{p}}t\right)\right)\right].$$

Therefore, we successfully establish the right-hand inequality in the estimate (24). □

In the case of a semilinear equation where $\ell(u) \equiv 1$, condition (22) is derived from the convergence of the integral (8).

Remark 2 also applies equally to problem (1) under condition (3).

Remark 4. The estimates (12)–(16), (19)–(21), (24), and (25) hold for the solutions to the first initial-boundary-value problem in a half-strip. This is demonstrated through the construction of the proofs in Theorems 3, 6, 8, and 11 [3].

5 Conclusions

In brief, the exact local estimates provided in this study support solutions for non-linear parabolic p-Laplacian problems and address theoretical and practical queries related to non-linear diffusion equations. The shrinking of the support in Cauchy problems for non-linear parabolic p-Laplacian equations with a positive initial condition u_0 is investigated. The precise asymptotic formulas for the support border of the solutions to the Cauchy problems are proven.

Consider problem (2) under condition (3) for $\nu \geq 0$ and $0 < \alpha < 1$. Assume that u_0 is a positive even function satisfying (11) and is not restricted by the rate of decrease. Further, assume that there is a value for $\hat{x} > 0$ such that $u_0'(x) < 0$ for $x \geq \hat{x}$.

- In Theorem 3, it is shown that if $0 < \alpha < 1$ and $\nu \geq 1 - \alpha(p-1)$, then for any small value $\varepsilon > 0$, there is a positive value δ such that the functions that characterize the bound of the solution's support satisfy the local estimate (12). As a consequence, the precise asymptotic formula (13) follows. Furthermore, exact local estimates for the boundary of the solution's support to problem (2) under condition (3) are established in the proof of Theorem 3, and for all $\varepsilon > 0$ that is sufficiently small, $x_\varepsilon > 0$ and $\delta > 0$ exist such that the solution of this problem satisfies estimate (14).
- In Theorem 6, it is shown that if $0 < \alpha < 1$ and $0 \leq \nu < 1 - \alpha(p-1)$, then for each positive ε that is sufficiently small and $\rho > 0$, there exists a δ that depends on ε and ρ, such that the functions that characterize the bounds of the solution's support satisfy the local estimate (15). Based on the conditions stated in Theorem 6, we establish while proving estimate (15) that for each positive ε that is sufficiently small, there exist $x_\varepsilon > 0$ and $\delta > 0$ such that the solution of problem (2) under condition (3) satisfies estimate (16).

Consider problem (1) under condition (3). Suppose the case where both conditions (17) and (18) hold. (If $\nu \geq 1$ and $0 < \alpha < 1$, then conditions (17) and (18) are valid for equation (2).) We set

$$\Phi(u) = \int_0^u \frac{\mathrm{d}\zeta}{H(\zeta)}, \quad u \geq 0.$$

Assume that the initial function $u_0(x)$ meets the aforementioned conditions.

- In Theorem 8, it is shown that for any sufficiently small value $\varepsilon > 0$, there is a positive value δ such that the functions that characterize the bounds of the solution's support satisfy the local estimate (19). As a consequence, the precise asymptotic formula (20) follows. Furthermore, we show that, under the conditions of Theorem 8, for every $\varepsilon > 0$ that is sufficiently small, $x_\varepsilon > 0$ and $\delta > 0$ exist such that the solution to problem (1) under condition (3) meets estimate (21).

Assume that conditions (8), (17), and (22) are met. (This requirement implies for equation (2) that $0 < \alpha < 1$ and $0 \leq \nu < 1 - \alpha(p-1)$.) We set

$$\Psi(u) = \int_0^1 \left[\int_0^\zeta H(\psi^{-1}(\mu))\mathrm{d}\mu \right]^{-\frac{1}{p}} \mathrm{d}\zeta < \infty.$$

- In Theorem 8, it is shown that for all $\varepsilon > 0$ that is sufficiently small, there is a positive value δ such that the functions that characterize the bounds of the solution's support satisfy the local estimate (24). Based on the conditions stated in Theorem 6, we establish while proving estimate (24) that for each positive ε that is sufficiently small, there exist $x_\varepsilon > 0$ and $\delta > 0$ such that the solution to problem (2) under condition (3) satisfies estimate (25).

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