Research Article

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Existence of three solutions for two quasilinear Laplacian systems on graphs

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Abstract: We deal with the existence of three distinct solutions for a poly-Laplacian system with a parameter on finite graphs and a (p, q)-Laplacian system with a parameter on locally finite graphs. The main tool is an abstract critical point theorem in [G. Bonanno and S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal. **89** (2010), no. 1, 1–10]. A key point in this study is that we overcome the difficulty to prove that the Gâteaux derivative of the variational functional for poly-Laplacian operator admits a continuous inverse, which is caused by the special definition of the poly-Laplacian operator on graph and mutual coupling of two variables in system.

Keywords: critical point theorem, ploy-Laplacian system, finite graph, locally finite graph, nontrivial solution

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1 Introduction

Let $G = (V, E)$ be a graph with the vertex set V and the edges set E. If both V and E are finite set, then G is called as a finite graph. If for any $x \in V$, there are finite vertexes $y \in V$ such that $xy \in E$ (xy denotes an edge connecting x with y), then G is called as a locally finite graph. For any edge $xy \in E$ with two vertexes x, y, let ω_{xy} (> 0) denote its weight and suppose that $\omega_{xy} = \omega_{yx}$. For any $x \in V$, let $\deg(x) = \sum_{y\sim x}\omega_{xy}$, where $y \sim x$ denotes those y connecting x with $xy \in E$. Suppose that $\mu : V \to \mathbb{R}^+$ is a finite measure. Define the directional derivative of a function $u : V \to \mathbb{R}$ by

$$D_{w,y}u(x) \coloneqq \frac{1}{\sqrt{2}}(u(x) - u(y))\sqrt{\frac{w_{xy}}{\mu(x)}}. \tag{1.1}$$

Define the gradient of u as a vector

$$\nabla u(x) \coloneqq (D_{w,y}u(x))_{y\in V}, \tag{1.2}$$

which is indexed by the vertices $y \in V$. It is easy to obtain that $\nabla(u_1 + u_2) = \nabla u_1 + \nabla u_2$ and

$$\begin{aligned}\nabla u\nabla v &= (D_{w,y}u(x))_{y\in V}(D_{w,y}v(x))_{y\in V}\\ &= \sum_{y\sim x} D_{w,y}u(x)D_{w,y}v(x)\\ &= \frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)).\end{aligned}$$

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Let

$$\Gamma(u,v)(x) = \frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(u(y)-u(x))(v(y)-v(x)). \tag{1.3}$$

Then,

$$\Gamma(u,v) = \nabla u \nabla v. \tag{1.4}$$

Let $\Gamma(u) = \Gamma(u,u)$ and define the length of the gradient by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(u(y)-u(x))^2\right)^{\frac{1}{2}}. \tag{1.5}$$

The Laplacian operator of $u : V \to \mathbb{R}$ is defined by

$$\Delta u(x) \coloneqq \frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(u(y)-u(x)). \tag{1.6}$$

Let $|\nabla^m u|$ denote the length of m-order gradient of u, which is defined by

$$|\nabla^m u| = \begin{cases} |\nabla \Delta^{\frac{m-1}{2}} u|, & \text{when } m \text{ is odd,} \\ |\Delta^{\frac{m}{2}} u|, & \text{when } m \text{ is even,} \end{cases} \tag{1.7}$$

where $\nabla \Delta^{\frac{m-1}{2}} u$ is defined as in (1.2) with u replaced by $\Delta^{\frac{m-1}{2}} u$, and $\Delta^{\frac{m}{2}} u$ is defined by $\Delta^{\frac{m}{2}} u = \Delta(\Delta^{\frac{m}{2}-1} u)$ which means that u is replaced by $\Delta^{\frac{m}{2}-1} u$ in (1.6). By mathematical induction, we can obtain that

$$\Delta^{\frac{m}{2}}(u_1 + u_2) = \Delta^{\frac{m}{2}} u_1 + \Delta^{\frac{m}{2}} u_2, \quad \text{if } m \text{ is even.} \tag{1.8}$$

For any given $p > 1$, the p-Laplacian operator is defined by

$$\Delta_p u(x) \coloneqq \frac{1}{2\mu(x)}\sum_{y\sim x} (|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x))\omega_{xy}(u(y)-u(x)). \tag{1.9}$$

It is easy to see that p-Laplacian operator reduces to the Laplacian operator of u if $p = 2$.

For any function $u : V \to \mathbb{R}$, we denote

$$\int_V u(x)\,\mathrm{d}\mu = \sum_{x\in V} \mu(x)u(x). \tag{1.10}$$

For any given real number $r \geq 1$, we define

$$L^r(V) = \left\{ u : V \to \mathbb{R} \,\middle|\, \int_V |u(x)|^r \mathrm{d}\mu < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^r(V)} = \left(\int_V |u(x)|^r \mathrm{d}\mu\right)^{\frac{1}{r}}. \tag{1.11}$$

In the distributional sense, $\Delta_p u$ can be written as follows. For any $u \in C_c(V)$,

$$\int_V (\Delta_p u) v\,\mathrm{d}\mu = -\int_V |\nabla u|^{p-2}\Gamma(u,v)\,\mathrm{d}\mu, \tag{1.12}$$

where $C_c(V)$ is the set of all real functions with compact support. Furthermore, a more general operator can be introduced, denoted by $\pounds_{m,p}$, as follows: for any function $\phi : V \to \mathbb{R}$,

$$\int_V (\pounds_{m,p}u)\phi \mathrm{d}\mu = \begin{cases} \int_V |\nabla^m u|^{p-2}\Gamma(\Delta^{\frac{m-1}{2}}u, \Delta^{\frac{m-1}{2}}\phi)\mathrm{d}\mu, & \text{if } m \text{ is odd}, \\ \int_V |\nabla^m u|^{p-2}\Delta^{\frac{m}{2}}u\Delta^{\frac{m}{2}}\phi \mathrm{d}\mu, & \text{if } m \text{ is even}, \end{cases} \tag{1.13}$$

where $m \geq 1$ are integers and $p > 1$. The operator $\pounds_{m,p}$ is called as the poly-Laplacian operator of u if $p = 2$, and obviously, the operator $\pounds_{m,p}$ reduces to the p-Laplacian operator if $m = 1$. The results above are taken from [1,2].

In this study, we study the existence of three solutions for the following poly-Laplacian system:

$$\begin{cases} \pounds_{m_1,p}u + h_1(x)|u|^{p-2}u = \lambda F_u(x, u, v), & x \in V, \\ \pounds_{m_2,q}v + h_2(x)|v|^{q-2}v = \lambda F_v(x, u, v), & x \in V, \end{cases} \tag{1.14}$$

where $G = (V, E)$ is a finite graph, $m_i \geq 1$ are integers, $h_i : V \to \mathbb{R}$, $i = 1, 2$, $p, q > 1$, $\lambda > 0$, $F : V \times \mathbb{R}^2 \to \mathbb{R}$, and $\pounds_{m_1,p}$ and $\pounds_{m_2,q}$ are defined by (1.13).

Moreover, if $G = (V, E)$ is a locally finite graph, we consider the existence of three solutions for the following (p, q)-Laplacian system:

$$\begin{cases} -\Delta_p u + h_1(x)|u|^{p-2}u = \lambda F_u(x, u, v), & x \in V, \\ -\Delta_q v + h_2(x)|v|^{q-2}v = \lambda F_v(x, u, v), & x \in V, \end{cases} \tag{1.15}$$

where $-\Delta_p$ and $-\Delta_q$ are defined by (1.9) with $p \geq 2$ and $q \geq 2$, $F : V \times \mathbb{R}^2 \to \mathbb{R}$, $h_i : V \to \mathbb{R}$, $i = 1, 2$, and $\lambda > 0$.

In recent years, the study of equations on graphs attracted much attention. We refer readers to [2–8] and references therein. Grigior'yan et al. [2] investigated the following poly-Laplacian equation on graph $G = (V, E)$:

$$\pounds_{m,p}u + h(x)|u|^{p-2}u = \lambda f(x, u) \quad \text{in } x \in V, \tag{1.16}$$

where $p > 1$, $h : V \to \mathbb{R}$, and $f : V \times \mathbb{R} \to \mathbb{R}$. They considered the case that the graph $G = (V, E)$ is a locally finite graph, $h(x) \equiv 0$ and equation (1.16) satisfies the Dirichlet boundary condition, and the case that the graph $G = (V, E)$ is a finite graph. They established some existence results of a positive solution for equation (1.16) with $\lambda = 1$ by mountain pass theorem. Grigior'yan et al. [3] studied (1.16) with $m = 1$ and $p = 2$, where V is a locally finite graph. They obtained two existence results of positive solutions for equation (1.16) by mountain pass theorem. In [4], when $m = 1$ and $p = 2$, by applying a three critical point theorem from [9], Imbesi et al. established some existence results of at least two solutions for equation (1.16) when the parameter λ locates at some concrete range. Pinamonti and Stefani [5] investigated (1.16) with $h \equiv 0$ and Dirichlet boundary value condition. They established some existence and uniqueness results. Yu et al. [6] studied system (1.14) with $\lambda = 1$, $p = q$, and $F(x, u, v)$ satisfying asymptotically-p-linear growth at infinity with respect to (u, v). By using the mountain pass theorem, they obtained that system (1.14) has a nontrivial solution and they also presented some corresponding results for equation (1.16) with $\lambda = 1$. Yang and Zhang [7] investigated system (1.15) with perturbations and two parameters λ_1 and λ_2. When F possesses sub-(p, q) growth on (u, v), an existence result of one nontrivial solution was established by Ekeland's variational principle, and when F possesses super-(p, q) growth on (u, v), one solution of positive energy and one solution of negative energy were obtained by mountain pass theorem and Ekeland's variational principle, respectively. Zhang et al. [10] considered system (1.14) with $\lambda = 1$. They established an existence result and a multiplicity result of nontrivial solutions when F satisfies the super-(p, q) growth conditions on (u, v) via mountain pass theorem and symmetric mountain pass theorem, respectively.

In the present study, our work are mainly motivated by [10–12]. Bonanno and Marano [11] established an existence result of three critical points for $f_\lambda := \Phi - \lambda\Psi$ with $\lambda \in \mathbb{R}$, and obtained a well-determined large interval of parameters for which f_λ possesses at least three critical points under weaker regularity and compactness conditions. Furthermore, by using the three critical points theorem, Bonanno and Bisci [12] obtained that a non-autonomous elliptic Dirichlet problem possesses at least three weak solutions.

Based on the works in [10–12], the motivation of our work is to consider whether the three critical points theorem due to Bonanno and Marano in [11] can be applied to systems (1.14) and (1.15). The main difficulty of such problem is to prove that the Gâteaux derivative of the variational functional Φ admits a continuous inverse. The main reason to cause this difficulty is that the special definition of $\pounds_{m,p}$ and mutual coupling of u and v in systems (1.14) and (1.15) make proving the uniformly monotone of Φ' difficult. To overcome this difficulty, we discuss the case that m is even and the case that m is odd, respectively, and sufficiently use the formulas (1.4) and (1.8), and in order to deal with mutual coupling of u and v, we divide into four cases about the norms of u and v and then, make some careful arguments in Lemma 3.5.

We call that (u, v) is a non-trivial solution of system (1.14) (or (1.15)) if (u, v) satisfies (1.14) (or (1.15)) and $(u, v) \neq (0, 0)$. Next, we state our results.

(I) For the poly-Laplacian system on finite graph

Theorem 1.1. *Let $G = (V, E)$ be a finite graph. Assume that the following conditions hold:*

(H) $h_i(x) > 0$ *for all* $x \in V$, $i = 1, 2$;

(F_0) $F(x, s, t)$ *is continuously differentiable in* $(s, t) \in \mathbb{R}^2$ *for all* $x \in V$;

(F_1) $\int_V F(x, 0, 0)\mathrm{d}\mu = 0$;

(F_2) *there exist two constants* $\alpha \in [0, p)$, $\beta \in [0, q)$ *and functions* $f_i, g : V \to \mathbb{R}$, $i = 1, 2$, *such that*

$$F(x, s, t) \leq f_1(x)|s|^\alpha + f_2(x)|t|^\beta + g(x)$$

for all $(x, s, t) \in V \times \mathbb{R} \times \mathbb{R}$;

(F_3) *there are positive constants* γ_1, γ_2, δ_1, *and* δ_2 *with* $\delta_i > \gamma_i \kappa_i$, $i = 1, 2$, *such that*

$$\Lambda_1 := \frac{1}{\gamma_1^p + \gamma_2^q} \max_{x \in V, |s| \leq \frac{(p\gamma_1^p + p\gamma_2^q)^{\frac{1}{p}}}{h_{1,\min}^{1/p}\mu_{\min}^{1/p}}, |t| \leq \frac{(q\gamma_1^p + q\gamma_2^q)^{\frac{1}{q}}}{h_{2,\min}^{1/q}\mu_{\min}^{1/q}}} F(x, s, t)|V|$$
$$< \frac{\inf_{x \in V} F(x, \delta_1, \delta_2)|V|}{\frac{\delta_1^p}{p}\int_V h_1(x)\mathrm{d}\mu + \frac{\delta_2^q}{q}\int_V h_2(x)\mathrm{d}\mu} := \Lambda_2,$$

where $|V| = \sum_{x \in V}\mu(x)$, $h_{i,\min} = \min_{x \in V} h_i(x)$, $i = 1, 2$, $\mu_{\min} = \min_{x \in V}\mu(x)$, *and*

$$\kappa_1 = \left(\frac{1}{p}\int_V h_1(x)\mathrm{d}\mu\right)^{-\frac{1}{p}}, \quad \kappa_2 = \left(\frac{1}{q}\int_V h_2(x)\mathrm{d}\mu\right)^{-\frac{1}{q}}.$$

Then, for each parameter λ *belonging to* $(\Lambda_2^{-1}, \Lambda_1^{-1})$, *system* (1.14) *has at least three distinct solutions.*

(II) For the (p,q)-Laplacian system on locally finite graph

Theorem 1.2. *Let $G = (V, E)$ be a locally finite graph. Assume that the following conditions hold:*

(M) *there exists a* $\mu_0 > 0$ *such that* $\mu(x) \geq \mu_0$ *for all* $x \in V$;

(H_1) *there exists a constant* $h_0 > 0$ *such that* $h_i(x) \geq h_0 > 0$ *for all* $x \in V$, $i = 1, 2$;

$(F_0)'$ $F(x, s, t)$ *is continuously differentiable in* $(s, t) \in \mathbb{R}^2$ *for all* $x \in V$, *and there exists a function* $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and a function* $b : V \to \mathbb{R}^+$ *with* $b \in L^1(V)$ *such that*

$$|F_s(x, s, t)| \leq a(|(s, t)|)b(x), \ |F_t(x, s, t)| \leq a(|(s, t)|)b(x), \ |F(x, s, t)| \leq a(|(s, t)|)b(x),$$

for all $x \in V$ *and all* $(s, t) \in \mathbb{R}^2$;

$(F_1)'$ $\int_V F(x, 0, 0)\mathrm{d}\mu = 0$ *and there exists a* $x_0 \in V$ *such that* $F(x_0, 0, 0) = 0$;

$(F_2)'$ *there exist two constants* $\alpha \in [0, p)$, $\beta \in [0, q)$ *and functions* $f_i, g : V \to \mathbb{R}$, $i = 1, 2$, *with* $f_i \in L^\infty(V)$, $i = 1, 2$ *and* $g \in L^1(V)$, *such that*

$$F(x, s, t) \leq f_1(x)|s|^\alpha + f_2(x)|t|^\beta + g(x)$$

for all $(x, s, t) \in V \times \mathbb{R} \times \mathbb{R}$;

$(F_3)'$ *there are positive constants* γ_1, γ_2, δ_1, *and* δ_2 *with* $\delta_i > \gamma_i\kappa_i$, $i = 1, 2$, *such that*

$$\Theta_1 \coloneqq \frac{1}{\gamma_1^p + \gamma_2^q} \max_{|(s,t)| \le \frac{1}{h_0^{1/p}\mu_0^{1/p}}(p\gamma_1^p + p\gamma_2^q)^{\frac{1}{p}} + \frac{1}{h_0^{1/q}\mu_0^{1/q}}(q\gamma_1^p + q\gamma_2^q)^{\frac{1}{q}}} a(|(s,t)|)\int_V b(x)\mathrm{d}\mu$$
$$< \frac{F(x_0, \delta_1, \delta_2)}{\frac{\delta_1^p M_1}{p} + \frac{\delta_2^q M_2}{q}} \coloneqq \Theta_2,$$

where $\kappa_1 = \left(\frac{M_1}{p}\right)^{-\frac{1}{p}}$, $\kappa_2 = \left(\frac{M_2}{q}\right)^{-\frac{1}{q}}$, *and*

$$M_1 = \left(\frac{\deg(x_0)}{2\mu(x_0)}\right)^{\frac{p}{2}}\mu(x_0) + h_1(x_0)\mu(x_0) + \sum_{y\sim x_0}\left(\frac{w_{x_0y}}{2\mu(y)}\right)^{\frac{p}{2}}\mu(y),$$
$$M_2 = \left(\frac{\deg(x_0)}{2\mu(x_0)}\right)^{\frac{q}{2}}\mu(x_0) + h_2(x_0)\mu(x_0) + \sum_{y\sim x_0}\left(\frac{w_{x_0y}}{2\mu(y)}\right)^{\frac{q}{2}}\mu(y).$$

Then, for each parameter λ *belonging to* $(\Theta_2^{-1}, \Theta_1^{-1})$, *system* (1.15) *has at least three distinct solutions.*

Remark 1.3. In Theorems 1.1 and 1.2, all three solutions are nontrivial solutions if we furthermore assume that $F_s(x, 0, 0) \neq 0$ or $F_t(x, 0, 0) \neq 0$ for some $x \in V$.

2 Preliminaries

In this section, we recall the Sobolev space on graph and some embedding relationships [2,7,10]. We also recall an abstract critical point theorem in [11], which is the main tool to prove our main results.

Let $G = (V, E)$ be a graph. For any given integer $m \geq 1$ and any given real number $l > 1$, we define

$$W^{m,l}(V) = \left\{u : V \to \mathbb{R} \;\middle|\; \int_V (|\nabla^m u(x)|^l + h(x)|u(x)|^l)\mathrm{d}\mu < \infty\right\}$$

endowed with the norm

$$||u||_{W^{m,l}(V)} = \left(\int_V (|\nabla^m u(x)|^l + h(x)|u(x)|^l)\mathrm{d}\mu\right)^{\frac{1}{l}},$$

where $h(x) > 0$ for all $x \in V$. If V is a finite graph, then $W^{m,l}(V)$ is of finite dimension.

Lemma 2.1. [2,10] *Let* $G = (V, E)$ *be a finite graph. For all* $\psi \in W^{m,l}(V)$, *there is*

$$||\psi||_\infty \leq d_l||\psi||_{W^{m,l}(V)},$$

where $||\psi||_\infty = \max_{x\in V}|\psi(x)|$ *and* $d_l = \left(\frac{1}{\mu_{\min}h_{\min}}\right)^{\frac{1}{l}}$.

Lemma 2.2. [2,10] *Let* $G = (V, E)$ *be a finite graph. Then,* $W^{m,l}(V) \hookrightarrow L^r(V)$ *for all* $1 \leq r \leq +\infty$. *Particularly, if* $1 < r < +\infty$, *then for all* $\psi \in W^{m,l}(V)$,

$$||\psi||_{L^r(V)} \leq K_{l,r}||\psi||_{W^{m,l}(V)},$$

where

$$K_{l,r} = \frac{\left(\sum_{x\in V}\mu(x)\right)^{\frac{1}{r}}}{\mu_{\min}^{\frac{1}{l}} h_{\min}^{\frac{1}{l}}}.$$

Lemma 2.3. [7] *Let $G = (V, E)$ be a locally finite graph. If $\mu(x) \geq \mu_0$ and (H_1) holds, then $W_h^{1,l}(V)$ is continuously embedded into $L^r(V)$ for all $1 < l \leq r \leq \infty$, and the following inequalities hold:*

$$\|u\|_\infty \leq \frac{1}{h_0^{1/l}\mu_0^{1/l}}\|u\|_{W^{1,l}(V)}$$

and

$$\|u\|_{L^r(V)} \leq \mu_0^{\frac{l-r}{lr}} h_0^{-\frac{1}{l}}\|u\|_{W^{1,l}(V)} \quad \textit{for all } \; l \leq r < \infty.$$

Lemma 2.4. [11] *Let W be a real reflexive Banach space, $\Phi : W \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on W^*, $\Psi : W \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:

(a_1) $\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a_2) *for each $\lambda \in \Lambda_r := \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right)$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\varphi_\lambda := \Phi - \lambda\Psi$ has at least three distinct critical points in W.

3 Proofs for the poly-Laplacian system (1.14)

Let $G = (V, E)$ be a finite graph. In order to investigate the poly-Laplacian system (1.14), we work in the space $W := W^{m_1,p}(V) \times W^{m_2,q}(V)$ with the norm endowed with $\|(u, v)\| = \|u\|_{W^{m_1,p}(V)} + \|v\|_{W^{m_2,q}(V)}$. Then, $(W, \|\cdot\|)$ is a Banach space of finite dimension. Consider the functional $\varphi : W \to \mathbb{R}$ as

$$\varphi(u, v) = \frac{1}{p}\int_V(|\nabla^{m_1}u|^p + h_1(x)|u|^p)\mathrm{d}\mu + \frac{1}{q}\int_V(|\nabla^{m_2}v|^q + h_2(x)|v|^q)\mathrm{d}\mu - \lambda\int_V F(x, u, v)\mathrm{d}\mu. \tag{3.1}$$

Then, under the assumptions of Theorem 1.1, $\varphi \in C^1(W, \mathbb{R})$ and

$$\begin{aligned}\langle\varphi'(u, v), (\phi_1, \phi_2)\rangle = &\int_V[\pounds_{m_1,p}u\phi_1 + h_1(x)|u|^{p-2}u\phi_1 - \lambda F_u(x, u, v)\phi_1]\mathrm{d}\mu \\ &+ \int_V[\pounds_{m_2,q}v\phi_2 + h_2(x)|v|^{q-2}v\phi_2 - \lambda F_v(x, u, v)\phi_2]\mathrm{d}\mu\end{aligned} \tag{3.2}$$

for any $(u, v), (\phi_1, \phi_2) \in W$. In order to apply Lemma 2.4, we will use the functionals $\Phi : W \to \mathbb{R}$ and $\Psi : W \to \mathbb{R}$ defined by setting

$$\begin{aligned}\Phi(u, v) &= \frac{1}{p}\int_V(|\nabla^{m_1}u|^p + h_1(x)|u|^p)\mathrm{d}\mu + \frac{1}{q}\int_V(|\nabla^{m_2}v|^q + h_2(x)|v|^q)\mathrm{d}\mu \\ &= \frac{1}{p}\|u\|^p_{W^{m_1,p}(V)} + \frac{1}{q}\|v\|^q_{W^{m_2,q}(V)}\end{aligned}$$

and

$$\Psi(u, v) = \int_V F(x, u, v)\mathrm{d}\mu.$$

Then, $\varphi(u, v) = \Phi - \lambda\Psi$. Moreover, it is easy to see that $(u, v) \in W$ is a critical point of φ if and only if

$$\int_V (\pounds_{m_1,p}u + h_1(x)|u|^{p-2}u - \lambda F_u(x, u, v))\phi_1 \mathrm{d}\mu = 0$$

and

$$\int_V (\pounds_{m_2,q}v + h_2(x)|v|^{q-2}v - \lambda F_v(x, u, v))\phi_2 \mathrm{d}\mu = 0$$

for all $(\phi_1, \phi_2) \in W$. By the arbitrariness of ϕ_1 and ϕ_2, we conclude that

$$\pounds_{m_1,p}u + h_1(x)|u|^{p-2}u = \lambda F_u(x, u, v),$$
$$\pounds_{m_2,q}v + h_2(x)|v|^{q-2}v = \lambda F_v(x, u, v).$$

Thus, the problem of finding the solutions for system (1.14) is reduced to seek the critical points of functional φ on W.

Lemma 3.1. *Assume that* (F_0) *holds. Then, for any given* $r > 0$, *the following inequality holds*:

$$\frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,r]}\Psi(u, v)}{r} \le \frac{1}{r} \max_{x\in V,|s|\le\frac{(pr)^{\frac{1}{p}}}{h_{\min}^{1/p}\mu_{\min}^{1/p}},|t|\le\frac{(qr)^{\frac{1}{q}}}{h_{\min}^{1/q}\mu_{\min}^{1/q}}} F(x, u(x), v(x))|V|.$$

Proof. By (F_0), we have

$$\begin{aligned}\Psi(u, v) &= \int_V F(x, u(x), v(x))\mathrm{d}\mu \\ &= \sum_V F(x, u(x), v(x))\mathrm{d}\mu \\ &\le \max_{x\in V,|s|\le\|u\|_\infty,|t|\le\|v\|_\infty} F(x, s, t)|V|\end{aligned}$$

for every $(u, v) \in W$. Furthermore, for all $(u, v) \in W$ with $\Phi(u, v) \le r$, by Lemma 2.1, we obtain

$$\|u\|_\infty \le \frac{1}{h_{1,\min}^{1/p}\mu_{\min}^{1/p}}\|u\|_{W^{m_1,p}(V)} \le \frac{1}{h_{1,\min}^{1/p}\mu_{\min}^{1/p}}(pr)^{\frac{1}{p}}, \quad \|v\|_\infty \le \frac{1}{h_{2,\min}^{1/q}\mu_{\min}^{1/q}}\|v\|_{W^{m_2,q}(V)} \le \frac{1}{h_{2,\min}^{1/q}\mu_{\min}^{1/q}}(qr)^{\frac{1}{q}}.$$

Hence, we obtain

$$\sup_{(u,v)\in\Phi^{-1}(-\infty,r]} \Psi(u, v) \le \max_{x\in V,|s|\le\frac{(pr)^{\frac{1}{p}}}{h_{1,\min}^{1/p}\mu_{\min}^{1/p}},|t|\le\frac{(qr)^{\frac{1}{q}}}{h_{2,\min}^{1/q}\mu_{\min}^{1/q}}} F(x, s, t)|V|. \tag{3.3}$$

Then, the proof is completed by multiplying $\frac{1}{r}$ on both sides of (3.3). □

Lemma 3.2. *Assume that* (F_0) *and* (F_3) *hold. Then, there exists* $(u_{\delta_1}, v_{\delta_2}) \in W$ *such that*

$$\frac{\sup_{(u,v)\in\Phi^{-1}(-\infty,\gamma_1^p+\gamma_2^q]}\Psi(u, v)}{\gamma_1^p + \gamma_2^q} < \frac{\Psi(u_{\delta_1}, v_{\delta_2})}{\Phi(u_{\delta_1}, v_{\delta_2})}.$$

Proof. Let

$$u_{\delta_1}(x) = \delta_1, \quad v_{\delta_2}(x) = \delta_2, \quad \forall x \in V,$$

where $\delta_i, i = 1, 2$ are given in (F_3). It is easy to verify that $(u_{\delta_1}, v_{\delta_2}) \in W$, $|\nabla^{m_1} u_{\delta_1}| = 0$, and $|\nabla^{m_1} v_{\delta_1}| = 0$ for all $m_i \geq 1, i = 1, 2$. Then,

$$\begin{aligned}\Phi(u_{\delta_1}, v_{\delta_2}) &= \frac{1}{p}\int_V (|\nabla^{m_1} u_{\delta_1}|^p + h_1(x)|u_{\delta_1}|^p)\mathrm{d}\mu + \frac{1}{q}\int_V (|\nabla^{m_2} v_{\delta_2}|^q + h_2(x)|v_{\delta_2}|^q)\mathrm{d}\mu \\ &= \frac{\delta_1^p}{p}\int_V h_1(x)\mathrm{d}\mu + \frac{\delta_2^q}{q}\int_V h_2(x)\mathrm{d}\mu.\end{aligned} \tag{3.4}$$

Note that $\delta_i > \gamma_i \kappa_i, i = 1, 2$, where

$$\kappa_1 = \left(\frac{1}{p}\int_V h_1(x)\mathrm{d}\mu\right)^{-\frac{1}{p}}, \quad \kappa_2 = \left(\frac{1}{q}\int_V h_2(x)\mathrm{d}\mu\right)^{-\frac{1}{q}}.$$

Then, (3.4) implies that $\Phi(u_{\delta_1}, v_{\delta_2}) \geq \gamma_1^p + \gamma_2^q$. Moreover,

$$\Psi(u_{\delta_1}, v_{\delta_2}) = \int_V F(x, u_{\delta_1}, v_{\delta_2})\mathrm{d}\mu = \int_V F(x, \delta_1, \delta_2)\mathrm{d}\mu \geq \inf_{x \in V} F(x, \delta_1, \delta_2)|V|. \tag{3.5}$$

Hence, by (3.4) and (3.5), we obtain

$$\frac{\Psi(u_{\delta_1}, v_{\delta_2})}{\Phi(u_{\delta_1}, v_{\delta_2})} \geq \frac{\inf_{x \in V} F(x, \delta_1, \delta_2)|V|}{\frac{\delta_1^p}{p}\int_V h_1(x)\mathrm{d}\mu + \frac{\delta_2^q}{q}\int_V h_2(x)\mathrm{d}\mu}. \tag{3.6}$$

In view of Lemma 3.1, (3.6), and (F_3), we obtain

$$\begin{aligned}&\frac{\sup_{(u,v) \in \Phi^{-1}(-\infty, \gamma_1^p + \gamma_2^q]} \Psi(u, v)}{\gamma_1^p + \gamma_2^q} \\ &\leq \frac{1}{\gamma_1^p + \gamma_2^q} \max_{x \in V, |s| \leq \frac{1}{h_{\min}^{1/p}\mu_{\min}^{1/p}}(p(\gamma_1^p + \gamma_2^q))^{\frac{1}{p}}, |t| \leq \frac{1}{h_{\min}^{1/q}\mu_{\min}^{1/q}}(q(\gamma_1^p + \gamma_2^q))^{\frac{1}{q}}} F(x, s, t)|V| \\ &< \frac{\inf_{x \in V} F(x, \delta_1, \delta_2)|V|}{\frac{\delta_1^p}{p}\int_V h_1(x)\mathrm{d}\mu + \frac{\delta_2^q}{q}\int_V h_2(x)\mathrm{d}\mu} \\ &\leq \frac{\Psi(u_{\delta_1}, v_{\delta_2})}{\Phi(u_{\delta_1}, v_{\delta_2})}.\end{aligned}$$

The proof is complete. □

Lemma 3.3. *Assume that* (F_2) *holds. Then, for each* $\lambda \in (0, +\infty)$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Proof. By (F_2), we have

$$\begin{aligned}\varphi(u, v) &= \frac{1}{p}\|u\|_{W^{m_1,p}(V)}^p + \frac{1}{q}\|v\|_{W^{m_2,q}(V)}^q - \lambda\int_V F(x, u, v)\mathrm{d}\mu \\ &\geq \frac{1}{p}\|u\|_{W^{m_1,p}(V)}^p + \frac{1}{q}\|v\|_{W^{m_2,q}(V)}^q - \lambda\|f_1\|_\infty \|u\|_\infty^\alpha - \lambda\|f_2\|_\infty \|v\|_\infty^\beta - \lambda\int_V g(x)\mathrm{d}\mu \\ &\geq \frac{1}{p}\|u\|_{W^{m_1,p}(V)}^p + \frac{1}{q}\|v\|_{W^{m_2,q}(V)}^q - \lambda\|f_1\|_\infty d_p^\alpha \|u\|_{W^{m_1,p}(V)}^\alpha - \lambda\|f_2\|_\infty d_q^\beta \|v\|_{W^{m_2,q}(V)}^\beta - \lambda\int_V g(x)\mathrm{d}\mu.\end{aligned}$$

Note that $\alpha \in [0, p)$ and $\beta \in [0, q)$. Therefore, $\varphi(u, v)$ is a coercive functional for every $\lambda \in (0, +\infty)$. □

Lemma 3.4. *The Gâteaux derivative of* Φ *admits a continuous inverse on* W^**, where* W^* *is the dual space of* W.

Proof. First, we prove that Φ' is uniformly monotone. Via (2.2) of [8], there exists a positive constant c_p such that

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq c_p\,|x - y|^p, \quad \text{for all } x, y \in \mathbb{R}^N, \tag{3.7}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$. Note that

$$\begin{aligned}
&\langle \Phi'(u_1, v_1) - \Phi'(u_2, v_2), (u_1 - u_2, v_1 - v_2)\rangle \\
&= \int_V [\pounds_{m_1,p}u_1(u_1 - u_2) + h_1(x)|u_1|^{p-2}u_1(u_1 - u_2)]\mathrm{d}\mu + \int_V [\pounds_{m_2,q}v_1(v_1 - v_2) + h_2(x)|v_1|^{q-2}v_1(v_1 - v_2)]\mathrm{d}\mu \\
&\quad - \int_V [\pounds_{m_1,p}u_2(u_1 - u_2) + h_1(x)|u_2|^{p-2}u_2(u_1 - u_2)]\mathrm{d}\mu - \int_V [\pounds_{m_2,q}v_2(v_1 - v_2) + h_2(x)|v_2|^{q-2}v_2(v_1 - v_2)]\mathrm{d}\mu.
\end{aligned}$$

First, we prove that

$$\begin{aligned}
I &:= \int_V [\pounds_{m_1,p}u_1(u_1 - u_2) + h_1(x)|u_1|^{p-2}u_1(u_1 - u_2) - \pounds_{m_1,p}u_2(u_1 - u_2) + h_1(x)|u_2|^{p-2}u_2(u_1 - u_2)]\mathrm{d}\mu \\
&\geq c_p\|u_1 - u_2\|^p_{W^{m_1,p}(V)}.
\end{aligned}$$

When m_1 is odd, by (1.13), (1.4), (1.7), (1.8), and (3.7), we have

$$\begin{aligned}
&\int_V [\pounds_{m_1,p}u_1(u_1 - u_2) - \pounds_{m_1,p}u_2(u_1 - u_2)]\mathrm{d}\mu \\
&= \int_V \left[|\nabla^{m_1}u_1|^{p-2}\Gamma\left(\Delta^{\frac{m_1-1}{2}}u_1, \Delta^{\frac{m_1-1}{2}}(u_1 - u_2)\right) - |\nabla^{m_1}u_2|^{p-2}\Gamma\left(\Delta^{\frac{m_1-1}{2}}u_2, \Delta^{\frac{m_1-1}{2}}(u_1 - u_2)\right)\right]\mathrm{d}\mu \\
&= \int_V \left[|\nabla^{m_1}u_1|^{p-2}\nabla\Delta^{\frac{m_1-1}{2}}u_1 \cdot \nabla\Delta^{\frac{m_1-1}{2}}(u_1 - u_2) - |\nabla^{m_1}u_2|^{p-2}\nabla\Delta^{\frac{m_1-1}{2}}u_2 \cdot \nabla\Delta^{\frac{m_1-1}{2}}(u_1 - u_2)\right]\mathrm{d}\mu \\
&= \int_V \left[|\nabla\Delta^{\frac{m_1-1}{2}}u_1|^{p-2}\nabla\Delta^{\frac{m_1-1}{2}}u_1 \cdot \nabla\Delta^{\frac{m_1-1}{2}}(u_1 - u_2) - |\nabla\Delta^{\frac{m_1-1}{2}}u_2|^{p-2}\nabla\Delta^{\frac{m_1-1}{2}}u_2 \cdot \nabla\Delta^{\frac{m_1-1}{2}}(u_1 - u_2)\right]\mathrm{d}\mu \\
&= \int_V \left[\nabla\Delta^{\frac{m_1-1}{2}}(u_1 - u_2) \cdot (|\nabla\Delta^{\frac{m_1-1}{2}}u_1|^{p-2}\nabla\Delta^{\frac{m_1-1}{2}}u_1 - |\nabla\Delta^{\frac{m_1-1}{2}}u_2|^{p-2}\nabla\Delta^{\frac{m_1-1}{2}}u_2)\right]\mathrm{d}\mu \\
&\geq \int_V c_p\left|\nabla\Delta^{\frac{m_1-1}{2}}(u_1 - u_2)\right|^p\mathrm{d}\mu = \int_V c_p\,|\nabla^{m_1}(u_1 - u_2)|^p\mathrm{d}\mu.
\end{aligned}$$

When m_1 is even, by (1.13), (1.7), (1.8), and (3.7), we also have

$$\begin{aligned}
&\int_V [\pounds_{m_1,p}u_1(u_1 - u_2) - \pounds_{m_1,p}u_2(u_1 - u_2)]\mathrm{d}\mu \\
&= \int_V \left[|\nabla^{m_1}u_1|^{p-2}\Delta^{\frac{m_1}{2}}u_1\Delta^{\frac{m_1}{2}}(u_1 - u_2) - |\nabla^{m_1}u_2|^{p-2}\Delta^{\frac{m_1}{2}}u_2\Delta^{\frac{m_1}{2}}(u_1 - u_2)\right]\mathrm{d}\mu \\
&= \int_V \left[\left|\Delta^{\frac{m_1}{2}}u_1\right|^{p-2}\Delta^{\frac{m_1}{2}}u_1\Delta^{\frac{m_1}{2}}(u_1 - u_2) - \left|\Delta^{\frac{m_1}{2}}u_2\right|^{p-2}\Delta^{\frac{m_1}{2}}u_2\Delta^{\frac{m_1}{2}}(u_1 - u_2)\right]\mathrm{d}\mu \\
&= \int_V \left[\Delta^{\frac{m_1}{2}}(u_1 - u_2)\left(\left|\Delta^{\frac{m_1}{2}}u_1\right|^{p-2}\Delta^{\frac{m_1}{2}}u_1 - \left|\Delta^{\frac{m_1}{2}}u_2\right|^{p-2}\Delta^{\frac{m_1}{2}}u_2\right)\right]\mathrm{d}\mu \\
&\geq \int_V c_p\left|\Delta^{\frac{m_1}{2}}(u_1 - u_2)\right|^p\mathrm{d}\mu = \int_V c_p\,|\nabla^{m_1}(u_1 - u_2)|^p\mathrm{d}\mu.
\end{aligned}$$

Thus, for all positive integers m, we obtain

$$\int_V [\pounds_{m_1,p}u_1(u_1 - u_2) - \pounds_{m_1,p}u_2(u_1 - u_2)]\mathrm{d}\mu \geq \int_V c_p\,|\nabla^{m_1}(u_1 - u_2)|^p\mathrm{d}\mu. \tag{3.8}$$

By (3.7), we have

$$\int_V [h_1(x)|u_1|^{p-2}u_1(u_1-u_2) - h_1(x)|u_2|^{p-2}u_2(u_1-u_2)]\mathrm{d}\mu \ge \int_V h_1(x)c_p\,|u_1-u_2|^p\mathrm{d}\mu. \tag{3.9}$$

So, by (3.8) and (3.9), we obtain that

$$I \ge c_p\|u_1-u_2\|^p_{W^{m_1,p}(V)}.$$

Similarly, we can prove that there exists a positive constant c_q such that

$$\begin{aligned}II &:= \int_V [\pounds_{m_2,q}v_1(v_1-v_2) + h_2(x)|v_1|^{q-2}v_1(v_1-v_2) - \pounds_{m_2,q}v_2(v_1-v_2) + h_2(x)|v_2|^{q-2}v_2(v_1-v_2)]\mathrm{d}\mu\\ &\ge c_q\|v_1-v_2\|^q_{W^{m_2,q}(V)}.\end{aligned}$$

Hence,

$$\langle\Phi'(u_1,v_1)-\Phi'(u_2,v_2),(u_1-u_2,v_1-v_2)\rangle \ge c_p\|u_1-u_2\|^p_{W^{m_1,p}(V)} + c_q\|v_1-v_2\|^q_{W^{m_2,q}(V)}. \tag{3.10}$$

Next we consider the following four cases if we let $\max\{p,q\}=p$.

(1) Assume that $\|u_1-u_2\|_{W^{m_1,p}(V)} > 1$ and $\|v_1-v_2\|_{W^{m_2,q}(V)} > 1$. Then, $\|(u_1-u_2,v_1-v_2)\| > 2$ and

$$\begin{aligned}&c_p\|u_1-u_2\|^p_{W^{m_1,p}(V)} + c_q\|v_1-v_2\|^q_{W^{m_2,q}(V)}\\ &\ge \min\{c_p,c_q\}(\|u_1-u_2\|^q_{W^{m_1,p}(V)} + \|v_1-v_2\|^q_{W^{m_2,q}(V)})\\ &\ge \frac{\min\{c_p,c_q\}}{2^{q-1}}\|(u_1-u_2,v_1-v_2)\|^q\\ &\ge \frac{\min\{c_p,c_q\}}{2^{p-1}}\|(u_1-u_2,v_1-v_2)\|^q.\end{aligned} \tag{3.11}$$

Let

$$a_1(t) = \frac{\min\{c_p,c_q\}}{2^{p-1}}t^{q-1}, \qquad t>2. \tag{3.12}$$

(2) Assume that $\|u_1-u_2\|_{W^{m_1,p}(V)} \le 1$ and $\|v_1-v_2\|_{W^{m_2,q}(V)} \le 1$. Then, $\|(u_1-u_2,v_1-v_2)\| \le 2$ and

$$\begin{aligned}&c_p\|u_1-u_2\|^p_{W^{m_1,p}(V)} + c_q\|v_1-v_2\|^q_{W^{m_2,q}(V)}\\ &\ge \min\{c_p,c_q\}(\|u_1-u_2\|^p_{W^{m_1,p}(V)} + \|v_1-v_2\|^p_{W^{m_2,q}(V)})\\ &\ge \frac{\min\{c_p,c_q\}}{2^{p-1}}\|(u_1-u_2,v_1-v_2)\|^p\\ &\ge \begin{cases}\dfrac{\min\{c_p,c_q\}}{2^{p-1}}\|(u_1-u_2,v_1-v_2)\|^p, & \|(u_1-u_2,v_1-v_2)\|\le 1,\\ \dfrac{\min\{c_p,c_q\}}{2^{p-1}}\|(u_1-u_2,v_1-v_2)\|^q, & 1<\|(u_1-u_2,v_1-v_2)\|\le 2.\end{cases}\end{aligned} \tag{3.13}$$

Let

$$a_2(t) = \begin{cases}\dfrac{\min\{c_p,c_q\}}{2^{p-1}}t^{p-1}, & 0\le t\le 1,\\ \dfrac{\min\{c_p,c_q\}}{2^{p-1}}t^{q-1}, & 1<t\le 2.\end{cases} \tag{3.14}$$

(3) Assume that $\|u_1 - u_2\|_{W^{m_1,p}(V)} > 1$ and $\|v_1 - v_2\|_{W^{m_2,q}(V)} \le 1$. Then, $\|(u_1 - u_2, v_1 - v_2)\|^q > 1$ and

$$\begin{aligned} & c_p\|u_1 - u_2\|^p_{W^{m_1,p}(V)} + c_q\|v_1 - v_2\|^q_{W^{m_2,q}(V)} \\ \ge\ & \min\{c_p, c_q\}(\|u_1 - u_2\|^q_{W^{m_1,p}(V)} + \|v_1 - v_2\|^q_{W^{m_2,q}(V)}) \\ \ge\ & \frac{\min\{c_p, c_q\}}{2^{q-1}}\|(u_1 - u_2, v_1 - v_2)\|^q \\ \ge\ & \frac{\min\{c_p, c_q\}}{2^{p-1}}\|(u_1 - u_2, v_1 - v_2)\|^q. \end{aligned} \tag{3.15}$$

Let

$$a_3(t) = \frac{\min\{c_p, c_q\}}{2^{p-1}} t^{q-1}, \qquad t > 1. \tag{3.16}$$

(4) Assume that $\|u_1 - u_2\|_{W^{m_1,p}(V)} \le 1$ and $\|v_1 - v_2\|_{W^{m_2,q}(V)} > 1$. Then, $\|(u_1 - u_2, v_1 - v_2)\|^q > 1$. Note that $q - p \le 0$. Thus, we have

$$\begin{aligned} & c_p\|u_1 - u_2\|^p_{W^{m_1,p}(V)} + c_q\|v_1 - v_2\|^q_{W^{m_2,q}(V)} \\ \ge\ & \min\{c_p, c_q\|v_1 - v_2\|^{q-p}_{W^{m_2,q}(V)}\}(\|u_1 - u_2\|^p_{W^{m_1,p}(V)} + \|v_1 - v_2\|^p_{W^{m_2,q}(V)}) \\ \ge\ & \min\{c_p, c_q(\|u_1 - u_2\|_{W^{m_1,p}(V)} + \|v_1 - v_2\|_{W^{m_2,q}(V)})^{q-p}\}(\|u_1 - u_2\|^p_{W^{m_1,p}(V)} + \|v_1 - v_2\|^p_{W^{m_2,q}(V)}) \\ \ge\ & \min\{c_p, c_q\|(u_1 - u_2, v_1 - v_2)\|^{q-p}\}\frac{1}{2^{p-1}}\|(u_1 - u_2, v_1 - v_2)\|^p \\ =\ & \min\left\{\frac{c_p}{2^{p-1}}\|(u_1 - u_2, v_1 - v_2)\|^p, \frac{c_q}{2^{p-1}}\|(u_1 - u_2, v_1 - v_2)\|^q\right\} \\ \ge\ & \frac{\min\{c_p, c_q\}}{2^{p-1}}\|(u_1 - u_2, v_1 - v_2)\|^q. \end{aligned} \tag{3.17}$$

Let

$$a_4(t) = \frac{\min\{c_p, c_q\}}{2^{p-1}} t^{q-1}, \qquad t > 1. \tag{3.18}$$

Combining (3.12), (3.14), (3.16), and (3.18), we define $a : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ by

$$a(t) = \begin{cases} \dfrac{\min\{c_p, c_q\}}{2^{p-1}} t^{p-1}, & 0 \le t \le 1, \\ \dfrac{\min\{c_p, c_q\}}{2^{p-1}} t^{q-1}, & t > 1. \end{cases} \tag{3.19}$$

Then, a is continuous and strictly monotone increasing with $a(0) = 0$ and $a(t) \to +\infty$ as $t \to +\infty$. Thus, by (3.11), (3.13), (3.15), and (3.17), (3.10) can be written as

$$\langle \Phi'(u_1, v_1) - \Phi'(u_2, v_2), (u_1 - u_2, v_1 - v_2)\rangle \ge a(\|(u_1 - u_2, v_1 - v_2)\|_W)\|(u_1 - u_2, v_1 - v_2)\|_W.$$

So Φ' is uniformly monotone in W if $\max\{p, q\} = p$. Similarly, if we let $\max\{p, q\} = q$, we can also obtain the same conclusion.

Next we show that Φ' is also hemicontinuous in W. Assume that $s \to s^*$ and $s, s^* \in [0, 1]$. Note that

$$\begin{aligned} & |\langle \Phi'((u_1, u_2) + s(v_1, v_2)), (w_1, w_2)\rangle - \langle \Phi'((u_1, u_2) + s^*(v_1, v_2)), (w_1, w_2)\rangle| \\ & \le \|\Phi'((u_1, u_2) + s(v_1, v_2)) - \Phi'((u_1, u_2) + s^*(v_1, v_2))\|_* \cdot \|(w_1, w_2)\| \end{aligned}$$

for all $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in W$, where $\|\cdot\|_*$ denotes the norm of the dual space W^*. Then, the continuity of Φ' implies that

$$\langle \Phi'((u_1, u_2) + s(v_1, v_2)), (w_1, w_2)\rangle \to \langle \Phi'((u_1, u_2) + s^*(v_1, v_2)), (w_1, w_2)\rangle, \quad \text{as } s \to s^*$$

for all $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in W$. Hence Φ' is hemicontinuous in W.

Moreover, for all $(u, v) \in W$, we have

$$\begin{aligned}\langle \Phi'(u,v),(u,v)\rangle &= \int_V [|\nabla^{m_1}u|^p + h_1(x)|u|^p]\mathrm{d}\mu + \int_V [|\nabla^{m_2}v|^q + h_2(x)|v|^q]\mathrm{d}\mu \\ &= \|u\|^p_{W^{m_1,p}(V)} + \|v\|^q_{W^{m_2,q}(V)}.\end{aligned}$$

So, Φ' is coercive in W. Thus, by Theorem 26.A in [13], we can obtain that Φ' admits a continuous inverse in W. □

Lemma 3.5. $\Phi : W \to \mathbb{R}$ *is sequentially weakly lower semi-continuous.*

Proof. Since Φ is continuously differentiable and Φ' is uniformly monotone, which implies that Φ' is monotone. It follows from Proposition 25.20 in [13] that Φ is sequentially weakly lower semi-continuous. □

Lemma 3.6. Ψ *has compact derivative.*

Proof. Obviously, Ψ is a C^1 functional on W. Assume that $\{(u_n, v_n)\} \subset W$ is bounded. Note that W is of finite dimension. Then, there exists a subsequence $\{(u_k, v_k)\}$ such that $(u_k, v_k) \to (u_0, v_0)$ for some $(u_0, v_0) \in W$. By the continuity of Ψ', it is easy to obtain that

$$\|\Psi'(u_k, v_k) - \Psi'(u_0, v_0)\|_* \to 0, \quad \text{as } n \to \infty.$$

Hence, Ψ' is compact in W. □

Proof of Theorem 1.1. Obviously, by (F_0) and (F_1), $\Phi(0) = \Psi(0) = 0$ and both Φ and Ψ are continuously differentiable. Moreover, it is easy to see that $\Phi : W \to \mathbb{R}$ is coercive. Lemmas 3.2–3.6 imply that all other conditions in Lemma 2.4 are satisfied. Hence, Lemma 2.4 implies that for each $\lambda \in (\Lambda_2^{-1}, \Lambda_1^{-1})$, the functional φ has at least three distinct critical points that are solutions of system (1.14). □

4 Proofs for the (p, q)-Laplacian system (1.15)

Let $G = (V, E)$ be a locally finite graph. In order to investigate the (p, q)-Laplacian system (1.15), we work in the space $W_1 := W^{1,p}(V) \times W^{1,q}(V)$ with the norm endowed with $\|(u, v)\|_1 = \|u\|_{W^{1,p}(V)} + \|v\|_{W^{1,q}(V)}$ and then, $(W_1, \|\cdot\|_1)$ is a Banach space which is of infinite dimension. Different from the case of finite graph in Section 2, the continuous differentiability of variational functional for (1.15) cannot be obtained just by (F_0). However, by using the condition $(F_0)'$, the difficulty has been overcome in [7] so that we can apply Lemma 2.4 to system (1.15).

We consider the functional $\bar{\varphi} : W_1 \to \mathbb{R}$ as

$$\bar{\varphi}(u, v) = \frac{1}{p}\int_V (|\nabla u|^p + h_1(x)|u|^p)\mathrm{d}\mu + \frac{1}{q}\int_V (|\nabla v|^q + h_2(x)|v|^q)\mathrm{d}\mu - \lambda\int_V F(x, u, v)\mathrm{d}\mu. \tag{4.1}$$

Then, by Appendix A.2 in [7], under the assumptions of Theorem 1.2, we have $\bar{\varphi} \in C^1(W_1, \mathbb{R})$, and

$$\begin{aligned}\langle \bar{\varphi}'(u,v),(\phi_1,\phi_2)\rangle &= \int_V [|\nabla u|^{p-2}\Gamma(u,\phi_1) + h_1(x)|u|^{p-2}u\phi_1 - \lambda F_u(x,u,v)\phi_1]\mathrm{d}\mu \\ &\quad + \int_V [|\nabla v|^{q-2}\Gamma(v,\phi_2) + h_2(x)|v|^{q-2}v\phi_2 - \lambda F_v(x,u,v)\phi_2]\mathrm{d}\mu\end{aligned} \tag{4.2}$$

for any $(u, v), (\phi_1, \phi_2) \in W_1$. Define $\bar{\Phi} : W_1 \to \mathbb{R}$ and $\bar{\Psi} : W_1 \to \mathbb{R}$ by

$$\begin{aligned}\bar{\Phi}(u,v) &= \frac{1}{p}\int_V (|\nabla u|^p + h_1(x)|u|^p)\mathrm{d}\mu + \frac{1}{q}\int_V (|\nabla v|^q + h_2(x)|v|^q)\mathrm{d}\mu \\ &= \frac{1}{p}\|u\|^p_{W^{1,p}(V)} + \frac{1}{q}\|v\|^q_{W^{1,q}(V)}\end{aligned}$$

and

$$\overline{\Psi}(u,v)=\int_V F(x,u,v)\mathrm{d}\mu.$$

Then, $\overline{\varphi}(u,v)=\overline{\Phi}-\lambda\overline{\Psi}$. Moreover, it is easy to see that $(u,v)\in W_1$ is a critical point of $\overline{\varphi}$ if and only if

$$\int_V[|\nabla u|^{p-2}\Gamma(u,\phi_1)+h_1(x)|u|^{p-2}u\phi_1-\lambda F_u(x,u,v)\phi_1]\mathrm{d}\mu=0$$

and

$$\int_V[|\nabla v|^{q-2}\Gamma(v,\phi_2)+h_2(x)|v|^{q-2}v\phi_2-\lambda F_v(x,u,v)\phi_2]\mathrm{d}\mu=0$$

for all $(\phi_1,\phi_2)\in W_1$.

Lemma 4.1. *Assume that (M), (H_1), and $(F_0)'$ hold. Then, for any given $r>0$, the following inequality holds*:

$$\frac{\sup_{(u,v)\in\overline{\Phi}^{-1}(-\infty,r]}\overline{\Psi}(u,v)}{r}\le\frac{1}{r}\max_{|(s,t)|\le\frac{1}{h_0^{1/p}\mu_0^{1/p}}(pr)^{\frac{1}{p}}+\frac{1}{h_0^{1/q}\mu_0^{1/q}}(qr)^{\frac{1}{q}}}a(|(s,t)|)\int_V b(x)\mathrm{d}\mu.$$

Proof. By $(F_0)'$, we have

$$\begin{aligned}\overline{\Psi}(u,v)&=\int_V F(x,u(x),v(x))\mathrm{d}\mu\\&\le\int_V a(|(u(x),v(x))|)b(x)\mathrm{d}\mu\\&\le\max_{|(s,t)|\le\|u\|_\infty+\|v\|_\infty}a(|(s,t)|)\int_V b(x)\mathrm{d}\mu\end{aligned}$$

for every $(u,v)\in W_1$. Furthermore, for all $(u,v)\in W_1$ with $\overline{\Phi}(u,v)\le r$, by Lemma 2.3, we obtain

$$\|u\|_\infty\le\frac{1}{h_0^{1/p}\mu_0^{1/p}}\|u\|_{W^{1,p}(V)}\le\frac{1}{h_0^{1/p}\mu_0^{1/p}}(pr)^{\frac{1}{p}},\quad\|v\|_\infty\le\frac{1}{h_0^{1/q}\mu_0^{1/q}}\|v\|_{W^{1,q}(V)}\le\frac{1}{h_0^{1/q}\mu_0^{1/q}}(qr)^{\frac{1}{q}}.$$

Then,

$$\sup_{(u,v)\in\overline{\Phi}^{-1}(-\infty,r]}\overline{\Psi}(u,v)\le\max_{|(s,t)|\le\frac{1}{h_0^{1/p}\mu_0^{1/p}}(pr)^{\frac{1}{p}}+\frac{1}{h_0^{1/q}\mu_0^{1/q}}(qr)^{\frac{1}{q}}}a(|(s,t)|)\int_V b(x)\mathrm{d}\mu.\tag{4.3}$$

Furthermore, the proof is completed by multiplying $\frac{1}{r}$ on both sides of (4.3). □

Lemma 4.2. *Assume that (M), (H_1), $(F_0)'$, and $(F_3)'$ hold. Then, there exists $(u_{\delta_1},v_{\delta_2})\in W_1$ such that*

$$\frac{\sup_{(u,v)\in\overline{\Phi}^{-1}(-\infty,\gamma_1^p+\gamma_2^q]}\overline{\Psi}(u,v)}{\gamma_1^p+\gamma_2^q}<\frac{\overline{\Psi}(u_{\delta_1},v_{\delta_2})}{\overline{\Phi}(u_{\delta_1},v_{\delta_2})}.$$

Proof. Let

$$u_{\delta_1}(x)=\begin{cases}\delta_1,& x=x_0\\0,& x\ne x_0\end{cases},\quad v_{\delta_2}(x)=\begin{cases}\delta_2,& x=x_0\\0,& x\ne x_0,\end{cases}$$

where δ_i, $i = 1, 2$ are defined in $(F_3)'$. Then, a simple calculation implies that

$$|\nabla u_{\delta_1}|(x) = \begin{cases} \sqrt{\dfrac{\deg(x_0)}{2\mu(x_0)}}\,\delta_1, & x = x_0, \\ \sqrt{\dfrac{w_{x_0y}}{2\mu(y)}}\,\delta_1, & x = y \quad \text{with } y \sim x_0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$|\nabla v_{\delta_2}|(x) = \begin{cases} \sqrt{\dfrac{\deg(x_0)}{2\mu(x_0)}}\,\delta_2, & x = x_0, \\ \sqrt{\dfrac{w_{x_0y}}{2\mu(y)}}\,\delta_2, & x = y \quad \text{with } y \sim x_0, \\ 0, & \text{otherwise}. \end{cases}$$

Then,

$$\begin{aligned} &\int_V (|\nabla u_{\delta_1}|^p + h_1(x)|u_{\delta_1}|^p)\mathrm{d}\mu \\ &= \sum_{x\in V} (|\nabla u_{\delta_1}|^p + h_1(x)|u_{\delta_1}|^p)\mu(x) \\ &= (|\nabla u_{\delta_1}|^p(x_0) + h_1(x_0)|u_{\delta_1}|^p(x_0))\mu(x_0) + \sum_{y\sim x_0} (|\nabla u_{\delta_1}|^p(y) + h_1(y)|u_{\delta_1}|^p(y))\mu(y) \\ &= \left(\frac{\deg(x_0)}{2\mu(x_0)}\right)^{\frac{p}{2}} \delta_1^p\mu(x_0) + h_1(x_0)\delta_1^p\mu(x_0) + \delta_1^p \sum_{y\sim x_0}\left(\frac{w_{x_0y}}{2\mu(y)}\right)^{\frac{p}{2}}\mu(y) \\ &= \delta_1^p M_1 \quad (\text{defined in}\,(F_3)'), \end{aligned} \tag{4.4}$$

and similarly,

$$\begin{aligned} &\int_V (|\nabla v_{\delta_2}|^q + h_2(x)|v_{\delta_2}|^q)\mathrm{d}\mu \\ &= \left(\frac{\deg(x_0)}{2\mu(x_0)}\right)^{\frac{q}{2}} \delta_2^q\mu(x_0) + h_2(x_0)\delta_2^q\mu(x_0) + \delta_2^q \sum_{y\sim x_0}\left(\frac{w_{x_0y}}{2\mu(y)}\right)^{\frac{q}{2}}\mu(y) \\ &= \delta_2^q M_2 \quad (\text{defined in}\,(F_3)'). \end{aligned} \tag{4.5}$$

Note that $\{y|y \sim x_0\}$ is a finite set. Then, (4.4) and (4.5) imply that $(u_{\delta_1}, v_{\delta_2}) \in W_1$. Moreover,

$$\bar{\Phi}(u_{\delta_1}, v_{\delta_2}) = \frac{1}{p}\int_V (|\nabla u_{\delta_1}|^p + h_1(x)|u_{\delta_1}|^p)\mathrm{d}\mu + \frac{1}{q}\int_V (|\nabla v_{\delta_2}|^q + h_2(x)|v_{\delta_2}|^q)\mathrm{d}\mu = \frac{\delta_1^p M_1}{p} + \frac{\delta_2^q M_2}{q}. \tag{4.6}$$

Note that $\delta_i > \gamma_i\kappa_i$, $i = 1, 2$, where

$$\kappa_1 = \left(\frac{M_1}{p}\right)^{-\frac{1}{p}}, \quad \kappa_2 = \left(\frac{M_2}{q}\right)^{-\frac{1}{q}}.$$

Then, (4.6) implies that $\bar{\Phi}(u_{\delta_1}, v_{\delta_2}) \geq \gamma_1^p + \gamma_2^q$. Moreover, $(F_1)'$ implies that

$$\begin{aligned} \bar{\Psi}(u_{\delta_1}, v_{\delta_2}) &= \int_V F(x, u_{\delta_1}, v_{\delta_2})\mathrm{d}\mu \\ &= F(x_0, \delta_1, \delta_2) + \int_{V/\{x_0\}} F(x, 0, 0)\mathrm{d}\mu \\ &= F(x_0, \delta_1, \delta_2). \end{aligned} \tag{4.7}$$

Hence, by (4.6) and (4.7), we obtain

$$\frac{\bar{\Psi}(u_{\delta_1}, v_{\delta_2})}{\bar{\Phi}(u_{\delta_1}, v_{\delta_2})} = \frac{F(x_0, \delta_1, \delta_2)}{\frac{\delta_1^p M_1}{p} + \frac{\delta_2^q M_2}{q}}.$$

In view of Lemma 4.1 and $(F_3)'$, we obtain

$$\begin{aligned}
&\frac{\sup_{(u,v)\in\bar{\Phi}^{-1}(-\infty,\gamma_1^p+\gamma_2^q]}\bar{\Psi}(u,v)}{\gamma_1^p + \gamma_2^q}\\
&\le \frac{1}{\gamma_1^p + \gamma_2^q} \max_{|(s,t)|\le \frac{1}{h_0^{1/p}\mu_0^{1/p}}(p\gamma_1^p+p\gamma_2^q)^{\frac{1}{p}} + \frac{1}{h_0^{1/q}\mu_0^{1/q}}(q\gamma_1^p+q\gamma_2^q)^{\frac{1}{q}}} a(|(s,t)|)\int_V b(x)\mathrm{d}\mu\\
&< \frac{F(x_0, \delta_1, \delta_2)}{\frac{\delta_1^p M_1}{p} + \frac{\delta_2^q M_2}{q}}\\
&= \frac{\bar{\Psi}(u_{\delta_1}, v_{\delta_2})}{\bar{\Phi}(u_{\delta_1}, v_{\delta_2})}.
\end{aligned}$$

The proof is complete. □

Lemma 4.3. *Assume that* (H_1) *and* $(F_2)'$ *hold. Then, for each* $\lambda \in (0, +\infty)$, *the functional* $\bar{\Phi} - \lambda\bar{\Psi}$ *is coercive.*

Proof. By $(\bar{F}_2)'$, we have

$$\begin{aligned}
\varphi(u,v) &= \frac{1}{p}\|u\|^p_{W^{1,p}(V)} + \frac{1}{q}\|v\|^q_{W^{1,q}(V)} - \lambda\int_V F(x,u,v)\mathrm{d}\mu\\
&\ge \frac{1}{p}\|u\|^p_{W^{1,p}(V)} + \frac{1}{q}\|v\|^q_{W^{1,q}(V)} - \lambda\|f_1\|_\infty\|u\|^\alpha_\infty - \lambda\|f_2\|_\infty\|v\|^\beta_\infty - \lambda\int_V g(x)\mathrm{d}\mu\\
&\ge \frac{1}{p}\|u\|^p_{W^{1,p}(V)} + \frac{1}{q}\|v\|^q_{W^{1,q}(V)} - \lambda\|f_1\|_\infty h_0^{-\frac{\alpha}{p}}\mu_0^{-\frac{\alpha}{p}}\|u\|^\alpha_{W^{1,p}(V)}\\
&\quad - \lambda\|f_2\|_\infty h_0^{-\frac{\beta}{q}}\mu_0^{-\frac{\beta}{q}}\|v\|^\beta_{W^{1,q}(V)} - \lambda\int_V g(x)\mathrm{d}\mu.
\end{aligned}$$

Note that $\alpha \in [0, p)$ and $\beta \in [0, q)$. Therefore, $\bar{\varphi}(u, v)$ is a coercive functional for every $\lambda \in (0, +\infty)$. □

Lemma 4.4. *The Gâteaux derivative of* $\bar{\Phi}$ *admits a continuous inverse on* W_1^*, *where* W_1^* *is the dual space of* W_1.

Proof. In the proof of Lemma 3.4, we only need to take $m_i = 1, i = 1, 2$ and let $G = (V, E)$ be a locally finite graph. Then, the proof is essentially the same as that in Lemma 3.4. □

Lemma 4.5. $\bar{\Phi} : W_1 \to \mathbb{R}$ *is sequentially weakly lower semi-continuous.*

Proof. The proof is essentially the same as that in Lemma 3.5, taking $m_i = 1, i = 1, 2$ and letting $G = (V, E)$ be a locally finite graph. □

Lemma 4.6. $\bar{\Psi}$ *has compact derivative.*

Proof. Obviously, $\bar{\Psi}$ is a C^1 functional on W_1. Assume that $\{(u_n, v_n)\} \subset W_1$ is bounded. Then, by Lemma 2.3, there exists a positive constant $M > 0$ such that $\|u_k\|_\infty \le M$ and $\|v_k\|_\infty \le M$ and there exists a subsequence $\{(u_k, v_k)\}$ such that $(u_k, v_k) \rightharpoonup (u_0, v_0)$ for some $(u_0, v_0) \in W_1$. In particular,

$$\lim_{k\to\infty}\int_V u_k\varphi\mathrm{d}\mu = \int_V u_0\varphi\mathrm{d}\mu, \quad \forall\varphi \in C_c(V),$$

which implies that

$$\lim_{k\to\infty} u_k(x) = u_0(x) \quad \text{for any fixed } x \in V \tag{4.8}$$

by taking

$$\varphi(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x. \end{cases}$$

Similarly, we have

$$\lim_{k\to\infty} v_k(x) = v_0(x) \quad \text{for any fixed } x \in V. \tag{4.9}$$

Note that

$$\begin{aligned} &\|\bar{\Psi}'(u_k, v_k) - \bar{\Psi}'(u_0, v_0)\|_* \\ &= \sup_{\|(\phi_1,\phi_2)\|=1} |\langle \bar{\Psi}'(u_k, v_k) - \bar{\Psi}'(u_0, v_0), (\phi_1, \phi_2)\rangle| \\ &= \sup_{\|(\phi_1,\phi_2)\|=1} \left| \int_V [F_{u_k}(x, u_k, v_k) - F_{u_0}(x, u_0, v_0)]\phi_1 \mathrm{d}\mu + \int_V [F_{v_k}(x, u_k, v_k) - F_{v_0}(x, u_0, v_0)]\phi_2 \mathrm{d}\mu \right| \\ &\le \sup_{\|(\phi_1,\phi_2)\|=1} \left[\left| \int_V [F_{u_k}(x, u_k, v_k) - F_{u_0}(x, u_0, v_0)]\phi_1 \mathrm{d}\mu \right| + \left| \int_V [F_{v_k}(x, u_k, v_k) - F_{v_0}(x, u_0, v_0)]\phi_2 \mathrm{d}\mu \right| \right]. \end{aligned}$$

By $(F_0)'$, we have

$$\begin{aligned} &|F_{u_k}(x, u_k, v_k) - F_{u_0}(x, u_0, v_0)| \\ &\le [a(|(u_k, v_k)|) + a(|(u_0, v_0)|)]b(x) \\ &\le \left[\max_{|(s,t)|\le\|u_0\|_\infty+\|v_0\|_\infty} a(|(s, t)|) + \max_{|(s,t)|\le 2M} a(|(s, t)|) \right] b(x) \\ &\coloneqq l(x). \end{aligned}$$

Note that $b \in L^1(V)$. Hence, $l(x) \in L^1(V)$ and so $\int_V |F_{u_k}(x, u_k, v_k) - F_{u_0}(x, u_0, v_0)|\mathrm{d}\mu$ is uniformly convergent. Thus, by (4.8), (4.9), and the continuity of F_u, we have

$$\begin{aligned} &\int_V [F_{u_k}(x, u_k, v_k) - F_{u_0}(x, u_0, v_0)]\phi_1 \mathrm{d}\mu \\ &\le \left(\int_V |F_{u_k}(x, u_k, v_k) - F_{u_0}(x, u_0, v_0)|\mathrm{d}\mu \right) \|\phi_1\|_\infty \\ &\le \left(\int_V |F_{u_k}(x, u_k, v_k) - F_{u_0}(x, u_0, v_0)|\mathrm{d}\mu \right) h_0^{-\frac{1}{p}} \mu_0^{-\frac{1}{p}} \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Similarly, we also have

$$\int_V [F_{v_k}(x, u_k, v_k) - F_{v_0}(x, u_0, v_0)]\phi_2 \mathrm{d}\mu \to 0 \quad \text{as } k \to \infty.$$

So,

$$\|\bar{\Psi}'(u_k, v_k) - \bar{\Psi}'(u_0, v_0)\|_* \to 0 \quad \text{as } k \to \infty.$$

Hence, $\bar{\Psi}'$ is compact in W_1. □

Proof of Theorem 1.2. Obviously, by $(F_0)'$ and $(F_1)'$, $\bar{\Phi}(0) = \bar{\Psi}(0) = 0$ and both $\bar{\Phi}$ and $\bar{\Psi}$ are continuously differentiable. Moreover, it is easy to see that $\bar{\Phi} : W_1 \to \mathbb{R}$ is coercive. Lemmas 4.2–4.6 imply that all other conditions in Lemma 2.4 are satisfied. Hence, Lemma 2.4 implies that for each $\lambda \in (\Theta_2^{-1}, \Theta_1^{-1})$, the functional $\bar{\varphi}$ has at least three distinct critical points that are solutions of system (1.15). □

Remark 4.7. On the locally finite graph, we do not consider the more general poly-Laplacian system. That is because it is difficult to obtain the continuous differentiability of the variational functional φ when $m_i > 1$, $i = 1, 2$, which is caused by the special definition of $\mathcal{L}_{m,p}$.

5 Results of the scalar equation

By using the similar arguments of Theorem 1.1, we can also obtain a similar result for the following scalar equation on finite graph $G = (V, E)$:

$$\pounds_{m,p}u + h(x)|u|^{p-2}u = \lambda f(x, u), \qquad x \in V, \tag{5.1}$$

where $m \geq 1$ is an integer, $h : V \to \mathbb{R}$, $p > 1$, $\lambda > 0$, and $f : V \times \mathbb{R} \to \mathbb{R}$.

Theorem 5.1. *Let $G = (V, E)$ be a finite graph and $F(x, s) = \int_0^s f(x, \tau)\mathrm{d}\tau$ for all $x \in V$. Assume that the following conditions hold*:
(h) $h(x) > 0$ *for all* $x \in V$;
(f_0) $F(x, s)$ *is continuously differentiable in* $s \in \mathbb{R}$ *for all* $x \in V$;
(f_1) $\int_V F(x, 0)\mathrm{d}\mu = 0$;
(f_2) *there exist a constant* $\alpha \in [0, p)$ *and functions* $g_1, g_2 : V \to \mathbb{R}$ *such that*

$$F(x, s) \leq g_1(x)|s|^\alpha + g_2(x)$$

for all $(x, s) \in V \times \mathbb{R}$;
(f_3) *there are positive constants* γ *and* δ *with* $\delta > \gamma\kappa$, *such that*

$$\Lambda_1' := \frac{1}{\gamma^p} \max_{x \in V, |s| \leq \frac{(p\gamma^p)^{\frac{1}{p}}}{h_{\min}^{1/p}\mu_{\min}^{1/p}}} F(x, s)|V| < \frac{\inf_{x \in V} F(x, \delta)|V|}{\frac{\delta^p}{p}\int_V h(x)\mathrm{d}\mu} := \Lambda_2',$$

where $|V| = \sum_{x \in V}\mu(x)$ *and* $\kappa = \left(\frac{1}{p}\int_V h(x)\mathrm{d}\mu\right)^{-\frac{1}{p}}$.

Then, for each parameter λ belonging to $(\Lambda_2'^{-1}, \Lambda_1'^{-1})$, equation (5.1) *has at least three distinct solutions.*

By using similar arguments of Theorem 1.2, we can also obtain a similar result for the following scalar equation on a locally finite graph $G = (V, E)$:

$$-\Delta_p u + h(x)|u|^{p-2}u = \lambda f(x, u), \qquad x \in V, \tag{5.2}$$

where $p \geq 2$, $h : V \to \mathbb{R}$, $\lambda > 0$, and $f : V \times \mathbb{R} \to \mathbb{R}$.

Theorem 5.2. *Let $G = (V, E)$ be a locally finite graph and $F(x, s) = \int_0^s f(x, \tau)\mathrm{d}\tau$ for all $x \in V$. Assume that (M) and the following conditions hold*:
$(h)'$ *there exist a constant* $h_0 > 0$ *such that* $h(x) \geq h_0 > 0$ *for all* $x \in V$;
$(f_0)'$ $F(x, s)$ *is continuously differentiable in* $s \in \mathbb{R}$ *for all* $x \in V$, *and there exist a function* $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and a function* $b : V \to \mathbb{R}^+$ *with* $b \in L^1(V)$ *such that*

$$|f(x, s)| \leq a(|s|)b(x), \quad |F(x, s)| \leq a(|s|)b(x),$$

for all $x \in V$ *and all* $s \in \mathbb{R}$;

$(f_1)'$ $\int_V F(x,0)\mathrm{d}\mu = 0$ and there exists a $x_0 \in V$ such that $F(x_0, 0) = 0$;
$(f_2)'$ there exist a constant $\alpha \in [0, p)$ and functions $g_i : V \to \mathbb{R}$, $i = 1, 2$, with $g_1 \in L^\infty(V)$ and $g_2 \in L^1(V)$, such that

$$F(x,s) \le g_1(x)|s|^\alpha + g_2(x)$$

for all $(x,s) \in V \times \mathbb{R}$;
$(f_3)'$ there are positive constants γ and δ with $\delta > \gamma\kappa$, such that

$$\Theta_1' := \frac{1}{\gamma^p} \max_{|s| \le \frac{1}{h_0^{1/q}\mu_0^{1/q}}(p\gamma^p)^{\frac{1}{p}}} a(|s|) \int_V b(x)\mathrm{d}\mu < \frac{F(x,\delta)\mathrm{d}\mu}{\frac{\delta^p}{p} M} =: \Theta_2',$$

where $\kappa = \left(\frac{1}{p}\int_V h(x)\mathrm{d}\mu\right)^{-p}$ and

$$M = \left(\frac{\deg(x_0)}{2\mu(x_0)}\right)^{\frac{p}{2}} \mu(x_0) + h(x_0)\mu(x_0) + \sum_{y \sim x_0} \left(\frac{w_{x_0 y}}{2\mu(y)}\right)^{\frac{p}{2}} \mu(y).$$

Then, for each parameter λ belonging to $(\Theta_2'^{-1}, \Theta_1'^{-1})$, equation (5.2) has at least three solutions.

Remark 5.3. In Theorems 5.1 and 5.2, all three solutions are nontrivial solutions if we furthermore assume that $f(x,0) \neq 0$ for some $x \in V$.

6 Examples

In this section, we present two examples as applications of Theorems 1.1 and 5.2.

Example 6.1. Let $p = 2$, $q = 3$, and $m = 2$ in (1.14). Consider the following system:

$$\begin{cases} \pounds_{2,2}u + h_1(x)u = \lambda F_u(x,u,v), & x \in V, \\ \pounds_{2,3}v + h_2(x)v = \lambda F_v(x,u,v), & x \in V, \end{cases} \tag{6.1}$$

where $G = (V, E)$ is a finite graph of 9 vertexes, i.e., $V = \{x_1, x_2, \ldots, x_9\}$, the measure $\mu(x_i) = 1$, $i = 1, 2, \ldots, 9$, $h_i : V \to \mathbb{R}^+$ with $h_i \equiv 9$, $i = 1, 2$, for all $x \in V$ and put

$$\omega_1 = \frac{(p\gamma_1^p + p\gamma_2^q)^{\frac{1}{p}}}{h_{1,\min}^{1/p}\mu_{\min}^{1/p}}, \quad \omega_2 = \frac{(q\gamma_1^p + q\gamma_2^q)^{\frac{1}{q}}}{h_{2,\min}^{1/q}\mu_{\min}^{1/q}}.$$

Let $\lambda > 0$ and $F : V \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\frac{\partial F(x,s,t)}{\partial s} = \begin{cases} \omega_1 - |s|, & 0 \le |s| \le \omega_1, \\ |s|^3 - \omega_1^3, & \omega_1 < |s| < 4\omega_1, \\ (4\omega_1)^{r_1}|s|^{3-r_1} - \omega_1^3, & 4\omega_1 \le |s|, \end{cases}$$

and

$$\frac{\partial F(x,s,t)}{\partial t} = \begin{cases} \omega_2 - |t|, & 0 \le |t| \le \omega_2, \\ |t|^4 - \omega_2^4, & \omega_2 < |t| < 5\omega_2, \\ (5\omega_2)^{r_2}|t|^{4-r_2} - \omega_2^4, & 5\omega_2 \le |t|. \end{cases}$$

Then,

$$F(x,s,t)=\begin{cases}\frac{1}{2}\omega_1^2+\frac{1}{4}(4\omega_1)^4+\frac{3}{4}\omega_1^4+\frac{1}{4-r_1}(4\omega_1)^{r_1}|s|^{4-r_1}-\omega_1^3|s| & \\ \quad -\frac{1}{4-r_1}(4\omega_1)^4+\frac{1}{2}\omega_2^2+\frac{1}{5}(5\omega_2)^5+\frac{4}{5}\omega_2^5 & \\ \quad +\frac{1}{5-r_2}(5\omega_2)^{r_2}|t|^{5-r_2}-\omega_2^4|t|-\frac{1}{5-r_2}(5\omega_2)^5, & 4\omega_1\le|s|, 5\omega_2\le|t|,\\ \omega_1|s|-\frac{1}{2}|s|^2+\frac{1}{2}\omega_2^2+\frac{1}{5}|t|^5-\omega_2^4|t|+\frac{4}{5}\omega_2^5, & 0\le|s|\le\omega_1, \omega_2<|t|<5\omega_2,\\ \omega_1|s|-\frac{1}{2}|s|^2+\frac{1}{2}\omega_2^2+\frac{1}{5}(5\omega_2)^5+\frac{4}{5}\omega_2^5 & \\ \quad +\frac{1}{5-r_2}(5\omega_2)^{r_2}|t|^{5-r_2}-\omega_2^4|t|-\frac{1}{5-r_2}(5\omega_2)^5, & 0\le|s|\le\omega_1, 5\omega_2\le|t|,\\ \frac{1}{2}\omega_1^2+\frac{1}{4}|s|^4-\omega_1^3|s|+\frac{3}{4}\omega_1^4+\omega_2|t|-\frac{1}{2}|t|^2, & \omega_1<|s|<4\omega_1, 0\le|t|\le\omega_2,\\ \frac{1}{2}\omega_1^2+\frac{1}{4}|s|^4-\omega_1^3|s|+\frac{3}{4}\omega_1^4+\frac{1}{2}\omega_2^2+\frac{1}{5}|t|^5-\omega_2^4|t|+\frac{4}{5}\omega_2^5, & \omega_1<|s|<4\omega_1, \omega_2<|t|<5\omega_2,\\ \frac{1}{2}\omega_1^2+\frac{1}{4}|s|^4-\omega_1^3|s|+\frac{3}{4}\omega_1^4+\frac{1}{2}\omega_2^2+\frac{1}{5}(5\omega_2)^5+\frac{4}{5}\omega_2^5 & \\ \quad +\frac{1}{5-r_2}(5\omega_2)^{r_2}|t|^{5-r_2}-\omega_2^4|t|-\frac{1}{5-r_2}(5\omega_2)^5, & \omega_1<|s|<4\omega_1, 5\omega_2\le|t|,\\ \frac{1}{2}\omega_1^2+\frac{1}{4}(4\omega_1)^4+\frac{3}{4}\omega_1^4+\frac{1}{4-r_1}(4\omega_1)^{r_1}|s|^{4-r_1}-\omega_1^3|s| & \\ \quad -\frac{1}{4-r_1}(4\omega_1)^4+\omega_2|t|-\frac{1}{2}|t|^2, & 4\omega_1\le|s|, 0\le|t|\le\omega_2,\\ \frac{1}{2}\omega_1^2+\frac{1}{4}(4\omega_1)^4+\frac{3}{4}\omega_1^4+\frac{1}{4-r_1}(4\omega_1)^{r_1}|s|^{4-r_1}-\omega_1^3|s| & \\ \quad -\frac{1}{4-r_1}(4\omega_1)^4+\frac{1}{2}\omega_2^2+\frac{1}{5}|t|^5-\omega_2^4|t|+\frac{4}{5}\omega_2^5, & 4\omega_1\le|s|, \omega_2<|t|<5\omega_2,\\ \omega_1|s|-\frac{1}{2}|s|^2+\omega_2|t|-\frac{1}{2}|t|^2, & 0\le|s|\le\omega_1, 0\le|t|\le\omega_2,\end{cases}$$

where $(r_1,r_2)\in(1,2]\times(1,3]$. Next we verify that h_1, h_2, and F satisfy the conditions in Theorem 1.1.

- Obviously, h_i satisfy (H), $i=1,2$, and F satisfies (F_0) and (F_1).
- Let

$$f_1(x)\equiv\frac{(4\omega_1)^{r_1}}{4-r_1},\quad f_2(x)\equiv\frac{(5\omega_2)^{r_2}}{5-r_2}$$

  and

$$g(x)\equiv\frac{1}{2}\omega_1^2+\frac{1}{4}(4\omega_1)^4+\frac{3}{4}\omega_1^4+\frac{1}{2}\omega_2^2+\frac{1}{5}(5\omega_2)^5+\frac{4}{5}\omega_2^5,\quad \text{for all } x\in V.$$

  Then,

$$F(x,s,t)\le f_1(x)|s|^\alpha+f_2(x)|t|^\beta+g(x),$$

  where $\alpha\in[0,p)$, $\beta\in[0,q)$. Hence, F satisfies (F_2).
- Let

$$\delta_1=4\omega_1=4\times15^{\frac{1}{2}},\quad \delta_2=5\omega_2=5\times\left(\frac{45}{2}\right)^{\frac{1}{3}},\quad \gamma_1=\left(\frac{81}{2}\right)^{\frac{1}{2}},\quad \gamma_2=\left(\frac{81}{3}\right)^{\frac{1}{3}}.$$

Then, $\delta_1 > \gamma_1\kappa_1 = 1$, $\delta_2 > \gamma_2\kappa_2 = 1$,

$$\begin{aligned}
\Lambda_2^{-1} &= \frac{\frac{\delta_1^p}{p}\int_V h_1(x)\mathrm{d}\mu + \frac{\delta_2^q}{q}\int_V h_2(x)\mathrm{d}\mu}{\inf_{x\in V}F(x,\delta_1,\delta_2)|V|} \\
&= \frac{\frac{81}{2}\cdot(4\cdot 15^{\frac{1}{2}})^2 + \frac{81}{3}\cdot\left(5\cdot\left(\frac{45}{2}\right)^{\frac{1}{3}}\right)^3}{\left(\frac{1}{2}\cdot 15^{\frac{2}{2}} + \frac{1}{4}(4\cdot 15^{\frac{1}{2}})^4 - 4\cdot 15^{\frac{4}{2}} + \frac{3}{4}\cdot 15^{\frac{4}{2}} + \frac{1}{2}\cdot\left(\frac{45}{2}\right)^{\frac{2}{3}} + \frac{1}{5}\left(5\cdot\left(\frac{45}{2}\right)^{\frac{1}{3}}\right)^5 - 5\cdot\left(\frac{45}{2}\right)^{\frac{5}{3}} + \frac{4}{5}\cdot\left(\frac{45}{2}\right)^{\frac{5}{3}}\right)\times 9} \\
&\approx 0.07614
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_1^{-1} &= \frac{\gamma_1^p + \gamma_2^q}{\max\limits_{x\in V,|s|\le\frac{(p\gamma_1^p+p\gamma_2^q)^{\frac{1}{p}}}{h_{1,\min}^{1/p}\mu_{\min}^{1/p}},|t|\le\frac{(q\gamma_1^p+q\gamma_2^q)^{\frac{1}{q}}}{h_{2,\min}^{1/q}\mu_{\min}^{1/q}}} F(x,s,t)|V|} \\
&= \frac{\frac{81}{2} + \frac{81}{3}}{\left(\frac{1}{2}\cdot(15)^{\frac{2}{2}} + \frac{1}{2}\cdot\left(\frac{45}{2}\right)^{\frac{2}{3}}\right)\times 9} \\
&\approx 0.65303 > \Lambda_2^{-1}.
\end{aligned}$$

Hence, (F_3) holds. Thus, by Theorem 1.1, for each $\lambda \in (\Lambda_2^{-1}, \Lambda_1^{-1}) \approx (0.07614, 0.65303)$, system (6.1) has at least three distinct solutions.

Example 6.2. Let $p = 3$ in (5.2). Consider the following scalar equation on locally finite graph $G = (V, E)$:

$$-\Delta_3 u + h(x)u = \lambda f(x,u), \qquad x \in V, \tag{6.2}$$

where the measure $\mu(x) \equiv 1$ and $h(x) \equiv 4$ for all $x \in V$. For some fixed $x_0 \in V$, there are four edges $x_0y \in E$ with $w_{x_0y} = 2$. Put

$$\omega = \frac{(p\gamma^p)^{\frac{1}{p}}}{h_0^{1/p}\mu_0^{1/p}}.$$

Let $\lambda > 0$ and $F : V \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x_0,s) = \frac{\partial F(x_0,s)}{\partial s} = \begin{cases} \omega - |s|, & |s| \le \omega, \\ |s|^5 - \omega^5, & \omega < |s| \le 6\omega, \\ (6\omega)^r s^{5-r} - \omega^5, & 6\omega < |s|, \end{cases}$$

and

$$f(x,s) = 0, \quad \text{for all } x \in V/\{x_0\}.$$

Then,

$$F(x_0,s) = \begin{cases} \omega|s| - \dfrac{1}{2}|s|^2, & |s| \le \omega, \\ \dfrac{1}{2}\omega^2 + \dfrac{1}{6}|s|^6 - \omega^5|s| + \dfrac{5}{6}\omega^6, & \omega < |s| \le 6\omega, \\ \dfrac{1}{2}\omega^2 + \dfrac{6^6+5}{6}\omega^6 - \dfrac{1}{6-r}(6\omega)^6 + \dfrac{1}{6-r}(6\omega)^r|s|^{6-r} - \omega^5|s|, & 6\omega < |s| \end{cases}$$

and

$$F(x,s)=0, \quad \text{for all } x\in V/\{x_0\},$$

where $r\in(3,5]$ and $\lambda>0$. Next we verify that h and F satisfy the conditions in Theorem 5.2.

- Obviously, h satisfies $(h)'$.
- Let

$$g_1(x)=\begin{cases}\frac{1}{6-r}(6\omega)^r, & x=x_0,\\ 0, & x\neq x_0,\end{cases}$$

$$g_2(x)=\begin{cases}\frac{1}{2}\omega^2+\frac{6^6+5}{6}\omega^6, & x=x_0,\\ 0, & x\neq x_0,\end{cases}$$

$$a(|s|)=\begin{cases}\omega|s|-\frac{1}{2}|s|^2+1, & |s|\leq\omega,\\ \frac{1}{2}\omega^2+\frac{1}{6}|s|^6-\omega^5|s|+\frac{5}{6}\omega^6+1, & \omega<|s|\leq 6\omega,\\ \frac{1}{2}\omega^2+\frac{6^6+5}{6}\omega^6-\frac{1}{6-r}(6\omega)^6+\frac{1}{6-r}(6\omega)^r|s|^{6-r}-\omega^5|s|+1, & 6\omega<|s|,\end{cases}$$

and

$$b(x)=\begin{cases}1, & x=x_0,\\ 0, & x\neq x_0.\end{cases}$$

Then,

$$\|g_1\|_\infty=\frac{1}{6-r}(6\omega)^r, \quad \|g_2\|_{L^1(V)}=\frac{1}{2}\omega^2+\frac{6^6+5}{6}\omega^6$$

and

$$F(x,s)\leq g_1(x)|s|^\alpha+g_2(x).$$

Moreover,

$$f(x,s)\leq a(|s|)b(x), \quad F(x,s)\leq a(|s|)b(x),$$

for all $x\in V$ and all $s\in\mathbb{R}$. Hence, F satisfies $(f_0)'$, $(f_1)'$, and $(f_2)'$.

- Let

$$\delta=6\omega=6\cdot 4^{\frac{1}{3}}, \quad \gamma=\left(\frac{16}{3}\right)^{\frac{1}{3}}.$$

Then, $\delta>\gamma\kappa=1$,

$$\begin{aligned}\Theta_2^{-1}&=\frac{\frac{\delta^p}{p}M}{\inf_{x\in V}F(x,\delta)}\\ &=\frac{(6\cdot 4^{\frac{1}{3}})^3\times\frac{1}{3}\times 16}{\frac{1}{2}\cdot 4^{\frac{2}{3}}+6^5\cdot 4^2-6\cdot 4^2+\frac{5}{6}\cdot 4^2}\\ &\approx 0.0371\end{aligned}$$

and

$$\Theta_1^{-1} = \frac{\gamma^p}{\max\limits_{x\in V,|s|\le \frac{(p\gamma^p)^{\frac{1}{p}}}{h_{1,\min}^{1/p}\mu_{\min}^{1/p}}} a(|s|)\int_V b(x)\mathrm{d}\mu} = \frac{\frac{16}{3}}{\frac{1}{2}\cdot 4^{\frac{2}{3}}+1} \approx 2.36 > \Theta_2^{-1}.$$

Hence, F satisfies $(f_3)'$. Thus, by Theorem 5.2, for each $\lambda \in (\Theta_1^{-2}, \Theta_1^{-1}) \approx (0.0371, 2.36)$, equation (6.2) has at least three distinct solutions.

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