DE GRUYTER

Demonstratio Mathematica 2024; 57: 20240061

## Research Article

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# A higher-dimensional categorical perspective on 2-crossed modules

https://doi.org/10.1515/dema-2024-0061
received August 29, 2023; accepted July 30, 2024

**Abstract:** In this study, we will express the 2-crossed module of groups from a higher-dimensional categorical perspective. According to simplicial homotopy theory, a 2-crossed module is the Moore complex of a 2-truncated simplicial group. Therefore, the 2-crossed module is an algebraic homotopy model for the homotopy 3-types. Tricategories are a three-dimensional generalization of the bicategory concept. Any tricategory is triequivalent to the **Gray** category, where **Gray** is a category enriched over the monoidal category **2Cat** equipped with the Gray tensor product. Briefly, a **Gray** category is a semi-strict 3-category for homotopy 3-types. Naturally, the tricategory perspective is used in homotopy theory. The 2-crossed module is associated with the concept of the **Gray** category. The aim of this study is to obtain a single object tricategory from any 2-crossed module of groups.

**Keywords:** crossed module, 2-crossed module, categorification, sestertius category, tricategory

**MSC 2020:** 18G45, 18N10, 18N20, 18N25

## 1 Introduction

Whitehead described crossed modules in [1,2] for modeling homotopy 2-types from the algebraic homotopy theory perspective. Roughly speaking, in terms of nonabelian algebraic homotopy 2-types, any crossed module of groups $\mathcal{G}_1 = (E \overset{\partial}{\to} G)$ has a classification space $X$ such that $\pi_2(X) \cong \ker(\partial)$ and $\pi_1(X) \cong \operatorname{coker}(\partial)$, while each of the homotopy groups $\pi_i(X)$ for $i > 2$ is trivial. Loday introduced cat$^1$-groups, a similar structure for homotopy 2-types in [3]. From a categorical point of view, Brown and Spencer [4,5] associated crossed modules of groupoids with double groupoids. In terms of structures in categories, a crossed module of groups is equivalent to an internal category in the category of groups **Grp** [6,7]. From the point of view of monoidal categories in [8], crossed modules are equivalent to strict 2-groups.

Gordon et al. [9] introduced the tricategory concept, the natural three-dimensional generalization of bicategory [10]. Also, bicategories are called weak 2-categories [8]. Any tricategory is triequivalent to the **Gray** category, where **Gray** is a category enriched over the monoidal category **2Cat** equipped with the **Gray** tensor product [9]. Briefly, the **Gray** category means exactly a strict cubical tricategory. Furthermore, a strict cubical tricategory is a semi-strict 3-category for homotopy 3-types, as a **Gray** category [11,12].

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Conduché [13] defined a 2-crossed module for the algebraic homotopy 3-type models using simplicial homotopy theory. In brief, the 2-crossed module is the Moore complex of a 2-truncated simplicial group. In [14,15], the 2-crossed module is associated with the concept of the **Gray** category.

The aim of this study is to obtain a single object tricategory from any 2-crossed module of groups. For this, we will first construct a strict 2-group from a crossed module of groups by a categorical point of view. In contrast to crossed modules, the categorical structure must change as one moves towards higher dimensions. Because the 2-crossed module defined on the pre-crossed module does not form a strict 3-category structure. The algebraic homotopy 3-type models related to the concept of semi-strict 3-group are sestertius categories. The word sestertius is derived from the Latin phrase "SEMIS TERTIVS," which is translated as "2 1/2." Mathematically, a sestertius category is defined by a three-dimensional generalization of a strict 2-category, called a sesquicategory, for which there is no interchange law (for a different perspective, refer [16]). Therefore, we will obtain a sestertius category with one object in which all 1-, 2-, and 3-morphisms are invertible from a 2-crossed module of groups. Afterwards, we will construct a tricategory from this sestertius category.

## 2 Preliminaries

Before giving the definition of a tricategory, we will simply give an overview of the two- and three-dimensional categories.

Roughly speaking, a category consists of objects and the morphisms between them, as follows:

$$a \xrightarrow{f} b.$$

For these structures, we will use the nomenclature 0- and 1-cells, respectively. If we use the 0-source and 0-target functions to describe the 0-cells, we obtain $s_0(f) = a$ and $t_0(f) = b$. Naturally, this structure has to satisfy unit and associative composition:

$$a \xrightarrow{1_a} a, \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend right, "g\sharp_0 f"'] & b \arrow[r, "g"] & c. \end{tikzcd}$$

Note that the composition of the 1-cells is "$\sharp_0$." We may write this composition as $g\sharp_0 f$ (meaning first $g$ and $f$). Throughout our work, we will describe all compositions in this way. Of course, the composite of $g\sharp_0 f$ of two 1-cells is defined if and only if $t_0(f) = s_0(g)$. Thus, for composition $g\sharp_0 f$, $s_0(g\sharp_0 f) = s_0(f)$, and $t_0(g\sharp_0 f) = t_0(g)$. In addition, if the morphisms of 1-cells are invertible, this structure is a groupoid.

If we want to add 2-cell to this structure, we have the following:

$$\begin{tikzcd} a \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "f'"', ""{name=D}] & b. \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

As the diagram shows, the 1-source and 1-target of the 2-cell $\alpha$ are $s_1(\alpha) = f$ and $t_1(\alpha) = f'$, respectively.

Let us have 2-cells with source and target compatibility as follows:

$$\begin{tikzcd} a \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "f'"', ""{name=D}] & b, \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r, bend left, "g", ""{name=U, below}] \arrow[r, bend right, "g'"', ""{name=D}] & c. \arrow[Rightarrow, from=U, to=D, "\beta"] \end{tikzcd}$$

Thus, we can define whiskering diagrams as follows:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, "g"] & c \end{tikzcd} = \begin{tikzcd} a \arrow[rr, bend left, "g\sharp_0 f"] \arrow[rr, bend right, "g\sharp_0 f'"'] \arrow[rr, phantom, "\Downarrow g\sharp_0\alpha"] & & c, \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "f'"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & c \end{tikzcd} = \begin{tikzcd} a \arrow[rr, bend left, "g\sharp_0 f'"] \arrow[rr, bend right, "g'\sharp_0 f'"'] \arrow[rr, phantom, "\Downarrow \beta\sharp_0 f'"] & & c. \end{tikzcd}$$

In addition, the horizontal composition of 2-cells “$\sharp_0$” is defined as follows:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & c \end{tikzcd} = \begin{tikzcd} a \arrow[r, bend left, "g\sharp_0 f"] \arrow[r, bend right, "g'\sharp_0 f'"'] \arrow[r, phantom, "\Downarrow \beta\sharp_0\alpha"] & c. \end{tikzcd}$$

Further, if we have 2-cells $\alpha$ and $\alpha'$ where the 1-target of $\alpha$ is $f'$ and the 1-source of $\alpha'$ is $f'$, then the vertical composition of these 2-cells, “$\sharp_1$,” is defined as follows:

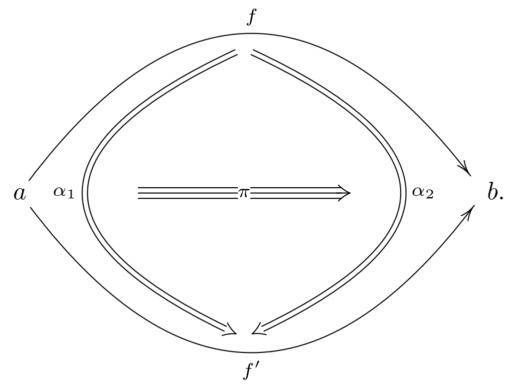
$$\begin{array}{c} \begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd} \\ \begin{tikzcd} a \arrow[r, bend left, "f'"] \arrow[r, bend right, "f''"'] \arrow[r, phantom, "\Downarrow\alpha'"] & b \end{tikzcd} \end{array} = \begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f''"'] \arrow[r, phantom, "\Downarrow \alpha'\sharp_1\alpha"] & b. \end{tikzcd}$$

Additionally, the vertical composition $\sharp_1$ of 2-cells is compatible with the horizontal composition $\sharp_0$ of 2-cells. If we show the interchange law diagrammatically, we have

$$\begin{tikzcd} a \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] & b \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] & c \end{tikzcd} = \begin{tikzcd} a \arrow[r, bend left=60, "g\sharp_0 f"] \arrow[r, bend right=60, "g''\sharp_0 f''"'] \arrow[r, phantom, "\Downarrow (\beta'\sharp_1\beta)\sharp_0(\alpha'\sharp_1\alpha)=(\beta'\sharp_0\alpha')\sharp_1(\beta\sharp_0\alpha)."] & c; \end{tikzcd}$$

$$(\beta'\sharp_1\beta)\sharp_0(\alpha'\sharp_1\alpha) = (\beta'\sharp_0\alpha')\sharp_1(\beta\sharp_0\alpha).$$

We can upgrade this structure as much as we want. If we add 3-cells, we have




Note that the 2-source and 2-target of 3-cell $\pi$ are $s_2(\pi) = \alpha_1$ and $t_2(\pi) = \alpha_2$, respectively. Thus, we would expect to have a vertical composition $\sharp_2$ for 3-cells. Of course, it is necessary to satisfy the interchange law for 3-cells. If the morphisms of these 1-, 2-, and 3-cells are invertible, then this structure is a 3-groupoid.

We will consider the tricategory structure, which is a particular type of the 3-category structure. Tricategory is some kind of algebraic notation for weak 3-categories. We recall the definition of tricategory given in [9,17].

**Definition 2.1.** A tricategory $\mathfrak{T}$ consists of following data subject to the following axioms.
**DATA:**
(1) A set $Ob\mathfrak{T}$ of objects of $\mathfrak{T}$.
(2) For $(a, b) \in Ob\mathfrak{T} \times Ob\mathfrak{T}$, a bicategory (weak 2-category) $\mathfrak{T}(a, b)$, called the hom-bicategory of $\mathfrak{T}$ at $a$ and $b$, exists. The objects of $\mathfrak{T}(a, b)$ will be referred to as the 1-cells of $\mathfrak{T}$ with source $a$ and target $b$, the arrows of $\mathfrak{T}(a, b)$ will be referred to as 2-cells of $\mathfrak{T}$ (with their same source and target), and the 2-cells of $\mathfrak{T}(a, b)$ will be referred to as 3-cells of $\mathfrak{T}$ (also with their same source and target).
(3) For objects $a$, $b$, and $c$ of $Ob\mathfrak{T}$, a functor $\otimes:\mathfrak{T}(b, c) \times \mathfrak{T}(a, b) \longrightarrow \mathfrak{T}(a, c)$ exists called composition.
(4) For an object $a$ of $Ob\mathfrak{T}$, a functor $I_a : 1 \longrightarrow \mathfrak{T}(a, a)$, exists where 1 denotes the unit bicategory.
(5) For objects $a$, $b$, $c$, and $d$ of $Ob\mathfrak{T}$, an adjoint equivalence $\alpha$

$$\begin{tikzcd}
\mathfrak{T}(c,d) \times \mathfrak{T}(b,c) \times \mathfrak{T}(a,b) \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathfrak{T}(b,d) \times \mathfrak{T}(a,b) \arrow[d, "\otimes"] \\
\mathfrak{T}(c,d) \times \mathfrak{T}(a,c) \arrow[r, "\otimes"'] & \mathfrak{T}(a,d)
\end{tikzcd}$$

in **Bicat** $(\mathfrak{T}(c, d) \times \mathfrak{T}(b, c) \times \mathfrak{T}(a, b), \mathfrak{T}(a, d))$ exists.
(6) For objects $a$, $b$ of $Ob\mathfrak{T}$, adjoint equivalences $\mathfrak{l}$ and $\mathfrak{r}$

$$\begin{tikzcd}
 & \mathfrak{T}(b,b) \times \mathfrak{T}(a,b) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{l}"] & \\
\mathfrak{T}(a,b) \arrow[ur, "I_b \times 1"] \arrow[rr, "1"'] & {} & \mathfrak{T}(a,b)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathfrak{T}(a,b) \times \mathfrak{T}(a,a) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{r}"] & \\
\mathfrak{T}(a,b) \arrow[ur, "1 \times I_a"] \arrow[rr, "1"'] & {} & \mathfrak{T}(a,b)
\end{tikzcd}$$

in **Bicat** $(\mathfrak{T}(a, b), \mathfrak{T}(a, b))$ exists.

(7) For $a, b, c, d, e \in Ob\mathfrak{T}$, an isomorphism 2-cell $\pi$ ( i.e., an invertible modification)

$$
\begin{tikzcd}
& \mathfrak{T}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, "1\times\otimes\times1"] & & \mathfrak{T}^3 \arrow[dr, "\otimes\times1"] \arrow[dl, phantom, "\Downarrow\alpha\times1"] & \\
\mathfrak{T}^3 \arrow[dr, "1\times\otimes"'] \arrow[rr, phantom, "\overset{1\times\alpha}{\Longleftarrow}"] & & \mathfrak{T}^3 \arrow[rr, "\otimes\times1"] \arrow[dl, "1\times\otimes"] \arrow[dr, phantom, "\Downarrow\alpha\times1"] & & \mathfrak{T}^2 \arrow[dl, "\otimes"] \\
& \mathfrak{T}^2 \arrow[rr, "\otimes"'] & & \mathfrak{T} &
\end{tikzcd}
\overset{\pi}{\Rrightarrow}
\begin{tikzcd}
& \mathfrak{T}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, phantom, "="] & & \mathfrak{T}^3 \arrow[dr, "\otimes\times1"] \arrow[dl, "1\times\otimes"'] & \\
\mathfrak{T}^3 \arrow[rr, "\otimes\times1"'] \arrow[dr, "1\times\otimes"'] \arrow[drr, phantom, "\Downarrow\alpha"] & & \mathfrak{T}^2 \arrow[dr, "\otimes"] \arrow[rr, phantom, "\overset{\alpha}{\Longleftarrow}"] & & \mathfrak{T}^2 \arrow[dl, "\otimes"] \\
& \mathfrak{T}^2 \arrow[rr, "\otimes"'] & & \mathfrak{T} &
\end{tikzcd}
$$

in the bicategory **Bicat** $(\mathfrak{T}^4(a, b, c, d, e), \mathfrak{T}(a, e))$ exists, where $\mathfrak{T}^4 = \mathfrak{T}^4(a, b, c, d, e)$ is abbreviation for $\mathfrak{T}(d, e) \times \mathfrak{T}(c, d) \times \mathfrak{T}(b, c) \times \mathfrak{T}(a, b)$.

$$
\begin{tikzcd}
\mathfrak{T}^2 \arrow[drr, bend left, "1"] \arrow[dr, "1\times I\times1"] \arrow[ddr, bend right, "1"'] & \arrow[d, phantom, "\Downarrow\mathfrak{r}\times1"] & \\
\arrow[r, phantom, "\overset{1\times\mathfrak{l}}{\Longleftarrow}"] & \mathfrak{T}^3 \arrow[r, "\otimes\times1"] \arrow[d, "1\times\otimes"] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathfrak{T}^2 \arrow[d, "\otimes"] \\
& \mathfrak{T}^2 \arrow[r, "\otimes"'] & \mathfrak{T}
\end{tikzcd}
\overset{\mu}{\Rrightarrow}
\begin{tikzcd}
\mathfrak{T}^2 \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] \arrow[dr, phantom, "\Downarrow 1"] & & \\
& & \mathfrak{T}^2 \arrow[d, "\otimes"] \\
& \mathfrak{T}^2 \arrow[r, "\otimes"'] & \mathfrak{T}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathfrak{T}^3 \arrow[dr, "\otimes\times1"] \arrow[d, phantom, "\Downarrow\mathfrak{l}\times1"] & \\
\mathfrak{T}^2 \arrow[ur, "I\times1\times1"] \arrow[rr, "1"'] \arrow[d, "\otimes"'] & & \mathfrak{T}^2 \arrow[d, "\otimes"] \\
\mathfrak{T} \arrow[rr, "1"'] & = & \mathfrak{T}
\end{tikzcd}
\overset{\lambda}{\Rrightarrow}
\begin{tikzcd}
& \mathfrak{T}^3 \arrow[dr, "\otimes\times1"] \arrow[dd, "1\times\otimes"] & \\
\mathfrak{T}^2 \arrow[ur, "I\times1\times1"] \arrow[ddd, "\otimes"'] \arrow[r, phantom, "="] & & \mathfrak{T}^2 \arrow[ddd, "\otimes"] \arrow[l, phantom, "\Downarrow\alpha"] \\
& \mathfrak{T}^2 \arrow[ddr, "\otimes"] \arrow[dd, phantom, "\Downarrow\mathfrak{l}"] & \\
& & \\
\mathfrak{T} \arrow[uur, "I\times1"] \arrow[rr, "1"'] & & \mathfrak{T}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathfrak{T} \arrow[rr, "1"] & & \mathfrak{T} \\
\mathfrak{T}^2 \arrow[u, "\otimes"] \arrow[rr, "1"] \arrow[dr, "1\times1\times I"'] & = & \mathfrak{T}^2 \arrow[u, "\otimes"'] \\
& \mathfrak{T}^3 \arrow[ur, "1\times\otimes"'] \arrow[u, phantom, "\Downarrow1\times r"] &
\end{tikzcd}
\overset{\rho}{\Rrightarrow}
\begin{tikzcd}
\mathfrak{T} \arrow[rr, "1"] \arrow[dr, "1\times I"'] & \arrow[d, phantom, "r"] & \mathfrak{T} \\
& \mathfrak{T}^2 \arrow[ur, "\otimes"'] & \\
\mathfrak{T}^2 \arrow[uu, "\otimes"] \arrow[dr, "1\times1\times I"'] \arrow[r, phantom, "="] & & \mathfrak{T}^2 \arrow[uu, "\otimes"'] \arrow[l, phantom, "\Rightarrow\alpha"] \\
& \mathfrak{T}^3 \arrow[uu, "\otimes\times1"] \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}
$$

(8) For objects $a, b, c$ of $\mathfrak{T}$, invertible modifications exist.

**AXIOMS:**

(1) [Non-abelian 4-cocycle condition] The following equation of 2-cells holds in the bicategory $\mathfrak{T}(a_1, a_5)$, where we will denote the 2-cell elements by $a_i$, $1 \le i \le 14$ for easy reading of the diagrams. For elements $k, j, h, g$, and $f$ of 2-cells in $\mathfrak{T}$

$$
\begin{aligned}
a_1 &= (k\otimes\ (j\otimes\ (h\otimes\ g)))\otimes\ f, & a_2 &= k\otimes\ ((j\otimes\ (h\otimes\ g))\otimes\ f),\\
a_3 &= k\otimes\ (j\otimes\ ((h\otimes\ g)\otimes\ f)), & a_4 &= k\otimes\ (j\otimes\ (h\otimes\ (g\otimes\ f))),\\
a_5 &= (k\otimes\ j)\otimes\ (h\otimes\ (g\otimes\ f)), & a_6 &= ((k\otimes\ j)\otimes\ h)\otimes\ (g\otimes\ f),\\
a_7 &= (((k\otimes\ j)\otimes\ h)\otimes\ g)\otimes\ f, & a_8 &= ((k\otimes\ (j\otimes\ h))\otimes\ g)\otimes\ f\\
a_9 &= (k\otimes\ ((j\otimes\ h)\otimes\ g))\otimes\ f, & a_{10} &= k\otimes\ (((j\otimes\ h)\otimes\ g)\otimes\ f),\\
a_{11} &= (k\otimes\ (j\otimes\ h))\otimes\ (g\otimes\ f), & a_{12} &= k\otimes\ ((j\otimes\ h)\otimes\ (g\otimes\ f)),\\
a_{13} &= ((k\otimes\ j)\otimes\ (h\otimes\ g))\otimes\ f, & a_{14} &= (k\otimes\ j)\otimes\ ((h\otimes\ g)\otimes\ f);
\end{aligned}
$$

$$
\begin{tikzcd}
 & & & a_1 \arrow[drr, "\alpha"] & & & \\
 & a_9 \arrow[urr, "(1\alpha)1"] \arrow[drr, "\alpha"] \arrow[rrrr, phantom, "\cong"] & & & & a_2 \arrow[dr, "1\alpha"] & \\
a_8 \arrow[ur, "\alpha 1"] \arrow[drr, "\alpha"] \arrow[rrr, phantom, "\Downarrow\pi"] & & & a_{10} \arrow[urr, "1(\alpha 1)"] \arrow[dr, "1\alpha"] \arrow[rrr, phantom, "\Downarrow 1\pi"] & & & a_3 \arrow[dd, "1(1\alpha)"] \\
 & & a_{11} \arrow[rr, "\alpha"] \arrow[ll, phantom, "\cong"] & & a_{12} \arrow[drr, "1\alpha"] & & \\
a_7 \arrow[uu, "(\alpha 1)1"] \arrow[drr, "\alpha"] & & & & & & a_4 \\
 & & a_6 \arrow[uu, "\alpha(11)"] \arrow[rr, "\alpha"] \arrow[urrrr, phantom, "\Downarrow\pi"] & & a_5 \arrow[urr, "\alpha"] & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & & & a_1 \arrow[drr, "\alpha"] & & & \\
 & a_9 \arrow[urr, "(1\alpha)1"] & & & & a_2 \arrow[dr, "1\alpha"] & \\
a_8 \arrow[ur, "\alpha 1"] \arrow[rr, phantom, "\Downarrow 1\pi"] & & & & & & a_3 \arrow[ddd, "1(1\alpha)"] \arrow[llll, phantom, "\Downarrow\pi"] \\
 & & a_{13} \arrow[uuur, "\alpha 1"] \arrow[drr, "\alpha"] & & & & \\
 & & & & a_{14} \arrow[uurr, "\alpha"] \arrow[dd, "(11)\alpha"] \arrow[drr, phantom, "\cong"] & & \\
a_7 \arrow[uuu, "(\alpha 1)1"] \arrow[uurr, "\alpha 1"] \arrow[drr, "\alpha"] \arrow[rrrr, phantom, "\Downarrow\pi"] & & & & & & a_4 \\
 & & a_6 \arrow[rr, "\alpha"] & & a_5. \arrow[urr, "\alpha"] & &
\end{tikzcd}
$$

(2) [Left normalization] The following equation of 2-cells holds in the bicategory $\mathfrak{T}(a_1, a_4)$, where the unmarked isomorphisms are either naturality isomorphisms for $\alpha$ or unique coherence isomorphisms from the *Hom*-bicategory.

$$a_1 = (h \otimes (l \otimes g)) \otimes f, \quad a_2 = (h \otimes g) \otimes f, \quad a_3 = h \otimes (g \otimes f),$$
$$a_4 = ((h \otimes l) \otimes g) \otimes f, \quad a_5 = (h \otimes l) \otimes (g \otimes f), \quad a_6 = h \otimes ((l \otimes g) \otimes f),$$
$$a_7 = h \otimes (l \otimes (g \otimes f));$$

$$\begin{tikzcd}
 & & a_1 \arrow[dd, "\alpha"] \arrow[dr, "(1\mathfrak{l})1"] & \\
 & a_4 \arrow[ur, "\alpha 1"] \arrow[dd, "\alpha"] \arrow[r, phantom, "\Downarrow\pi"] & & a_2 \arrow[dddd, "\alpha"] \\
 & & a_6 \arrow[dd, "1\alpha"] \arrow[dddr, "1(\mathfrak{l}1)"] \arrow[ur, phantom, "\cong"] \arrow[ddr, phantom, "\Downarrow\lambda"] & \\
a_2 \arrow[uur, "(\mathfrak{r}1)1"] \arrow[ddr, "\alpha"'] \arrow[r, phantom, "\cong"] & a_5 \arrow[dr, "\alpha"] & & \\
 & & a_7 \arrow[dr, "1\mathfrak{l}"] \arrow[d, phantom, "\Downarrow\mu"] & \\
 & a_3 \arrow[uu, "\mathfrak{r}(11)"] \arrow[rr, bend right, "1"'] & \, & a_3
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
 & & a_1 \arrow[dr, "(1\mathfrak{l})1"] \arrow[d, phantom, "\Downarrow\mu"] & \\
 & a_4 \arrow[ur, "\alpha 1"] & \, & a_2 \arrow[dd, "\alpha"] \\
a_2 \arrow[ur, "(\mathfrak{r}1)1"] \arrow[urrr, "11"] \arrow[dr, "\alpha"'] & & \cong & \\
 & a_3 \arrow[rr, "1"'] & & a_3 .
\end{tikzcd}$$

(3) [Right normalization] The following equation of 2-cells holds in the bicategory $\mathfrak{T}(a_1, a_4)$.

$$a_1 = h \otimes ((g \otimes l) \otimes f), \quad a_2 = h \otimes (g \otimes (l \otimes f)), \quad a_3 = h \otimes (g \otimes f),$$
$$a_4 = (h \otimes g) \otimes f, \quad a_5 = (h \otimes (g \otimes l)) \otimes f, \quad a_6 = ((h \otimes g) \otimes l) \otimes f,$$
$$a_7 = (h \otimes g) \otimes (l \otimes f);$$

$$\begin{tikzcd}
 & a_1 \arrow[dr, "1\alpha"] & & \\
a_3 \arrow[ur, "1(\mathfrak{r}1)"] \arrow[r, phantom, "\cong"] & \, & a_2 \arrow[ddr, "1(1\mathfrak{l})"] & \\
 & a_5 \arrow[uu, "\alpha"] \arrow[r, phantom, "\Downarrow\pi"] & \, & \\
 & & a_7 \arrow[uu, "\alpha"] \arrow[dd, "(11)\mathfrak{l}"'] \arrow[r, phantom, "\cong"] & a_3 \\
 & a_6 \arrow[uu, "\alpha 1"] \arrow[ur, "\alpha"] \arrow[d, phantom, "\Downarrow\mu"] & & \\
a_4 \arrow[uuuu, "\alpha"] \arrow[uuur, "(1\mathfrak{r})1"] \arrow[ur, "\mathfrak{r}1"'] \arrow[u, phantom, "\Downarrow\rho 1"] \arrow[rr, bend right, "1"'] & \, & a_4 \arrow[uur, "\alpha"'] &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
 & a_1 \arrow[dr, "1\alpha"] \arrow[d, phantom, "\Downarrow\mu"] & & \\
a_3 \arrow[ur, "1(\mathfrak{r}1)"] \arrow[drrr, "11"] & \, & a_2 \arrow[dr, "1(1\mathfrak{l})"] & \\
 & \cong & & a_3 \\
a_4 \arrow[uu, "\alpha"] \arrow[rr, "1"'] & & a_4 . \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

**Example 2.2.**
(1) A monoidal bicategory is a tricategory with a single object.
(2) **BiCat** is a tricategory with objects being bicategories; 1-cells are homomorphisms between these bicategories; 2-cells are pseudonatural transformations; and 3-cells are modifications. As seen in [18], **BiCat** is not triequivalent to **Gray**, but it can be **Gray**-categorized in a special way [17].
(3) Fundamental 3-groupoids are tricategories. 3-groupoids are a generalization of 2-groupoids in higher-dimensional category theory. If we look at fundamental groupoids with a higher categorical approach, the fundamental 3-groupoid is a tricategory whose objects are spaces, 1-cells are continuous maps, and higher cells are the appropriate mediator homotopy types. This structure is explored in detail in [19]. In [17], in terms of the fundamental 3-groupoid, the objects of an $X$ space are points of $X$, and the higher cells are paths, two-dimensional disks, and equivalence classes of three-dimensional disks in $X$. This structure is also a tricategory.

**Remark 2.3.** A tricategory is strict if all of its constraints are identities; that is, if the invertible modifications (5)–(8) in Definition 2.1 are identities [9].

**Lemma 2.4.** *Let $\mathfrak{T}$ be a tricategory; if it satisfies the following conditions, it is a cubical tricategory* [17].
(1) *Each bicategory $\mathfrak{T}(a, b)$ is a strict 2-category.*
(2) *Each functor $I_a : 1 \longrightarrow \mathfrak{T}(a, a)$ is a cubical functor.*
(3) *For objects $a, b$, and $c$ of $Ob\mathfrak{T}$, each $\otimes : \mathfrak{T}(b, c) \times \mathfrak{T}(a, b) \longrightarrow \mathfrak{T}(a, c)$ is a cubical functor.*

**Definition 2.5.** Let $\mathfrak{T}$ and $\mathfrak{T}'$ be tricategories. A trihomomorphism $\mathfrak{F}$ from $\mathfrak{T}$ to $\mathfrak{T}'$ is a map that consists of the following data and axioms:
**DATA:**
(1) A morphism $Ob\mathfrak{T} \rightarrow Ob\mathfrak{T}'$.
(2) For $a, b \in Ob\mathfrak{T}$, a functor $\mathfrak{F}_{ab} : \mathfrak{T}(a, b) \rightarrow \mathfrak{T}'(\mathfrak{F}a, \mathfrak{F}b)$.
(3) For $a, b$, and $c \in Ob\mathfrak{T}$, an adjoint equivalence $\xi : \otimes' \circ (\mathfrak{F} \times \mathfrak{F}) \Rightarrow \mathfrak{F} \circ \otimes$ with left adjoint is illustrated in the following diagram:

$$\begin{tikzcd}
\mathfrak{T}(b,c) \times \mathfrak{T}(a,b) \arrow[r, "\mathfrak{F}\times\mathfrak{F}"] \arrow[d, "\otimes"'] & \mathfrak{T}'(\mathfrak{F}b, \mathfrak{F}c) \times \mathfrak{T}'(\mathfrak{F}a, \mathfrak{F}b) \arrow[d, "\otimes'"] \arrow[dl, phantom, "\Swarrow\xi"] \\
\mathfrak{T}(a,c) \arrow[r, "\mathfrak{F}"'] & \mathfrak{T}'(\mathfrak{F}a, \mathfrak{F}c);
\end{tikzcd}$$

(4) For all $a \in Ob\mathfrak{T}$, an adjoint equivalence $\iota : I'_{\mathfrak{F}a} \Rightarrow \mathfrak{F} \circ I_a$ with left adjoint is illustrated in the following diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "I'_{\mathfrak{F}a}"] \arrow[dr, "I_a"'] & {} \arrow[d, phantom, "\Downarrow\iota"] & \mathfrak{T}'(\mathfrak{F}a, \mathfrak{F}a) \\
& \mathfrak{T}(a,a); \arrow[ur, "\mathfrak{F}"'] &
\end{tikzcd}$$

(5) For $a, b, c, d \in Ob\mathfrak{T}$, an invertible modification $\theta$ is illustrated in the following diagram:

$$
\begin{tikzcd}
& \mathfrak{T}^3 \arrow[rr, "\mathfrak{F}\times\mathfrak{F}\times\mathfrak{F}"] \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"] & & \mathfrak{T}'^3 \arrow[dr, "\otimes'\times 1"] & \\
\mathfrak{T}^2 \arrow[dr, "\otimes"'] \arrow[rr, phantom, "\overset{\alpha}{\Longleftarrow}"] & & \mathfrak{T}^2 \arrow[rr, "\mathfrak{F}\times\mathfrak{F}"] \arrow[dl, "\otimes"] \arrow[ur, phantom, "\Downarrow\xi\times 1"] \arrow[dr, phantom, "\Downarrow\xi"] & & \mathfrak{T}'^2 \arrow[dl, "\otimes'"] \\
& \mathfrak{T} \arrow[rr, "\mathfrak{F}"'] & & \mathfrak{T}' &
\end{tikzcd}
\quad \overset{\theta}{\Rrightarrow} \quad
\begin{tikzcd}
& \mathfrak{T}^3 \arrow[rr, "\mathfrak{F}\times\mathfrak{F}\times\mathfrak{F}"] \arrow[dl, "1\times\otimes"'] & & \mathfrak{T}'^3 \arrow[dl, "1\times\otimes'"'] \arrow[dr, "\otimes'\times 1"] & \\
\mathfrak{T}^2 \arrow[dr, "\otimes"'] \arrow[rr, "\mathfrak{F}\times\mathfrak{F}"'] \arrow[ur, phantom, "\Downarrow 1\times\xi"] & & \mathfrak{T}'^2 \arrow[dr, "\otimes'"] \arrow[dl, phantom, "\Downarrow\xi"] \arrow[rr, phantom, "\overset{\alpha'}{\Longleftarrow}"] & & \mathfrak{T}'^2 \arrow[dl, "\otimes'"] \\
& \mathfrak{T} \arrow[rr, "\mathfrak{F}"'] & & \mathfrak{T}'; &
\end{tikzcd}
$$

(6) For $a, b \in Ob\mathfrak{T}$, invertible modifications $\kappa$ and $\chi$ is illustrated in the following diagram:

$$
\begin{tikzcd}
& \mathfrak{T}'^2 \arrow[dr, "\otimes'"] & \\
\mathfrak{T}' \arrow[ur, "I'\times 1"] \arrow[r, phantom, "\overset{\iota\times 1}{\Longrightarrow}"] & \phantom{x} \arrow[r, phantom, "\overset{\xi}{\Rightarrow}"] & \mathfrak{T}' \\
& \mathfrak{T}^2 \arrow[uu, "\mathfrak{F}\times\mathfrak{F}"] \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{l}"] & \\
\mathfrak{T} \arrow[uu, "\mathfrak{F}"] \arrow[ur, "I\times 1"] \arrow[rr, "1"'] & \phantom{x} & \mathfrak{T} \arrow[uu, "\mathfrak{F}"']
\end{tikzcd}
\quad \overset{\kappa}{\Longrrightarrow} \quad
\begin{tikzcd}
& \mathfrak{T}'^2 \arrow[dr, "\otimes'"] \arrow[d, phantom, "\Downarrow\mathfrak{l}'"] & \\
\mathfrak{T}' \arrow[ur, "I'\times 1"] \arrow[rr, "1"] & \phantom{x} & \mathfrak{T}' \\
\mathfrak{T} \arrow[u, "\mathfrak{F}"] \arrow[rr, "1"'] \arrow[urr, phantom, "="] & & \mathfrak{T}, \arrow[u, "\mathfrak{F}"']
\end{tikzcd}
$$

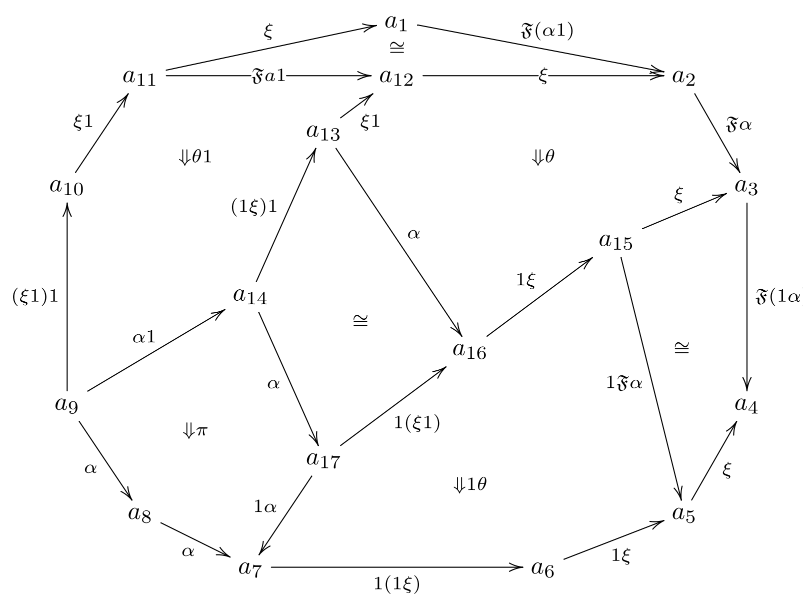
and

$$
\begin{tikzcd}
\mathfrak{T}' \arrow[rr, "1"] & & \mathfrak{T}' \\
\mathfrak{T} \arrow[u, "\mathfrak{F}"] \arrow[rr, "1"] \arrow[dr, "1\times I"'] \arrow[urr, phantom, "="] & \phantom{x} \arrow[d, phantom, "\Downarrow\mathfrak{r}"] & \mathfrak{T} \arrow[u, "\mathfrak{F}"'] \\
& \mathfrak{T}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}
\quad \overset{\chi}{\Rrightarrow} \quad
\begin{tikzcd}
\mathfrak{T}' \arrow[rr, "1"] \arrow[dr, "1\times I'"'] & \phantom{x} \arrow[d, phantom, "\Downarrow\mathfrak{r}'"] & \mathfrak{T}' \\
\phantom{x} \arrow[r, phantom, "\overset{1\times\iota}{\Longrightarrow}"] & \mathfrak{T}'^2 \arrow[ur, "\otimes'"] \arrow[r, phantom, "\overset{\xi}{\Rightarrow}"] & \phantom{x} \\
\mathfrak{T} \arrow[uu, "\mathfrak{F}"] \arrow[dr, "1\times I"'] & & \mathfrak{T} \arrow[uu, "\mathfrak{F}"'] \\
& \mathfrak{T}^2. \arrow[uu, "\mathfrak{F}\times\mathfrak{F}"] \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

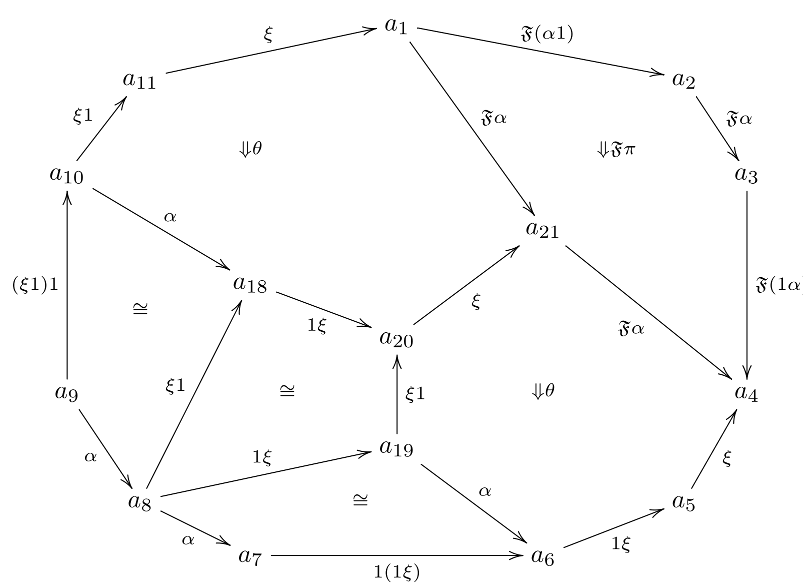
**AXIOMS:**

(1) For all 1-morphisms $(x, y, z, w) \in \mathfrak{T}(d, e) \times \mathfrak{T}(c, d) \times \mathfrak{T}(b, c) \times \mathfrak{T}(a, b)$, the following equation of modifications holds:

$$
\begin{array}{lll}
a_1 = \mathfrak{F}(((fg)j)k), & a_2 = \mathfrak{F}((f(gj))k), & a_3 = \mathfrak{F}(f((gj)k)), \\
a_4 = \mathfrak{F}(f(g(jk))), & a_5 = \mathfrak{F}f\,\mathfrak{F}(g(jk)), & a_6 = \mathfrak{F}f(\mathfrak{F}g\,\mathfrak{F}(jk)), \\
a_7 = \mathfrak{F}f(\mathfrak{F}g(\mathfrak{F}j\mathfrak{F}k)), & a_8 = (\mathfrak{F}f\,\mathfrak{F}g)(\mathfrak{F}j\mathfrak{F}k), & a_9 = ((\mathfrak{F}f\,\mathfrak{F}g)\mathfrak{F}j)\mathfrak{F}k, \\
a_{10} = (\mathfrak{F}(fg)\mathfrak{F}j)\mathfrak{F}k, & a_{11} = \mathfrak{F}((fg)j)\mathfrak{F}k, & a_{12} = \mathfrak{F}(f(gj))\mathfrak{F}k, \\
a_{13} = (\mathfrak{F}f\,\mathfrak{F}(gj))\mathfrak{F}k, & a_{14} = (\mathfrak{F}f(\mathfrak{F}g\mathfrak{F}j))\mathfrak{F}k, & a_{15} = \mathfrak{F}f\,\mathfrak{F}((gj))k, \\
a_{16} = \mathfrak{F}f(\mathfrak{F}(gj)\mathfrak{F}k), & a_{17} = \mathfrak{F}f(\mathfrak{F}(gj)\mathfrak{F}k), & a_{18} = \mathfrak{F}(fg)(\mathfrak{F}j\mathfrak{F}k), \\
a_{19} = (\mathfrak{F}f\,\mathfrak{F}g)\mathfrak{F}(jk), & a_{20} = \mathfrak{F}(fg)\mathfrak{F}(jk), & a_{21} = \mathfrak{F}((fg)(jk));
\end{array}
$$




$$\|$$




(2) For all 1-cells $(x, y) \in \mathfrak{T}(b, c) \times \mathfrak{T}(a, b)$, the following equation of modifications holds:

$$a_1 = \mathfrak{F}((fI)g), \quad a_2 = \mathfrak{F}(f(Ig)), \quad a_3 = \mathfrak{F}(fg),$$
$$a_4 = \mathfrak{F}f\,\mathfrak{F}g, \quad a_5 = \mathfrak{F}f\,\mathfrak{F}(Ig), \quad a_6 = \mathfrak{F}f(\mathfrak{F}I\,\mathfrak{F}g),$$
$$a_7 = (\mathfrak{F}f\,\mathfrak{F}I)\mathfrak{F}g, \quad a_8 = \mathfrak{F}(fI)\mathfrak{F}g, \quad a_9 = (\mathfrak{F}fI)\mathfrak{F}g,$$
$$a_{10} = \mathfrak{F}f(I\mathfrak{F}g);$$

$$
\begin{tikzcd}
 & & a_1 \arrow[rrr, "\mathfrak{F}\alpha"] \arrow[drr, phantom, "\Downarrow\theta"] & & & a_2 \arrow[dddr, "\mathfrak{F}(1I)"] & \\
 & & & & a_5 \arrow[ur, "\xi"] \arrow[dddddrr, "1\mathfrak{F}I"] \arrow[ddrr, phantom, "\cong"] & & \\
 & & & a_6 \arrow[ur, "1\xi"] & & & \\
a_3 \arrow[uuurr, "\mathfrak{F}(\mathfrak{r}1)"] \arrow[r, phantom, "\cong"] & & & & & & a_3 \\
 & a_8 \arrow[uuuur, "\xi"] & a_7 \arrow[l, "\xi 1"] \arrow[uur, "\alpha"] \arrow[dr, phantom, "\cong"] & & & & \\
 & \arrow[r, phantom, "\overset{\chi 1}{\Rrightarrow}"] & a_9 \arrow[u, "(1\iota)1"] \arrow[r, "\alpha"] & a_{10} \arrow[uuu, "1(\iota 1)"] \arrow[drrr, "1I"] \arrow[uur, phantom, "\Downarrow 1\kappa"] \arrow[d, phantom, "\Downarrow\mu"] & & & \\
a_4 \arrow[uuu, "\xi"] \arrow[uur, "\mathfrak{F}\mathfrak{r}I"] \arrow[urr, "\mathfrak{r}1"] \arrow[rrrrrr, "1"] & & & & & & a_4 \arrow[uuu, "\xi"]
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & a_1 \arrow[rr, "\mathfrak{F}\alpha"] \arrow[dr, phantom, "\Downarrow\mathfrak{F}\mu"] & & a_2 \arrow[dr, "\mathfrak{F}(1I)"] & \\
a_3 \arrow[ur, "\mathfrak{F}(\mathfrak{r}1)"] \arrow[rrrr, "1"] \arrow[drr, phantom, "\cong"] & & & & a_3 \\
a_4 \arrow[u, "\xi"] \arrow[rrrr, "1"] & & & & a_4 . \arrow[u, "\xi"]
\end{tikzcd}
$$

# 3 Algebraic type models for homotopy theory

## 3.1 Algebraic 2-types

In this section, we will give information about crossed modules.

**Definition 3.1.** A crossed module of groups, $\mathcal{G}_1 = (E \xrightarrow{\partial} G)$, consists of two groups $G$ and $E$ with a left group action of $G$ on $E$, and group homomorphim $\partial : E \to G$ satisfying the following conditions:
$\mathcal{XMOD}$-1 $\partial({}^{g}e) = g\partial(e)g^{-1}$, $\forall g \in G$ and $e \in E$,
$\mathcal{XMOD}$-2 ${}^{\partial(e)}e' = ee'e^{-1}$, $\forall e, e' \in E$.

If $\mathcal{G}_1 = (E \xrightarrow{\partial} G)$ only satisfies the $\mathcal{XMOD}$-1 condition, it is a pre-crossed module.

**Example 3.2.**

(1) Let $G$ be a group and $\mathrm{Aut}(G)$ be the automorphism group of group $G$. With the conjugation homomorphism $\phi$, the following structure is a crossed module:

$$\phi : G \longrightarrow Aut(G).$$

This structure is called the automorphism crossed module of group $G$.

(2) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration sequence of pointed spaces. Then, $p$ is a fibration that satisfies $F = p^{-1}(b_0)$, where $b_0$ is the basepoint of $B$. Say that the fibre $F$ is pointed at $f_0$, and $f_0$ is also considered to be the basepoint of $E$. Then, we have an induced map of fundamental groups as follows:

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E).$$

Triple $(\pi_1(F), \pi_1(E), \pi_1(i))$ is a crossed module. Refer [20] for detailed information.

**Definition 3.3.** Let $\mathcal{G}_1 = (E \xrightarrow{\partial} G)$ and $\mathcal{G}_1' = (E' \xrightarrow{\partial'} G')$ be two crossed modules of groups. A $f = (f_1, f_0)$ morphism of crossed modules from $\mathcal{G}_1$ to $\mathcal{G}_1'$ consists of group homomorphisms $f_1 : E \to E'$ and $f_0 : G \to G'$, also the following equations are satisfied:

(1) $f_0\partial = \partial' f_1$,

(2) $f_1({}^g e) = {}^{f_0(g)} f_1(e)$,

for each $g \in G$ and $e \in E$.

Thus, we can define the category $\mathcal{XMOD}_{Grp}$, formed by the crossed modules of groups and their morphisms.

Now let us categorically express the crossed module $\mathcal{G}_1 = (E \xrightarrow{\partial} G)$. Kamps and Porter [14] barely mentioned that one converts a crossed module to a 2-groupoid, but we will give in detail how to construct a 2-group $\mathfrak{G}(\mathcal{G}_1)$ (strict 2-groupoid with one object) as follows.

$\{\bullet\}$ is the 0-cells of the $\mathfrak{G}(\mathcal{G}_1)$ 2-groupoid that we will construct. The bottom group of the crossed module, $G$, forms the 1-cells (elements of group $G$) of this 2-groupoid. The composition "$\sharp_0$" is defined as follows:

$$\begin{tikzcd} \bullet \arrow[r, "g_2"] \arrow[rr, bend right, "g_1\sharp_0 g_2 = g_1 g_2"'] & \bullet \arrow[r, "g_1"] & \bullet. \end{tikzcd}$$

The semi-direct product group $E \rtimes G$ will give the 2-cells of $\mathfrak{G}(\mathcal{G}_1)$. For the 2-cell $(e, g)$, ($g \in G$ and $e \in E$), the 1-source and 1-target are $s_1((e, g)) = g$ and $t_1((e, g)) = \partial(e)g$, respectively. In addition, for each $g \in G$, $i_1(g) = (1_E, g)$. This is the picture of 2-cell $(e, g)$

$$(e, g) : g \Longrightarrow \partial(e)g : \bullet \longrightarrow \bullet,$$

from the 2-categorical point of view, as follows:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "\partial(e)g"'{name=D}] & \bullet. \arrow[Rightarrow, from=U, to=D, "(e,g)"] \end{tikzcd}$$

### 3.1.1 Horizontal composition of 2-cells

Suppose $(e_2, g_2) : g_2 \Longrightarrow \partial(e_2)g_2 : \bullet \longrightarrow \bullet$ and $(e_1, g_1) : g_1 \Longrightarrow \partial(e_1)g_1 : \bullet \longrightarrow \bullet \in E \rtimes G$, then $(e_1, g_1)\sharp_0(e_2, g_2) = (e_1, g_1)(e_2, g_2) = (e_1 e_2, g_1 g_2)$. Thus, the horizontal composition "$\sharp_0$" for 2-cells is defined as follows. First, let us define whiskering compositions as follows.

Whiskering as

$$\begin{tikzcd} \bullet \arrow[r, "g_2"] & \bullet \arrow[r, bend left=50, "g_1"{name=U}] \arrow[r, bend right=50, "\partial(e_1)g_1"'{name=D}] \arrow[from=U, to=D, Rightarrow, "(e_1,g_1)"] & \bullet \end{tikzcd} \quad resp. \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_2"{name=U}] \arrow[r, bend right=50, "\partial(e_2)g_2"'{name=D}] \arrow[from=U, to=D, Rightarrow, "(e_2,g_2)"] & \bullet \arrow[r, "g_1"] & \bullet \end{tikzcd}$$

clearly gives $(e_1, g_1g_2)$ and $({}^{g_1}e_2, g_1g_2)$, respectively. The duality of whiskerings is defined similarly. Naturally, the horizontal composition of 2-cells of 2-groupoid $\mathfrak{G}(\mathcal{G}_1)$ is shown from the 2-categorical point of view as follows:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_2"{name=U1}] \arrow[r, bend right=50, "\partial(e_2)g_2"'{name=D1}] \arrow[from=U1, to=D1, Rightarrow, "(e_2,g_2)"] & \bullet \arrow[r, bend left=50, "g_1"{name=U2}] \arrow[r, bend right=50, "\partial(e_1)g_1"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "(e_1,g_1)"] & \bullet \end{tikzcd} = \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=30, "g_1\sharp_0 g_2=g_1g_2"{name=U}] \arrow[r, bend right=30, "\partial(e_1)g_1\sharp_0\partial(e_2)g_2=\partial(e_1)g_1\partial(e_2)g_2"'{name=D}] \arrow[from=U, to=D, Rightarrow, "(e_1{,}g_1)\sharp_0(e_2{,}g_2)=(e_1\,{}^{g_1}e_2{,}g_1g_2)"] & \bullet \end{tikzcd}.$$

From the definition of the 1-source $s_1$ and 1-target $t_1$, we have

$$s_1((e_1, g_1)\sharp_0(e_2, g_2)) = s_1(e_1^{g_1}e_2, g_1g_2) = g_1g_2$$

and

$$\begin{aligned} t_1((e_1, g_1)\sharp_0(e_2, g_2)) &= t_1((e_1{}^{g_1}e_2, g_1g_2)) \\ &= \partial(e_1{}^{g_1}e_2)g_1g_2 \\ &= \partial(e_1)\partial({}^{g_1}e_2)g_1g_2 \\ &= \partial(e_1)g_1\partial(e_2)g_1^{-1}g_1g_2 \\ &= \partial(e_1)g_1\partial(e_2)g_2. \end{aligned}$$

In the last calculations, we used $\mathcal{XMOD}$-1 in Definition 3.1.

### 3.1.2 Vertical composition of 2-cells

Suppose $(e', \partial(e)g)$ is another element in $(E \rtimes G)$. The vertical composition "$\sharp_1$" of 2-cells is given by

$$(e', \partial(e)g)\sharp_1(e, g) = (e'e, g).$$

The vertical composition of 2-cells is illustrated as follows:

$$\begin{array}{c} \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=30, "g"{name=U}] \arrow[r, bend right=30, "\partial(e)g"'{name=D}] \arrow[from=U, to=D, Rightarrow, "(e,g)"] & \bullet \end{tikzcd} \\ \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=30, "\partial(e)g"{name=U}] \arrow[r, bend right=30, "\partial(e')\partial(e)g"'{name=D}] \arrow[from=U, to=D, Rightarrow, "(e'{,}\partial(e)g)"] & \bullet \end{tikzcd} \end{array} = \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=30, "g"{name=U}] \arrow[r, bend right=30, "\partial(e'e)g"'{name=D}] \arrow[from=U, to=D, Rightarrow, "(e'{,}\partial(e)g)\sharp_1(e{,}g)=(e'e{,}g)"] & \bullet \end{tikzcd},$$

$$s_1((e', \partial(e)g)\sharp_1(e, g)) = s_1((e'e, g)) = g,$$

and

$$t_1((e', \partial(e)g)\sharp_1(e, g)) = t_1((e'e, g)) = \partial(e'e)g.$$

### 3.1.3 Interchange law of 2-cells

The $\mathcal{XMOD}$-2 condition in the definition of crossed module of groups 3.1 establishes the interchange law between the horizontal and vertical compositions, and as a result, we have a strict 2-groupoid. The interchange law holds. It is explicitly expressed as

$$
\begin{aligned}
((e_1', \partial(e_1)g_1)\sharp_1(e_1, g_1))\sharp_0((e_2', \partial(e_2)g_2)\sharp_1(e_2, g_2)) &= (e_1'e_1, g_1)\sharp_0(e_2'e_2, g_2)\\
&= (e_1'e_1g_1(e_2'e_2), g_1g_2)\\
&= (e_1'e_1^{g_1}e_2'e_1^{-1}e_1{}^{g_1}e_2, g_1g_2)\\
&= (e_1'^{\partial(e_1)}({}^{g_1}e_2')e_1{}^{g_1}e_2, g_1g_2)\\
&= (e_1'^{(\partial(e_1)g_1)}e_2'e_2'e_1{}^{g_1}e_2, g_1g_2)\\
&= (e_1'^{(\partial(e_1)g_1)}e_2', \partial(e_1)g_1\partial(e_2)g_2)\sharp_1(e_1{}^{g_1}e_2, g_1g_2)\\
&= ((e_1', \partial(e_1)g_1)\sharp_0(e_2', \partial(e_2)g_2))\sharp_1((e_1, g_1)\sharp_0(e_2, g_2)),
\end{aligned}
$$

and it can be drawn as follows:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, bend left=70, "g_2"{name=A}] \arrow[r, "\partial(e_2)g_2"{name=B, description}] \arrow[r, bend right=70, "\partial(e_2'e_2)g_2"'{name=C}] \arrow[from=A, to=B, Rightarrow, "(e_2,g_2)"] \arrow[from=B, to=C, Rightarrow, "(e_2',\partial(e_2)g_2)"] & \bullet \arrow[r, bend left=70, "g_1"{name=D}] \arrow[r, "\partial(e_1)g_1"{name=E, description}] \arrow[r, bend right=70, "\partial(e_1'e_1)g_1"'{name=F}] \arrow[from=D, to=E, Rightarrow, "(e_1,g_1)"] \arrow[from=E, to=F, Rightarrow, "(e_1',\partial(e_1)g_1)"] & \bullet \arrow[r, phantom, "="] & \bullet \arrow[r, bend left=50, "g_1\sharp_0 g_2"{name=G}] \arrow[r, bend right=50, "\partial(e_1'e_1)g_1\sharp_0\partial(e_2'e_2)g_2"'{name=H}] \arrow[from=G, to=H, Rightarrow, "\begin{array}{c}\left((e_1',\partial(e_1)g_1)\sharp_1(e_1,g_1)\right)\sharp_0\left((e_2',\partial(e_2)g_2)\sharp_1(e_2,g_2)\right)\\ =\left((e_1',\partial(e_1)g_1)\sharp_0(e_2',\partial(e_2)g_2)\right)\sharp_1\left((e_1,g_1)\sharp_0(e_2,g_2)\right)\end{array}"] & \bullet
\end{tikzcd}
$$

Consequently, we construct the following strict 2-group structure:

$$
\mathfrak{G}(\mathcal{G}_1) = \left(\begin{tikzcd} E \rtimes G \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & G \arrow[l, bend left=60, "i_1"] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & \{\bullet\} \arrow[l, bend left=60, "i_0"] \end{tikzcd}\right).
$$

Conversely, any strict 2-group is a crossed module of groups. More specifically, let $\mathfrak{G}$ be a 2-groupoid with a single object as follows:

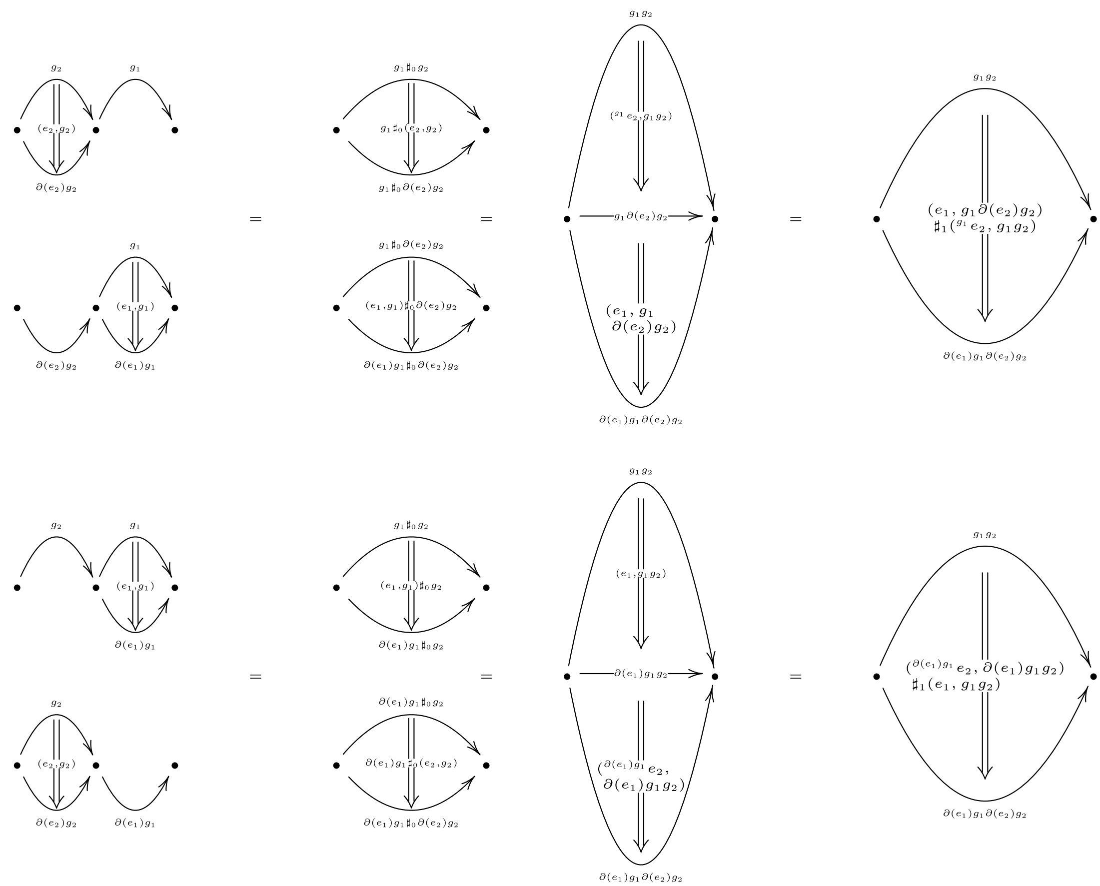
$$
\mathfrak{G} = \left(\begin{tikzcd} G_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & G_1 \arrow[l, bend left=60, "i_1"] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & G_0 = \{\bullet\} \arrow[l, bend left=60, "i_0"] \end{tikzcd}\right).
$$

To obtain the crossed module of groups from a strict 2-group $\mathfrak{G}$, we have $\{\bullet\} = G_0$, $G = G_1$, and $E = \operatorname{Ker} s_1 \subseteq G_2$. Let $\partial : E \to G$ be the restriction of the 1-target map $t_1 : G_2 \to G_1$, and the left action is as follows:

$$
{}^{g}e = i_1(g)\sharp_0 e\sharp_0 i_1(g)^{-1}
$$

for each $g \in G$ and $e \in E$.

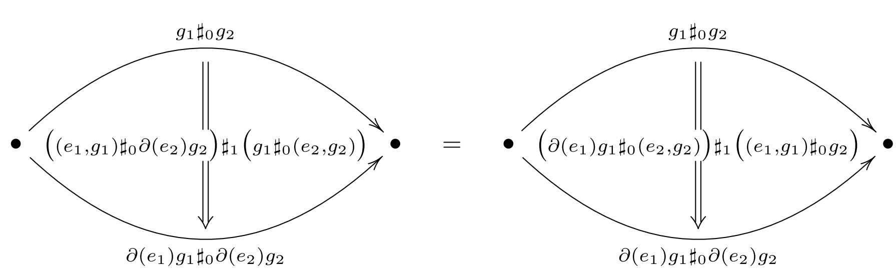
Naturally, any crossed module is a sesquicategory. If we look at it in more detail as follows, it satisfies the condition of being a sesquicategory becoming a 2-category in [12,21]. For all $g \in G$ and $e \in E$, we have:



Therefore, we have

$$
\begin{aligned}
((e_1, g_1)\sharp_0 \partial(e_2)g_2)\sharp_1(g_1\sharp_0(e_2, g_2)) &= (e_1, g_1\partial(e_2)g_2)\sharp_1({}^{g_1}e_2, g_1g_2) \\
&= (e_1{}^{g_1}e_2, g_1g_2) \\
&= (e_1{}^{g_1}e_2e_1^{-1}e_1, g_1g_2) \\
&= ({}^{\partial(e_1)}({}^{g_1}e_2)e_1, g_1g_2) \\
&= ({}^{\partial(e_1)g_1}e_2e_1, g_1g_2) \\
&= ({}^{\partial(e_1)g_1}e_2, \partial(e_1)g_1g_2)\sharp_1(e_1, g_1g_2) \\
&= (\partial(e_1)g_1\sharp_0(e_2, g_2))\sharp_1((e_1, g_1)\sharp_0 g_2).
\end{aligned}
$$

We can diagrammatically represent this sesquicategory condition as follows:



The 2-group construction that we obtained above can also be constructed from the bicategory perspective that Jean Bénabou gives in [10]. Roughly, in the bicategory $\mathfrak{B}(\mathcal{G}_1)$, we construct from the crossed module $\mathcal{G}_1$, $Ob\mathfrak{B}(\mathcal{G}_1) = \{\bullet\}$ is the set of objects, and for $\bullet \in Ob\mathfrak{B}(\mathcal{G}_1)$, $\mathfrak{B}(\mathcal{G}_1)(\bullet, \bullet)$ is a category, as shown below:

$$\mathfrak{B}(\mathcal{G}_1)(\bullet,\bullet) = \left( \begin{tikzcd} \bullet \arrow[rr, bend left, "g"{name=U}] \arrow[rr, bend right, "\partial(e)g"'{name=D}] & & \bullet . \arrow[from=U, to=D, Rightarrow, "(e,g)"] \end{tikzcd} \right)$$

- **Objects:** Every 1-cell, $g : \bullet \to \bullet \in G$, is an object.
- **1-Morphisms:** Every 2-cell, $(e, g) : g \Rightarrow \partial(e)g \in E \rtimes G$, is a 1-morphism. $\mathfrak{B}(\mathcal{G}_1)(\bullet, \bullet)$ is a strict category with the composition $\sharp_1$.

## 3.2 Algebraic 3-types

First of all, we give related knowledge about the 2-crossed modules of groups given in [13,22,23].

**Definition 3.4.** A 2-crossed module of groups $\mathcal{G}_2 = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \{-,-\})$ is a semi-exact sequence of G-groups, together with left group actions of $G$ on $E$ and $L$ (and on $G$ by conjugation), and a $G$- equivariant function,

$$\{-,-\} : E \times E \longrightarrow L\ ,$$

called the Peiffer lifting. Here $G$-equivariance means:

$${}^{g}\{e_1, e_2\} = \{{}^{g}e_1, {}^{g}e_2\}.$$

For each $g \in G$ and $e_1, e_2 \in E$ the following should be satisfied:
$2\mathcal{XMOD}$-1 $\partial\delta = 1$,
$2\mathcal{XMOD}$-2 $\delta(\{e_1, e_2\}) = \langle e_1, e_2\rangle = e_1e_2e_1^{-1}\,{}^{\partial(e_1)}e_2^{-1}$,
$2\mathcal{XMOD}$-3 $\{\delta(l_1), \delta(l_2)\} = [l_1, l_2] = l_1l_2l_1^{-1}l_2^{-1}$,
$2\mathcal{XMOD}$-4 $\{\delta(l), e\}\{e, \delta(l)\} = l^{\partial(e)}l_{-1}$,
$2\mathcal{XMOD}$-5 $\{e_1e_2, e_3\} = \{e_1, e_2e_3e_2^{-1}\}^{\partial(e_1)}\{e_2, e_3\}$,
$2\mathcal{XMOD}$-6 $\{e_1, e_2e_3\} = \{e_1, e_2\}^{(\partial(e_1)e_2)}\{e_1, e_3\}$,
for each $g \in G$, $e \in E$, and $l \in L$.

**Remark 3.5.** It should be noted that $(L \xrightarrow{\delta} E)$ is a crossed module, with

$${}_{e}l = l\{\delta(l)^{-1}, e\}$$

for each $e \in E$ and $l \in L$. However, $(E \xrightarrow{\partial} G)$ is just a pre-crossed module.

**Lemma 3.6.** *Let $\mathcal{G}_2 = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \{-,-\})$ be a 2-crossed module of groups. Then, ${}^{g}({}^{e}l) = {}^{g}e({}^{g}l)$, for each $g \in G$, $e \in E$, and $l \in L$* [22,23].

**Example 3.7.**

(1) From a functorial point of view, any crossed module of groups $E \xrightarrow{\partial} G$ is a 2-crossed module $1 \xrightarrow{0} E \xrightarrow{\partial} G$, with trivial Pfeifer lifting [20].
(2) The quadratic modules that Baues [24] defined are a special version of the 2-crossed modules, satisfying certain additional nilpotency conditions. Refer [25] for its relation to 2-crossed modules.

**Definition 3.8.** Let $\mathcal{G}_2 = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \{-,-\})$ and $\mathcal{G}' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} G', \{-,-\}')$ be two 2-crossed modules of groups. A morphism of 2-crossed modules from $\mathcal{G}$ to $\mathcal{G}'$ is illustrated by the following commutative diagram:

$$\begin{tikzcd}
L \arrow[r, "\delta"] \arrow[d, "f_2"'] & E \arrow[r, "\partial"] \arrow[d, "f_1"'] & G \arrow[d, "f_0"] \\
L' \arrow[r, "\delta'"'] & E' \arrow[r, "\partial'"'] & G',
\end{tikzcd}$$

where $f_2 : L \to L'$, $f_1 : E \to E'$, and $f_0 : G \to G'$ are group homomorphisms, and also the following equations are satisfied:

$$f_1({}^{g}e) = {}^{f_0(g)}f_1(e),$$
$$f_2({}^{g}l) = {}^{f_0(g)}f_2(l),$$
$$f_2(\{e_1, e_2\}) = \{f_1(e_1), f_1(e_2)\},$$

for each $g \in G$, $e \in E$, and $l \in L$.

We shall let $2\mathcal{XMOD}_{Grp}$, denote the category whose objects are 2-crossed modules of groups and morphisms are morphisms of 2-crossed modules, as defined above.

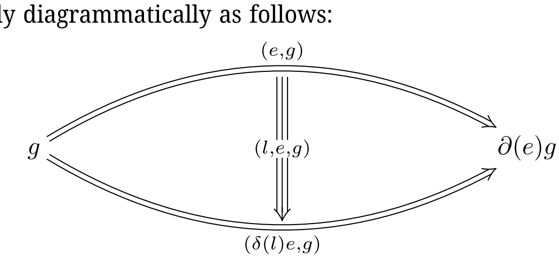
Now let us construct a 2-crossed module $\mathcal{G}_2 = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \{-,-\})$ as a categorical $\mathfrak{G}(\mathcal{G}_2)$. We have:

$$\mathfrak{G}(\mathcal{G}_2)_0 = \{\bullet\}, \quad \mathfrak{G}(\mathcal{G}_2)_1 = G, \quad \mathfrak{G}(\mathcal{G}_2)_2 = E \rtimes G, \quad \mathfrak{G}(\mathcal{G}_2)_3 = L \rtimes E \rtimes G.$$

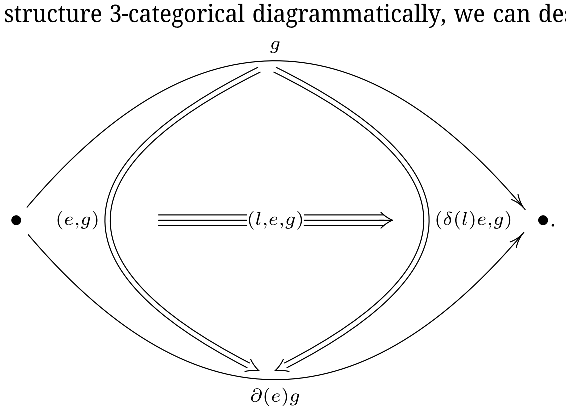
The 1-, 2-, and 3-cells are as follows:

$$(l,e,g) : (e,g) \Rrightarrow (\delta(l)e,g) : g \Longrightarrow \partial(e)g : \bullet \longrightarrow \bullet,$$

for $g \in G$, $e \in E$, and $l \in L$. For the 3-cell $(l, e, g)$, the 2-source and 2-target are $s_2((l, e, g)) = (e, g)$ and $t_2((l, e, g)) = (\delta(l)e, g)$, respectively. In addition, for each $g \in G$ and $e \in E$, $i_2((e, g)) = (1_L, e, g)$. We can show it 2-categorically diagrammatically as follows:




If we want to picture this structure 3-categorical diagrammatically, we can describe it as follows:



### 3.2.1 Vertical composition of 3-cells

Suppose $(l', \delta(l)e, g)$ is another element in $(L \rtimes E \rtimes G)$. The vertical composition "$\sharp_2$" of 3-cells is given by

$$(l', \delta(l)e, g)\sharp_2(l, e, g) = (l'l, e, g).$$

The vertical composition of 3-cells is illustrated as follows:

$$\begin{array}{c} \begin{tikzcd} g \arrow[rr, bend left, Rightarrow, "{(e,g)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l)e,g)}"'] \arrow[rr, phantom, "{(l,e,g)}"] & & \partial(e)g \end{tikzcd} \\ \begin{tikzcd} g \arrow[rr, bend left, Rightarrow, "{(\delta(l)e,g)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l')\delta(l)e,g)}"'] \arrow[rr, phantom, "{(l',\delta(l)e,g)}"] & & \partial(e)g \end{tikzcd} \end{array} = \begin{tikzcd} g \arrow[rr, bend left, Rightarrow, "{(g,h)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l'l)e,g)}"'] \arrow[rr, phantom, "{(l',\delta(l)e,g)\sharp_2(l,e,g)=(l'l,e,g)}"] & & \partial(e)g, \end{tikzcd}$$

$$s_2((l', \delta(l)e, g)\sharp_2(l, e, g)) = s_2((l'l, e, g)) = (e, g)$$

and

$$t_2((l', \delta(l)e, g)\sharp_2(l, e, g)) = t_2((l'l, e, g)) = (\delta(l'l)e, g).$$

### 3.2.2 Horizontal composition of 3-cells

Whiskering as

$$\begin{tikzcd} g \arrow[r, Rightarrow, "{(e,g)}"] & \partial(e)g \arrow[rr, bend left, Rightarrow, "{(e',\partial(e)g)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l')e',\partial(e)g)}"'] \arrow[rr, phantom, "{(l',e',\partial(e)g)}"] & & \partial(e'e)g \end{tikzcd} \quad resp. \quad \begin{tikzcd} g \arrow[rr, bend left, Rightarrow, "{(e,g)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l)e,g)}"'] \arrow[rr, phantom, "{(l,e,g)}"] & & \partial(e)g \arrow[r, Rightarrow, "{(e',\partial(e)g)}"] & \partial(e'e)g \end{tikzcd}$$

clearly gives $(l', e', \partial(e)g)\sharp_1(e, g) = (l', e'e, g)$ and $(e', \partial(e)g)\sharp_1(l, e, g) = ({}^{e'}l, e'e, g)$, respectively. Naturally, 2-source and 2-target are defined as follows: $s_2((l', e'e, g)) = (e'e, g) = s_2(({}^{e'}l, e'e, g))$, $t_2((l', e'e, g)) = (\delta(l')e'e, g)$, and $t_2(({}^{e'}l, e'e, g)) = (e'\delta(l)e, g)$. The horizontal composition "$\sharp_1$" of 3-cells is given by

$$(l', e', \partial(e)g)\sharp_1(l, e, g) = (l'^{e'}l, e'e, g).$$

$$\begin{tikzcd} g \arrow[rr, bend left, Rightarrow, "{(e,g)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l)e,g)}"'] \arrow[rr, phantom, "{(l,e,g)}"] & & \partial(e)g \arrow[rr, bend left, Rightarrow, "{(e',\partial(e)g)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l')e',\partial(e)g)}"'] \arrow[rr, phantom, "{(l',e',\partial(e)g)}"] & & \partial(e'e)g \end{tikzcd} = \begin{tikzcd} g \arrow[rr, bend left, Rightarrow, "{(e',\partial(e)g)\sharp_1(e,g)=(e'e,g)}"] \arrow[rr, bend right, Rightarrow, "{(\delta(l')e',\partial(e)g)\sharp_1(\delta(l)e,g)=(\delta(l')e'\delta(l)e,g)}"'] \arrow[rr, phantom, "{(l',e',\partial(e)g)\sharp_1(l,e,g)=(l'\ {}^{e'}l,e'e,g)}"] & & \partial(e'e)g \end{tikzcd}$$

$$s_2((l', e', \partial(e)g)\sharp_1(l, e, g)) = s_2((l'\ {}^{e'}l, e'e, g)) = (e'e, g)$$

and

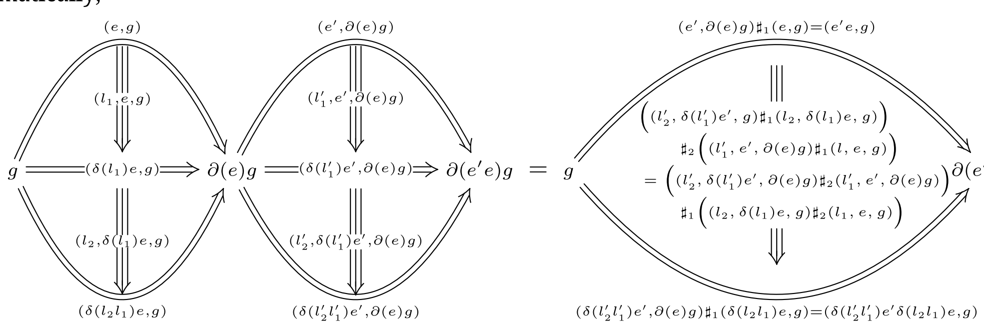
$$\begin{aligned}
t_2((l', e', \partial(e)g)\sharp_1(l, e, g)) &= t_2((l'\,{}^{e'}l, e'e, g) \\
&= (\delta(l'\,{}^{e'}l)e'e, g)) \\
&= (\delta(l')\delta({}^{e'}l)e'e, g) \\
&= (\delta(l')\delta(l\{\delta(l)^{-1}, e'\})e'e, g) \\
&= (\delta(l')\delta(l)\delta(\{\delta(e)^{-1}, g'\})e'e, g) \\
&= (\delta(l')\delta(l)\langle\delta(l)^{-1}, e'\rangle e'e, g) \\
&= (\delta(l')\delta(l)\delta(l)^{-1}e'(\delta(l)^{-1})^{-1}\ {}^{\partial(\delta(l)^{-1})}e'^{-1}e'e, g) \\
&= (\delta(l')e'\delta(l)e, g).
\end{aligned}$$

In the last calculations, we used $2\mathcal{XMOD}$-1,-2 in Definition 3.4 and Remark 3.5.

The compositions $\sharp_2$ and $\sharp_1$ of 3-cells are compatibles

$$\begin{aligned}
&((l_2', \delta(l_1')e', \partial(e)g)\sharp_1(l_2, \delta(l_1)e, g))\sharp_2((l_1', e', \partial(e)g)\sharp_1(l_1, e, g)) \\
&\quad = (l_2'\,{}^{(\delta(l_1')e')}l_2, \delta(l_1)e'\delta(l_1)e, g)\sharp_2(l_1'\,{}^{e'}l_1, e'e, g) \\
&\quad = (l_2'\,{}^{\delta(l_1')}({}^{e'}l_2\,{}^{e'}l_2)l_1'\,{}^{e'}l_1, e'e, g) \\
&\quad = (l_2'l_1'\,{}^{e'}l_2l_1'^{-1}l_1'\,{}^{e'}l_1, e'e, g) \\
&\quad = (l_2'l_1'\,{}^{e'}l_2\,{}^{e'}l_1, e'e, g) \\
&\quad = (l_2'l_1'\,{}^{e'}(l_2l_1), e'e, g) \\
&\quad = (l_2'l_1', e', \partial(e)g)\sharp_1(l_2l_1, e, g) \\
&\quad = ((l_2', \delta(l_1')e', \partial(e)g)\sharp_2(l_1', e', \partial(e)g))\sharp_1((l_2, \delta(l_1)e, g)\sharp_2(l_1, e, g)).
\end{aligned}$$

Diagrammatically,




As it can be seen, this equality is satisfied because $(L \xrightarrow{\delta} E)$ is a crossed module, see Remark 3.5. Interchange law,

$$\begin{aligned}
&((l_2', \delta(l_1')e', \partial(e)g)\sharp_1(l_2, \delta(l_1)e, g))\sharp_2((l_1', e', \partial(e)g)\sharp_1(l, e, g)) \\
&\quad = ((l_2', \delta(l_1')e', \partial(e)g)\sharp_2(l_1', e', \partial(e)g))\sharp_1((l_2, \delta(l_1)e, g)\sharp_2(l_1, e, g))
\end{aligned}$$

is satisfied for vertical $\sharp_2$ and horizontal $\sharp_1$ compositions of 3-cells.

### 3.2.3 0-Horizontal composition of 3-cells

Furthermore, it is necessary to specify the 0-horizontal composition between the *Hom*-sets of 1-cells whose 0-sources and 0-targets are compatible. The 0-horizontal composition “$\sharp_0$” of 3-cells is given by

$$(l_1, e_1, g_1)\sharp_0(l_2, e_2, g_2) = (l_1^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2).$$

Whiskering as

$$\begin{tikzcd} \bullet \arrow[r, Rightarrow, "{(e_2,g_2)}"] & \bullet \arrow[r, bend left=50, Rightarrow, "{(e_1,g_1)}"] \arrow[r, bend right=50, Rightarrow, "{(\delta(l_1)e_1,g_1)}"'] \arrow[r, phantom, "{(l_1,e_1,g_1)}"] & \bullet \end{tikzcd} \qquad resp. \qquad \begin{tikzcd} \bullet \arrow[r, bend left=50, Rightarrow, "{(e_2,g_2)}"] \arrow[r, bend right=50, Rightarrow, "{(\delta(l_2)e_2,g_2)}"'] \arrow[r, phantom, "{(l_2,e_2,g_2)}"] & \bullet \arrow[r, Rightarrow, "{(e_1,g_1)}"] & \bullet \end{tikzcd}$$

gives $(l_1, e_1{}^{g_1}e_2, g_1g_2)$ and $(e_1({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2)$, respectively. The composition "$\sharp_0$" of 3-cells from a 3-categorical perspective is as follows:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, Rightarrow, "{(e_2,g_2)}"] \arrow[r, bend right=50, Rightarrow, "{(\delta(l_2)e_2,g_2)}"'] \arrow[r, phantom, "{(l_2,e_2,g_2)}"] & \bullet \arrow[r, bend left=50, Rightarrow, "{(e_1,g_1)}"] \arrow[r, bend right=50, Rightarrow, "{(\delta(l_1)e_1,g_1)}"'] \arrow[r, phantom, "{(l_1,e_1,g_1)}"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=30, Rightarrow, "{(e_1,g_1)\sharp_0(e_2,g_2)=(e_1{}^{g_1}e_2,g_1g_2)}"] \arrow[r, bend right=30, Rightarrow, "{(\delta(l_1)e_1,g_1)\sharp_0(\delta(l_2)e_2,g_2)=(\delta(l_1)e_1{}^{g_1}(\delta(l_2)e_2),g_1g_2))}"'] \arrow[r, phantom, "{(l_1,e_1,g_1)\sharp_0(l_2,e_2,g_2)=(l_1^{e_1}({}^{g_1}l_2),e_1{}^{g_1}e_2,g_1g_2)}"] & \bullet \end{tikzcd},$$

$$s_2((l_1, e_1, g_1)\sharp_0(l_2, e_2, g_2)) = s_2((l_1^{e_1g_1}l_2, e_1{}^{g_1}e_2, g_1g_2)) = (e_1{}^{g_1}e_2, g_1g_2) = (e_1, g_1)\sharp_0(e_2, g_2)$$

and

$$\begin{aligned} t_2((l_1, e_1, g_1)\sharp_0(l_2, e_2, g_2)) &= t_2((l_1^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2)) \\ &= (\delta(l_1)^{e_1}({}^{g_1}l_2)e_1{}^{g_1}e_2, g_1g_2) \\ &= (\delta(l_1)\delta({}^{e_1}({}^{g_1}l_2))e_1{}^{g_1}e_2, g_1g_2) \\ &= (\delta(l_1)\delta({}^{g_1}l_2\{\delta({}^{g_1}l_2)^{-1}, e_1\})e_1{}^{g_1}e_2, g_1g_2) \\ &= (\delta(l_1)\delta({}^{g_1}l_2)\delta(\{\delta({}^{g_1}l_2)^{-1}, e_1\})e_1{}^{g_1}e_2, g_1g_2) \\ &= (\delta(l_1)\delta({}^{g_1}l_2)\delta({}^{g_1}l_2)^{-1}e_1(\delta({}^{g_1}l_2)^{-1})^{-1}\ {}^{\partial(\delta({}^{g_1}l_2)^{-1})}e_1^{-1}e_1{}^{g_1}e_2, g_1g_2) \\ &= (\delta(l_1)e_1\delta({}^{g_1}l_2){}^{g_1}e_2, g_1g_2) \\ &= (\delta(l_1)e_1^{g_1}(\delta(l_2)e_2), g_1g_2) \\ &= (\delta(l_1)e_1, g_1)\sharp_0(\delta(l_2)e_2, g_2). \end{aligned}$$

In the last calculations, we used $2\mathcal{XMOD}$-1,-2 in Definition 3.4 and Remark 3.5.
Furthermore, for 3-cells in the following diagram:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, Rightarrow, "{(e_3,g_3)}"] \arrow[r, bend right=50, Rightarrow, "{(\delta(l_3)e_3,g_3)}"'] \arrow[r, phantom, "{(l_3,e_3,g_3)}"] & \bullet \arrow[r, bend left=50, Rightarrow, "{(e_2,g_2)}"] \arrow[r, bend right=50, Rightarrow, "{(\delta(l_2)e_2,g_2)}"'] \arrow[r, phantom, "{(l_2,e_2,g_2)}"] & \bullet \arrow[r, bend left=50, Rightarrow, "{(e_1g_1)}"] \arrow[r, bend right=50, Rightarrow, "{(\delta(l_1)e_1,g_1)}"'] \arrow[r, phantom, "{(l_1,e_1,g_1)}"] & \bullet \end{tikzcd},$$

the composition $\sharp_0$ is associative as follows:

$$\begin{aligned} (l_1, e_1, g_1)\sharp_0((l_2, e_2, g_2)\sharp_0(l_3, e_3, g_3)) &= (l_1, e_1, g_1)\sharp_0(l_2^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3) \\ &= (l_1^{e_1}({}^{g_1}(l_2^{e_2}({}^{g_2}l_3)), e_1^{g_1}(e_2{}^{g_2}e_3), g_1g_2g_3)) \\ &= (l_1^{e_1}({}^{g_1}l_2)^{e_1}({}^{g_1e_2}({}^{g_2}l_3)), e_1{}^{g_1}e_2g_1({}^{g_2}e_3), g_1g_2g_3) \\ &= (l_1^{e_1}({}^{g_1}l_2)^{e_1{}^{g_1}e_2}({}^{g_1g_2}l_3), e_1{}^{g_1}e_2{}^{g_1g_2}e_3, g_1g_2g_3) \\ &= (l_1^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2)\sharp_0(l_3, e_3, g_3) \\ &= ((l_1, e_1, g_1)\sharp_0(l_2, e_2, g_2))\sharp_0(l_3, e_3, g_3). \end{aligned}$$

We used Lemma 3.6 to show that the composition $\sharp_0$ is associative.

The interchange law between compositions $\sharp_1$ and $\sharp_0$ is not satisfied as follows:

$$\begin{aligned}&((l_1', e_1', \partial(e_1)g_1)\sharp_0(l_2', e_2', \partial(e_2)g_2))\sharp_1((l_1, e_1, g_1)\sharp_0(l_2, e_2, g_2))\\ &\neq ((l_1', e_1', \partial(e_1)g_1)\sharp_1(l_1, e_1, g_1))\sharp_0((l_2', e_2', \partial(e_2)g_2)\sharp_1(l_2, e_2, g_2)).\end{aligned}$$

Because the tail of the 2-crossed module $\mathcal{G}_2$, $(E \xrightarrow{\partial} G)$, is just a pre-crossed module, see Remark 3.5. Therefore, the following structure is not a strict 3-groupoid with a single object. We have sestertius category (sestertius groupoid):

$$\mathfrak{G}(\mathcal{G}_2) = \left(\begin{tikzcd} L \rtimes E \rtimes G \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] & E \rtimes G \arrow[l, bend left=60, "i_2"] \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & G \arrow[l, bend left=60, "i_1"] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & \{\bullet\} \arrow[l, bend left=60, "i_0"] \end{tikzcd}\right).$$

Also, this structure is 3-globular (globularity means $s_n s_{n+1} = s_n t_{n+1}$, $t_n s_{n+1} = t_n t_{n+1}$.)

$$s_1 s_2((l, e, g)) = s_1((e, g)) = g = s_1((\delta(l)e, g)) = s_1 t_2((l, e, g))$$

and

$$\begin{aligned} t_1 s_2((l, e, g)) &= t_1((e, g))\\ &= \partial(e)g\\ &= \partial\delta(l)\partial(e)g\\ &= \partial(\delta(l)e)g\\ &= t_1((\delta(l)e, g))\\ &= t_1 t_2((l, e, g)),\end{aligned}$$

and reflexive (reflexive means $Id = s_n i_n = t_n i_n$.)

$$s_2 i_2((e, g)) = s_2((1_L, e, g)) = (e, g) = (\delta(1_L)e, g) = t_2((1_L, e, g)) = t_2 i_2((e, g)) = Id((e, g))$$

for each $g \in G$, $e \in E$, and $l \in L$.

It is also possible to proceed the other way for similar construction. Let us have a globular, reflexive, and strict sestertius category with one object as follows:

$$\mathfrak{G} = \left(\begin{tikzcd} G_3 \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] & G_2 \arrow[l, bend left=60, "i_2"] \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & G_1 \arrow[l, bend left=60, "i_1"] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & \{\bullet\} \arrow[l, bend left=60, "i_0"] \end{tikzcd}\right).$$

Then, the following left actions exist

$$\begin{aligned} G_1 \times G_2 &\longrightarrow G_2\\ (a, b) &\longmapsto {}^a b = i_1(a)\sharp_0 b\sharp_0 i_1(a)^{-1},\\ G_2 \times G_3 &\longrightarrow G_3\\ (b, c) &\longmapsto {}^b c = i_2(b)\sharp_0 c\sharp_0 i_2(b)^{-1},\\ G_1 \times G_3 &\longrightarrow G_3\\ (a, c) &\longmapsto {}^a c = i_2 i_1(a)\sharp_0 c\sharp_0 i_2 i_1(a)^{-1},\end{aligned}$$

for each $a \in G_1$, $b \in G_2$, and $c \in G_3$. With the help of these left actions, the following isomorphisms are defined:

$$\begin{aligned} G_2 &\longleftrightarrow Ker s_1 \rtimes G_1\\ b &\longmapsto (b\sharp_1 i_1 s_1(b^{-1}), s_1(b))\\ b\sharp_1 i_1(a) &\longleftarrow\!\mapsfromchar\ (b, a),\end{aligned}$$

For each $b \in G_2$,

$$b \mapsto (b\sharp_1 i_1s_1(b^{-1}), s_1(b)) \mapsto b\sharp_1 i_1s_1(b^{-1})\sharp_1 i_1s_1(b) = b.$$

Similarly, for each $b \in \mathrm{Ker}s_1$ and $a \in G_1$,

$$(b, a) \mapsto b\sharp_1 i_1(a) \mapsto (b\sharp_1 i_1(a))\sharp_1 i_1s_1(b\sharp_1 i_1(a))^{-1}, s_1(b\sharp_1 i_1(a)) = (b, a).$$

As a result, $G_2 \cong \mathrm{Ker}s_1 \rtimes G_1$. In the same way,

$$\begin{aligned}
G_3 &\longleftrightarrow Kers_2 \rtimes Kers_1 \rtimes G_1 \\
c &\longmapsto (c\sharp_2 i_2s_2(c^{-1}), s_2(c)\sharp_1 i_1s_1(s_2(c)^{-1}), s_1s_2(c)) \\
c\sharp_2 i_2(b)\sharp_2 i_2i_1(a) &\longleftarrow\!\shortmid (c, b, a),
\end{aligned}$$

for each $c \in G_3$,

$$c \mapsto (c\sharp_2 i_2s_2(c^{-1}), s_2(c)\sharp_1 i_1s_1(s_2(c)^{-1}), s_1s_2(c)) \mapsto c\sharp_2 i_2s_2(c^{-1})\sharp_2 i_2(s_2(c)\sharp_1 i_1s_1(s_2(c)^{-1}))\sharp_2 i_2i_1(s_1s_2(c)) = c,$$

and for $c \in \mathrm{Ker}s_2$, $b \in \mathrm{Ker}s_1$, and $a \in G_1$,

$$\begin{aligned}
&(c, b, a) \mapsto c\sharp_2 i_2(b)\sharp_2 i_2i_1(a) \mapsto (c\sharp_2 i_2(b)\sharp_2 i_2i_1(a)\sharp_2 i_2s_2(c\sharp_2 i_2(b)\sharp_2 i_2i_1(a)))^{-1}, s_2(c\sharp_2 i_2(b)\sharp_2 i_2i_1(a)) \\
&\sharp_1 i_1s_1(s_2(c\sharp_2 i_2(b)\sharp_2 i_2i_1(a)))^{-1}, s_1s_2(c\sharp_2 i_2(b)\sharp_2 i_2i_1(a)) = (c\sharp_2 i_2(b)\sharp_2 i_2i_1(a)\sharp_2 i_2i_1(a)^{-1}\sharp_2 i_2(b)^{-1} \\
&\sharp_2 i_2s_2(c)^{-1}, s_2(c)\sharp_1 s_2i_2(b)\sharp_1 i_1(a)\sharp_1 i_1s_1s_2i_2i_1(a)^{-1}\sharp_1 i_1s_1s_2i_2(b)^{-1}\sharp_1 i_1s_1s_2(c)^{-1}, s_1s_2(c) \\
&\sharp_0 s_1s_2i_2(b)\sharp_0 s_1s_2i_2i_1(a)) = (c, b, a).
\end{aligned}$$

Consequently, $G_3 \cong \mathrm{Ker}s_2 \rtimes \mathrm{Ker}s_1 \rtimes G_1$. Therefore, $L = \mathrm{Ker}s_2 \subseteq G_3$, $E = \mathrm{Ker}s_1 \subseteq G_2$, $G = G_1$, and target transformations have the following restrictions:

$$\mathcal{G}_2(\mathfrak{G}) = \left( Kers_2 \xrightarrow{\bar{t}_2} Kers_1 \xrightarrow{\bar{t}_1} G_1 \right).$$

$\mathcal{G}_2(\mathfrak{G})$ is a semi-exact sequence of $G$-groups. ($\mathrm{Ker}s_1 \xrightarrow{\bar{t}_1} G_1$) is a pre-crossed module of groups with left group action. In more detail, the axiom $\mathcal{XMOD}$-1 in Definition 3.1 is satisfied:

$$\bar{t}_1({}^a b) = \bar{t}_1(i_1(a)\sharp_0 b\sharp_0 i_1(a)^{-1}) = \bar{t}_1 i_1(a)\sharp_0 \bar{t}_1(b)\sharp_0 \bar{t}_1 i_1(a)^{-1} = a\sharp_0 \bar{t}_1(b)\sharp_0 a^{-1},$$

for each $a \in G_1$ and $b \in \mathrm{Ker}s_1$. Note that the reflexivity feature ($\bar{t}_1 i_1 = Id$) is used in the above steps. Similarly, ($\mathrm{Ker}s_2 \xrightarrow{\bar{t}_2} \mathrm{Ker}s_1$) is a crossed module of groups with left group action. Because the interchange law is satisfied between the $\sharp_2$ and $\sharp_1$ compositions of the sestertius category $\mathfrak{G}$, the $\mathcal{XMOD}$-2 axiom in Definition 3.1 is satisfied. Refer also [6].

There is also the Pfeifer lifting map $\{-,-\}_*$, defined as follows:

$$\begin{aligned}
\{-,-\}_* : Kers_1 \times Kers_1 &\longrightarrow Kers_2 \\
(b_1, b_2) &\longmapsto \{b_1, b_2\}_* = i_2(b_1\sharp_0 b_2\sharp_0 b_1^{-1}\sharp_1 {}^{\bar{t}_1(b_1)}b_2^{-1}).
\end{aligned}$$

Additionally, for each $a \in G_1$, $b_1, b_2 \in \mathrm{Ker}s_1$;

$$\begin{aligned}
{}^a\{b_1, b_2\}_* &= {}^a i_2(b_1\sharp_0 b_2\sharp_0 b_1^{-1}\sharp_1 {}^{\bar{t}_1(b_1)}b_2^{-1}) \\
&= {}^a i_2(b_1)\sharp_0^a i_2(b_2)\sharp_0^a i_2(b_1)^{-1}\sharp_1 {}^a i_2({}^{\bar{t}_1(b_1)}b_2^{-1}) \\
&= i_2i_1(a)\sharp_0 i_2(b_1)\sharp_0 i_2i_1(a)^{-1}\sharp_0 i_2i_1(a)\sharp_0 i_2(b_2)\sharp_0 i_2i_1(a)^{-1} \\
&\quad \sharp_0 i_2i_1(a)\sharp_0 i_2(b_1^{-1})\sharp_0 i_2i_1(a)^{-1}\sharp_1 i_2i_1(a)\sharp_0 i_2i_1\bar{t}_1(b_1)\sharp_0 i_2(b_2^{-1})\sharp_0 i_2i_1\bar{t}_1(b_1)^{-1}\sharp_0 i_2i_1(a)^{-1} \\
&= i_2(i_1(a)\sharp_0 b_1\sharp_0 i_1(a)^{-1}\sharp_0 i_1(a)\sharp_0 b_2\sharp_0 i_1(a)^{-1}\sharp_0 i_1(a)\sharp_0 b_1^{-1} \\
&\quad \sharp_0 i_1(a)^{-1}\sharp_1 i_1(a)\sharp_0 i_1\bar{t}_1(b_1)\sharp_0 b_2^{-1}\sharp_0 i_1\bar{t}_1(b_1)^{-1}\sharp_0 i_1(a)^{-1}) \\
&= i_2({}^a b_1\sharp_0 {}^a b_2\sharp_0 {}^a b_1^{-1}\sharp_1 {}^{a\sharp_0 \bar{t}_1(b_1)}b_2^{-1}) \\
&= i_2({}^a b_1\sharp_0 {}^a b_2\sharp_0 {}^a b_1^{-1}\sharp_1 {}^{a\sharp_0 \bar{t}_1(b_1)\sharp_0 a^{-1}}({}^a b_2^{-1})) \\
&= i_2({}^a b_1\sharp_0 {}^a b_2\sharp_0 {}^a b_1^{-1}\sharp_1 {}^{\bar{t}_1({}^a b_1)}({}^a b_2^{-1})) \\
&= \{{}^a b_1, {}^a b_2\}_*.
\end{aligned}$$

Moreover, there exist the left action equality ${}^{(b,a)}c = {}^{b}({}^{i_1(a)}c)$ and

$$\begin{aligned}
{}^{a}({}^{b}c) &= {}^{a}(i_2(b)\sharp_0 c\sharp_0 i_2(b)^{-1}) \\
&= {}^{a}i_2(b)\sharp_0 {}^{a}c\sharp_0 {}^{a}i_2(b)^{-1} \\
&= i_2i_1(a)\sharp_0 i_2(b)\sharp_0 i_2i_1(a)^{-1}\sharp_0 {}^{a}c\sharp_0 i_2i_1(a)\sharp_0 i_2(b)^{-1}\sharp_0 i_2i_1(a)^{-1} \\
&= i_2(i_1(a)\sharp_0 b\sharp_0 i_1(a)^{-1})\sharp_0 {}^{a}c\sharp_0 i_2(i_1(a)\sharp_0 b^{-1}\sharp_0 i_1(a)^{-1}) \\
&= i_2({}^{a}b)\sharp_0 {}^{a}c\sharp_0 i_2({}^{a}b)^{-1} \\
&= {}^{a}b({}^{a}c),
\end{aligned}$$

for each $a \in G_1$, $b \in \operatorname{Ker} s_1$, and $c \in \operatorname{Ker} s_2$. These definitions satisfy the axioms given in Definition 3.4. Now we will only satisfy axioms $2\mathcal{XMOD}$-1 and $2\mathcal{XMOD}$-2. We leave the remaining axioms to the readers. It is pretty fun to prove.
$2\mathcal{XMOD}$-1 for each $c \in \operatorname{Ker} s_2$, $\bar{t}_1\bar{t}_2(c) = \bar{t}_1 s_2(c) = 1$, ($\because$ the globularity),
$2\mathcal{XMOD}$-2 we have

$$\begin{aligned}
\bar{t}_2(\{b_1, b_2\}_*) &= \bar{t}_2 i_2(b_1\sharp_0 b_2\sharp_0 b_1^{-1}\sharp_1 {}^{\bar{t}_1(b_1)}b_2^{-1}) \\
&= \bar{t}_2(i_2(b_1)\sharp_0 i_2(b_2)\sharp_0 i_2(b_1^{-1})\sharp_1 i_2({}^{\bar{t}_1(b_1)}b_2^{-1})) \\
&= (\bar{t}_2 i_2)(b_1)\sharp_0(\bar{t}_2 i_2)(b_2)\sharp_0(\bar{t}_2 i_2)(b_1^{-1})\sharp_1(\bar{t}_2 i_2)({}^{\bar{t}_1(b_1)}b_2^{-1}) \\
&= b_1\sharp_0 b_2\sharp_0 b_1^{-1}\sharp_1 {}^{\bar{t}_1(b_1)}b_2^{-1}
\end{aligned}$$

for each $b_1, b_2 \in \operatorname{Ker} s_1$, ($\because$ the reflexivity $t_n i_n = Id_n$).

Along with these definitions and constructions, the following theorem is given.

**Theorem 3.9.** *A 2-crossed module of groups is a globular, reflexive, and strict sestertius groupoid with one object (sestertius group).*

Now, we will obtain a tricategory in light of these knowledges.

### 3.2.4 From 2-crossed modules of groups to tricategories

In this section, we will construct tricategory from a 2-crossed module. Let $\mathcal{G}_2 = (L \overset{\delta}{\to} E \overset{\partial}{\to} G, \{--\})$ be a 2-crossed module of groups. We construct the $\mathfrak{T}(\mathcal{G}_2)$ tricategory as follows:

**DATA:**

(1) $Ob\mathfrak{T}(\mathcal{G}_2) = \{\bullet\}$

(2) For $\bullet \in Ob\mathfrak{T}(\mathcal{G}_2)$. $\mathfrak{T}(\mathcal{G}_2)(\bullet, \bullet)$ is a bicategory as follows:

$$\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) = \left(\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=80, "g"] \arrow[r, bend right=80, "\partial(e)g"'] \arrow[r, phantom, "(e,g)\overset{(l,e,g)}{\Rrightarrow}(\delta(l)e,g)"] & \bullet
\end{tikzcd}\right).$$

- **Objects:** Every 1-cell, $g : \bullet \to \bullet \in G$, is an object.
- **1-Morphisms:** Every 2-cell, $(e, g) : g \Rightarrow \partial(e)g \in E \rtimes G$, is a 1-morphism.
- **2-Morphisms:** Every 3-cell, $(l, e, g) : (e, g) \Rrightarrow (\delta(l)e, g) \in L \rtimes E \rtimes G$, is a 2-morphism.

The vertical composition of $\mathfrak{T}(\mathcal{G}_2)$ category is $\sharp_2$, and the horizontal composition is $\sharp_1$. Note that $\mathfrak{T}(\mathcal{G}_2)$ is a strict 2-category (see in Remark 2.3).

(3) The functor

$$\begin{aligned}
\otimes : \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) &\longrightarrow \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)\\
\big((l_1,e_1,g_1),(l_2,e_2,g_2)\big) &\longmapsto (l_1,e_1,g_1)\otimes(l_2,e_2,g_2) = (l_1,e_1,g_1)\sharp_0(l_2,e_2,g_2)\\
&\qquad = ({l_1}^{e_1}({}^{g_1}l_2), e_1\, {}^{g_1}e_2, g_1g_2)
\end{aligned}$$

is called composition:

$$\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, Rightarrow, bend left=60, "{(e_2,g_2)}"] \arrow[r, Rightarrow, bend right=60, "{(\delta(l_2)e_2,g_2)}"'] \arrow[r, phantom, "{\Downarrow (l_2,e_2,g_2)}"] & \bullet \arrow[r, Rightarrow, bend left=60, "{(e_1,g_1)}"] \arrow[r, Rightarrow, bend right=60, "{(\delta(l_1)e_1,g_1)}"'] \arrow[r, phantom, "{\Downarrow (l_1,e_1,g_1)}"] & \bullet \arrow[r, phantom, "="] & \bullet \arrow[r, Rightarrow, bend left=40, "{(e_1,g_1)\sharp_0(e_2,g_2)=(e_1{}^{g_1}e_2,g_1g_2)}"] \arrow[r, Rightarrow, bend right=40, "{(\delta(l_1)e_1,g_1)\sharp_0(\delta(l_2)e_2,g_2)=(\delta(l_1)e_1{}^{g_1}(\delta(l_2)e_2),g_1g_2))}"'] \arrow[r, phantom, "{\Downarrow (l_1,e_1,g_1)\sharp_0(l_2,e_2,g_2)=({l_1}^{e_1}({}^{g_1}l_2),e_1\, {}^{g_1}e_2,g_1g_2)}"] & \bullet
\end{tikzcd}$$

For composable 2-morphisms of $\mathfrak{T}(\mathcal{G}_2)$, the functor $\otimes$ is cubical [17, page 38, Definition 3.1]. Therefore, we have either

$$\begin{aligned}
&((e_1, g_1)\sharp_0(e_2', \partial(e_2)g_2))\sharp_1((1_E, g_1)\sharp_0(e_2, g_2))\\
&\quad = (e_1{}^{g_1}e_2', g_1\partial(e_2)g_2)\sharp_1({}^{g_1}e_2, g_1g_2)\\
&\quad = (e_1{}^{g_1}e_2'{}^{g_1}e_2, g_1g_2)\\
&\quad = (e_1{}^{g_1}(e_2'e_2), g_1g_2)\\
&\quad = (e_1, g_1)\sharp_0(e_2'e_2, g_2)\\
&\quad = ((e_1, g_1)\sharp_1(1_E, g_1))\sharp_0((e_2', \partial(e_2)g_2)\sharp_1(e_2, g_2)),
\end{aligned}$$

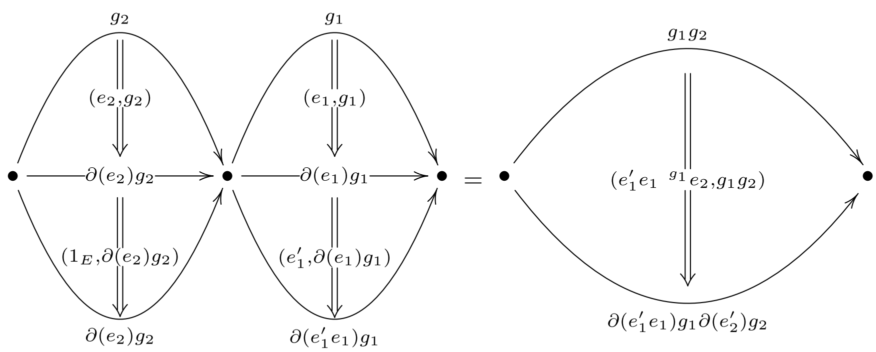
diagrammatically,

$$\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=70, "g_2"] \arrow[r, "\partial(e_2)g_2" description] \arrow[r, bend right=70, "\partial(e_2'e_2)g_2"'] \arrow[r, phantom, shift left=4, "{\Downarrow (e_2,g_2)}"] \arrow[r, phantom, shift right=4, "{\Downarrow (e_2',\partial(e_2)g_2)}"] & \bullet \arrow[r, bend left=70, "g_1"] \arrow[r, "g_1" description] \arrow[r, bend right=70, "\partial(e_1)g_1"'] \arrow[r, phantom, shift left=4, "{\Downarrow (1_E,g_1)}"] \arrow[r, phantom, shift right=4, "{\Downarrow (e_1,g_1)}"] & \bullet \arrow[r, phantom, "="] & \bullet \arrow[r, bend left=60, "g_1g_2"] \arrow[r, bend right=60, "\partial(e_1)g_1\partial(e_2'e_2)g_2"'] \arrow[r, phantom, "{\Downarrow ({}^{g_1}(e_2'e_2),g_1g_2)}"] & \bullet,
\end{tikzcd}$$

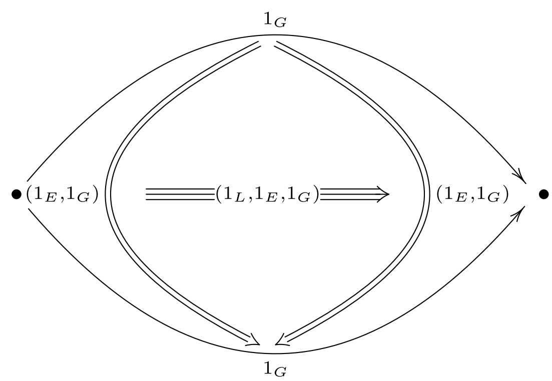
or

$$\begin{aligned}
&((e_1', \partial(e_1)g_1)\sharp_0(1_E, \partial(e_2)g_2))\sharp_1((e_1, g_1)\sharp_0(e_2, g_2))\\
&\quad = (e_1', \partial(e_1)g_1\partial(e_2)g_2)\sharp_1(e_1{}^{g_1}e_2, g_1g_2)\\
&\quad = (e_1'e_1{}^{g_1}e_2, g_1g_2)\\
&\quad = (e_1'e_1, g_1)\sharp_0(e_2, g_2)\\
&\quad = ((e_1', \partial(e_1)g_1)\sharp_1(e_1, g_1))\sharp_0((1_E, \partial(e_2)g_2)\sharp_1(e_2, g_2)),
\end{aligned}$$

diagrammatically,

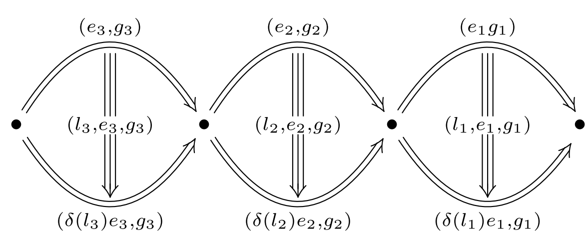



(4) The functor $I_{\bullet} : \mathbf{1} \longrightarrow \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)$, where $\mathbf{1}$ denotes the unit bicategory:




$I_{\bullet}$ is also cubical.

(5) For object $\bullet$ of $Ob\mathfrak{T}(\mathcal{G}_2)$ as follows:




an adjoint equivalence $\alpha$;

$$\alpha : (l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes (l_3, e_3, g_3)) \longrightarrow ((l_1, e_1, g_1) \otimes (l_2, e_2, g_2)) \otimes (l_3, e_3, g_3).$$

Since every $\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)$ is a strict 2-category, we have

$$(l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes (l_3, e_3, g_3)) = ((l_1, e_1, g_1) \otimes (l_2, e_2, g_2)) \otimes (l_3, e_3, g_3).$$

Note that $\otimes$ is associative

$$\begin{tikzcd}
\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathfrak{T}(\mathcal{G}_2)(x,z) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[d, "\otimes"] \\
\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[r, "\otimes"'] & \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet);
\end{tikzcd}$$

$$\begin{tikzcd}
(l_1,e_1,g_1)\sharp_0\big((l_2,e_2,g_2)\sharp_0(l_3,e_3,g_3)\big) \arrow[r, mapsto, "\otimes\times 1"] \arrow[d, mapsto, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow\alpha"] & (l_1\,{}^{e_1}({}^{g_1}l_2), e_1\,{}^{g_1}e_2, g_1g_2)\sharp_0(l_3,e_3,g_3) \arrow[d, mapsto, "\otimes"] \\
(l_1,e_1,g_1)\sharp_0(l_2\,{}^{e_2}({}^{g_2}l_3), e_2\,{}^{g_2}e_3, g_2g_3) \arrow[r, mapsto, "\otimes"'] & (l_1\,{}^{e_1}({}^{g_1}l_2)\,{}^{e_1}\,{}^{g_1e_2}({}^{g_1g_2}l_3), e_1\,{}^{g_1}e_2\,{}^{g_1g_2}e_3, g_1g_2g_3);
\end{tikzcd}$$

in **Bicat** $(\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)\times\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)\times\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet), \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet))$.

(6) For each $g\in G$, $e\in E$, and $l\in L$, adjoint equivalences $\mathfrak{l}$ and $\mathfrak{r}$

$$\begin{tikzcd}
& \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)\times\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{l}"] & \\
\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[ur, "I_\bullet\times 1"] \arrow[rr, "1"'] & {} & \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet);
\end{tikzcd}$$

$$\begin{tikzcd}
& (1_L,1_E,1_G)\otimes(l,e,g) \arrow[dr, mapsto, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{l}"] & \\
(l,e,g) \arrow[ur, "I_\bullet\times 1"] \arrow[rr, mapsto, "1"'] & {} & (l,e,g),
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)\times\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{r}"] & \\
\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[ur, "1\times I_\bullet"] \arrow[rr, "1"'] & {} & \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet);
\end{tikzcd}$$

$$\begin{tikzcd}
& (l,e,g)\otimes(1_L,1_E,1_G) \arrow[dr, mapsto, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{r}"] & \\
(l,e,g) \arrow[ur, mapsto, "1\times I_\bullet"] \arrow[rr, mapsto, "1"'] & {} & (l,e,g).
\end{tikzcd}$$

in **BiCat** $(\mathfrak{T}(\mathcal{G})(\bullet,\bullet), \mathfrak{T}(\mathcal{G})(\bullet,\bullet))$.

(7) For $\bullet\in Ob\mathfrak{T}(\mathcal{G})$ as follows:

$$\begin{tikzcd}
\bullet \arrow[r, Rightarrow, bend left, "{(e_4,g_4)}"] \arrow[r, Rightarrow, bend right, "{(\delta(l_4)e_4,g_4)}"'] \arrow[r, phantom, "{\Downarrow(e_4,g_4,h_4)}"] & \bullet \arrow[r, Rightarrow, bend left, "{(e_3,g_3)}"] \arrow[r, Rightarrow, bend right, "{(\delta(l_3)e_3,g_3)}"'] \arrow[r, phantom, "{\Downarrow(e_3,g_3,h_3)}"] & \bullet \arrow[r, Rightarrow, bend left, "{(e_2,g_2)}"] \arrow[r, Rightarrow, bend right, "{(\delta(l_2)e_2,g_2)}"'] \arrow[r, phantom, "{\Downarrow(e_2,g_2,h_2)}"] & \bullet \arrow[r, Rightarrow, bend left, "{(e_1g_1)}"] \arrow[r, Rightarrow, bend right, "{(\delta(l_1)e_1,g_1)}"'] \arrow[r, phantom, "{\Downarrow(e_1,g_1,h_1)}"] & \bullet,
\end{tikzcd}$$

and a 2-cell isomorphism $\pi$ (Instead of $\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)$, we use $\mathfrak{T}^4(\mathcal{G}_2)$ as its abbreviation):

$$
\begin{tikzcd}
 & \mathfrak{T}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, "1\times\otimes\times1"] & \arrow[d, phantom, "\Downarrow\alpha\times1"] & \mathfrak{T}^3_1 \arrow[dr, "\otimes\times1"] & \\
\mathfrak{T}^3_2 \arrow[dr, "1\times\otimes"'] & \arrow[l, phantom, "\overset{1\times\alpha}{\Longleftarrow}"] & \mathfrak{T}^3_3 \arrow[rr, "\otimes\times1"] \arrow[dl, "1\times\otimes"] & \arrow[d, phantom, "\Downarrow\alpha\times1"] & \mathfrak{T}^2_1 \arrow[dl, "\otimes"] \\
 & \mathfrak{T}^2_2 \arrow[rr, "\otimes"'] & & \mathfrak{T} &
\end{tikzcd}
\quad \overset{\pi}{\Rrightarrow} \quad
\begin{tikzcd}
 & \mathfrak{T}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] & = & \mathfrak{T}^3_1 \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times1"] & \\
\mathfrak{T}^3_2 \arrow[rr, "\otimes\times1"'] \arrow[dr, "1\times\otimes"'] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathfrak{T}^2_3 \arrow[dr, "\otimes"] & \arrow[l, phantom, "\overset{\alpha}{\Longleftarrow}"] & \mathfrak{T}^2_1 \arrow[dl, "\otimes"] \\
 & \mathfrak{T}^2_2 \arrow[rr, "\otimes"'] & & \mathfrak{T} &
\end{tikzcd}
$$

$$
\begin{aligned}
a_1 &= (l_1, e_1, g_1) \otimes (l_2, e_2, g_2) \otimes (l_3, e_3, g_3) \otimes (l_4, e_4, g_4)\\
a_2 &= (l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3, e_3, g_3) \otimes (l_4, e_4, g_4)\\
a_3 &= (l_1, e_1, g_1) \otimes (l_2, e_2, g_2) \otimes (l_3{}^{e_3}({}^{g_3}l_4), e_3{}^{g_3}e_4, g_3g_4)\\
a_4 &= (l_1, e_1, g_1) \otimes (l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3) \otimes (l_4, e_4, g_4)\\
a_5 &= (l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3{}^{e_3}({}^{g_3}l_4), e_3{}^{g_3}e_4, g_3g_4)\\
a_6 &= (l_1{}^{e_1}({}^{g_1}l_2)^{e_1{}^{g_1}e_2}({}^{g_1g_2}l_3), e_1{}^{g_1}e_2{}^{(g_1g_2)}e_3, g_1g_2g_3) \otimes (l_4, e_4, g_4)\\
a_7 &= (l_1, e_1, g_1) \otimes (l_2{}^{e_2}({}^{g_2}l_3)e_2{}^{g_2}e_3({}^{g_3}l_4), e_2{}^{g_2}e_3{}^{(g_2g_3)}e_4, g_2g_3g_4)\\
a_8 &= (l_1{}^{e_1}({}^{g_1}l_2)^{e_1{}^{g_1}e_2}({}^{g_1g_2}l_3)^{e_1{}^{g_1}e_2{}^{g_1g_2}g_3}({}^{g_1g_2g_3}l_4), e_1{}^{g_1}e_2{}^{(g_1g_2)}g_3{}^{(g_1g_2g_3)}e_4, g_1g_2g_3g_4)\\
&= (l_1{}^{e_1}({}^{g_1}(l_2{}^{e_2}({}^{g_2}l_3)^{e_2{}^{g_2}e_3}({}^{g_3}l_4))), e_1{}^{g_1}(e_2{}^{g_2}e_3{}^{(g_2g_3)}e_4), g_1g_2g_3g_4).
\end{aligned}
$$

$$
\begin{tikzcd}
 & a_1 \arrow[rr, mapsto, "\otimes\times1\times1"] \arrow[dl, mapsto, "1\times1\times\otimes"'] \arrow[dr, mapsto, "1\times\otimes\times1"] & \arrow[d, phantom, "\Downarrow\alpha\times1"] & a_2 \arrow[dr, mapsto, "\otimes\times1"] & \\
a_3 \arrow[dr, mapsto, "1\times\otimes"'] & \arrow[l, phantom, "\overset{1\times\alpha}{\Longleftarrow}"] & a_4 \arrow[rr, mapsto, "\otimes\times1"] \arrow[dl, mapsto, "1\times\otimes"] & \arrow[d, phantom, "\Downarrow\alpha\times1"] & a_6 \arrow[dl, mapsto, "\otimes"] \\
 & a_7 \arrow[rr, mapsto, "\otimes"'] & & a_8 &
\end{tikzcd}
\quad \overset{\pi}{\Rrightarrow} \quad
\begin{tikzcd}
 & a_1 \arrow[rr, mapsto, "\otimes\times1\times1"] \arrow[dl, mapsto, "1\times1\times\otimes"'] & = & a_2 \arrow[dl, mapsto, "1\times\otimes"'] \arrow[dr, mapsto, "\otimes\times1"] & \\
a_3 \arrow[rr, mapsto, "\otimes\times1"'] \arrow[dr, mapsto, "1\times\otimes"'] & \arrow[d, phantom, "\Downarrow\alpha"] & a_5 \arrow[dr, mapsto, "\otimes"] & \arrow[l, phantom, "\overset{\alpha}{\Longleftarrow}"] & a_6 \arrow[dl, mapsto, "\otimes"] \\
 & a_7 \arrow[rr, mapsto, "\otimes"'] & & a_8 &
\end{tikzcd}
$$

in the bicategory **Bicat** $(\mathfrak{T}^4(\mathcal{G}_2), \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet))$

(8) For invertible modifications,

$$
\begin{tikzcd}
\mathfrak{T}^2(\mathcal{G}_2) \arrow[drr, bend left, "1"] \arrow[dr, "1\times I\times1"] \arrow[ddr, bend right, "1"'] & \arrow[d, phantom, "\Downarrow\mathfrak{r}\times1"] & \\
\arrow[r, phantom, "\overset{1\times\mathfrak{l}}{\Longleftarrow}"] & \mathfrak{T}^3(\mathcal{G}_2) \arrow[r, "\otimes\times1"] \arrow[d, "1\times\otimes"] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathfrak{T}^2(\mathcal{G}_2) \arrow[d, "\otimes"] \\
 & \mathfrak{T}^2(\mathcal{G}_2) \arrow[r, "\otimes"'] & \mathfrak{T}(\mathcal{G}_2)
\end{tikzcd}
\quad \overset{\mu}{\Rrightarrow} \quad
\begin{tikzcd}
\mathfrak{T}^2(\mathcal{G}_2) \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] \arrow[dr, phantom, "\Downarrow 1"] & & \\
 & & \mathfrak{T}^2(\mathcal{G}_2) \arrow[d, "\otimes"] \\
 & \mathfrak{T}^2(\mathcal{G}_2) \arrow[r, "\otimes"'] & \mathfrak{T}(\mathcal{G}_2)
\end{tikzcd}
$$

$a_1 = (l_1, e_1, g_1) \otimes (l_2, e_2, g_2)$, $\quad a_2 = (l_1, e_1, g_1) \otimes (1_L, 1_E, 1_G) \otimes (l_2, e_2, g_2)$, $\quad a_3 = (l_1^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2)$;

$$
\begin{tikzcd}
a_1 \arrow[dr, "1\times I\times 1"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] \arrow[r, phantom, "\Downarrow\mathfrak{r}\times 1"] & & \\
\arrow[r, phantom, "\overset{1\times\mathfrak{l}}{\Longleftarrow}"] & a_2 \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow\alpha"] & a_1 \arrow[d, "\otimes"] \\
& a_1 \arrow[r, "\otimes"'] & a_3
\end{tikzcd}
\overset{\mu}{\Longrightarrow}
\begin{tikzcd}
a_1 \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] \arrow[dr, Rightarrow, "1"] & & \\
& & a_1 \arrow[d, "\otimes"] \\
& a_1 \arrow[r, "\otimes"'] & a_3.
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathfrak{T}^3(\mathcal{G}_2) \arrow[dr, "\otimes\times 1"] \arrow[d, phantom, "\Downarrow\mathfrak{l}\times 1"] & \\
\mathfrak{T}^2(\mathcal{G}_2) \arrow[ur, "I\times 1\times 1"] \arrow[rr, "1"'] \arrow[d, "\otimes"'] & {} \arrow[d, phantom, "="] & \mathfrak{T}^2(\mathcal{G}_2) \arrow[d, "\otimes"] \\
\mathfrak{T}(\mathcal{G}_2) \arrow[rr, "1"'] & {} & \mathfrak{T}(\mathcal{G}_2)
\end{tikzcd}
\overset{\lambda}{\Longrightarrow}
\begin{tikzcd}
& \mathfrak{T}^3(\mathcal{G}_2) \arrow[dr, "\otimes\times 1"] \arrow[dd, "1\times\otimes"] & \\
\mathfrak{T}^2(\mathcal{G}_2) \arrow[ur, "I\times 1\times 1"] \arrow[dd, "\otimes"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "\Downarrow\alpha"] & \mathfrak{T}^2(\mathcal{G}_2) \arrow[dd, "\otimes"] \\
& \mathfrak{T}^2(\mathcal{G}_2) \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{l}"] & \\
\mathfrak{T}(\mathcal{G}_2) \arrow[ur, "I\times 1"] \arrow[rr, "1"'] & {} & \mathfrak{T}(\mathcal{G}_2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (1_L, 1_E, 1_G)\otimes(l_1, e_1, g_1)\otimes(l_2, e_2, g_2) \arrow[dr, mapsto, "\otimes\times 1"] \arrow[d, phantom, "\Downarrow\mathfrak{l}\times 1"] & \\
(l_1, e_1, g_1)\otimes(l_2, e_2, g_2) \arrow[ur, mapsto, "I\times 1\times 1"] \arrow[rr, mapsto, "1"'] \arrow[d, mapsto, "\otimes"'] & {} \arrow[d, phantom, "="] & (l_1, e_1, g_1)\otimes(l_2, e_2, g_2) \arrow[d, mapsto, "\otimes"] \\
(l_1\,{}^{e_1}({}^{g_1}l_2), e_1\,{}^{g_1}e_2, g_1g_2) \arrow[rr, mapsto, "1"'] & {} & (l_1\,{}^{e_1}({}^{g_1}l_2), e_1\,{}^{g_1}e_2, g_1g_2)
\end{tikzcd}
$$

$$
\Big\Downarrow\lambda
$$

$$
\begin{tikzcd}
& (1_L, 1_E, 1_G)\otimes(l_1, e_1, g_1)\otimes(l_2, e_2, g_2) \arrow[dr, mapsto, "\otimes\times 1"] \arrow[dd, mapsto, "1\times\otimes"] & \\
(l_1, e_1, g_1)\otimes(l_2, e_2, g_2) \arrow[ur, mapsto, "I\times 1\times 1"] \arrow[dd, mapsto, "\otimes"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "\Downarrow\alpha"] & (l_1, e_1, g_1)\otimes(l_2, e_2, g_2) \arrow[dd, "\otimes"] \\
& (1_L, 1_E, 1_G)\otimes(l_1, e_1, g_1)\otimes(l_2, e_2, g_2) \arrow[dr, mapsto, "\otimes"] \arrow[d, phantom, "\Downarrow\mathfrak{l}"] & \\
(l_1\,{}^{e_1}({}^{g_1}l_2), e_1\,{}^{g_1}e_2, g_1g_2) \arrow[ur, mapsto, "I\times 1"] \arrow[rr, mapsto, "1"'] & {} & (l_1\,{}^{e_1}({}^{g_1}l_2), e_1\,{}^{g_1}e_2, g_1g_2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathfrak{T}(\mathcal{G}_2) \arrow[rr, "1"] & & \mathfrak{T}(\mathcal{G}_2) \\
{} \arrow[rr, phantom, "="] & & {} \\
\mathfrak{T}^2(\mathcal{G}_2) \arrow[uu, "\otimes"] \arrow[rr, "1"'] \arrow[dr, "1\times1\times I"'] & {} \arrow[d, phantom, "\Downarrow 1\times r"] & \mathfrak{T}^2(\mathcal{G}_2) \arrow[uu, "\otimes"'] \\
& \mathfrak{T}^3(\mathcal{G}_2) \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}
\overset{\rho}{\Longrightarrow}
\begin{tikzcd}
\mathfrak{T}(\mathcal{G}_2) \arrow[rrrr, "1"] \arrow[drr, "1\times I"'] & & {} \arrow[d, phantom, "r"] & & \mathfrak{T}(\mathcal{G}_2) \\
& & \mathfrak{T}^2(\mathcal{G}_2) \arrow[urr, "\otimes"'] & & \\
\mathfrak{T}^2(\mathcal{G}_2) \arrow[uu, "\otimes"] \arrow[drr, "1\times1\times I"'] & {} \arrow[u, phantom, "="] & & {} \arrow[u, phantom, "\Rightarrow\alpha"] & \mathfrak{T}^2(\mathcal{G}_2) \arrow[uu, "\otimes"'] \\
& & \mathfrak{T}^3(\mathcal{G}_2) \arrow[uu, "\otimes\times1"] \arrow[urr, "1\times\otimes"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(l_1\, {}^{e_1}({}^{g_1}l_2), e_1\, {}^{g_1}e_2, g_1g_2) \arrow[rr, mapsto, "1"] & & (l_1\, {}^{e_1}({}^{g_1}l_2), e_1\, {}^{g_1}e_2, g_1g_2) \\
{} \arrow[rr, phantom, "="] & & {} \\
(l_1,e_1,g_1)\otimes(l_2,e_2,g_2) \arrow[uu, mapsto, "\otimes"] \arrow[rr, mapsto, "1"'] \arrow[dr, mapsto, "1\times1\times I"'] & {} \arrow[d, phantom, "\Downarrow 1\times r"] & (l_1,e_1,g_1)\otimes(l_2,e_2,g_2) \arrow[uu, mapsto, "\otimes"'] \\
& (l_1,e_1,g_1)\otimes(l_2,e_2,g_2)\otimes(1_L,1_E,1_G) \arrow[ur, mapsto, "1\times\otimes"'] &
\end{tikzcd}
$$

$$
\Big\Downarrow \rho
$$

$$
\begin{tikzcd}
(l_1\, {}^{e_1}({}^{g_1}l_2), e_1\, {}^{g_1}e_2, g_1g_2) \arrow[rrrr, mapsto, "1"] \arrow[drr, mapsto, "1\times l"'] & & {} \arrow[d, phantom, "r"] & & (l_1\, {}^{e_1}({}^{g_1}l_2), e_1\, {}^{g_1}e_2, g_1g_2) \\
& & (l_1\, {}^{e_1}({}^{g_1}l_2), e_1\, {}^{g_1}e_2, g_1g_2)\otimes(1_L,1_E,1_G) \arrow[urr, mapsto, "\otimes"'] & & \\
(l_1,e_1,g_1)\otimes(l_2,e_2,g_2) \arrow[uu, mapsto, "\otimes"] \arrow[drr, mapsto, "1\times1\times I"'] & {} \arrow[u, phantom, "="] & & {} \arrow[u, phantom, "\Rightarrow\alpha"] & (l_1,e_1,g_1)\otimes(l_2,e_2,g_2) \arrow[uu, mapsto, "\otimes"'] \\
& & (l_1,e_1,g_1)\otimes(l_2,e_2,g_2)\otimes(1_L,1_E,1_G) \arrow[uu, mapsto, "\otimes\times1"] \arrow[urr, mapsto, "1\times\otimes"'] & &
\end{tikzcd}
$$

Note that the tricategory $\mathfrak{T}(\mathcal{G}_2)$ we have defined provides Remark 2.3; that is why it is strict.

**AXIOMS:**

(1) [Non-abelian 4-cocycle condition] For any strict 2-category $\mathfrak{T}(\mathcal{G}_2)(-,-)$, the following 2-cells equation diagram preserves the natural isomorphism of $\alpha$. We have 2-cells elements as follows:

$$
\begin{tikzcd}[column sep=large]
\bullet \arrow[r, Rightarrow, bend left=50, "{(e_5,g_5)}", ""{name=U5, below}] \arrow[r, Rightarrow, bend right=50, "{(\delta(l_5)e_5,g_5)}"', ""{name=D5}] &
\bullet \arrow[r, Rightarrow, bend left=50, "{(e_4,g_4)}", ""{name=U4, below}] \arrow[r, Rightarrow, bend right=50, "{(\delta(l_4)e_4,g_4)}"', ""{name=D4}] &
\bullet \arrow[r, Rightarrow, bend left=50, "{(e_3,g_3)}", ""{name=U3, below}] \arrow[r, Rightarrow, bend right=50, "{(\delta(l_3)e_3,g_3)}"', ""{name=D3}] &
\bullet \arrow[r, Rightarrow, bend left=50, "{(e_2,g_2)}", ""{name=U2, below}] \arrow[r, Rightarrow, bend right=50, "{(\delta(l_2)e_2,g_2)}"', ""{name=D2}] &
\bullet \arrow[r, Rightarrow, bend left=50, "{(e_1g_1)}", ""{name=U1, below}] \arrow[r, Rightarrow, bend right=50, "{(\delta(l_1)e_1,g_1)}"', ""{name=D1}] &
\bullet.
\arrow[Rightarrow, from=U5, to=D5, "{(l_5,e_5,g_5)}"]
\arrow[Rightarrow, from=U4, to=D4, "{(l_4,e_4,g_4)}"]
\arrow[Rightarrow, from=U3, to=D3, "{(l_3,e_3,g_3)}"]
\arrow[Rightarrow, from=U2, to=D2, "{(l_2,e_2,g_2)}"]
\arrow[Rightarrow, from=U1, to=D1, "{(l_1,e_1,g_1)}"]
\end{tikzcd}
$$

We will use the following elements as abbreviations for the objects of $\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)$;

$$
\begin{aligned}
a_1 &= ((l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes (l_3{}^{e_3}({}^{g_3}l_4), e_3{}^{g_3}e_4, g_3g_4))) \otimes (l_5, e_5, g_5)\\
a_2 &= (l_1, e_1, g_1) \otimes (((l_2, e_2, g_2) \otimes (l_3{}^{e_3}({}^{g_3}l_4), e_3{}^{g_3}e_4, g_3g_4)) \otimes (l_5, e_5, g_5))\\
a_3 &= (l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes ((l_3{}^{e_3}({}^{g_3}l_4), e_3{}^{g_3}e_4, g_3g_4) \otimes (l_5, e_5, g_5)))\\
a_4 &= (l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes ((l_3, e_3, g_3) \otimes (l_4{}^{e_4}({}^{g_4}l_5), e_4{}^{g_4}e_5, g_4g_5)))\\
a_5 &= (l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3, e_3, g_3 \otimes (l_4{}^{e_4}({}^{g_4}l_5), e_4{}^{g_4}e_5, g_4g_5))\\
a_6 &= ((l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3, e_3, g_3)) \otimes (l_4{}^{e_4}({}^{g_4}l_5), e_4{}^{g_4}e_5, g_4g_5)\\
a_7 &= (((l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3, e_3, g_3)) \otimes (l_4, e_4, g_4)) \otimes (l_5, e_5, g_5)\\
a_8 &= (((l_1, e_1, g_1) \otimes (l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3)) \otimes (l_4, e_4, g_4)) \otimes (l_5, e_5, g_5)\\
a_9 &= ((l_1, e_1, g_1) \otimes ((l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3) \otimes (l_4, e_4, g_4))) \otimes (l_5, e_5, g_5)\\
a_{10} &= (l_1, e_1, g_1) \otimes (((l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3) \otimes (l_4, e_4, g_4))(l_5, e_5, g_5))\\
a_{11} &= ((l_1, e_1, g_1) \otimes (l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3)) \otimes (l_4{}^{e_4}({}^{g_4}l_5), e_4{}^{g_4}e_5, g_4g_5)\\
a_{12} &= (l_1, e_1, g_1) \otimes ((l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3) \otimes (l_4{}^{e_4}({}^{g_4}l_5), e_4{}^{g_4}e_5, g_4g_5))\\
a_{13} &= ((l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3{}^{e_3}({}^{g_3}l_4), e_3{}^{g_3}e_4, g_3g_4)) \otimes (l_5, e_5, g_5)\\
a_{14} &= (l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes ((l_3{}^{e_3}({}^{g_3}l_4), e_3{}^{g_3}e_4, g_3g_4) \otimes (l_5, e_5, g_5)).
\end{aligned}
$$

Thus, we have the following commutative diagrams:

$$
\begin{tikzcd}
 & & & a_1 \arrow[drr, "\alpha"] \arrow[dd, phantom, "="] & & & \\
 & a_9 \arrow[urr, "(1\alpha)1"] \arrow[drr, "\alpha"'] \arrow[ddr, phantom, "\Downarrow\pi"] & & & & a_2 \arrow[dr, "1\alpha"] & \\
a_8 \arrow[ur, "\alpha 1"] \arrow[drr, "\alpha"'] & & & a_{10} \arrow[urr, "1(\alpha 1)"'] \arrow[dr, "1\alpha"] \arrow[rrr, phantom, "\Downarrow 1\pi"] & & & a_3 \arrow[dd, "1(1\alpha)"] \\
 & & a_{11} \arrow[rr, "\alpha"'] \arrow[ddrr, phantom, "\Downarrow\pi"] & & a_{12} \arrow[drr, "1\alpha"] & & \\
a_7 \arrow[uu, "(\alpha 1)1"] \arrow[drr, "\alpha"'] \arrow[urr, phantom, "="] & & & & & & a_4 \\
 & & a_6 \arrow[uu, "\alpha(11)"'] \arrow[rr, "\alpha"'] & & a_5 \arrow[urr, "\alpha"'] & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & & & a_1 \arrow[drr, "\alpha"] & & & \\
 & a_9 \arrow[urr, "(1\alpha)1"] & & & & a_2 \arrow[dr, "1\alpha"] & \\
a_8 \arrow[ur, "\alpha 1"] \arrow[drr, phantom, "\Downarrow 1\pi"] & & & & & & a_3 \arrow[ddd, "1(1\alpha)"] \\
 & & a_{13} \arrow[uuur, "\alpha 1"] \arrow[drr, "\alpha"'] \arrow[rrrr, phantom, "\Downarrow\pi"] \arrow[ddrr, phantom, "\Downarrow\pi"] & & & & \\
 & & & & a_{14} \arrow[uurr, "\alpha"] \arrow[dd, "(11)\alpha"'] \arrow[drr, phantom, "="] & & \\
a_7 \arrow[uuu, "(\alpha 1)1"] \arrow[uurr, "\alpha 1"'] \arrow[drr, "\alpha"'] & & & & & & a_4 \\
 & & a_6 \arrow[rr, "\alpha"'] & & a_5. \arrow[urr, "\alpha"'] & &
\end{tikzcd}
$$

(2) [Left normalization] The following equation of 2-cells in any strict 2-category $\mathfrak{T}(\mathcal{G}_2)(-,-)$, where the unmarked isomorphisms are either naturality isomorphisms for $\alpha$ or unique coherence isomorphisms

from the hom-strict 2-category. The 2-cell elements are as follows:

$$
\begin{tikzcd}
\bullet \arrow[r, Rightarrow, bend left] \arrow[r, Rightarrow, bend right] \arrow[r, phantom, "(l_3,e_3,g_3)"] & \bullet \arrow[r, Rightarrow, bend left] \arrow[r, Rightarrow, bend right] \arrow[r, phantom, "(l_2,e_2,g_2)"] & \bullet \arrow[r, Rightarrow, bend left] \arrow[r, Rightarrow, bend right] \arrow[r, phantom, "(1_L,1_E,1_G)"] & \bullet \arrow[r, Rightarrow, bend left] \arrow[r, Rightarrow, bend right] \arrow[r, phantom, "(l_1,e_1,g_1)"] & \bullet.
\end{tikzcd}
$$

We will use the following elements as abbreviations for the objects of $\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)$;

$$
\begin{aligned}
a_1 &= ((l_1,e_1,g_1) \otimes ((1_L,1_E,1_G) \otimes (l_2,e_2,g_2))) \otimes (l_3,e_3,g_3)\\
&= ((l_1,e_1,g_1) \otimes (l_2,e_2,g_2)) \otimes (l_3,e_3,g_3)\\
a_2 &= (l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3,e_3,g_3)\\
a_3 &= (l_1,e_1,g_1) \otimes (l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3)\\
a_4 &= ((l_1,e_1,g_1) \otimes ((1_L,1_E,1_G)) \otimes (l_2,e_2,g_2)) \otimes (l_3,e_3,g_3)\\
&= ((l_1,e_1,g_1) \otimes (l_2,e_2,g_2)) \otimes (l_3,e_3,g_3)\\
a_5 &= ((l_1,e_1,g_1) \otimes (1_L,1_E,1_G)) \otimes (l_2,e_2,g_2) \otimes (l_3,e_3,g_3)\\
&= (l_1,e_1,g_1) \otimes (l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3)\\
a_6 &= (l_1,e_1,g_1) \otimes (((1_L,1_E,1_G) \otimes (l_2,e_2,g_2)) \otimes (l_3,e_3,g_3))\\
&= (l_1,e_1,g_1) \otimes ((l_2,e_2,g_2) \otimes (l_3,e_3,g_3))\\
a_7 &= (l_1,e_1,g_1) \otimes ((1_L,1_E,1_G) \otimes (l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3)).
\end{aligned}
$$

Thus, we have the following commutative diagrams:

$$
\begin{tikzcd}
 & & a_1 \arrow[r, "(1\mathfrak{l})1"] \arrow[d, "\alpha"'] & a_2 \arrow[dddd, "\alpha"] \\
 & a_4 \arrow[ur, "\alpha 1"] \arrow[dd, "\alpha"'] \arrow[r, phantom, "\Downarrow\pi"] & a_6 \arrow[dd, "1\alpha"'] \arrow[dddr, "1(\mathfrak{l}1)"] \arrow[ur, phantom, "="] & \\
 & & & \\
a_2 \arrow[uur, "(\mathfrak{r}1)1"] \arrow[ddr, "\alpha"'] \arrow[r, phantom, "="] & a_5 \arrow[r, "\alpha"] & a_7 \arrow[dr, "1\mathfrak{l}"'] \arrow[ur, phantom, "\Downarrow\lambda"] & \\
 & a_3 \arrow[u, "\mathfrak{r}(11)"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\Downarrow\mu"] & & a_3
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & & a_1 \arrow[r, "(1\mathfrak{l})1"] \arrow[d, phantom, "\Downarrow\mu"] & a_2 \arrow[dd, "\alpha"] \\
 & a_4 \arrow[ur, "\alpha 1"] & \phantom{x} & \\
a_2 \arrow[ur, "(\mathfrak{r}1)1"] \arrow[uurrr, "11"'] \arrow[dr, "\alpha"'] & & = & \\
 & a_3. \arrow[rr, "1"'] & & a_3
\end{tikzcd}
$$

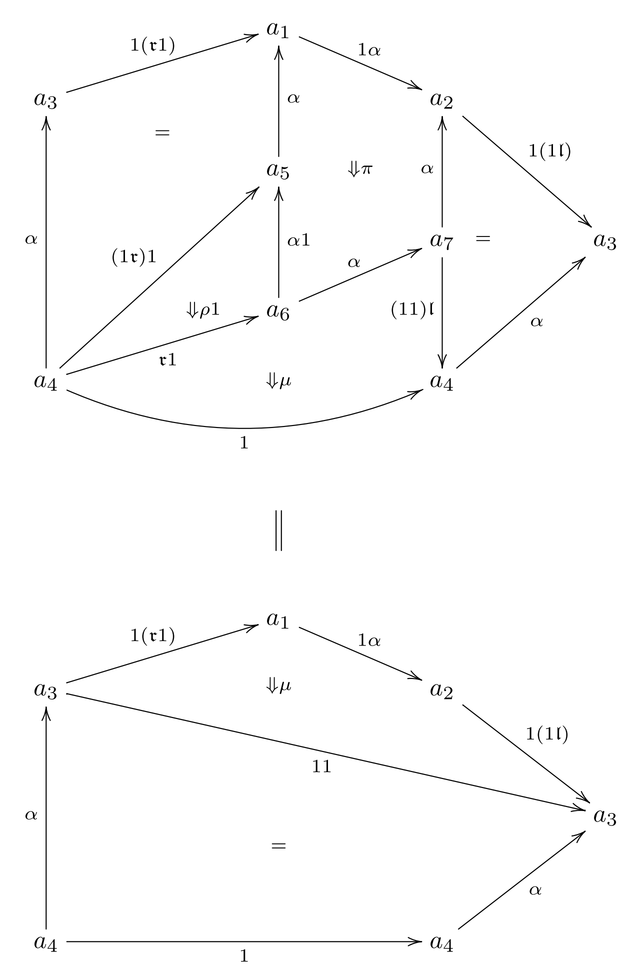
(3) [Right normalization] Similarly, the following equation of 2-cells holds in any strict 2-category $\mathfrak{T}(\mathcal{G}_2)(-,-)$. The 2-cell elements are as follows:

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (l_3,e_3,g_3)"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (1_L,1_E,1_G)"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (l_2,e_2,g_2)"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (l_1,e_1,g_1)"] & \bullet.
\end{tikzcd}$$

We will use the following elements as abbreviations for the objects of $\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet)$;

$$\begin{aligned}
a_1 &= (l_1, e_1, g_1) \otimes (((l_2, e_2, g_2) \otimes (1_L, 1_E, 1_G)) \otimes (l_3, e_3, g_3)) \\
&= (l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes (l_3, e_3, g_3)) \\
a_2 &= (l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes ((1_L, 1_E, 1_G) \otimes (l_3, e_3, g_3))) \\
&= (l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes (l_3, e_3, g_3)) \\
a_3 &= (l_1, e_1, g_1) \otimes (l_2{}^{e_2}({}^{g_2}l_3), e_2{}^{g_2}e_3, g_2g_3) \\
a_4 &= (l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3, e_3, g_3) \\
a_5 &= ((l_1, e_1, g_1) \otimes ((l_2, e_2, g_2) \otimes (1_L, 1_E, 1_G))) \otimes (l_3, e_3, g_3) \\
&= ((l_1, e_1, g_1) \otimes (l_2, e_2, g_2)) \otimes (l_3, e_3, g_3) \\
a_6 &= ((l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (1_L, 1_E, 1_G)) \otimes (e_3, g_3, h_3) \\
a_7 &= ((l_1, e_1, g_1) \otimes (l_2, e_2, g_2)) \otimes ((1_L, 1_E, 1_G) \otimes (l_3, e_3, g_3)) \\
&= (l_1{}^{e_1}({}^{g_1}l_2), e_1{}^{g_1}e_2, g_1g_2) \otimes (l_3, e_3, g_3).
\end{aligned}$$

Thus, we have the following commutative diagrams:



Note that this structure is a strict cubical tricategory with a single object, as it satisfies Remark 2.3. and Lemma 2.4. We can also obtain reverse construction; we can derive a 2-crossed module from any strict cubical tricategory with a single object (sestertius groupoid with one object). See how to obtain a 2-crossed module of groups from a sestertius category with one object in Section 3.2.

In this way, we can construct the morphism of 2-crossed modules in the form of trihomomorphism. Let $f$ be a 2-crossed module morphism between $\mathcal{G}_2$ and $\mathcal{G}'_2$ as shown below:

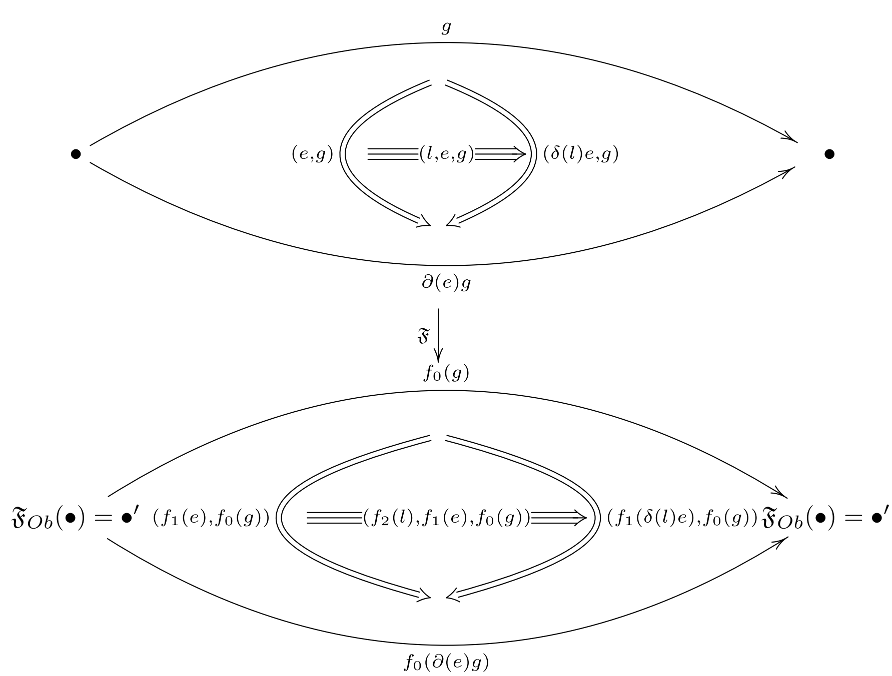
$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "f_2"'] & E \arrow[r, "\partial"] \arrow[d, "f_1"'] & G \arrow[d, "f_0"] \\ L' \arrow[r, "\delta'"'] & E' \arrow[r, "\partial'"'] & G'. \end{tikzcd}$$

Roughly speaking, trihomomorphism consists of the following data and provides the axioms in Definition 2.5:

$$\begin{tikzcd} \mathfrak{T}(\mathcal{G}_2) \arrow[dd, "\mathfrak{F}=(\mathfrak{F}_2,\mathfrak{F}_1,\mathfrak{F}_0,\mathfrak{F}_{Ob})"'] & L \rtimes E \rtimes G \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] \arrow[d, "\mathfrak{F}_2"] & E \rtimes G \arrow[l, bend left=50, "i_2"] \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] \arrow[d, "\mathfrak{F}_1"] & G \arrow[l, bend left=50, "i_1"] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] \arrow[d, "\mathfrak{F}_0"] & \{\bullet\} \arrow[l, bend left=50, "i_0"] \arrow[d, "\mathfrak{F}_{Ob}"] \\ {} \arrow[r, phantom, "="] & L' \rtimes E' \rtimes G' \arrow[r, shift left, "s'_2"] \arrow[r, shift right, "t'_2"'] & E' \rtimes G' \arrow[l, bend left=50, "i'_2"] \arrow[r, shift left, "s'_1"] \arrow[r, shift right, "t'_1"'] & G' \arrow[l, bend left=50, "i'_1"] \arrow[r, shift left, "s'_0"] \arrow[r, shift right, "t'_0"'] & \{\bullet'\}; \arrow[l, bend left=50, "i'_0"] \\ \mathfrak{T}'(\mathcal{G}'_2) & & & & \end{tikzcd}$$

(1) $\{\bullet\} \xrightarrow{\mathfrak{F}_{Ob}} \mathfrak{F}_{Ob}(\{\bullet\}) = \{\bullet'\}$,
(2) For $\bullet \in Ob\mathfrak{T}(\mathcal{G}_2)$, the following diagram represents

$$\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \xrightarrow{\mathfrak{F}} \mathfrak{T}'(\mathcal{G}'_2)(\mathfrak{F}_{Ob}(\bullet),\mathfrak{F}_{Ob}(\bullet)) :$$



For each $g \in G$, $e \in E$, and $l \in L$;

$$\begin{aligned}
\mathfrak{F}_0(g) &= f_0(g),\\
\mathfrak{F}_0(\partial(e)g) &= f_0(\partial(e)g),\\
\mathfrak{F}_1((e,g)) &= (f_1(e), f_0(g)),\\
\mathfrak{F}_1((\delta(l)e,g)) &= (f_1(\delta(l)e), f_0(g)),\\
\mathfrak{F}_2((l,e,g)) &= (f_2(l), f_1(e), f_0(g)).
\end{aligned}$$

The morphisms $\mathfrak{F}_n$ are compatible with $n$-source and $n$-target morphisms for $0 \leq n \leq 2$.

$$s_0'(\mathfrak{F}_0(g)) = t_0'(\mathfrak{F}_0(g)) = s_0'(\mathfrak{F}_0(\partial(e)g)) = t_0'(\mathfrak{F}_0(\partial(e)g)) = \mathfrak{F}_{Ob}(\bullet) = \bullet',$$

$$s_1'(\mathfrak{F}_1((e,g))) = s_1'((f_1(e), f_0(g))) = f_0(g),$$

$$t_1'(\mathfrak{F}_1((e,g))) = t_1'((f_1(e), f_0(g))) = \partial'(f_1(e))f_0(g) = \partial' f_1(e) f_0(g) = f_0\partial(e) f_0(g),$$

$$s_1'(\mathfrak{F}_1((\delta(l)e,g))) = s_1'((f_1(\delta(l)e), f_0(g))) = f_0(g),$$

$$\begin{aligned}
t_1'(\mathfrak{F}_1((\delta(l)e,g))) &= t_1'((f_1(\delta(l)e), f_0(g)))\\
&= \partial'(f_1(\delta(l)e))f_0(g)\\
&= \partial'(f_1(\delta(l)))\partial'(f_1(e))f_0(g)\\
&= \partial'\delta'(f_2(l))\partial' f_1(e) f_0(g)\\
&= \partial' f_1(e) f_0(g)\\
&= f_0\partial(e) f_0(g),
\end{aligned}$$

and

$$s_2'(\mathfrak{F}_2((l,e,g))) = s_2'((f_2(l), f_1(e), f_0(g))) = (f_1(e), f_0(g)),$$

$$\begin{aligned}
t_2'(\mathfrak{F}_2((l,e,g))) &= t_2'((f_2(l), f_1(e), f_0(g)))\\
&= (\delta'(f_2(l)) f_1(e), f_0(g))\\
&= (f_1\delta(l) f_1(e), f_0(g))\\
&= (f_1(\delta(l)e), f_0(g)).
\end{aligned}$$

(3) For $\bullet \in Ob\mathfrak{T}(\mathcal{G}_2)$, an adjoint equivalence $\xi : \otimes' \circ (\mathfrak{F} \times \mathfrak{F}) \Rightarrow \mathfrak{F} \circ \otimes$ with left adjoint is illustrated in the following diagram:

$$\begin{tikzcd}
\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \times \mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[r, "\mathfrak{F}\times\mathfrak{F}"] \arrow[d, "\otimes"'] & \mathfrak{T}'(\mathcal{G}_2')(\mathfrak{F}(\bullet),\mathfrak{F}(\bullet)) \arrow[d, "\otimes'"] \arrow[dl, phantom, "\Swarrow\xi"] \\
\mathfrak{T}(\mathcal{G}_2)(\bullet,\bullet) \arrow[r, "\mathfrak{F}"'] & \mathfrak{T}'(\mathcal{G}_2')(\mathfrak{F}(\bullet),\mathfrak{F}(\bullet));
\end{tikzcd}$$

$$\begin{tikzcd}
(l_1,e_1,g_1)\otimes(l_2,e_2,g_2) \arrow[r, mapsto, "\mathfrak{F}\times\mathfrak{F}"] \arrow[d, mapsto, "\otimes"'] & (f_2(l_1),f_1(e_1),f_0(g_1))\otimes(f_2(l_2),f_1(e_2),f_0(g_2)) \arrow[d, mapsto, "\otimes'"] \arrow[dl, phantom, "\Swarrow\xi"] \\
(l_1\ {}^{e_1}({}^{g_1}l_2), e_1\ {}^{g_1}e_2, g_1g_2) \arrow[r, mapsto, "\mathfrak{F}"'] & (f_2(l_1)\ {}^{f_1(e_1)}({}^{f_0(g_1)}f_2(l_2)), f_1(e_1)\ {}^{f_0(g_1)}f_1(e_2), f_0(g_1)f_0(g_2)),
\end{tikzcd}$$

(4) For $\bullet \in Ob\mathfrak{T}(\mathcal{G}_2)$, an adjoint equivalence $\iota : I'_{\mathfrak{F}_{Ob}(\bullet)} \Rightarrow \mathfrak{F} \circ I_{\bullet}$ with left adjoint illustrated in the following diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "I'_{\mathfrak{F}_{Ob}(\bullet)}"] \arrow[dr, "I_{\bullet}"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathfrak{T}'(\mathcal{G}'_2)(\mathfrak{F}_{Ob}(\bullet), \mathfrak{F}_{Ob}(\bullet)) \\
& \mathfrak{T}(\mathcal{G}_2)(\bullet, \bullet), \arrow[ur, "\mathfrak{F}"'] &
\end{tikzcd}$$

Since $f_n$s are group homomorphisms, unit elements are preserved.

Other data (invertible modifications) and axioms in Definition 2.5 are easily provided, because $\otimes$ is associative and $I_{\bullet}$ is preserved under $\mathfrak{F}$.

# 4 Conclusion

In this study, we obtained a strict cubical tricategory with a single object by looking at the 2-crossed module from a higher-dimensional perspective, and we also defined the reverse construction. With this type of definition, non-abelian objects and other algebraic 3-type models can be defined. The important point here is the difference in understanding between the crossed module and the 2-crossed module.

**Acknowledgement:** The authors would like to thank the handling editor and the referees for their helpful and useful comments and suggestions.

**Funding information:** The authors state no funding involved.

**Author contributions:** All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

**Conflict of interest:** The authors state no conflict of interest.

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