Research Article

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The sequential Henstock-Kurzweil delta integral on time scales

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Abstract: In this study, the basic theory of the sequential Henstock-Kurzweil delta integral on time scales will be discussed. First, we give the notion and the elementary properties of this integral; then we show the equivalence of the Henstock-Kurzweil delta integral and the sequential Henstock-Kurzweil delta integral on time scales. In addition, we consider the Cauchy criterion and the Fundamental Theorems of Calculus. Finally, we prove Henstock's lemma and give some convergence theorems. As an application, we consider the existence theorem of a kind of functional dynamic equations.

Keywords: sequential Henstock-Kurzweil integral, time scales, delta integral, convergence theorems, functional dynamic equations

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1 Introduction

As far as we know, we need to consider both continuous and discrete cases simultaneously when we solve some problems in many fields. But this often leads to a lot of repetitive theories. For the sake of unifying the theories of continuous and discrete analysis, Hilger and Aulbach introduced the calculus on time scales first in [1–3]. Later, it was soon applied to the study of differential equations [4–7]. Subsequently, some researchers noted the need for a more general theory of integration to study the dynamic equations. Guseinov [8–10] introduced the Riemann delta integral; then Park et al. [11] gave some conditions for the Riemann delta integrability of functions on $[a, b]_{\mathbb{T}}$. Mozyrska et al. [12] introduced the Riemann-Stieltjes delta integral. Liu and Zhao [13] studied the McShane delta integral; then Park et al. [14] discussed the relationships between the McShane delta integral and the McShane integral. You and Zhao [15] gave some convergence theorems for the McShane delta integral. Guseinov [9] also introduced the Lebesgue delta integral. Recently, Qin and Wang [16] studied the Lebesgue-Stieltjes-$\diamond_\alpha$ delta integral and delta measure. In fact, integrals can be studied not only on one-dimensional time scales but also in the context of multiple integrals [17]. And more details about time scales can be found in [5–7].

As already known, the Henstock-Kurzweil integral is a more general integral including Newton integral, Riemann integral, improper Riemann integral, and Lebesgue integral. It was studied by Kurzweil [18] in 1957 and Henstock [19] in 1963. For more details about the Henstock-Kurzweil integral and its applications, we can refer to [20–29]. Peterson and Thompson [30] introduced the Henstock-Kurzweil delta integral, where they defined a gauge function as $\delta = (\delta_L, \delta_R)$ to ensure that a δ-fine partition $\mathcal{P}$ for $[a, b]_{\mathbb{T}}$ can be found. Then, Thomson [31] studied the Henstock-Kurzweil integral on time scales using covering theory, which differs from [30].

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Later, Avsec et al. [32] generalized the Henstock-Kurzweil integral to unbounded time scales using the same method as [30] and Park et al. [33] discussed the relationships between the Henstock-Kurzweil delta integral and the Henstock-Kurzweil integral.

In fact, we know that the Henstock-Kurzweil integral is more difficult to transfer into abstract spaces than the Lebesgue integral. In order to solve this issue, Lee [34] considered a new approach to the Henstock-Kurzweil integral called the sequential Henstock-Kurzweil integral, but he did not develop it. Recently, Paxton [35] studied the sequential Henstock-Kurzweil integral and gave some properties about it. Then, Iluebe and Mogbademu [36–38] gave some convergence theorems for the sequential Henstock-Stieltjes integral, generalized the sequential Henstock-Kurzweil integral to set valued functions and studied the relationships between the sequential Henstock-Kurzweil integral and the topological Henstock-Kurzweil integral. In our work, we study the sequential Henstock-Kurzweil integral on time scales using the same method as [30] and extend the results in [30,35].

The structure of this study is as follows. In Section 2, we give some notions about time scales and other preliminaries. In Section 3, we give the notion of sequential Henstock-Kurzweil delta integrals; then we show the equivalence of the sequential Henstock-Kurzweil delta integral and the Henstock-Kurzweil delta integral. Then, we discuss the basic properties and prove Cauchy criterion of this integral. In Section 4, we give the fundamental theorems of calculus and some examples. In Section 5, we give Henstock's lemma and some convergence theorems. As an application, we consider the existence theorem of a kind of functional dynamic equation (FDE) in Section 6. Finally, we give some conclusions.

2 Preliminaries

We give some notions and other preliminaries in this section.

Definition 2.1. (Definition 1.1, [17]) We say $\mathbb{T}$ is a time scale, provided $\mathbb{T} \subset \mathbb{R}(\mathbb{T} \neq \varnothing)$ and $\mathbb{T}$ is closed.

Definition 2.2. (Definition 1.6, 1.7, 1.8, [17]) We define the forward and backward operator on $\mathbb{T}$ by $\sigma(s) = \inf\{q \in \mathbb{T} : q > s\}$ and $\rho(s) = \sup\{q \in \mathbb{T} : q < s\}$, where $\inf\varnothing = \sup\mathbb{T}$, $\sup\varnothing = \inf\mathbb{T}$.

Definition 2.3. (Definition 1.1, [5]) For any $s \in \mathbb{T}$, s is called right-scattered (left-scattered), provided $\sigma(s) > s(\rho(s) < s)$. $s \in \mathbb{T}$ is called right-dense if $s < \sup\mathbb{T}$ and $\sigma(s) = s$, while if $s > \inf\mathbb{T}$ and $\rho(s) = s$, we say $s \in \mathbb{T}$ is left-dense.

Definition 2.4. (Definition 1.25, [17]) For any $s \in \mathbb{T}$, we define the right-graininess and left-graininess function on $\mathbb{T}$ by $\mu(s) = \sigma(s) - s$, $\nu(s) = s - \rho(s)$.

Definition 2.5. (Definition 1.1.9, [6]) If $\sup\mathbb{T} = \infty$, let $\mathbb{T}^{\mathcal{K}} := \mathbb{T}$, while if $\sup\mathbb{T} < \infty$, let $\mathbb{T}^{\mathcal{K}} := \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}]$.

Definition 2.6. (Definition 1.3, [30]) Let $h : \mathbb{T} \to \mathbb{R}$ and $s \in \mathbb{T}^{\mathcal{K}}$, for any $\varepsilon > 0$, there exists $h^{\Delta}(s) \in \mathbb{R}$ and there is a neighborhood $N(s, \varepsilon)$ of s, we have

$$|[h(\sigma(s)) - h(x)] - h^{\Delta}(s)[\sigma(s) - x]| \leqslant \varepsilon|\sigma(s) - x|,$$

for all $x \in N(s, \varepsilon)$. We say $h^{\Delta}(s)$ is the delta derivative of h at s and we call h is delta differentiable on $\mathbb{T}$ if $h^{\Delta}(s)$ exists for every $s \in \mathbb{T}^{\mathcal{K}}$.

Theorem 2.1. (Theorem 1.16, [5]) *Assume $h : \mathbb{T} \to \mathbb{R}$, for any $q \in \mathbb{T}^{\mathcal{K}}$, we have the following properties:*
(*a*) *If h is delta differentiable at q, then h is continuous at q.*
(*b*) *If q is right-scattered and h is continuous at q, then h is delta differentiable at q with*

$$h^{\Delta}(q) = \frac{h(\sigma(q)) - h(q)}{\sigma(q) - q}.$$

(c) *If q is right-dense, then h is delta differentiable at q iff the limit*

$$\lim_{u \to q} \frac{h(u) - h(q)}{u - q}$$

is finite. Further

$$h^{\Delta}(q) = \lim_{u \to q} \frac{h(u) - h(q)}{u - q}.$$

Definition 2.7. (Definition 1.15, [7]) Assume $h : \mathbb{T} \to \mathbb{R}$, we say that h is right-dense (rd)-continuous if h is continuous at all right-dense points in $\mathbb{T}$ and all its left-dense points' left side limits are finite, we denote all rd-continuous functions on $\mathbb{T}$ as $C_{rd}(\mathbb{T})$. If h is delta differentiable on $\mathbb{T}$ and $h^{\Delta}(s)$ is rd-continuous, we denote $h \in C^1_{rd}(\mathbb{T})$.

Throughout this study, we assume that $a, b \in \mathbb{T}$, then we define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}.$$

Definition 2.8. (Definition 1.5, [30]) A partition $\mathcal{P}$ for $[a, b]_{\mathbb{T}}$ is defined by

$$\mathcal{P} = \{a = s_0 \leqslant \xi_1 \leqslant s_1 \leqslant \ldots \leqslant s_{n-1} \leqslant \xi_n \leqslant s_n = b\}$$

with $s_i > s_{i-1}$, we call s_i end points, ξ_i tag points, and $s_i, \xi_i \in [a, b]_{\mathbb{T}}$, sometimes we write $\mathcal{P} = \{([s_{i-1}, s_i], \xi_i)\}_{i=1}^{n}$.

Let $h : [a, b] \to \mathbb{R}$, we denote that $S(h, \mathcal{P}) = \sum_{i=1}^{n} h(\xi_i)(s_i - s_{i-1})$.

Definition 2.9. (Definition 1.4, 1.6, [30]) For any $s \in [a, b)_{\mathbb{T}}$, we say $\delta = (\delta_L, \delta_R)$ is a δ-gauge for $[a, b]_{\mathbb{T}}$ if $\delta_R(s) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_R(b) \geqslant 0$, $\delta_R(s) \geqslant \mu(s)$, and $\delta_L(s) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_L(a) \geqslant 0$. For any δ-gauge for $[a, b]_{\mathbb{T}}$, we say a partition $\mathcal{P}$ is a δ-fine partition, provided

$$\xi_i - \delta_L(\xi_i) \leqslant s_{i-1} < s_i \leqslant \xi_i + \delta_R(\xi_i), i \in \{1, 2, \ldots, n\}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then $\delta_L = \delta_R$.

Theorem 2.2. (Lemma 1.9, [30]) *If δ is a δ-gauge for $[a, b]_{\mathbb{T}}$, there exists a δ-fine partition $\mathcal{P}$ for $[a, b]_{\mathbb{T}}$.*

Definition 2.10. (Definition 1.2.2, [21]) Assume $h : [a, b] \to \mathbb{R}$, we say that h is Henstock-Kurzweil integrable on $[a, b]$ if there exists a number $V \in \mathbb{R}$ and for any $\varepsilon > 0$, there exists a $\delta(x) > 0$ for every δ-fine partition $\mathcal{P}$, we have $|S(h, \mathcal{P}) - V| < \varepsilon$, and we write $(H)\int_a^b h\mathrm{d}x = V$.

Definition 2.11. (Definition 9, [35]) Let $h : [a, b] \to \mathbb{R}$, we say that h is sequential Henstock-Kurzweil integrable on $[a, b]$ if there exists a number $V \in \mathbb{R}$ and for any $\varepsilon > 0$, there exists a sequence of positive functions $\{\delta_k\}_{k=1}^{\infty}$ such that for every δ_k-fine partition $\mathcal{P}_k$, its end points are $\{\{a = s_{0k} < s_{1k} < \ldots < s_{m_k k} = b\}\}_{k=1}^{\infty}$, its tag points are $\{\{\xi_{1k}, \xi_{2k}, \ldots, \xi_{m_k k}\}\}_{k=1}^{\infty}$, we have $|S(h, \mathcal{P}_k) - V| < \varepsilon (k \to \infty)$, and we write $(SH)\int_a^b h\mathrm{d}x = V$.

Remark 2.1. We say that h is not sequential Henstock-Kurzweil integrable on $[a, b]$ if for any $V \in \mathbb{R}$ and there exists a $\varepsilon_0 > 0$, we cannot find a sequence of positive functions $\{\delta_k\}_{k=1}^{\infty}$ satisfying $|S(h, \mathcal{P}_k) - V| < \varepsilon_0 (k \to \infty)$, where $\mathcal{P}_k$ is a δ_k-fine partition for $[a, b]$.

Example 2.1. Let $h : [0,1] \to \mathbb{R}$, then we define

$$h(x) = \begin{cases} \frac{1}{n}, & x = \frac{m}{n}, (m,n) = 1, \\ 0, & x \in (0,1)\backslash\mathbb{Q} \cup \{0,1\}. \end{cases}$$

We claim that h is sequential Henstock-Kurzweil integrable with

$$(SH)\int_0^1 h(x)\mathrm{d}x = 0.$$

Let $\varepsilon > 0$ be given, assume $\{\delta_k\}_{k=1}^{\infty}$ is a decreasing sequence on [0, 1], let K be large enough such that $\frac{1}{K} < \varepsilon$, we define

$$\delta_K(x) = \begin{cases} \frac{n\varepsilon}{(n-1)2^n}, & x = \frac{m}{n}, (m,n) = 1, n \geqslant 2, \\ 1, & \text{otherwise}. \end{cases}$$

When $k > K$, for every δ_k-fine partition $\mathcal{P}_k$, the tag points ξ have two cases:
If $\xi = \frac{m}{n}$, $h(\xi) = \frac{1}{n}$, if $\xi \in (0,1)\backslash\mathbb{Q} \cup \{0,1\}$, $h(\xi) = 0$, thus

$$|S(h,\mathcal{P}_k)| = \left|\sum_{n=1}^{m_k\in\mathbb{N}} h(\xi_{nk})(x_{nk} - x_{(n-1)k})\right| < \left|\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}\right| = \varepsilon.$$

Therefore, $(SH)\int_0^1 h(x)\mathrm{d}x = 0$.

Theorem 2.3. (Theorem 1, [35]) *Let $h : [a,b] \to \mathbb{R}$, h is Henstock-Kurzweil integrable on $[a,b]$ if and only if h is sequential Henstock-Kurzweil integrable on $[a,b]$.*

Remark 2.2. When h is Henstock-Kurzweil integrable on $[a,b]$, by Definition 2.10, there exists a function $\delta(x) > 0$ on $[a,b]$. Then, we let $\delta_k = \delta$. It is easy to show that h is sequential Henstock-Kurzweil integrable on $[a,b]$. Conversely, if h is sequential Henstock-Kurzweil integrable on $[a,b]$, by Definition 2.11, there exists a sequence of positive functions $\{\delta_k\}_{k=1}^{\infty}$ and a positive integer K, let $\delta(x) = \delta_{K+1}(x)$. Obviously, h is Henstock-Kurzweil integrable on $[a,b]$.

Definition 2.12. (Definition 1.7, [30]) Let $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$, we say that h is Henstock-Kurzweil delta integrable on $[a,b]_{\mathbb{T}}$ if there exists a number $V \in \mathbb{R}$ and for any $\varepsilon > 0$, there exists a δ-gauge $\delta = (\delta_L, \delta_R)$, for each δ-fine partition $\mathcal{P}$, we have $|S(h,\mathcal{P}) - V| < \varepsilon$, and we write $(HD)\int_a^b h\Delta s = V$.

3 Sequential Henstock-Kurzweil delta integral on time scales

We now give the definition of the sequential Henstock-Kurzweil delta integral and show its equivalence to the Henstock-Kurzweil delta integral. Then, we discuss the basic properties about it and give the Cauchy criterion.

Definition 3.1. Assume $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$, we say that h is sequential Henstock-Kurzweil delta integrable on $[a,b]_{\mathbb{T}}$ if there exists a number $V \in \mathbb{R}$ and for any $\varepsilon > 0$, there exists a sequence of δ-gauges $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ such that for every δ_k-fine partition $\mathcal{P}_k$, its end points are

$$\{\{a = s_{0k} < s_{1k} < s_{2k} < \ldots < s_{m_k k} = b\}\}_{k=1}^{\infty},$$

its tag points are $\{\{\xi_{1k}, \xi_{2k}, \ldots, \xi_{m_k k}\}\}_{k=1}^{\infty}$, we have $|S(h,\mathcal{P}_k) - V| < \varepsilon (k \to \infty)$, we write $(SHD)\int_a^b h\Delta s = V$ and $h \in SHD([a,b]_{\mathbb{T}})$.

Remark 3.1. We say that h is not sequential Henstock-Kurzweil delta integrable on $[a,b]_{\mathbb{T}}$ if for any $V \in \mathbb{R}$ and there exists a $\varepsilon_0 > 0$, we cannot find a sequence of δ-gauges $\{\delta_k\}_{k=1}^{\infty}$ satisfy $|S(h,\mathcal{P}_k) - V| < \varepsilon_0 (k \to \infty)$, where $\mathcal{P}_k$ is a δ_k-fine partition for $[a,b]_{\mathbb{T}}$.

Example 3.1. Let $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and $S = \bigcup_{i=1}^{\infty} q_i$, q_i is a rational number in $[a,b]_{\mathbb{T}}$. We define h by

$$h(s) = \begin{cases} 1, & s \in S, \\ 0, & s \in [a,b]_{\mathbb{T}} \backslash S. \end{cases}$$

For any $\varepsilon > 0$, let K be large enough such that $\frac{1}{K} < \varepsilon$, then we define $\delta_L^K(q_i) = \delta_R^K(q_i) = \frac{\varepsilon}{2^{i+1}}$, $i \geqslant 1$, $\delta_L^K(s) = 1$, $\delta_R^K(s) = \max\{1, \mu(s)\}$ for $s \in [a,b]_{\mathbb{T}} \backslash S$, then we assume $\{\delta_k\}_{k=1}^{\infty}$ is a decreasing sequence, so when $k > K$, for every δ_k-fine partition $\mathcal{P}_k$, we have

$$|S(h,\mathcal{P}_k)| = \left| \sum_{i=1}^{m_k \in \mathbb{N}} h(\xi_{ik})(s_{ik} - s_{(i-1)k}) \right| \leqslant \left| \sum_{i=1}^{\infty} \delta_L^K(q_i) + \delta_R^K(q_i) \right| < \varepsilon.$$

Therefore, $(SHD)\int_a^b h\Delta s = 0$.

Theorem 3.1. *Let $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$, h is Henstock-Kurzweil delta integrable on $[a,b]_{\mathbb{T}}$ if and only if $h \in SHD([a,b]_{\mathbb{T}})$.*

Proof. ($\Rightarrow$) Since h is Henstock-Kurzweil delta integrable on $[a,b]_{\mathbb{T}}$, for any $\varepsilon > 0$, there exists a $\delta = (\delta_L, \delta_R)$, for every δ-fine partition $\mathcal{P}$, we have

$$\left| S(h,\mathcal{P}) - (HD)\int_a^b h\Delta s \right| < \varepsilon.$$

Let $\varepsilon = \frac{1}{k}(k = 1, 2, \ldots)$, for one ε, we have one δ_k, so we obtain a sequence of δ-gauges $\{\delta_k\}_{k=1}^{\infty}$, for every δ_k-fine partition $\mathcal{P}_k$, we have

$$\left| S(h,\mathcal{P}_k) - (HD)\int_a^b h\Delta s \right| < \frac{1}{k}.$$

Therefore, there exists a $\delta_K \in \{\delta_k\}_{k=1}^{\infty}$, when $k > K$, for every δ_k-fine partition $\mathcal{P}_k$, we have

$$\left| S(h,\mathcal{P}_k) - (HD)\int_a^b h\Delta s \right| < \frac{1}{k} \to 0(k \to \infty).$$

So, $(SHD)\int_a^b h\Delta s = (HD)\int_a^b h\Delta s$.

($\Leftarrow$) For any $\varepsilon > 0$, since $h \in SHD([a,b]_{\mathbb{T}})$, we have a $\delta_N \in \{\delta_n = (\delta_L^n, \delta_R^n)\}_{n=1}^{\infty}$, when $n > N$, for every δ_n-fine partition $\mathcal{P}_n$ we have

$$\left| S(h,\mathcal{P}_n) - (SHD)\int_a^b h\Delta s \right| < \varepsilon.$$

We choose $\delta = \delta_{N+1}$, denote $\mathcal{P} = \mathcal{P}_{N+1}$, so every δ_{N+1}-fine partition $\mathcal{P}_{N+1}$ is a δ-fine partition. So

$$\left| S(h,\mathcal{P}) - (SHD)\int_a^b h\Delta s \right| = \left| S(h,\mathcal{P}_{N+1}) - (SHD)\int_a^b h\Delta s \right| < \varepsilon.$$

Therefore, $(HD)\int_a^b h\Delta s = (SHD)\int_a^b h\Delta s$, and the proof is complete. □

Theorem 3.2. *If $h \in SHD([a,b]_{\mathbb{T}})$, the value of this integral is unique.*

Proof. Suppose the value of this integral is not unique and $(SHD)\int_a^b h\Delta s = V_1 \in \mathbb{R}$. Let $V_2 \in \mathbb{R}$, $V_2 \neq V_1$ such that $(SHD)\int_a^b h\Delta s = V_2$.

For any $\varepsilon > 0$, since $h \in SHD([a,b]_{\mathbb{T}})$, we have a $\delta_N \in \{\delta_n = (\delta_L^n, \delta_R^n)\}_{n=1}^{\infty}$, when $n > N$, for every δ_n-fine partition $\mathcal{P}_n$, we have

$$|S(h, \mathcal{P}_n) - V_1| < \frac{\varepsilon}{2}.$$

Similarly, we have another $\delta_M \in \{\delta_m = (\delta_L^m, \delta_R^m)\}_{m=1}^{\infty}$, when $m > M$, for every δ_m-fine partition $\mathcal{P}_m$, we have

$$|S(h, \mathcal{P}_m) - V_2| < \frac{\varepsilon}{2}.$$

For every $\delta_n \in \{\delta_n = (\delta_L^n, \delta_R^n)\}_{n=1}^{\infty}$ and $\delta_m \in \{\delta_m = (\delta_L^m, \delta_R^m)\}_{m=1}^{\infty}$, we choose

$$\delta_L^k = \min\{\delta_L^n, \delta_L^m\},\ \delta_R^k = \min\{\delta_R^n, \delta_R^m\}(k = m = n = 1, 2, \ldots).$$

Therefore, $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ is a sequence of δ-gauges on $[a,b]_{\mathbb{T}}$, and for each δ_k-fine partition $\mathcal{P}_k$, it is also δ_n and δ_m-fine partition for $[a,b]_{\mathbb{T}}$.

Then, for the above ε, there exists a $\delta_K \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > \max\{M, N\}$ and $\mathcal{P}_k$ is a δ_k-fine partition, we have

$$\begin{aligned} |V_1 - V_2| &= |V_1 - S(h, \mathcal{P}_k) + S(h, \mathcal{P}_k) - V_2| \\ &\leqslant |V_1 - S(h, \mathcal{P}_k)| + |S(h, \mathcal{P}_k) - V_2| \\ &< \varepsilon. \end{aligned}$$

Therefore, the value of this integral is unique. □

Theorem 3.3. *Let $h \in SHD([a,b]_{\mathbb{T}})$, if $h(s) \geqslant 0$ for any $s \in [a,b]_{\mathbb{T}}$, then*

$$(SHD)\int_a^b h\Delta s \geqslant 0.$$

Proof. Since $h \in SHD([a,b]_{\mathbb{T}})$, for any $\varepsilon > 0$, we have a $\delta_K \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > K$, for every δ_k-fine partition $\mathcal{P}_k$, we have

$$\left| S(h, \mathcal{P}_k) - (SHD)\int_a^b h\Delta s \right| < \varepsilon.$$

Therefore,

$$\lim_{k\to\infty} S(h, \mathcal{P}_k) = (SHD)\int_a^b h\Delta s,$$

and we know $S(h, \mathcal{P}_k) = \sum_{i=1}^{m_k \in \mathbb{N}} h(\xi_{ik})(s_{ik} - s_{(i-1)k})$, $h(\xi_{ik}) \geqslant 0$, $s_{ik} - s_{(i-1)k} \geqslant 0$, so $S(h, \mathcal{P}_k) \geqslant 0$, thus we have

$$(SHD)\int_a^b h\Delta s \geqslant 0.$$ □

Theorem 3.4. *Let $h_1, h_2 \in SHD([a,b]_{\mathbb{T}})$, for any $\alpha, \beta \in \mathbb{R}$, we have $\alpha h_1 + \beta h_2 \in SHD([a,b]_{\mathbb{T}})$ and*

$$(SHD)\int_a^b \alpha h_1 + \beta h_2 \Delta s = (SHD)\alpha\int_a^b h_1\Delta s + (SHD)\beta\int_a^b h_2\Delta s.$$

Proof. The case where $\alpha = 0, \beta = 0$ is trivial, so we assume $\alpha \neq 0, \beta \neq 0$.

For any $\varepsilon > 0$, since $h_1 \in SHD([a,b]_{\mathbb{T}})$, we have a $\delta_N \in \{\delta_n = (\delta_L^n, \delta_R^n)\}_{n=1}^{\infty}$, when $n > N$, for every δ_n-fine partition $\mathcal{P}_n$ we have

$$\left| S(h_1, \mathcal{P}_n) - (SHD)\int_a^b h_1 \Delta s \right| < \frac{\varepsilon}{2|\alpha|}.$$

Similarly, we have another $\delta_M \in \{\delta_m = (\delta_L^m, \delta_R^m)\}_{m=1}^{\infty}$, when $m > M$, for every δ_m-fine partition $\mathcal{P}_m$, we have

$$\left| S(h_2, \mathcal{P}_m) - (SHD)\int_a^b h_2 \Delta s \right| < \frac{\varepsilon}{2|\beta|}.$$

As in the proof of Theorem 3.2, we choose $\delta_L^k = \min\{\delta_L^n, \delta_L^m\}$, $\delta_R^k = \min\{\delta_R^n, \delta_R^m\}$, so $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ is a sequence of δ-gauges on $[a,b]_{\mathbb{T}}$, and for every δ_k-fine partition $\mathcal{P}_k$, it is also δ_n and δ_m-fine partition for $[a,b]_{\mathbb{T}}$.

For the above ε, we let $K = \max\{M, N\}$, when $k > K$ and $\mathcal{P}_k$ is a δ_k-fine partition, we have

$$\begin{aligned}
&\left| S(\alpha h_1 + \beta h_2, \mathcal{P}_k) - \left((SHD)\alpha\int_a^b h_1 \Delta s + (SHD)\beta\int_a^b h_2 \Delta s \right) \right| \\
&= \left| \sum_{i=1}^{m_k \in \mathbb{N}} (\alpha h_1 + \beta h_2)(\xi_{ik})(s_{ik} - s_{(i-1)k}) - \left((SHD)\alpha\int_a^b h_1 \Delta s + (SHD)\beta\int_a^b h_2 \Delta s \right) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\alpha h_1 + \beta h_2 \in SHD([a,b]_{\mathbb{T}})$ and

$$(SHD)\int_a^b (\alpha h_1 + \beta h_2)\Delta s = (SHD)\alpha\int_a^b h_1 \Delta s + (SHD)\beta\int_a^b h_2 \Delta s.$$

□

Theorem 3.5. *Let $h_1, h_2 \in SHD([a,b]_{\mathbb{T}})$, and $h_1(s) \leqslant h_2(s)$ for any $s \in [a,b]_{\mathbb{T}}$, then $(SHD)\int_a^b h_1 \Delta s \leqslant (SHD)\int_a^b h_2 \Delta s$.* It is easy to show by Theorems 3.3 and 3.4.

Proof. Since $h_1, h_2 \in SHD([a,b]_{\mathbb{T}})$, by Theorem 3.4 we know that $h_2 - h_1 \in SHD([a,b]_{\mathbb{T}})$. In addition, $h_2(s) - h_1(s) \geqslant 0$ for any $s \in [a,b]_{\mathbb{T}}$. By Theorem 3.3 we have

$$(SHD)\int_a^b (h_2 - h_1)\Delta s \geqslant 0,$$

the proof is complete. □

Theorem 3.6. *Let $h, |h| \in SHD([a,b]_{\mathbb{T}})$, then*

$$(SHD)\left| \int_a^b h \Delta s \right| \leqslant (SHD)\int_a^b |h| \Delta s.$$

We can prove it right away using Theorem 3.5.

Theorem 3.7. *For any $c \in (a,b)_{\mathbb{T}}$, if $h \in SHD([a,c]_{\mathbb{T}})$, $h \in SHD([c,b]_{\mathbb{T}})$, then $h \in SHD([a,b]_{\mathbb{T}})$, and*

$$(SHD)\int_a^b h \Delta s = (SHD)\int_a^c h \Delta s + (SHD)\int_c^b h \Delta s.$$

Proof. Since $h \in SHD([a,c]_{\mathbb{T}})$, $h \in SHD([c,b]_{\mathbb{T}})$, for any $\varepsilon > 0$, we have a $\delta_N \in \{\delta_n = (\delta_L^n, \delta_R^n)\}_{n=1}^{\infty}$ on $[a,c]_{\mathbb{T}}$, when $n \geqslant N$, for every δ_n-fine partition $\mathcal{P}_n$, we have

$$\left| S(h, \mathcal{P}_n) - (SHD)\int_a^c h \Delta s \right| < \frac{\varepsilon}{2}.$$

In the same way, we have another $\delta_M \in \{\delta_m = (\delta_L^m, \delta_R^m)\}_{m=1}^{\infty}$ on $[c, b]_{\mathbb{T}}$, when $m \geqslant M$, for every δ_m-fine partition $\mathcal{P}_m$, we have

$$\left| S(h, \mathcal{P}_m) - (SHD)\int_c^b h\Delta s \right| < \frac{\varepsilon}{2}.$$

Assume $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ is a decreasing sequence, let K be large enough such that $\frac{1}{K} < \varepsilon$, then define a δ-gauge $\delta_K \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ on $[a, b]_{\mathbb{T}}$, let $\delta_L^K(s) = \delta_L^N(s)$, $s \in [a, c)_{\mathbb{T}}$, and

$$\delta_L^K(c) = \begin{cases} \delta_L^N(c), & \nu(c) = 0, \\ \min\left\{\delta_L^N(c), \dfrac{\nu(c)}{3}\right\}, & \nu(c) > 0, \end{cases}$$

$\delta_L^K(s) = \min\left\{\delta_L^M(s), \frac{s-c}{3}\right\}$, $s \in (c, b]_{\mathbb{T}}$, $\delta_R^K(s) = \min\left\{\delta_R^N(s), \max\left\{\mu(s), \frac{c-s}{3}\right\}\right\}$, $s \in [a, c)_{\mathbb{T}}$, $\delta_R^K(s) = \delta_R^N(s)$, $s \in [c, b]_{\mathbb{T}}$.

Then, let $\mathcal{P}_k$ be a δ_k-fine partition for $[a, b]_{\mathbb{T}}$, when $k > K$, by the way we define δ_k, c has two cases: c is an end point or a tag point.

(1) If c is an end point, we denote that

$$\mathcal{P}_k = \mathcal{P}' \cup \mathcal{P}'' = \{a = s_{0k} < s_{1k} < \ldots < s_{pk} = c = s_{(p+1)k} < \ldots < s_{m_k k} = b\},$$

where

$$\mathcal{P}' = \{a = s_{0k} < s_{1k} < \ldots < s_{pk} = c\},$$

$$\mathcal{P}'' = \{c = s_{(p+1)k} < s_{(p+2)k} < \ldots < s_{m_k k} = b\}.$$

We know, $\mathcal{P}'$ is a δ_N-fine partition and $\mathcal{P}''$ is a δ_M-fine partition by the way we define δ-gauge. Therefore, we have

$$\begin{aligned} &\left| S(h, \mathcal{P}_k) - \left((SHD)\int_a^c h\Delta s + (SHD)\int_c^b h\Delta s \right) \right| = \left| S(h, \mathcal{P}' \cup \mathcal{P}'') - \left((SHD)\int_a^c h\Delta s + (SHD)\int_c^b h\Delta s \right) \right| \\ &\leqslant \left| S(h, \mathcal{P}') - (SHD)\int_a^c h\Delta s \right| + \left| S(h, \mathcal{P}'') - (SHD)\int_c^b h\Delta s \right| < \varepsilon. \end{aligned}$$

(2) If c is a tag point, assume that $c = \xi_{pk}$, we use the following equation:

$$h(c)(s_{pk} - s_{(p-1)k}) = h(c)(s_{pk} - c) + h(c)(c - s_{(p-1)k}).$$

Then, we have

$$\begin{aligned} &\left| S(h, \mathcal{P}_k) - \left((SHD)\int_a^c h\Delta s + (SHD)\int_c^b h\Delta s \right) \right| \\ &= \left| \sum_{i=1}^{m_k \in \mathbb{N}} h(\xi_{ik})(s_{ik} - s_{(i-1)k}) - \left((SHD)\int_a^c h\Delta s + (SHD)\int_c^b h\Delta s \right) \right| \\ &\leqslant \left| \sum_{i=1}^{p-1} h(\xi_{ik})(s_{ik} - s_{(i-1)k}) + h(c)(c - s_{(p-1)k}) - (SHD)\int_a^c h\Delta s \right| \\ &\quad + \left| \sum_{i=p+1}^{m_k \in \mathbb{N}} h(\xi_{ik})(s_{ik} - s_{(i-1)k}) + h(c)(s_{pk} - c) - (SHD)\int_c^b h\Delta s \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $h \in SHD([a, b]_{\mathbb{T}})$ and $(SHD)\int_a^b h\Delta s = (SHD)\int_a^c h\Delta s + (SHD)\int_c^b h\Delta s$. □

Theorem 3.8. (Cauchy criterion) *Assume $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$, $h \in SHD([a,b]_{\mathbb{T}})$ if and only if for any $\varepsilon > 0$, there exists a $\delta_K \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > K$, for all δ_k-fine partitions $\mathcal{P}_k$ and $\mathcal{Q}_k$ for $[a,b]_{\mathbb{T}}$, we have*

$$|S(h,\mathcal{P}_k) - S(h,\mathcal{Q}_k)| < \varepsilon.$$

Proof. (⇒) If $h \in SHD([a,b]_{\mathbb{T}})$, for any $\varepsilon > 0$, there exists a $\delta_K \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > K$, we denote $\mathcal{P}_k$ and $\mathcal{Q}_k$ are δ_k-fine partitions, so

$$\left| S(h,\mathcal{P}_k) - (SHD)\int_a^b h\Delta s \right| < \frac{\varepsilon}{2}, \left| S(h,\mathcal{Q}_k) - (SHD)\int_a^b h\Delta s \right| < \frac{\varepsilon}{2}.$$

Then, we have

$$|S(h,\mathcal{P}_k) - S(h,\mathcal{Q}_k)| \leqslant \left| S(h,\mathcal{P}_k) - (SHD)\int_a^b h\Delta s \right| + \left| S(h,\mathcal{Q}_k) - (SHD)\int_a^b h\Delta s \right| < \varepsilon.$$

(⇐) Let $\varepsilon > 0$ be given, there exists a $\delta_{K_1} \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > K_1$, for all δ_k-fine partitions $\mathcal{P}_k$ and $\mathcal{Q}_k$ for $[a,b]_{\mathbb{T}}$, we have

$$|S(h,\mathcal{P}_k) - S(h,\mathcal{Q}_k)| < \frac{\varepsilon}{2}.$$

Without loss of generality we assume that $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ is a decreasing sequence and for any $n > k$, $\mathcal{Q}_n$ is a δ_k-fine partition, so $|S(h,\mathcal{Q}_k) - S(h,\mathcal{Q}_n)| < \frac{\varepsilon}{2}$, therefore $\{S(h,\mathcal{Q}_k)\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Thus, there exists $A \in \mathbb{R}$, $S(h,\mathcal{Q}_k) \to A (k \to \infty)$, we denote $A = (SHD)\int_a^b h\Delta s$.

For the above ε, there exists $K_2 > 0$, when $k > K_2$, we have

$$\left| S(h,\mathcal{Q}_k) - (SHD)\int_a^b h\Delta s \right| < \frac{\varepsilon}{2}.$$

Then, let $K = \max\{K_1, K_2\}$, so there exists a $\delta_K \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > K$, we have

$$\begin{aligned} &\left| S(h,\mathcal{P}_k) - (SHD)\int_a^b h\Delta s \right| \\ &\leqslant |S(h,\mathcal{P}_k) - S(h,\mathcal{Q}_k)| + \left| S(h,\mathcal{Q}_k) - (SHD)\int_a^b h\Delta s \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $h \in SHD([a,b]_{\mathbb{T}})$. □

Example 3.2. Let $\mathbb{T} = [2,4] \cap \mathbb{Z}$ and $h(s) = \sigma(s)^2 + s\sigma(s) + s^2$.

For any $\varepsilon > 0$, we define the following sequence of δ-gauges $\{\delta_k\}_{k=1}^{\infty}$:

$$\begin{aligned} &\delta_L^k(2) = 1 + \frac{1}{k}, \quad \delta_R^k(2) = \frac{3}{2} + \frac{1}{k}, \\ &\delta_L^k(3) = \frac{1}{2} + \frac{1}{k}, \quad \delta_R^k(3) = \frac{3}{2} + \frac{1}{k}, \\ &\delta_L^k(4) = \frac{1}{2} + \frac{1}{k}, \quad \delta_R^k(4) = 1 + \frac{1}{k}, \end{aligned}$$

for every positive integer k, we choose K large enough such that $\frac{1}{K} < \varepsilon$, when $k > K$, by the way we define δ-gauge, it is easy to show that the δ_k-fine partition $\mathcal{P}_k$ only can be $\{([2,3],2),([3,4],3)\}$, it is obvious that

$$|S(h,\mathcal{P}_k) - S(h,\mathcal{Q}_k)| = 0 < \varepsilon,$$

where $\mathcal{P}_k, \mathcal{Q}_k$ are δ_k-fine partitions, by Theorem 3.8, we know h is sequential Henstock-Kurzweil delta integrable on $\mathbb{T}$.

4 Fundamental theorems of the sequential Henstock-Kurzweil delta integral on time scales

The fundamental theorems of the sequential Henstock-Kurzweil delta integral are given in this section, which are very useful when we calculate integrals.

Definition 4.1. (Definition 3.5, [17]) Assume $h : \mathbb{T} \to \mathbb{R}$, we say that h is pre-differentiable on L if h is continuous on $\mathbb{T}$ and there exists a subset $L \subset \mathbb{T}^{\mathcal{K}}$ such that h is delta differentiable on L and $\mathbb{T}^{\mathcal{K}}\backslash L$ is countable and contains no right-scattered points of $\mathbb{T}$.

Theorem 4.1. *Let $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$, $H : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is continuous, and H is pre-differentiable on $L \subset [a,b]_{\mathbb{T}}^{\mathcal{K}}$ such that $H^{\Delta}(s) = h(s)$, $s \in L$, then $h \in SHD([a,b]_{\mathbb{T}})$ and*

$$(SHD)\int_a^b h\Delta s = H(b) - H(a).$$

Proof. We denote $O = \{s \in [a,b]_{\mathbb{T}} : \mu(s) > 0\}$, O contains all right-scattered points in $[a,b]_{\mathbb{T}}$, by Definition 4.1, $O \subset L$ and $[a,b]_{\mathbb{T}}\backslash L$ is a countable set, we denote it as $B = \{r_1, r_2, \ldots\}$, which is countable or finite. Now, we define a sequence of gauges $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ on $[a,b]_{\mathbb{T}}$ in the following way:

Let $g_i \in O$, we define $\delta_R^k(g_i) = \mu(g_i)$, because h is delta differentiable at g_i, so there exists $\{\delta_{L_1}^k(g_i)\}_{k=1}^{\infty}$, we have

$$|[H(\sigma(g_i)) - H(u)] - H^{\Delta}(g_i)[\sigma(g_i) - u]| < \frac{\varepsilon}{4(b-a)}|\sigma(g_i) - u|, \tag{1}$$

for all $u \in [g_i - \delta_{L_1}^k(g_i), g_i]_{\mathbb{T}}$, and due to the continuity of H, we have $\{\delta_{L_2}^k(g_i)\}_{k=1}^{\infty}$ such that

$$|[H(g_i) - H(u)] - H^{\Delta}(g_i)[g_i - u]| < \frac{\varepsilon}{2^{i+2}}, \tag{2}$$

for all $u \in [g_i - \delta_{L_2}^k(g_i), g_i]_{\mathbb{T}}$, so we define

$$\delta_L^k(g_i) = \min\{\delta_{L_1}^k(g_i), \delta_{L_2}^k(g_i)\}, \quad g_i \in O.$$

Then, we consider $g \in L\backslash O$, because H is differentiable at g, so there is a $\delta(g) > 0$ such that

$$|[H(g) - H(u)] - H^{\Delta}(g)[g - u]| < \frac{\varepsilon}{4(b-a)}|g - u|, \tag{3}$$

for all $u \in [g - \delta(g), g + \delta(g)]_{\mathbb{T}}$, so we define

$$\delta_L^k(g) = \delta_R^k(g) = \delta(g), \quad g \in L\backslash O.$$

Finally, we consider the points in B, let $r_i \in B$, because H is continuous at r_i, there is a $N(r_i) > 0$ such that

$$|[H(s) - H(u)] - h(r_i)[s - u]| < \frac{\varepsilon}{2^{i+2}}, \tag{4}$$

for any $s, u \in [r_i - N(r_i), r_i + N(r_i)]_{\mathbb{T}}$. Therefore, we define

$$\delta_L^k(r_i) = \delta_R^k(r_i) = N(r_i), \quad r_i \in B.$$

In conclusion, $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$ be a sequence of δ-gauges on $[a, b]_{\mathbb{T}}$, let $\mathcal{P}_k$ be a δ_k-fine partition for $[a, b]_{\mathbb{T}}$. Let K be large enough such that $\frac{1}{K} < \varepsilon$, when $k > K$, we have

$$\begin{aligned}
&\left|\sum_{j=1}^{m_k \in \mathbb{N}} h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(b) + H(a)\right| \\
&= \left|\sum_{j=1}^{m_k \in \mathbb{N}} h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(d_{jk}) + H(d_{(j-1)k})\right| \\
&\leqslant \sum_{j=1}^{m_k \in \mathbb{N}} |h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(d_{jk}) + H(d_{(j-1)k})|.
\end{aligned}$$

There are three possibilities for ξ_{jk}: $\xi_{jk} \in B$, $\xi_{jk} \in O$, $\xi_{jk} \in L \backslash O$.

If $\xi_{jk} \in O$, then we know $\sigma(\xi_{jk}) > \xi_{jk}$, and by the way we define δ we know that $d_{jk} = \xi_{jk}$ or $d_{jk} = \sigma(\xi_{jk})$, if $d_{jk} = \xi_{jk}$, using (2), we have

$$|h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(d_{jk}) + H(d_{(j-1)k})| < \frac{\varepsilon}{2^{j+2}}.$$

If $d_{jk} = \sigma(\xi_{jk})$, using (1), we have

$$|h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(d_{jk}) + H(d_{(j-1)k})| < \frac{\varepsilon}{4(b-a)}|d_{jk} - d_{(j-1)k}|.$$

If $\xi_{jk} \in B$, using (4), we have

$$|h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(d_{jk}) + H(d_{(j-1)k})| < \frac{\varepsilon}{2^{j+2}}.$$

If $\xi_{jk} \in L \backslash O$, using (3), we have

$$|h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(d_{jk}) + H(d_{(j-1)k})| < \frac{\varepsilon}{4(b-a)}|d_{jk} - d_{(j-1)k}|.$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^{m_k \in \mathbb{N}} |h(\xi_{jk})(d_{jk} - d_{(j-1)k}) - H(d_{jk}) + H(d_{(j-1)k})| \\
&< \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+2}} + \frac{\varepsilon}{2^{j+2}} + \frac{\varepsilon}{4(b-a)}(d_{jk} - d_{(j-1)k}) + \frac{\varepsilon}{4(b-a)}(d_{jk} - d_{(j-1)k}) \\
&= \varepsilon.
\end{aligned}$$

In conclusion, $h \in SHD([a, b]_{\mathbb{T}})$ and $(SHD)\int_a^b h\Delta s = H(b) - H(a)$. □

Example 4.1. Let $\mathbb{T} = \left\{\frac{n-1}{n} : n \in \mathbb{N}\right\} \cup \{1\}$, we define

$$h(q) = q + \sigma(q).$$

We claim that $h \in SHD([0, 1]_{\mathbb{T}})$ and $(SHD)\int_0^1 h\Delta q = 1$.

We define $H(q) = q^2$, $q \in \mathbb{T}$, and we know $\mathbb{T}^{\mathcal{K}} = \mathbb{T}$, let $L = \mathbb{T}^{\mathcal{K}}$, then we show $H^{\Delta}(q) = h(q)$ on L.

For any $q \in \mathbb{T}^{\mathcal{K}}$, let $\varepsilon > 0$ be given, for every $u \in (q - \varepsilon, q + \varepsilon)_{\mathbb{T}}$, we have

$$\begin{aligned}
&|[H(\sigma(q)) - H(u)] - h(q)[\sigma(q) - u]| \\
&= |\sigma^2(q) - u^2 - (q + \sigma(q))[\sigma(q) - u]| \\
&= |(\sigma(q) - u)(u - q)| = |\sigma(q) - u||u - q| < \varepsilon|\sigma(q) - u|.
\end{aligned}$$

Therefore, H is pre-differentiable on L, and $H^{\Delta}(q) = h(q)$ on L, thus

$$(SHD)\int_0^1 h\Delta q = H(1) - H(0) = 1.$$

Example 4.2. Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$, we define

$$h(s) = \begin{cases} \dfrac{(s^2 - 2s + 1)\sin\left(\frac{s}{1-s}\right) - (1-s)\sin s}{s^3}, & s \neq 0, \\ 0, & s = 0, 1. \end{cases}$$

Then, $h \in SHD([0,1]_{\mathbb{T}})$ and

$$(SHD)\int_0^1 h\Delta s = \sin 1 - 1.$$

We define

$$H(s) = \begin{cases} \dfrac{\sin s}{s}, & s \neq 0, \\ 1, & s = 0. \end{cases}$$

Then, we show $H^{\Delta}(s) = h(s)$ on $(0,1)_{\mathbb{T}}$, for every $s \in (0,1)_{\mathbb{T}}$, s is right-scattered and H is continuous at s. It is easy to show $\sigma(s) = \frac{s}{1-s}$, $s \in (0,1)_{\mathbb{T}}$, so by Theorem 2.1, we know

$$\begin{aligned} H^{\Delta}(s) &= \frac{H(\sigma(s)) - H(s)}{\sigma(s) - s} \\ &= \frac{\frac{1-s}{s}\sin\left(\frac{s}{1-s}\right) - \frac{\sin s}{s}}{\frac{s}{1-s} - s} \\ &= \frac{(s^2 - 2s + 1)\sin\left(\frac{s}{1-s}\right) - (1-s)\sin s}{s^3} = h(s). \end{aligned}$$

We denote $L = (0,1)_{\mathbb{T}}$, so H is pre-differentiable on L, and $H^{\Delta}(s) = h(s)$ on L, so by Theorem 4.1, we have

$$(SHD)\int_0^1 h\Delta s = H(1) - H(0) = \sin 1 - 1.$$

Lemma 4.1. *Let $h \in SHD([s, \sigma(s)]_{\mathbb{T}})$, we have*

$$(SHD)\int_s^{\sigma(s)} h(q)\Delta q = (\sigma(s) - s)h(s).$$

Proof. The case $\sigma(s) = s$ is easy, we suppose $\sigma(s) > s$, so the partition for $[s, \sigma(s))$ only can be $\mathcal{P} = \{s = s_0 < s_1 = \sigma(s)\}$, and the tag point is s, we define $\delta_L^k(s) = 1$, $\delta_R^k(s) = \mu(s)$, so $\mathcal{P}$ is a δ_k-fine partition, so $(SHD)\int_s^{\sigma(s)} h(q)\Delta q = (\sigma(s) - s)h(s)$. □

Theorem 4.2. *Let $h \in SHD([a,b]_{\mathbb{T}})$ and h is continuous on $[a,b]_{\mathbb{T}}$, then we define*

$$H(q) = (SHD)\int_a^q h(s)\Delta s,$$

for each $q \in [a,b]_{\mathbb{T}}$, then H is delta differentiable at every $q \in [a,b)_{\mathbb{T}}$ and $H^{\Delta}(q) = h(q)$.

Proof. Since h is continuous on $[a,b]_{\mathbb{T}}$, h is bounded. So there exists $M>0$, for every $q\in[a,b]_{\mathbb{T}}$, we have $|h(q)|\leqslant M$. We shall prove H is continuous on $[a,b]_{\mathbb{T}}$.

For any $\varepsilon>0$ and $q,u\in[a,b]_{\mathbb{T}}$, when $|q-u|<\frac{\varepsilon}{M}$, we have

$$\begin{aligned}|H(q)-H(u)| &= \left|(SHD)\int_a^q h(s)\Delta s-(SHD)\int_a^u h(s)\Delta s\right|\\ &= \left|(SHD)\int_q^u h(s)\Delta s\right| \leqslant (SHD)\int_q^u |h(s)|\Delta s\\ &\leqslant (SHD)\int_q^u M\Delta s<\varepsilon.\end{aligned}$$

Therefore, H is continuous.

By Theorem 2.1 and Lemma 4.1, for any $q\in[a,b)_{\mathbb{T}}$, if q is right-scattered, we have

$$\begin{aligned}H^{\Delta}(q) &= \frac{H(\sigma(q))-H(q)}{\sigma(q)-q}\\ &= \frac{(SHD)\int_a^{\sigma(q)} h(s)\Delta s-(SHD)\int_a^q h(s)\Delta s}{\sigma(q)-q}\\ &= \frac{(SHD)\int_q^{\sigma(q)} h(s)\Delta s}{\sigma(q)-q}=h(q).\end{aligned}$$

If q is right-dense, we have

$$\begin{aligned}H^{\Delta}(q) &= \lim_{m\to q}\frac{H(m)-H(q)}{m-q}\\ &= \lim_{m\to q}\frac{(SHD)\int_a^m h(s)\Delta s-(SHD)\int_a^q h(s)\Delta s}{m-q}\\ &= \lim_{m\to q}\frac{(SHD)\int_q^m h(s)\Delta s}{m-q}.\end{aligned}$$

For the above $\varepsilon>0$, since h is continuous at q, so for every $s\in[a,b)_{\mathbb{T}}$, there exists $\delta>0$, when $|s-q|<\delta$, $|h(s)-h(q)|<\varepsilon$, thus

$$\begin{aligned}&\left|\frac{(SHD)\int_q^m h(s)\Delta s}{m-q}-h(q)\right|\\ &= \left|\frac{1}{m-q}\left((SHD)\int_q^m h(s)\Delta s-(SHD)\int_q^m h(q)\Delta s\right)\right|\\ &\leqslant \frac{1}{|m-q|}(SHD)\int_q^m |h(s)-h(q)|\Delta s<\varepsilon.\end{aligned}$$

Therefore, H is delta differentiable at every $q\in[a,b)_{\mathbb{T}}$ and $H^{\Delta}(q)=h(q)$. □

Example 4.3. Let $\mathbb{T}=[2,4]\cap\mathbb{Z}$, we define $h(q)=\sigma(q)^2+q\sigma(q)+q^2$.

By Example 3.2, we know h is sequential Henstock-Kurzweil delta integrable on $\mathbb{T}$, then h is clearly continuous on $\mathbb{T}$. We consider $H(s)=s^3-8$.

For any $s \in \mathbb{T}\backslash\{4\}$, s is right-scattered, therefore

$$H^{\Delta}(s) = \frac{\sigma(s)^3 - s^3}{\sigma(s) - s} = \sigma(s)^2 + s\sigma(s) + s^2 = h(s).$$

By Theorem 4.2, we have

$$(SHD)\int_2^s h(q)\Delta q = (SHD)\int_2^s \sigma(q)^2 + q\sigma(q) + q^2\Delta q = s^3 - 8 = H(s).$$

5 Henstock's lemma and some convergence theorems

The proof of Henstock's lemma and some convergence theorems are given.

Lemma 5.1. *Let $h \in SHD([a,b]_{\mathbb{T}})$, then for any $c, d \in [a,b]_{\mathbb{T}} (d > c)$, $h \in SHD([c,d]_{\mathbb{T}})$.*

Proof. Since $h \in SHD([a,b]_{\mathbb{T}})$, by Theorem 3.8, for any $\varepsilon > 0$, there exists a $\delta_K \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > K$, for all δ_k-fine partitions $\mathcal{P}_k$ and $\mathcal{Q}_k$ for $[a,b]_{\mathbb{T}}$, we have

$$|S(h, \mathcal{P}_k) - S(h, \mathcal{Q}_k)| < \varepsilon.$$

For the above sequence of δ-gauges $\{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, we have δ_k-fine partitions $\{S_k\}_{k=1}^{\infty}$ for $[c,d]_{\mathbb{T}}$, δ_k-fine partitions $\{T_k\}_{k=1}^{\infty}$ for $[a,c]_{\mathbb{T}}$, δ_k-fine partitions $\{D_k\}_{k=1}^{\infty}$ for $[d,b]_{\mathbb{T}}$, when $k > K$, we take $\mathcal{P}_k^S, \mathcal{Q}_k^S \in \{S_k\}_{k=K+1}^{\infty}$, $\mathcal{P}_k^T \in \{T_k\}_{k=K+1}^{\infty}$, $\mathcal{P}_k^D \in \{D_k\}_{k=K+1}^{\infty}$, we know $\mathcal{P}_k^S \cup \mathcal{P}_k^T \cup \mathcal{P}_k^D$ and $\mathcal{Q}_k^S \cup \mathcal{P}_k^T \cup \mathcal{P}_k^D$ are δ_k-fine partitions, then we have

$$\begin{aligned}
&|S(h, \mathcal{P}_k^S) - S(h, \mathcal{Q}_k^S)| \\
&\quad = |S(h, \mathcal{P}_k^S \cup \mathcal{P}_k^T \cup \mathcal{P}_k^D) - S(h, \mathcal{Q}_k^S \cup \mathcal{P}_k^T \cup \mathcal{P}_k^D)| \\
&\quad = |S(h, \mathcal{P}_k) - S(h, \mathcal{Q}_k)| < \varepsilon.
\end{aligned}$$

Therefore, $h \in SHD([c,d]_{\mathbb{T}})$. □

Definition 5.1. A finite collection $\mathcal{P}_k^* = \{[s_{(i-1)k}, s_{ik}], \xi_{ik}\}_{i=1}^{m_k}$ is called a δ_k-fine subpartition for $[a,b]_{\mathbb{T}}$ if $\{[s_{0k}, s_{1k}], [s_{1k}, s_{2k}], \ldots, [s_{(m_k-1)k}, s_{m_k k}]\}$ are nonoverlapping subintervals of $[a,b]_{\mathbb{T}}$ except end points and $[s_{(i-1)k}, s_{ik}] \subset [\xi_{ik} - \delta_L^k(\xi_{ik}), \xi_{ik} + \delta_R^k(\xi_{ik})]$.

Lemma 5.2. (Henstock's lemma) *Let $h \in SHD([a,b]_{\mathbb{T}})$, for any $\varepsilon > 0$, there exists a sequence of δ-gauges $\{\delta_k\}_{k=1}^{\infty}$ and $K > 0$, when $k > K$, if $\mathcal{P}_k^* = \{[q_{(i-1)k}, q_{ik}], \xi_{ik}\}_{i=1}^{m_k}$ is a δ_k-fine subpartition for $[a,b]_{\mathbb{T}}$, then*

$$\left| S(h, \mathcal{P}_k^*) - \sum_{i=1}^{m_k} (SHD) \int_{q_{(i-1)k}}^{q_{ik}} h\Delta s \right| < \varepsilon.$$

Furthermore,

$$\sum_{i=1}^{m_k} \left| h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD) \int_{q_{(i-1)k}}^{q_{ik}} h\Delta s \right| < 2\varepsilon.$$

Proof. For any $\varepsilon > 0$, since $h \in SHD([a,b]_{\mathbb{T}})$, there exists a $\delta_{K_1} \in \{\delta_k = (\delta_L^k, \delta_R^k)\}_{k=1}^{\infty}$, when $k > K_1$, for every δ_k-fine partition $\mathcal{P}_k$, we have

$$\left| S(h, \mathcal{P}_k) - (SHD)\int_a^b h\Delta s \right| < \frac{\varepsilon}{2}.$$

The case $\mathcal{P}_k^* = \mathcal{P}_k$ is trivial, we consider the case $\mathcal{P}_k^* \neq \mathcal{P}_k$. We choose nonoverlapping intervals $\{[\gamma_{0k}, \gamma_{1k}]_{\mathbb{T}}, [\gamma_{1k}, \gamma_{2k}]_{\mathbb{T}}, \ldots, [\gamma_{(p_k-1)k}, \gamma_{p_k k}]_{\mathbb{T}}\}$ that satisfy

$$[a, b]_{\mathbb{T}} \backslash \bigcup_{i=1}^{m_k} (q_{(i-1)k}, q_{ik})_{\mathbb{T}} = \bigcup_{j=1}^{p_k} [\gamma_{(j-1)k}, \gamma_{jk}]_{\mathbb{T}}.$$

By Lemma 5.1, $h \in SHD([\gamma_{(j-1)k}, \gamma_{jk}]_{\mathbb{T}})(j = 1, 2, \ldots, p_k)$.

So there exists a $\delta_{jK_2} \in \{\delta_{jk} = (\delta_L^{jk}, \delta_R^{jk})\}_{k=1}^{\infty}(j = 1, 2, \ldots, p_k)$, when $k > K_2$, for every δ_{jk}-fine partition $\mathcal{P}_{jk}$, we have

$$\left| S(h, \mathcal{P}_{jk}) - (SHD) \int_{\gamma_{(j-1)k}}^{\gamma_{jk}} h \Delta s \right| < \frac{\varepsilon}{2p_k}.$$

Then, we define another sequence of gauges $\{\delta_{jk} = (\delta_L^{jk}, \delta_R^{jk})\}_{k=1}^{\infty}$ on $[\gamma_{(j-1)k}, \gamma_{jk}]_{\mathbb{T}}$ by the formula:

$$\delta_L^{jk}(q) = \min\{\delta_L^k(q), \delta_L^{jk}(q)\}, \delta_R^{jk}(q) = \min\{\delta_R^k(q), \delta_R^{jk}(q)\}.$$

For every δ_{jk}-fine partition $\mathcal{P'}_{jk}$, we define $\mathcal{P}_k = \mathcal{P}_k^* \cup (\bigcup_{j=1}^{p_k} \mathcal{P'}_{jk})$, when $k > \max\{K_1, K_2\}$, $\mathcal{P}_k$ is a δ_k-fine partition for $[a, b]_{\mathbb{T}}$ and we have

$$S(h, \mathcal{P}_k) = S(h, \mathcal{P}_k^*) + \sum_{j=1}^{p_k} S(h, \mathcal{P'}_{jk}).$$

Denote $A = \bigcup_{i=1}^{m_k} [q_{(i-1)k}, q_{ik}]_{\mathbb{T}}$, since A and $[\gamma_{(i-1)k}, \gamma_{ik}]_{\mathbb{T}} (i = 1, 2, \ldots, p_k)$ are collections of $[a, b]_{\mathbb{T}}$, by Theorem 3.7 and Lemma 5.1, we have

$$\begin{aligned}(SHD)\int_a^b h\Delta s &= \sum_{i=1}^{m_k} (SHD) \int_{q_{(i-1)k}}^{q_{ik}} h\Delta s + \sum_{j=1}^{p_k} (SHD) \int_{\gamma_{(j-1)k}}^{\gamma_{jk}} h\Delta s \\ &= (SHD)\int_A h\Delta s + \sum_{j=1}^{p_k} (SHD) \int_{\gamma_{(j-1)k}}^{\gamma_{jk}} h\Delta s.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left| S(h, \mathcal{P}_k^*) - \sum_{i=1}^{m_k} (SHD) \int_{q_{(i-1)k}}^{q_{ik}} h\Delta s \right| \\ &= \left| S(h, \mathcal{P}_k) - \sum_{i=1}^{m_k} (SHD) \int_{q_{(i-1)k}}^{q_{ik}} h\Delta s - \sum_{j=1}^{p_k} S(h, \mathcal{P'}_{jk}) \right| \\ &= \left| S(h, \mathcal{P}_k) - (SHD)\int_a^b h\Delta s + \sum_{j=1}^{p_k} (SHD) \int_{\gamma_{(j-1)k}}^{\gamma_{jk}} h\Delta s - \sum_{j=1}^{p_k} S(h, \mathcal{P'}_{jk}) \right| \\ &\leqslant \left| S(h, \mathcal{P}_k) - (SHD)\int_a^b h\Delta s \right| + \sum_{j=1}^{p_k} \left| (SHD) \int_{\gamma_{(j-1)k}}^{\gamma_{jk}} h\Delta s - S(h, \mathcal{P'}_{jk}) \right| \\ &< \varepsilon.\end{aligned}$$

The proof of the first part of this lemma is complete, then we consider next part.

We separate $\mathcal{P}_k^*$ into two parts:

$$\mathcal{P}_k^+ = \left\{([q_{(i-1)k}, q_{ik}], \xi_{ik}) \in \mathcal{P}_k^* : h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD)\int_{q_{(i-1)k}}^{q_{ik}} h\Delta s \geqslant 0\right\},$$

$$\mathcal{P}_k^- = \left\{([q_{(i-1)k}, q_{ik}], \xi_{ik}) \in \mathcal{P}_k^* : h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD)\int_{q_{(i-1)k}}^{q_{ik}} h\Delta s < 0\right\}.$$

Therefore, $\mathcal{P}_k^* = \mathcal{P}_k^+ \cup \mathcal{P}_k^-$, by the first part of this lemma, we have

$$\left|\sum_{([q_{(i-1)k}, q_{ik}], \xi_{ik}) \in \mathcal{P}_k^+} h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD)\int_{q_{(i-1)k}}^{q_{ik}} h\Delta s\right| < \varepsilon,$$

$$\left|\sum_{([q_{(i-1)k}, q_{ik}], \xi_{ik}) \in \mathcal{P}_k^-} h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD)\int_{q_{(i-1)k}}^{q_{ik}} h\Delta s\right| < \varepsilon.$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{m_k}\left| h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD)\int_{q_{(i-1)k}}^{q_{ik}} h\Delta s\right| &= \left|\sum_{([q_{(i-1)k}, q_{ik}], \xi_{ik}) \in \mathcal{P}_k^+} h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD)\int_{q_{(i-1)k}}^{q_{ik}} h\Delta s\right| \\
&\quad + \left|\sum_{([q_{(i-1)k}, q_{ik}], \xi_{ik}) \in \mathcal{P}_k^-} h(\xi_{ik})(q_{ik} - q_{(i-1)k}) - (SHD)\int_{q_{(i-1)k}}^{q_{ik}} h\Delta s\right| \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

□

Theorem 5.1. (Uniform convergence theorem) *Let $\{h_n\} \subset SHD([a,b]_{\mathbb{T}})$ and h_n is uniformly convergent to h on $[a,b]_{\mathbb{T}}$. Then, $h \in SHD([a,b]_{\mathbb{T}})$ and*

$$\lim_{n\to\infty}(SHD)\int_a^b h_n\Delta s = (SHD)\int_a^b \lim_{n\to\infty} h_n\Delta s = (SHD)\int_a^b h\Delta s.$$

Proof. Since h_n is uniformly convergent to h on $[a,b]_{\mathbb{T}}$, for any $\varepsilon > 0$, there exists $N_1 > 0$, when $n, m > N_1$, we have $|h_n - h| < \frac{\varepsilon}{3(b-a)}$, $|h_m - h| < \frac{\varepsilon}{3(b-a)}$, then

$$\begin{aligned}
|h_n - h_m| &= |h_n - h + h - h_m| \\
&\leqslant |h_n - h| + |h - h_m| \\
&< \frac{2\varepsilon}{3(b-a)}.
\end{aligned}$$

Thus, $-\frac{2\varepsilon}{3(b-a)} < h_n - h_m < \frac{2\varepsilon}{3(b-a)}$ and we know that $\{h_n\} \subset SHD([a,b]_{\mathbb{T}})$, it is easy to show that $(SHD)\int_a^b \frac{2\varepsilon}{3(b-a)}\Delta s = \frac{2\varepsilon}{3}$, by Theorem 3.4, we have

$$-\frac{2\varepsilon}{3} < (SHD)\int_a^b (h_n - h_m)\Delta s < \frac{2\varepsilon}{3}.$$

Thus, $|(SHD)\int_a^b h_n\Delta s - (SHD)\int_a^b h_m\Delta s| < \frac{2\varepsilon}{3}$, then $\left\{(SHD)\int_a^b h_n\Delta s\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, by the completeness of $\mathbb{R}$, there exists $A \in \mathbb{R}$, we have

$$\lim_{n\to\infty}(SHD)\int_a^b h_n\Delta s = A.$$

For the above $\varepsilon > 0$, there exists $N_2 > 0$, when $n > N_2$, we have

$$\left|(SHD)\int_a^b h_n\Delta s - A\right| < \frac{\varepsilon}{3}.$$

Then, we will show $h \in SHD([a,b]_{\mathbb{T}})$ and $(SHD)\int_a^b h\Delta s = A$.

Since $h_m \in SHD([a,b]_{\mathbb{T}})$, fix m, there exists a $\delta_{K_m}^m \in \{\delta_k^m = (\delta_{L,k}^m, \delta_{R,k}^m)\}_{k=1}^{\infty}$, when $k > K_m$, for every δ_k^m-fine partition $\mathcal{P}_k^m$, we have

$$\left|S(h_m, \mathcal{P}_k^m) - (SHD)\int_a^b h_m\Delta s\right| < \frac{\varepsilon}{3}.$$

Let $M = \max\{N_1+1, N_2+1\}$ and $\delta_k = \delta_k^M$. Therefore, for all δ_k-fine partitions $\mathcal{P}_k$, there are also δ_k^M-fine partitions, when $k > K_M$, we have

$$\begin{aligned}
&|S(h,\mathcal{P}_k) - S(h_M,\mathcal{P}_k)| \\
&= \left|\sum_{i=1}^{m_k\in\mathbb{N}} h(\xi_{ik})(s_{ik} - s_{(i-1)k})_{\delta_k^M} - \sum_{i=1}^{m_k\in\mathbb{N}} h_M(\xi_{ik})(s_{ik} - s_{(i-1)k})_{\delta_k^M}\right| \\
&= \left|\sum_{i=1}^{m_k\in\mathbb{N}} (h(\xi_{ik}) - h_M(\xi_{ik}))(s_{ik} - s_{(i-1)k})_{\delta_k^M}\right| \\
&\leqslant \sum_{i=1}^{m_k\in\mathbb{N}} |(h(\xi_{ik}) - h_M(\xi_{ik}))|(s_{ik} - s_{(i-1)k})_{\delta_k^M} < \frac{\varepsilon}{3}.
\end{aligned}$$

Therefore, we have

$$|S(h,\mathcal{P}_k) - A| \leqslant |S(h,\mathcal{P}_k) - S(h_M,\mathcal{P}_k)| + \left|S(h_M,\mathcal{P}_k) - (SHD)\int_a^b h_M\Delta s\right| + \left|(SHD)\int_a^b h_M\Delta s - A\right| < \varepsilon.$$

The result follows by the arbitrariness of ε. □

Example 5.1. Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$, we define

$$h_n(s) = \begin{cases} \sqrt{n}\,s, & s \in \left[0, \frac{1}{n}\right] \cap \mathbb{T}, \\ \frac{2}{\sqrt{n}} - \sqrt{n}\,s, & s \in \left(\frac{1}{n}, \frac{2}{n}\right] \cap \mathbb{T}, \\ 0, & s \in \left(\frac{2}{n}, 1\right] \cap \mathbb{T}. \end{cases}$$

We claim that $\lim_{n\to\infty}(SHD)\int_0^1 h_n(s)\Delta s = 0$.

When $s = 0$, we have $h_n(s) = 0$, then when $s \in (0,1]_{\mathbb{T}}$, let $n > \frac{2}{s}$, we have $h_n(s) = 0$, thus

$$\lim_{n\to\infty} h_n(s) = h(s) = 0.$$

Then, we show that this convergence is uniform. We know

$$\sup_{s\in\mathbb{T}}|h_n(s)-0| = h_n\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}.$$

Therefore, $\lim_{n\to\infty}\sup_{s\in\mathbb{T}}|h_n(s)-0| = 0$, h_n is uniformly convergent to 0, by Theorem 5.1, we have

$$\lim_{n\to\infty}(SHD)\int_0^1 h_n\Delta s = (SHD)\int_0^1 \lim_{n\to\infty} h_n\Delta s = 0.$$

Theorem 5.2. (Monotone convergence theorem) *Let $\{h_n\}\subset SHD([a,b]_{\mathbb{T}})$ and assume that*
(1) $\lim_{n\to\infty}h_n(s) = h(s)$ *for every* $s\in[a,b]_{\mathbb{T}}$;
(2) $h_n(s)\leqslant h_{n+1}(s)$ *for every* $s\in[a,b]_{\mathbb{T}}$, $n\in\mathbb{N}$;
(3) $\lim_{n\to\infty}(SHD)\int_a^b h_n(s)\Delta s = V$.

Then, $h\in SHD([a,b]_{\mathbb{T}})$ *and* $(SHD)\int_a^b h(s)\Delta s = V$.

Proof. For any $\varepsilon>0$, since $(SHD)\lim_{n\to\infty}\int_a^b h_n(s)\Delta s = V$, there exists $N_1>0$, when $n>N_1$, we have

$$\left|(SHD)\int_a^b h_n\Delta s - V\right| < \frac{\varepsilon}{3}.$$

Then, $\lim_{n\to\infty}h_n(s) = h(s)$ for every $s\in[a,b]_{\mathbb{T}}$, there exists $N_2(\varepsilon,s)>N_1$, then we have

$$|h_{N_2(\varepsilon,s)}(s) - h(s)| < \frac{\varepsilon}{3(b-a)}.$$

Let $\varepsilon'>0$ be given, since $\{h_n\}\subset SHD([a,b]_{\mathbb{T}})$, there exists a sequence of δ-gauges $\{\delta_k^n = (\delta_{Lk}^n,\delta_{Rk}^n)\}_{k=1}^{\infty}$ on $[a,b]_{\mathbb{T}}$, when $k>K_n$, for every δ_k^n-fine subpartition $\mathcal{P}_k^n$, we assume $\mathcal{P}_k^n$ has $m+1$ end points, by Lemma 5.2, we have

$$\sum_{i=1}^{m}\left|h_n(\xi_{ik})(s_{ik}-s_{(i-1)k}) - (SHD)\int_{s_{(i-1)k}}^{s_{ik}} h_n\Delta s\right| < \frac{\varepsilon'}{3}.$$

Therefore, we define

$$\delta_L^k(s) = \delta_{L(K_{N_2(\varepsilon,s)}+1)}^{N_2(\varepsilon,s)}(s),\ \delta_R^k(s) = \delta_{R(K_{N_2(\varepsilon,s)}+1)}^{N_2(\varepsilon,s)}(s).$$

Let $\mathcal{P}_k$ be a δ_k-fine partition, by the way we define δ_k, $\mathcal{P}_k$ is also a $\delta_{K_{N_2(\varepsilon,s)}+1}^{N_2(\varepsilon,s)}$-fine partition, we have

$$\begin{aligned}
&\left|\sum_{i=1}^{m_k}h(\xi_{ik})(s_{ik}-s_{(i-1)k}) - V\right| \\
&\leqslant \left|\sum_{i=1}^{m_k}h(\xi_{ik})(s_{ik}-s_{(i-1)k}) - \sum_{i=1}^{m_k}h_{N_2(\varepsilon,\xi_{ik})}(\xi_{ik})(s_{ik}-s_{(i-1)k})\right| \\
&\quad + \left|\sum_{i=1}^{m_k}h_{N_2(\varepsilon,\xi_{ik})}(\xi_{ik})(s_{ik}-s_{(i-1)k}) - V\right| \\
&\leqslant \sum_{i=1}^{m_k}|(h(\xi_{ik}) - h_{N_2(\varepsilon,\xi_{ik})}(\xi_{ik}))(s_{ik}-s_{(i-1)k})| \\
&\quad + \sum_{i=1}^{m_k}\left|h_{N_2(\varepsilon,\xi_{ik})}(\xi_{ik})(s_{ik}-s_{(i-1)k}) - (SHD)\int_{s_{(i-1)k}}^{s_{ik}} h_{N_2(\varepsilon,\xi_{ik})}\Delta s\right| \\
&\quad + \left|\sum_{i=1}^{m_k}(SHD)\int_{s_{(i-1)k}}^{s_{ik}} h_{N_2(\varepsilon,\xi_{ik})}\Delta s - V\right|.
\end{aligned}$$

It is easy to show that

$$\sum_{i=1}^{m_k} |(h(\xi_{ik}) - h_{N_2(\varepsilon,\xi_{ik})}(\xi_{ik}))(s_{ik} - s_{(i-1)k})| < \frac{\varepsilon}{3}.$$

Let K be large enough such that $\frac{1}{K} < \varepsilon$, when $k > K$, let $\varepsilon' = \frac{\varepsilon}{m_k}$, we have

$$\sum_{i=1}^{m_k} \left| h_{N_2(\varepsilon,\xi_{ik})}(\xi_{ik})(s_{ik} - s_{(i-1)k}) - (SHD) \int_{s_{(i-1)k}}^{s_{ik}} h_{N_2(\varepsilon,\xi_{ik})} \Delta s \right| < \frac{\varepsilon}{3}.$$

Then, we consider the last term, let

$$p = \min\{N_2(\varepsilon, \xi_{ik}) : 1 \leqslant i \leqslant m_k\},$$
$$q = \max\{N_2(\varepsilon, \xi_{ik}) : 1 \leqslant i \leqslant m_k\}.$$

Since $h_n(s) \leqslant h_{n+1}(s)$ for every $s \in [a, b]_{\mathbb{T}}$, $n \in \mathbb{N}$, thus

$$(SHD)\int_a^b h_n \Delta s \leqslant (SHD)\int_a^b h_{n+1} \Delta s.$$

Thus,

$$V - \frac{\varepsilon}{3} < (SHD)\int_a^b h_p \Delta s \leqslant \sum_{i=1}^{m_k} (SHD) \int_{s_{(i-1)k}}^{s_{ik}} h_{N_2(\varepsilon,\xi_{ik})} \Delta s,$$

$$\sum_{i=1}^{m_k} (SHD) \int_{s_{(i-1)k}}^{s_{ik}} h_{N_2(\varepsilon,\xi_{ik})} \Delta s \leqslant (SHD)\int_a^b h_q \Delta s < V + \frac{\varepsilon}{3}.$$

Therefore,

$$\left| \sum_{i=1}^{m_k} (SHD) \int_{s_{(i-1)k}}^{s_{ik}} h_{N_2(\varepsilon,\xi_{ik})} \Delta s - V \right| < \frac{\varepsilon}{3}.$$

The proof is complete. □

6 An application for the existence theorem of FDEs

In this section, we consider the existence theorem of the following FDEs:

$$\begin{cases} y^{\Delta}(s) = h(s, y(s), y(\tau(s))), & s \in [s_0, \phi(s_0)]_{\mathbb{T}}, \\ y(s) = \varphi(s), & s \in [\tau(s_0), s_0]_{\mathbb{T}}, \end{cases} \tag{5}$$

where $h \in C_{rd}([a, b]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C_{rd}([\tau(s_0), s_0])$, $\tau : [a, b]_{\mathbb{T}} \to [a, b]_{\mathbb{T}}$, and $\tau(s) \leqslant s, s \in [a, b]_{\mathbb{T}}$, $\phi(s_0) = \sup_{s_0 \leqslant s \leqslant b}\{s : \tau(s) \leqslant s_0\}$. We assume $[\tau(s_0), \phi(s_0)]_{\mathbb{T}} \subset [a, b]_{\mathbb{T}}$, and $[a, b]_{\mathbb{T}}$ is an infinite set.

Definition 6.1. Let $y : [\tau(s_0), \phi(s_0)]_{\mathbb{T}} \to \mathbb{R}$, y is called a solution of (5), provided $y \in C^1_{rd}([\tau(s_0), \phi(s_0)]_{\mathbb{T}})$ and it satisfies the conditions of (5).

Theorem 6.1. *Assume $[s_0, \phi(s_0)]_{\mathbb{T}}$ contains infinite points of $[a, b]_{\mathbb{T}}$ and $h \in SHD([a, b]_{\mathbb{T}})$, if there exists a number $M \in \mathbb{R}$ and for any $s \in [a, b]_{\mathbb{T}}$, $|h| \leqslant M$, then the initial value problem* (5) *has at least one solution y on $[\tau(s_0), \phi(s_0)]_{\mathbb{T}}$.*

Proof. Let

$$y_n(s) = \varphi(s_0) + (SHD)\int_{s_0}^{s} h(p, y_{n-1}(p), \varphi(\tau(p)))\Delta p, \quad s \in [s_0, \phi(s_0)]_{\mathbb{T}}.$$

For any $\varepsilon > 0$ and $s_1, s_2 \in [s_0, \phi(s_0)]_{\mathbb{T}}$, when $|s_1 - s_2| < \frac{\varepsilon}{M}$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}|y_n(s_1) - y_n(s_2)| &= \left|(SHD)\int_{s_1}^{s_2} h(p, y_{n-1}(p), \varphi(\tau(p)))\Delta p\right| \\ &\leqslant (SHD)\int_{s_1}^{s_2} |h(p, y_{n-1}(p), \varphi(\tau(p)))|\Delta p \\ &< \varepsilon.\end{aligned}$$

Therefore, $\{y_n\}$ is equi-continuous on $[s_0, \phi(s_0)]_{\mathbb{T}}$.

Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}|y_n(s)| &= \left|\varphi(s_0) + (SHD)\int_{s_0}^{s} h(p, y_{n-1}(p), \varphi(\tau(p)))\Delta p\right| \\ &\leqslant |\varphi(s_0)| + (SHD)\int_{s_0}^{s} |h(p, y_{n-1}(p), \varphi(\tau(p)))|\Delta p \\ &\leqslant |\varphi(s_0)| + Mb - Ms_0.\end{aligned}$$

Thus, $\{y_n\}$ is uniformly bounded on $[s_0, \phi(s_0)]_{\mathbb{T}}$, according to the Arzela-Ascoli Theorem, we know $\{y_n\}$ has a subsequence $\{y_{n_k}\}$, which uniformly converges to a continuous function y, and we know

$$y_{n_k}(s) = \varphi(s_0) + (SHD)\int_{s_0}^{s} h(p, y_{n_k-1}(p), \varphi(\tau(p)))\Delta p,$$

let $k \to \infty$, we have

$$y(s) = \varphi(s_0) + (SHD)\int_{s_0}^{s} h(p, y(p), \varphi(\tau(p)))\Delta p.$$

And we know $y^{\Delta}(s) = h(s, y(s), \varphi(\tau(p)))$, $s \in [s_0, \phi(s_0)]_{\mathbb{T}}$. Let

$$y(s) = \begin{cases} \varphi(s_0) + (SHD)\int_{s_0}^{s} h(p, y(p), \varphi(\tau(p)))\Delta p, & s \in [s_0, \phi(s_0)]_{\mathbb{T}}, \\ \varphi(s), & s \in [\tau(s_0), s_0]_{\mathbb{T}}. \end{cases}$$

It is clear that $y(s)$ is a solution of (5). □

Example 6.1. Let $\mathbb{T} = [1, 2] \cup \left\{2 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $[1, 2] \subset \mathbb{R}$, consider the following equation:

$$\begin{cases} y^{\Delta}(s) = \dfrac{\left(y\left(\frac{s}{2}\right)\right)^4 + 5\left(y\left(\frac{s}{2}\right)\right)^2 + (y(s))^2\left(\left(y\left(\frac{s}{2}\right)\right)^2 + 1\right)}{4 + (y(s))^2 + \left(y\left(\frac{s}{2}\right)\right)^2}, & s \in \{2\} \cup \left\{2 + \dfrac{1}{n}\right\}_{n\in\mathbb{N}}, \\ y(s) = 2s + 1, & s \in [1, 2]. \end{cases}$$

We claim this equation has at least one solution.

Let

$$h(s,u(s),v(s))=\frac{(v(s))^4+5(v(s))^2+(u(s))^2((v(s))^2+1)}{4+(u(s))^2+(v(s))^2}.$$

We know

$$\begin{aligned}|h(s,u(s),v(s))|&=\left|\frac{(v(s))^4+5(v(s))^2+(u(s))^2((v(s))^2+1)}{4+(u(s))^2+(v(s))^2}\right|\\&=\left|\frac{(u(s))^2+(v(s))^2}{4+(u(s))^2+(v(s))^2}+(v(s))^2\right|\\&\leqslant 1+16=17.\end{aligned}$$

By Theorem 6.1, the results follow.

7 Conclusion and future research

The aim of this study was to extend the theory of integrals on time scales. From the outset, we define the sequential Henstock-Kurzweil delta integral. Then, we give some properties and the Cauchy criterion of this integral. Finally, we consider the fundamental theorems of calculus and convergence theorems. As an application, we consider the existence theorem of a kind of FDE. The results of this study are the generalization of the results in [30,35]. As we know, vector-valued integrals are the direct generalization of ordinary numerical integrals [39–46]. In 1938, Pettis introduced the Pettis integral, which is an important integral when we consider the weak topology on the Banach space $\mathbb{X}$ [47–49]. If we replace Lebesgue integrable in the definition with Henstock-Kurzweil integrable, we obtain a more general integral called the Henstock-Kurzweil-Pettis integral [50–53]. Cichoń [54], the author introduced the Henstock-Kurzweil-Pettis integral on time scales and got some good results. For future research, we can replace Henstock-Kurzweil integrable with sequential Henstock-Kurzweil integrable and study the sequential Henstock-Kurzweil-Pettis integral on time scales.

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References

[1] B. Aulbach and S. Hilger, *Linear dynamic processes with inhomogeneous time scale*, Nonlinear dynamics and quantum dynamical systems: contributions to the International Seminar ISAM 90 held in Gaussig (GDR), March 19–23, 1990, G. A. Leonov, V. Reitmann, and W. Timmermann (Eds.), 1990, Akad.-Verlag, Berlin.

[2] S. Hilger, *Ein maßkettenkalkül mit anwendung auf zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, 1988.

[3] S. Hilger, *Analysis on measure chains-a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), no. 1–2, 18–56, DOI: https://doi.org/10.1007/BF03323153.

[4] E. M. Bonotto, M. Federson and J. G. Mesquita, *Generalized Ordinary Differential Equations in Abstract Spaces and Applications*, Wiley Online Library, Hoboken, NJ, 2021.

[5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Springer Science & Business Media, New York, 2001.

[6] G. S. Guseinov, *Integral Equations on Time Scales*, Atlantis Press, Paris, 2016.

[7] G. S. Guseinov, *Functional Dynamic Equations on Time Scales*, Springer, Cham, Switzerland, 2019.

[8] G. S. Guseinov and B. Kaymakçalan, *Basics of Riemann delta and nabla integration on time scales*, J. Difference Equ. Appl. **8** (2002), no. 11, 1001–1017, DOI: https://doi.org/10.1080/10236190290015272.

[9] G. S. Guseinov, *Integration on time scales*, J. Math. Anal. Appl. **285** (2003), no. 1, 107–127, DOI: https://doi.org/10.1016/S0022-247X(03)00361-5.

[10] G. S. Guseinov and B. Kaymakçalan, *On the Riemann integration on time scales*, Conference Proceedings of the Sixth International Conference on Difference Equations and Applications, CRC Press, Augsburg, 2001.

[11] J. M. Park, D. H. Lee, J. H. Yoon, Y. K. Kim, and J. T. Lim, *The Riemann Delta integral on time scales*, J. Chungcheong Math. Soc. **27** (2014), no. 2, 327–333, DOI: https://doi.org/10.14403/jcms.2014.27.2.327.

[12] D. Mozyrska, E. Pawłuszewicz, and D. F. M. Torres, *The Riemann-Stieltjes integral on time scales*, Aust. J. Math. Anal. Appl. **7** (2010), no. 1, 1–14.

[13] D. Liu and D. Zhao, *On the McShane integral on time scales, Chinese* Quart. J. Math. **27** (2012), no. 4, 556–561.

[14] J. M. Park, D. H. Lee, J. H. Yoon, Y. K. Kim, and J. T. Lim, *The relation between McShane integral and McShane delta integral*, J. Chungcheong Math. Soc. **27** (2014), no. 1, 113–121, DOI: https://doi.org/10.14403/jcms.2014.27.1.113.

[15] X. You and D. Zhao, *On convergence theorems for the McShane integral on time scales*, J. Chungcheong Math. Soc. **25** (2012), no. 3, 393–400, DOI: https://doi.org/10.14403/jcms.2012.25.3.393.

[16] G. Qin and C. Wang, *Lebesgue–Stieltjes combined $\diamond_\alpha$-measure and integral on time scales*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 2, 50, DOI: https://doi.org/10.1007/s13398-021-01000-y.

[17] M. Bohner and S. G. Georgiev, *Multivariable Dynamic Calculus on Time Scales*, Springer, Cham, Switzerland, 2016.

[18] J. Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J. **7** (1957), no. 3, 418–449, DOI: http://doi.org/10.21136/CMJ.1957.100258.

[19] R. Henstock, *Theory of Integration*, Butterworth, London, 1963.

[20] R. Bartle, *A Modern Theory of Integration*, vol. 32, American Mathematical Society, Providence, 2001.

[21] T. Y. Lee, *Henstock-Kurzweil Integration on Euclidean Spaces*, vol. 12, World Scientific, Singapore, 2011.

[22] P. Lee, *Lanzhou Lectures on Henstock Integration*, vol. 2, World Scientific, Singapore, 1989.

[23] E. Talvila, *The regulated primitive integral*, Illinois J. Math. **53** (2009), no. 4, 1187–1219, DOI: https://doi.org/10.1215/ijm/1290435346.

[24] M. Tvrdy, G. A. Monteiro, and A. Slavik, *Kurzweil-Stieltjes integral: Theory and Applications*, vol. 15. World Scientific, Singapore, 2018.

[25] C. W. Swartz and D. S. Kurtz, *Theories of Integration: The Integrals of Riemann, Lebesgue, Henstock-Kurzweil, and McShane*, vol. 13, World Scientific, Singapore, 2011.

[26] Y. Lu, G. Ye, W. Liu, and Y. Wang, *Existence of solutions of the wave equation involving the distributional Henstock-Kurzweil integral*, Differential Integral Equations **24** (2011), no. 11–12, 1063–1071, DOI: http://doi.org/10.57262/die/1356012876.

[27] V. Marraffa, *A characterization of strongly measurable Kurzweil-Henstock integrable functions and weakly continuous operators*, J. Math. Anal. Appl. **340** (2008), no. 2, 1171–1179.

[28] B. Satco, *Second order retarded differential inclusions in Banach spaces and Henstock-Lebesgue integral*, Fixed Point Theory **9** (2008), no. 1, 319–329.

[29] C. Ciufudean and B. Satco, *Performance evaluation of discrete event systems involving Henstock-Kurzweil integral, of discrete event systems involving Henstock-Kurzweil integral*, J. Syst. Sci. Syst. Eng. **18** (2009), no. 2, 243–256.

[30] A. Peterson and B. Thompson, *Henstock-Kurzweil delta and nabla integrals*, J. Math. Anal. Appl. **323** (2006), no. 1, 162–178, DOI: https://doi.org/10.1016/j.jmaa.2005.10.025.

[31] B. S. Thomson, *Henstock-Kurzweil integrals on time scales*, Panamer. Math. J. **18** (2008), no. 1, 1–19.

[32] S. Avsec, B. Bannish, B. Johnson, and S. Meckler, *The Henstock-Kurzweil delta integral on unbounded time scales*, Panamer. Math. J. **16** (2006), no. 3, 77–98.

[33] J. M. Park, D. H. Lee, J. H. Yoon, Y. K. Kim, and J. T. Lim, *The relation between Henstock integral and Henstock delta Integral on time scales*, J. Chungcheong Math. Soc. **26** (2013), no. 3, 625–630, DOI: https://doi.org/10.14403/jcms.2013.26.3.625.

[34] P. Lee, *The integrala la Henstock*, Sci. Math. Jpn. **67** (2008), no. 1, 13–21.

[35] L. Paxton, *A Sequential Approach to the Henstock Integral*, 2016, https://arxiv.org/abs/1609.05454.

[36] V. O. Iluebe and A. A. Mogbademu, *Dominated and bounded convergence results of sequential Henstock Stieltjes integral in real valued space*, J. Nepal Math. Soc. **3** (2020), no. 1, 17–20, DOI: https://doi.org/10.3126/jnms.v3i1.32999.
[37] V. O. Iluebe and A. A. Mogbademu, *Sequential Henstock integrals for interval valued functions*, Casp. J. Math. Sci. **11** (2022), no. 1, 358–367, DOI: https://doi.org/10.22080/CJMS.2022.23152.1620.
[38] V. O. Iluebe and A. A. Mogbademu, *Equivalence of sequential Henstock and topological Henstock integrals for interval valued functions*, J. Mahani Math. Res. **12** (2023), no. 2, 267–274.
[39] S. Cao, *The Henstock integral for Banach-valued functions*, SEA Bull. Math. **16** (1992), no. 1, 35–40.
[40] L. D. Piazza and K. Musiał, *Set-valued Kurzweil-Henstock-Pettis integral*, Set-Valued Analysis **13** (2005), no. 2, 167–179, DOI: https://doi.org/10.1007/s11228-004-0934-0.
[41] S. Heikkilä and G. Ye, *Convergence and comparison results for Henstock-Kurzweil and McShane integrable vector-valued functions*, Southeast Asian Bull. Math. **35** (2011), no. 3, 407–418.
[42] G. Riccobono, *An integral for a Banach valued function*, Tatra Mt. Math. Publ. **44** (2009), no. 1, 105–113, DOI: http://doi.org/10.2478/v10127-009-0051-4.
[43] S. Schwabik and G. Ye, *Topics in Banach Space Integration*, vol. 10. World Scientific, Singapore, 2005.
[44] L. D. Piazza and V. Marraffa, *An equivalent definition of the vector-valued McShane integral by means of partitions of unity*, Studia Math. **151** (2002), no. 2, 175–185.
[45] D. Candeloro and A. R. Sambucini, *Comparison between some norm and order gauge integrals in Banach lattices*, Panamer. Math. J. **25** (2015), no. 3, 1–16.
[46] A. Boccuto, D. Candeloro and A. R. Sambucini, *Henstock multivalued integrability in Banach lattices with respect to pointwise non atomic measures*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. **26** (2015), no. 4, 363–383.
[47] K. Musiał, *Topics in the theory of Pettis integration*, Rend. Istit. Mat. Univ. Trieste **23** (1991), no. 1, 177–262.
[48] L. D. Piazza and K. Musiał, *Henstock-Kurzweil-Pettis integrability of compact valued multifunctions with values in an arbitrary Banach space*, J. Math. Anal. Appl. **408** (2013), no. 2, 452–464.
[49] B. R. Satco and C. O. Turcu, *Henstock-Kurzweil-Pettis integral and weak topologies in nonlinear integral equations on time scales*, Math. Slovaca **63** (2013), no. 6, 1347–1360, DOI: https://doi.org/10.2478/s12175-013-0175-5.
[50] G. Ye, *On the Henstock-Kurzweil-Dunford and Kurzweil-Henstock-Pettis integrals*, Rocky Mountain J. Math. **39** (2009), no. 4, 1233–1244.
[51] S. B. Kaliaj, A. D. Tato, and F. D. Gumeni, *Controlled convergence theorems for Henstock-Kurzweil-Pettis integral on m-dimensional compact intervals*, Czechoslovak Math J. **62** (2012), 243–255, DOI: https://doi.org/10.1007/s10587-012-0009-6.
[52] G. Ye, *On Kurzweil-Henstock-Pettis and Kurzweil-Henstock integrals of Banach space-valued functions*, Taiwanese J. Math. **14** (2010), no. 1, 213–222, DOI: https://doi.org/10.11650/twjm/1500405736.
[53] A. Sikorska-Nowak, *Retarded functional differential equations in Banach spaces and Henstock-Kurzweil-Pettis integrals*, Discuss. Math. Differ. Incl. Control Optim. **27** (2007), no. 2, 315–327.
[54] M. Cichoń, *On integrals of vector-valued functions on time scales*, Commun. Math. Anal. **11** (2011), no. 1, 94–110.