Research Article

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On a discrete version of Fejér inequality for α-convex sequences without symmetry condition

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Abstract: In this study, we introduce the notion of α-convex sequences which is a generalization of the convexity concept. For this class of sequences, we establish a discrete version of Fejér inequality without imposing any symmetry condition. In our proof, we use a new approach based on the choice of an appropriate sequence, which is the unique solution to a certain second-order difference equation. Moreover, we obtain a refinement of the standard (right) Fejér inequality for convex sequences.

Keywords: α-convex sequences, convex sequences, Fejér-type inequalities, second-order difference equations

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1 Introduction

Convex functions constitute an important class of functions which is widely used in theoretical and applied mathematics. Due to this fact, much efforts have been devoted to the study of the properties of such functions, e.g., [1–11]. One of the important inequalities involving convex functions is the Hermite-Hadamard double inequality, which can be stated as follows: If $f:[a,b]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The above double inequality dates back to an observation made by Hermite [12] in 1883 with an independent use by Hadamard [13] in 1893. Hermite-Hadamard double inequality is very useful in the study of the properties of convex functions and their applications in optimization and approximation theory, e.g., [14–16]. This fact motivated the study of inequalities of type (1.1) in various directions. One of the first generalizations of (1.1) was obtained by Fejér (1906) in [17], where he proved that if $f:[a,b]\to\mathbb{R}$ is a convex function and $p:[a,b]\to\mathbb{R}$ is integrable, nonnegative, and symmetric function with respect to the midpoint $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\mathrm{d}x\le\int_a^b f(x)p(x)\mathrm{d}x\le\frac{f(a)+f(b)}{2}\int_a^b p(x)\mathrm{d}x. \tag{1.2}$$

For other generalizations and extensions of (1.1), see, e.g., [18–34] and references therein. On the other hand, it is natural to ask whether it is possible to weaken or withdraw the symmetry condition imposed on p in (1.2).

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In [35], a positive answer to this question has been provided. Namely, it was proved that if $f: [a, b] \to \mathbb{R}$ is a convex function and $p : [a, b] \to \mathbb{R}$ is integrable, positive, and normalized function (i.e., $\int_a^b p(x)\mathrm{d}x = 1$), then

$$f(\lambda a + \mu b) \le \int_a^b f(x)p(x)\mathrm{d}x \le \lambda f(a) + \mu f(b), \tag{1.3}$$

where

$$\lambda = \frac{1}{b-a}\int_a^b (b-x)p(x)\mathrm{d}x \quad \text{and} \quad \mu = \frac{1}{b-a}\int_a^b (x-a)p(x)\mathrm{d}x.$$

It can be easily seen that if addition of p is symmetric with respect to the midpoint $\frac{a+b}{2}$, then (1.3) reduces to (1.2).

The notion of convex sequences is a discrete version of the convexity concept. Several interesting inequalities involving convex sequences have been established, see, e.g., [6,36–41] and references therein. In particular, in [36], the authors established a discrete version of (1.2) involving convex sequences. Namely, it was proved that if $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$, $n \ge 3$, is a convex sequence and $p = (p_1, p_2, p_3, \ldots, p_n) \in \mathbb{R}^n$ is a symmetric sequence with respect to $\frac{n+1}{2}$ with $p_i \ge 0$ for all $i = 1, 2, 3, \ldots, n$, then

$$\left(\sum_{i=1}^n p_i\right)\frac{a_N + a_{n+1-N}}{2} \le \sum_{i=1}^n p_i a_i \le \left(\sum_{i=1}^n p_i\right)\frac{a_1 + a_n}{2}, \tag{1.4}$$

where $N = \left[\frac{n+1}{2}\right]$ is the integer part of $\frac{n+1}{2}$. In [39], using some matrix methods based on column stochastic and doubly stochastic matrices, the author (among many other results) extended the right inequality in (1.4) to sequences p that are not necessarily symmetric. Namely, he proved that, if $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n, n \ge 3$, is a convex sequence and $p = (p_1, p_2, p_3, \ldots, p_n) \in \mathbb{R}^n$ with $p_i \ge 0$ for all $i = 1, 2, 3, \ldots, n$, then

$$\sum_{i=1}^n p_i a_i \le \frac{1}{n-1}\left[\sum_{i=1}^n (n-i)p_i\right]a_1 + \frac{1}{n-1}\left[\sum_{i=1}^n (i-1)p_i\right]a_n. \tag{1.5}$$

Note that if p is symmetric with respect to $\frac{n+1}{2}$, then (1.5) reduces to the right inequality in (1.4).

In this study, we establish a refinement of (1.5). We also introduce the notion of α-convex sequences, where $\alpha \in \mathbb{R}^n$ is a positive sequence, i.e., $\alpha_i > 0$ for all $i = 1, 2, 3, \ldots, n$. In particular, if $\alpha_i = 1$ for all i, then an α-convex sequence reduces to a convex sequence. For this class of sequences, we establish an extension of (1.5) always without assuming any symmetry condition on p. Our approach is completely different to that used in [39]. Namely, our method is based on the choice of an appropriate sequence satisfying a certain second-order difference equation involving the two sequences α and p.

The rest of the work is organized as follows. In Section 2, we introduce the notion of α-convex sequences and provide some examples of such sequences. In particular, we give some examples of non-convex sequences that are α-convex. In Section 3, we state our main results and discuss some special cases. We finally prove the obtained results in Section 4.

2 α-convex sequences

We first recall the notion of convex sequences (e.g. [42]).

Definition 2.1. Let $n \ge 3$ and $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$. We say that a is a convex sequence, if

$$a_i \le \frac{a_{i-1} + a_{i+1}}{2} \tag{2.1}$$

for all $i = 2, \ldots, n-1$.

Throughout this study, we shall use the following notations. By $a = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq}$, we mean that $a_i \geq 0$ for all $i = 1, \ldots, n$. By $a = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>}$, we mean that $a_i > 0$ for all $i = 1, \ldots, n$.

We define α-convex sequences as follows.

Definition 2.2. Let $n \geq 3$ and $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$. We say that a is α-convex, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in \mathbb{R}^n_{>}$, if

$$\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1}) \geq 0 \tag{2.2}$$

for all $i = 2, \ldots, n-1$.

Note that if $n \geq 3$ and $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$ is a convex sequence, then a is α-convex, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ with $\alpha_i = 1$ for all $i = 1, 2, 3, \ldots, n$.

We provide below some examples of α-convex sequences.

Example 2.1. Let $n \geq 3$, $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$, and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in \mathbb{R}^n_{>}$. If a is a convex sequence and

$$(\alpha_i - \alpha_{i-1})(a_i - a_{i-1}) \geq 0 \tag{2.3}$$

for all $i = 2, \ldots, n-1$, then a is α-convex. Namely, for all $i = 2, \ldots, n-1$, we have

$$\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1}) = \alpha_i(a_{i+1} + a_{i-1} - 2a_i) + (\alpha_i - \alpha_{i-1})(a_i - a_{i-1}).$$

Since a is convex, we deduce from (2.3) that

$$\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1}) \geq 0.$$

Example 2.2. Let $n \geq 3$, $a = (a_1, a_2, a_3, \ldots, a_n)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$, where

$$a_i = -i(i-1), \quad i = 1, 2, 3, \ldots, n$$

and

$$\alpha_i = \frac{1}{i^2}, \quad i = 1, 2, 3, \ldots, n.$$

For all $i = 2, \ldots, n-1$, we have

$$a_{i+1} + a_{i-1} - 2a_i = -2 < 0,$$

which shows that a is not a convex sequence. On the other hand, an elementary calculation gives us that for all $i = 2, \ldots, n-1$,

$$\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1}) = \frac{2}{i(i-1)} > 0,$$

which shows that a is α-convex.

Example 2.3. Let $n > m$, where $m \geq 3$ is a fixed natural number. Let

$$a_i = (i - 3m + 1)(i-1)i, \quad i = 1, 2, 3, \ldots, m, \ldots, n.$$

Elementary calculation shows that

$$a_{i+1} + a_{i-1} - 2a_i = 6(i-m), \quad i = 2, 3, \ldots, m, \ldots, n-1,$$

which implies that, if $2 \leq i < m \leq n-1$, then

$$a_{i+1} + a_{i-1} - 2a_i < 0.$$

Consequently, the sequence $a = (a_1, a_2, a_3, \ldots, a_m, \ldots, a_n)$ is not convex. On the other hand, we have

$$\frac{1}{i}(a_{i+1} - a_i) - \frac{1}{i-1}(a_i - a_{i-1}) = 3, \quad i = 2, 3, \ldots, m, \ldots, n-1,$$

which shows that a is α-convex, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m, \ldots, \alpha_n)$ and

$$\alpha_i = \frac{1}{i}, \quad i = 1, 2, 3, \ldots, m, \ldots, n.$$

3 Main results

Our first main result is a refinement of inequality (1.5) obtained in [39].

Theorem 3.1. *Let $n \geq 3$. If $p = (p_1, p_2, p_3, \ldots, p_n) \in \mathbb{R}^n_{\geq}$ and $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$, then*

$$\sum_{i=1}^{n} p_i a_i \leq \frac{1}{n-1}\left[\sum_{i=1}^{n}(n-i)p_i\right]a_1 + \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]a_n - \frac{c_a}{2}\sum_{i=1}^{n}(i-1)(n-i)p_i, \tag{3.1}$$

where

$$c_a = \min\{a_{i+1} + a_{i-1} - 2a_i : \ i = 2, 3, \ldots, n-1\}.$$

Remark 3.1. Note that the sequence a in Theorem 3.1 is not assumed to be convex.

Remark 3.2. If in Theorem 3.1, we assume that a is a convex sequence, then $c_a \geq 0$. In this case, (3.1) is a refinement of (1.5).

In the special case $p_i = 1$ for all $i = 1, 2, 3, \ldots, n$, one has

$$\sum_{i=1}^{n}(i-1)(n-i)p_i = \sum_{i=1}^{n}(i-1)(n-i) = \frac{(n-2)(n-1)n}{6}$$

and

$$\sum_{i=1}^{n}(n-i)p_i = \sum_{i=1}^{n}(i-1)p_i = \frac{(n-1)n}{2}.$$

Hence, from Theorem 3.1, we deduce the following result.

Corollary 3.1. *Let $n \geq 3$. For all $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$, we have*

$$\sum_{i=1}^{n} a_i \leq \frac{n}{2}(a_1 + a_n) - \frac{(n-2)(n-1)nc_a}{12}. \tag{3.2}$$

A similar result to that given by Theorem 3.1 can be stated as follows.

Theorem 3.2. *Let $n \geq 3$. If $p = (p_1, p_2, p_3, \ldots, p_n) \in \mathbb{R}^n_{\geq}$ and $a = (a_1, a_2, a_3, \ldots, a_n) \in \mathbb{R}^n$, then*

$$\sum_{i=1}^{n} p_i a_i \geq \frac{1}{n-1}\left[\sum_{i=1}^{n}(n-i)p_i\right]a_1 + \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]a_n - \frac{C_a}{2}\sum_{i=1}^{n}(i-1)(n-i)p_i, \tag{3.3}$$

where

$$C_a = \max\{a_{i+1} + a_{i-1} - 2a_i : \ i = 2, 3, \ldots, n-1\}.$$

Our third main result is an extension of inequality (1.5) to the class of α-convex functions.

Theorem 3.3. *Let $n \geq 3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n) \in \mathbb{R}^n_>$. If $a = (a_1, a_2, a_3, \dots, a_n) \in \mathbb{R}^n$ is α-convex and $p = (p_1, p_2, p_3, \dots, p_n) \in \mathbb{R}^n_\geq$, then*

$$\sum_{i=1}^{n} a_i p_i \leq \left(\sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \left(a_1 \sum_{\ell=1}^{n-1} \sum_{k=\ell}^{n-1} \frac{p_\ell}{\alpha_k} + a_n \sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \frac{p_\ell}{\alpha_k}\right). \tag{3.4}$$

Remark 3.3. In the special case $\alpha_i = 1$ for all $i = 1, 2, 3, \dots, n$, one has

$$\left(\sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} = \frac{1}{n-1}, \quad \sum_{\ell=1}^{n-1} \sum_{k=\ell}^{n-1} \frac{p_\ell}{\alpha_k} = \sum_{\ell=1}^{n} (n-\ell) p_\ell, \quad \sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \frac{p_\ell}{\alpha_k} = \sum_{\ell=1}^{n} (\ell-1) p_\ell.$$

Thus, (3.4) reduces to (1.5). Hence, we recover the obtained result in [39].

If $p_i = 1$ for all $i = 1, 2, 3, \dots, n$, making use of Fubini's theorem, we obtain

$$\sum_{\ell=1}^{n-1} \sum_{k=\ell}^{n-1} \frac{p_\ell}{\alpha_k} = \sum_{k=1}^{n-1} \frac{1}{\alpha_k} \sum_{\ell=1}^{k} 1 = \sum_{k=1}^{n-1} \frac{k}{\alpha_k}$$

and

$$\sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \frac{p_\ell}{\alpha_k} = \sum_{k=1}^{n-1} \frac{1}{\alpha_k} \sum_{\ell=k+1}^{n} 1 = \sum_{k=1}^{n-1} \frac{n-k}{\alpha_k}.$$

Hence, from Theorem 3.3, we deduce the following result.

Corollary 3.2. *Let $n \geq 3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n) \in \mathbb{R}^n_>$. If $a = (a_1, a_2, a_3, \dots, a_n) \in \mathbb{R}^n$ is α-convex, then*

$$\sum_{i=1}^{n} a_i \leq \left(\sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \left(a_1 \sum_{k=1}^{n-1} \frac{k}{\alpha_k} + a_n \sum_{k=1}^{n-1} \frac{n-k}{\alpha_k}\right).$$

If $p_i = i$ for all $i = 1, 2, 3, \dots, n$, we obtain

$$\sum_{\ell=1}^{n-1} \sum_{k=\ell}^{n-1} \frac{p_\ell}{\alpha_k} = \sum_{k=1}^{n-1} \frac{1}{\alpha_k} \sum_{\ell=1}^{k} \ell = \sum_{k=1}^{n-1} \frac{k(k+1)}{2\alpha_k}$$

and

$$\sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \frac{p_\ell}{\alpha_k} = \sum_{k=1}^{n-1} \frac{1}{\alpha_k} \sum_{\ell=k+1}^{n} \ell = \sum_{k=1}^{n-1} \frac{(n-k)(n+k+1)}{2\alpha_k}.$$

Hence, from Theorem 3.3, we deduce the following result.

Corollary 3.3. *Let $n \geq 3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n) \in \mathbb{R}^n_>$. If $a = (a_1, a_2, a_3, \dots, a_n) \in \mathbb{R}^n$ is α-convex, then*

$$\sum_{i=1}^{n} i a_i \leq \frac{1}{2} \left(\sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \left(a_1 \sum_{k=1}^{n-1} \frac{k(k+1)}{\alpha_k} + a_n \sum_{k=1}^{n-1} \frac{(n-k)(n+k+1)}{\alpha_k}\right).$$

If $p_i = i^2$ for all $i = 1, 2, 3, \dots, n$, we obtain

$$\sum_{\ell=1}^{n-1} \sum_{k=\ell}^{n-1} \frac{p_\ell}{\alpha_k} = \sum_{k=1}^{n-1} \frac{1}{\alpha_k} \sum_{\ell=1}^{k} \ell^2 = \sum_{k=1}^{n-1} \frac{k(k+1)(2k+1)}{6\alpha_k}$$

and

$$\sum_{\ell=2}^{n}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}=\sum_{k=1}^{n-1}\frac{1}{\alpha_k}\sum_{\ell=k+1}^{n}\ell^2=\sum_{k=1}^{n-1}\frac{n(n+1)(2n+1)-k(k+1)(2k+1)}{6\alpha_k}.$$

Hence, from Theorem 3.3, we deduce the following result.

Corollary 3.4. *Let $n\geq 3$ and $\alpha=(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n)\in\mathbb{R}^n_>$. If $a=(a_1,a_2,a_3,\dots,a_n)\in\mathbb{R}^n$ is α-convex, then*

$$\sum_{i=1}^{n}i^2a_i\leq\frac{1}{6}\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\left(a_1\sum_{k=1}^{n-1}\frac{k(k+1)(2k+1)}{\alpha_k}+a_n\sum_{k=1}^{n-1}\frac{n(n+1)(2n+1)-k(k+1)(2k+1)}{\alpha_k}\right).$$

4 Proofs of the main results

This section is devoted to the proofs of Theorems 3.1 and 3.3. We first establish an auxiliary result which is crucial in the proof of Theorem 3.3.

4.1 Auxiliary result

Let $n\geq 4$, $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\dots,\alpha_n)\in\mathbb{R}^n_>$, and $p=(p_1,p_2,p_3,p_4,\dots,p_n)\in\mathbb{R}^n_\geq$. We consider the difference equation

$$\alpha_i(b_{i+1}-b_i)-\alpha_{i-1}(b_i-b_{i-1})=-p_i,\quad i=2,\dots,n-1 \tag{4.1}$$

under the boundary conditions

$$b_1=b_n=0. \tag{4.2}$$

Lemma 4.1. *Problem* (4.1) *under boundary conditions* (4.2) *admits a unique solution $b=(b_1,b_2,b_3,b_4,\dots,b_n)\in\mathbb{R}^n_\geq$ given by*

$$b_i=\begin{cases}0 & \text{if}\quad i\in\{1,n\},\\ \left(\alpha_1\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\sum_{k=2}^{n-1}\sum_{\ell=2}^{k}\frac{p_\ell}{\alpha_k} & \text{if}\quad i=2,\\ \left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\left(\sum_{k=i}^{n-1}\sum_{\ell=i}^{k}\frac{p_\ell}{\alpha_k}\sum_{j=1}^{i-1}\frac{1}{\alpha_j}+\sum_{\ell=2}^{i-1}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\sum_{j=i}^{n-1}\frac{1}{\alpha_j}\right) & \text{if}\quad i\in\{3,4,\dots,n-1\}.\end{cases} \tag{4.3}$$

Proof. Let us introduce the sequence

$$c_i=\alpha_i(b_{i+1}-b_i),\quad i=1,\dots,n-1. \tag{4.4}$$

From (4.1) and (4.4), we have

$$c_1=\alpha_1b_2 \tag{4.5}$$

and

$$c_i-c_{i-1}=-p_i,\quad i=2,\dots,n-1,$$

which implies by induction that

$$c_i=c_1-\sum_{k=2}^{i}p_k,\quad i=2,\dots,n-1. \tag{4.6}$$

In view of (4.4), we obtain

$$b_{i+1} - b_i = \frac{c_i}{\alpha_i}, \quad i = 2, \ldots, n-1,$$

which implies (with (4.5) and (4.6)) by induction that for all $i = 3, \ldots, n-1$,

$$b_i = b_2 + \sum_{k=2}^{i-1} \frac{c_k}{\alpha_k} = b_2 + \sum_{k=2}^{i-1} \frac{1}{\alpha_k}\left(c_1 - \sum_{\ell=2}^{k} p_\ell\right) = b_2 + \alpha_1 b_2 \sum_{k=2}^{i-1} \frac{1}{\alpha_k} - \sum_{k=2}^{i-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k},$$

that is,

$$b_i = b_2 + \alpha_1 b_2 \sum_{k=2}^{i-1} \frac{1}{\alpha_k} - \sum_{k=2}^{i-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k}, \quad i = 3, \ldots, n-1. \tag{4.7}$$

On the other hand, taking $i = n-1$ in (4.1) and using that $b_n = 0$, we obtain

$$\alpha_{n-1} b_{n-1} + \alpha_{n-2}(b_{n-1} - b_{n-2}) = p_{n-1}. \tag{4.8}$$

Similarly, taking, respectively, $i = n-1$ and $i = n-2$ in (4.7), we obtain

$$b_{n-1} - b_{n-2} = \frac{\alpha_2 b_2}{\alpha_{n-2}} - \sum_{\ell=2}^{n-2} \frac{p_\ell}{\alpha_{n-2}}. \tag{4.9}$$

Combining (4.8) with (4.9) and using (4.7) with $i = n-1$, we obtain

$$\alpha_{n-1}\left(b_2 + \alpha_1 b_2 \sum_{k=2}^{n-2} \frac{1}{\alpha_k} - \sum_{k=2}^{n-2} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k}\right) + \alpha_2 b_2 - \sum_{\ell=2}^{n-2} p_\ell = p_{n-1}.$$

After simplification, the above identity reduces to

$$b_2 = \left(\alpha_1 \sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \sum_{k=2}^{n-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k}. \tag{4.10}$$

We now use (4.7) and (4.10) to obtain

$$b_i = \left(\sum_{k=1}^{n-1} \frac{1}{\alpha_k}\right)^{-1} T_n, \quad i = 3, \ldots, n-1, \tag{4.11}$$

where

$$T_n = \sum_{k=2}^{n-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k} \sum_{j=1}^{i-1} \frac{1}{\alpha_j} - \sum_{j=1}^{n-1} \frac{1}{\alpha_j} \sum_{k=2}^{i-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k}.$$

After simplifications, we obtain

$$T_n = \sum_{k=i}^{n-1} \sum_{\ell=i}^{k} \frac{p_\ell}{\alpha_k} \sum_{j=1}^{i-1} \frac{1}{\alpha_j} + \sum_{\ell=2}^{i-1} \sum_{k=1}^{\ell-1} \frac{p_\ell}{\alpha_k} \sum_{j=i}^{n-1} \frac{1}{\alpha_j}. \tag{4.12}$$

Hence, by (4.11) and (4.12), we obtain

$$b_i = \left(\sum_{k=1}^{n-1} \frac{1}{\alpha_k}\right)^{-1} \left(\sum_{k=i}^{n-1} \sum_{\ell=i}^{k} \frac{p_\ell}{\alpha_k} \sum_{j=1}^{i-1} \frac{1}{\alpha_j} + \sum_{\ell=2}^{i-1} \sum_{k=1}^{\ell-1} \frac{p_\ell}{\alpha_k} \sum_{j=i}^{n-1} \frac{1}{\alpha_j}\right),$$

for all $i = 3, \ldots, n-1$. This completes the proof of Lemma 4.1. □

4.2 Proof of Theorem 3.1

By the definition of c_a, we have

$$a_{i+1} + a_{i-1} - 2a_i \geq c_a, \quad i = 2, 3, \ldots, n-1. \tag{4.13}$$

We introduce the sequences $A = (A_1, A_2, A_3, \ldots, A_n)$ and $B = (B_1, B_2, B_3, \ldots, B_n)$ defined by

$$B_i = (i-1)(n-i), \quad i = 1, 2, 3, \ldots, n \tag{4.14}$$

and

$$A_i = a_i + \frac{c_a}{2} B_i, \quad i = 1, 2, 3, \ldots, n.$$

For all $i = 2, 3, \ldots, n-1$, we have

$$A_{i+1} + A_{i-1} - 2A_i = (a_{i+1} + a_{i-1} - 2a_i) + \frac{c_a}{2}(B_{i+1} + B_{i-1} - 2B_i). \tag{4.15}$$

On the other hand, for all $i = 2, 3, \ldots, n-1$, we have

$$B_{i+1} + B_{i-1} - 2B_i = i(n-i-1) + (i-2)(n-i+1) - 2(i-1)(n-i) = -2,$$

which implies by (4.15) that

$$A_{i+1} + A_{i-1} - 2A_i = (a_{i+1} + a_{i-1} - 2a_i) - c_a, \quad i = 2, 3, \ldots, n-1. \tag{4.16}$$

Then, from (4.13) and (4.16), we deduce that A is a convex sequence. Applying inequality (1.5), we obtain

$$\sum_{i=1}^{n} p_i A_i \leq \frac{1}{n-1}\left[\sum_{i=1}^{n}(n-i)p_i\right]A_1 + \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]A_n.$$

Note that

$$A_1 = a_1 \quad \text{and} \quad A_n = a_n.$$

Hence, the above inequality is equivalent to

$$\sum_{i=1}^{n} p_i\left(a_i + \frac{c_a}{2}B_i\right) \leq \left[\sum_{i=1}^{n}(n-i)p_i\right]a_1 + \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]a_n,$$

that is,

$$\sum_{i=1}^{n} p_i a_i \leq \left[\sum_{i=1}^{n}(n-i)p_i\right]a_1 + \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]a_n - \frac{c_a}{2}\sum_{i=1}^{n} p_i B_i,$$

which proves (3.1). □

4.3 Proof of Theorem 3.2

The proof is similar to that of Theorem 3.1. Namely, by the definition of C_a, we have

$$a_{i+1} + a_{i-1} - 2a_i \leq C_a, \quad i = 2, 3, \ldots, n-1. \tag{4.17}$$

We introduce the sequence $A = (A_1, A_2, A_3, \ldots, A_n)$ defined by

$$A_i = -a_i - \frac{C_a}{2} B_i, \quad i = 1, 2, 3, \ldots, n,$$

where the sequence $B = (B_1, B_2, B_3, \dots, B_n)$ is defined by (4.14). For all $i = 2, 3, \dots, n-1$, we have by (4.17)

$$\begin{aligned} A_{i+1} + A_{i-1} - 2A_i &= -(a_{i+1} + a_{i-1} - 2a_i) - \frac{C_a}{2}(B_{i+1} + B_{i-1} - 2B_i) \\ &= -(a_{i+1} + a_{i-1} - 2a_i) + C_a \ge 0, \end{aligned} \tag{4.18}$$

which shows that A is a convex sequence. Applying the inequality (1.5), we obtain

$$\sum_{i=1}^{n} p_i A_i \le \frac{1}{n-1}\left[\sum_{i=1}^{n}(n-i)p_i\right]A_1 + \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]A_n.$$

Note that

$$A_1 = -a_1 \quad \text{and} \quad A_n = -a_n.$$

Hence, the above inequality is equivalent to

$$\sum_{i=1}^{n} p_i\left(-a_i - \frac{C_a}{2}B_i\right) \le -\frac{1}{n-1}\left[\sum_{i=1}^{n}(n-i)p_i\right]a_1 - \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]a_n,$$

that is,

$$\sum_{i=1}^{n} p_i a_i \ge \frac{1}{n-1}\left[\sum_{i=1}^{n}(n-i)p_i\right]a_1 + \frac{1}{n-1}\left[\sum_{i=1}^{n}(i-1)p_i\right]a_n - \frac{C_a}{2}\sum_{i=1}^{n}(i-1)(n-i)p_i,$$

which proves (3.3). □

4.4 Proof of Theorem 3.3

We study separately the cases $n = 3$ and $n \ge 4$.

- The case $n = 3$. In this case, we have

$$\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\left(a_1\sum_{\ell=1}^{n-1}\sum_{k=\ell}^{n-1}\frac{p_\ell}{\alpha_k} + a_n\sum_{\ell=2}^{n}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\right) - \sum_{i=1}^{n}a_i p_i = \left(\sum_{\tau=1}^{2}\frac{1}{\alpha_\tau}\right)^{-1}\left(a_1\sum_{\ell=1}^{2}\sum_{k=\ell}^{2}\frac{p_\ell}{\alpha_k} + a_3\sum_{\ell=2}^{3}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\right) - \sum_{i=1}^{3}a_i p_i. \tag{4.19}$$

On the other hand, we have

$$\begin{aligned} a_1\sum_{\ell=1}^{2}\sum_{k=\ell}^{2}\frac{p_\ell}{\alpha_k} + a_3\sum_{\ell=2}^{3}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k} &= a_1\left(p_1\sum_{k=1}^{2}\frac{1}{\alpha_k} + \frac{p_2}{\alpha_2}\right) + a_3\left(\frac{p_2}{\alpha_1} + p_3\sum_{k=1}^{2}\frac{1}{\alpha_k}\right) \\ &= \left(\sum_{k=1}^{2}\frac{1}{\alpha_k}\right)\left[a_1p_1 + a_3p_3 + p_2\left(\frac{a_1}{\alpha_2} + \frac{a_3}{\alpha_1}\right)\left(\sum_{k=1}^{2}\frac{1}{\alpha_k}\right)^{-1}\right]. \end{aligned} \tag{4.20}$$

Hence, in view of (4.19) and (4.20), after simplifications, we obtain

$$\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\left(a_1\sum_{\ell=1}^{n-1}\sum_{k=\ell}^{n-1}\frac{p_\ell}{\alpha_k} + a_n\sum_{\ell=2}^{n}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\right) - \sum_{i=1}^{n}a_i p_i = \left(\sum_{k=1}^{2}\frac{1}{\alpha_k}\right)^{-1}\frac{p_2}{\alpha_1\alpha_2}[\alpha_2(a_3-a_2) - \alpha_1(a_2-a_1)].$$

Furthermore, since a is α-convex, we have (with $n = 3$)

$$\alpha_2(a_3 - a_2) - \alpha_1(a_2 - a_1) \ge 0.$$

Consequently, we obtain (for $n = 3$)

$$\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\left(a_1\sum_{\ell=1}^{n-1}\sum_{k=\ell}^{n-1}\frac{p_\ell}{\alpha_k} + a_n\sum_{\ell=2}^{n}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\right) - \sum_{i=1}^{n}a_i p_i \ge 0,$$

which proves (3.4) in the case $n = 3$.

- The case $n \geq 4$. Let $b = (b_1, b_2, b_3, b_4, \dots, b_n) \in \mathbb{R}^n_{\geq}$ be the sequence defined by (4.3). By Lemma 4.1, we have

$$\sum_{i=1}^{n} a_i p_i = a_1 p_1 + a_n p_n - \sum_{i=2}^{n-1} a_i[\alpha_i(b_{i+1} - b_i) - \alpha_{i-1}(b_i - b_{i-1})]. \tag{4.21}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{i=2}^{n-1} a_i[\alpha_i(b_{i+1} - b_i) - \alpha_{i-1}(b_i - b_{i-1})] - \sum_{i=2}^{n-1} b_i[\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1})]\\
&= \sum_{i=2}^{n-1} a_i\alpha_i b_{i+1} - \sum_{i=2}^{n-1} a_i\alpha_i b_i - \sum_{i=2}^{n-1} a_i\alpha_{i-1} b_i + \sum_{i=2}^{n-1} a_i\alpha_{i-1} b_{i-1} - \sum_{i=2}^{n-1} \alpha_i b_i a_{i+1} + \sum_{i=2}^{n-1} b_i\alpha_i a_i\\
&\quad + \sum_{i=2}^{n-1} b_i\alpha_{i-1} a_i - \sum_{i=2}^{n-1} b_i\alpha_{i-1} a_{i-1}\\
&= \left(\sum_{i=2}^{n-1} a_i\alpha_i b_{i+1} - \sum_{i=2}^{n-1} b_i\alpha_{i-1} a_{i-1}\right) + \left(\sum_{i=2}^{n-1} b_i\alpha_{i-1} a_i - \sum_{i=2}^{n-1} a_i\alpha_{i-1} b_i\right)\\
&\quad + \left(\sum_{i=2}^{n-1} b_i\alpha_i a_i - \sum_{i=2}^{n-1} a_i\alpha_i b_i\right) + \left(\sum_{i=2}^{n-1} a_i\alpha_{i-1} b_{i-1} - \sum_{i=2}^{n-1} \alpha_i b_i a_{i+1}\right)\\
&= \left(a_{n-1}\alpha_{n-1} b_n + \sum_{i=3}^{n-1} a_{i-1}\alpha_{i-1} b_i - \sum_{i=3}^{n-1} a_{i-1}\alpha_{i-1} b_i - b_2\alpha_1 a_1\right)\\
&\quad + \left(a_2\alpha_1 b_1 + \sum_{i=2}^{n-2} a_{i+1}\alpha_i b_i - \sum_{i=2}^{n-2} a_{i+1}\alpha_i b_i - \alpha_{n-1} b_{n-1} a_n\right)\\
&= a_{n-1}\alpha_{n-1} b_n - b_2\alpha_1 a_1 + a_2\alpha_1 b_1 - \alpha_{n-1} b_{n-1} a_n.
\end{aligned}$$

Taking into consideration that $b_1 = b_n = 0$, we obtain

$$-\sum_{i=2}^{n-1} a_i[\alpha_i(b_{i+1} - b_i) - \alpha_{i-1}(b_i - b_{i-1})] = b_2\alpha_1 a_1 + \alpha_{n-1} b_{n-1} a_n - \sum_{i=2}^{n-1} b_i[\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1})]. \tag{4.22}$$

Then, combining (4.21) with (4.22), we obtain

$$\sum_{i=1}^{n} a_i p_i = a_1 p_1 + a_n p_n + b_2\alpha_1 a_1 + \alpha_{n-1} b_{n-1} a_n - \sum_{i=2}^{n-1} b_i[\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1})].$$

Since $b \in \mathbb{R}^n_{\geq}$ and $\alpha_i(a_{i+1} - a_i) - \alpha_{i-1}(a_i - a_{i-1}) \geq 0$, we deduce from the above inequality that

$$\sum_{i=1}^{n} a_i p_i \leq a_1(p_1 + \alpha_1 b_2) + a_n(p_n + \alpha_{n-1} b_{n-1}). \tag{4.23}$$

Now, from (4.3), we have

$$b_2 = \left(\alpha_1 \sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \sum_{k=2}^{n-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k}$$

and

$$b_{n-1} = \left(\sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \left(\frac{p_{n-1}}{\alpha_{n-1}} \sum_{j=1}^{n-2} \frac{1}{\alpha_j} + \frac{1}{\alpha_{n-1}} \sum_{\ell=2}^{n-2} \sum_{k=1}^{\ell-1} \frac{p_\ell}{\alpha_k}\right).$$

Then,

$$p_1 + \alpha_1 b_2 = p_1 + \left(\sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \sum_{k=2}^{n-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k} = \left(\sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau}\right)^{-1} \left(p_1 \sum_{\tau=1}^{n-1} \frac{1}{\alpha_\tau} + \sum_{k=2}^{n-1} \sum_{\ell=2}^{k} \frac{p_\ell}{\alpha_k}\right). \tag{4.24}$$

On the other hand, by Fubini's theorem, we have

$$p_1\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}+\sum_{k=2}^{n-1}\sum_{\ell=2}^{k}\frac{p_\ell}{\alpha_k}=p_1\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}+\sum_{\ell=2}^{n-1}p_\ell\sum_{k=\ell}^{n-1}\frac{1}{\alpha_k}=\sum_{\ell=1}^{n-1}\sum_{k=\ell}^{n-1}\frac{p_\ell}{\alpha_k},$$

which implies by (4.24) that

$$p_1+\alpha_1 b_2=\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\sum_{\ell=1}^{n-1}\sum_{k=\ell}^{n-1}\frac{p_\ell}{\alpha_k}. \tag{4.25}$$

Furthermore, we have

$$\begin{aligned}
p_n+\alpha_{n-1}b_{n-1}&=p_n+\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\left(p_{n-1}\sum_{j=1}^{n-2}\frac{1}{\alpha_j}+\sum_{\ell=2}^{n-2}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\right)\\
&=p_n+\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\sum_{\ell=2}^{n-1}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\\
&=\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\left(p_n\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}+\sum_{\ell=2}^{n-1}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}\right)\\
&=\left(\sum_{\tau=1}^{n-1}\frac{1}{\alpha_\tau}\right)^{-1}\sum_{\ell=2}^{n}\sum_{k=1}^{\ell-1}\frac{p_\ell}{\alpha_k}
\end{aligned} \tag{4.26}$$

Finally, (3.4) follows from (4.23), (4.25), and (4.26). □

5 Conclusion

The study of convex sequences is of great importance in many applications, such as combinatorics, algebra, geometry, analysis, probability, and statistics, e.g., [43–46]. Two main results are established in this study. The first one (Theorem 3.1) is a Fejér-type inequality that holds for any sequence $a=(a_1,a_2,a_3,\dots,a_n)\in\mathbb{R}^n$. In the particular case, when a is a convex sequence, the obtained inequality is a refinement of inequality (1.5) obtained in [39]. Our second main result (Theorem 3.3) is a Fejér-type inequality that holds for α-convex sequences without any symmetry condition imposed on the sequence p. In the particular case, when $\alpha_i=1$ for all $i=1,2,3,\dots,n$, an α-convex sequence is a convex sequence. In this case, our obtained inequality reduces to inequality (1.5). The approach used in the proof of Theorem 3.3 is completely different from that used in [39]. Namely, our method is based on the choice of an appropriate sequence b, which is the unique solution to a certain second-order difference equation (Lemma 4.1). We believe that the proposed approach can be useful for the study of Hermite-Hadamard-type and Fejér-type inequalities for other classes of sequences. In this work, we only studied right-Fejér-type inequalities for α-convex sequences. It would be interesting to study some possible extensions of the left-sided inequality in (1.4) to the class of α-convex sequences.

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