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Research Article

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Limiting dynamics for stochastic complex Ginzburg-Landau systems with time-varying delays on unbounded thin domains

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Abstract: This study deals with the limiting dynamics for stochastic complex Ginzburg-Landau systems with time-varying delays and multiplicative noise on unbounded thin domains. We first prove the existence and uniqueness of pullback tempered random attractors for the systems and then establish the upper semicontinuity of these attractors when the thin domains collapse onto $\mathbb{R}$.

Keywords: thin domain, complex Ginzburg-Landau equations, time-varying delays, random attractor, upper semicontinuity

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1 Introduction

Let $\mathbb{R}$ be the real region and $\mathcal{O}_\varepsilon$ be the region

$$\mathcal{O}_\varepsilon = \{x = (x_1, x_2) | x_1 \in \mathbb{R}, 0 < x_2 < \varepsilon g(x_1)\},$$

where $g \in C_b^2(\mathbb{R}, (0, +\infty))$ and $0 < \varepsilon \le 1$. Denote $\mathcal{O} = \mathbb{R} \times (0, 1)$.

In this work, we investigate the limiting dynamics for the following stochastic complex Ginzburg-Landau systems with time-varying delays and multiplicative noise on unbounded thin domains $\mathcal{O}_\varepsilon$: for $\tau \in \mathbb{R}$, $t > \tau$, and $x = (x_1, x_2) \in \mathcal{O}_\varepsilon$,

$$\begin{cases} \mathrm{d}\hat{u}^\varepsilon = (\lambda + i\alpha)\Delta \hat{u}^\varepsilon \mathrm{d}t - (\kappa + i\beta)|\hat{u}^\varepsilon|^2 \hat{u}^\varepsilon \mathrm{d}t - (\gamma + i\delta)\hat{u}^\varepsilon \mathrm{d}t + h(\hat{u}^\varepsilon(t - \tilde{\rho}(t)))\mathrm{d}t + f(t, x_1)\mathrm{d}t \\ \qquad + \displaystyle\sum_{j=1}^{m} c_j \hat{u}^\varepsilon \circ \mathrm{d}\omega_j, \\ \dfrac{\partial \hat{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, \quad x \in \partial \mathcal{O}_\varepsilon, \\ \hat{u}^\varepsilon(\tau + s, x) = \hat{\psi}^\varepsilon(s, x), \quad s \in [-\rho, 0], \end{cases} \tag{1.1}$$

where i is the unit of imaginary numbers, $\hat{u}^\varepsilon$ is the unknown complex valued function, ν_ε is the unit outward normal vector to $\partial\mathcal{O}_\varepsilon$, $\lambda, \alpha, \kappa, \beta, \gamma, \delta, \rho$ are real constants satisfying $\lambda, \kappa, \gamma, \rho > 0$, and $\kappa > |\beta|$. $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}))$, h is a nonlinear Lipschitz continuous function satisfying $h(0) = 0$, and the Lipschitz constant of h is denoted by L. $\tilde{\rho} \in C^1(\mathbb{R}, [0, \rho])$ is an adequate given delay function with $|\tilde{\rho}'(t)| \le \rho_0 < 1$. $c_j \in \mathbb{R}$ for $j = 1, \ldots, m$, and ω_j for

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$j = 1, \dots, m$ are independent two-sided real-valued Wiener processes on a probability space. The symbol $\circ$ is interpreted in the sense of Stratonovich's integration.

As $\varepsilon \to 0$, system (1.1) can be transformed into the following system on $\mathbb{R}$, for $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} \mathrm{d}u^0 = (\lambda + i\alpha)\dfrac{1}{g}(gu^0_{y_1})_{y_1}\mathrm{d}t - (\kappa + i\beta)|u^0|^2u^0\mathrm{d}t - (\gamma + i\delta)u^0\mathrm{d}t + h(u^0(t - \tilde{\rho}(t)))\mathrm{d}t + f(t, y_1)\mathrm{d}t \\ \qquad + \displaystyle\sum_{j=1}^{m} c_j u^0 \circ \mathrm{d}\omega_j, \\ \dfrac{\partial u^0}{\partial \nu_0} = 0, \quad y_1 \in \partial\mathcal{O}, \\ u^0(\tau + s, y_1) = \psi^0(s, y_1), \quad s \in [-\rho, 0]. \end{cases} \tag{1.2}$$

The study of global random attractors for stochastic systems was initially conducted by Ruelle in [1], and the foundational theory regarding these attractors were subsequently developed in [2–4]. The concept of pullback attractor for autonomous stochastic systems was introduced in [5], followed by extensive research in [6–12].

The systematic investigation of the asymptotic behavior of deterministic dissipative systems on thin domains was initiated by Hale and Raugel in [13]. Subsequently, their findings were expanded to encompass various systems (see, e.g., [14–20]). Recently, the authors addressed the limiting behavior of stochastic reaction-diffusion equations driven by multiplicative noise on both bounded thin domains and unbounded thin domains in [7] and [21], respectively. Concurrently, the asymptotic behavior of non-autonomous stochastic complex Ginzburg-Landau equations with multiplicative noise on bounded thin domains was examined in [22], while the case of unbounded thin domains was investigated in [23].

Time delays are commonly encountered in various systems, leading to potential instability, oscillation, and other system changes. Due to their practical and theoretical significance, numerous scholars have devoted themselves to the study of time-delay systems. Recently, researchers have explored random attractors for stochastic systems with fixed delay (as discussed in [24–27]) as well as those with time-varying delays (as mentioned in [28–32]).

However, to the best of our knowledge, there is a scarcity of literature regarding the existence of random attractors for stochastic complex Ginzburg-Landau equations with time-varying delays and multiplicative noise on unbounded thin domains.

Denote by X a Banach space with norm $\|\cdot\|_X$. Given $\rho > 0$, let $C([-\rho, 0], X)$ be the set of all continuous functions from $[-\rho, 0]$ to X with the maximum norm $\|\phi\|_{C([-\rho,0],X)} = \sup_{-\rho\le s\le 0}\|\phi\|_X$ for $\phi \in C([-\rho, 0], X)$. Denote $\mathcal{P} = C([-\rho, 0], L^2(\mathbb{R}))$ and $\mathcal{Q} = C([-\rho, 0], L^2(\mathcal{O}))$. Given $\tau \in \mathbb{R}$, $T > \tau$ and a function $u : [\tau - \rho, T) \to X$, for each $t \in [\tau, T)$, let $u_t : [-\rho, 0] \to X$ denote the function defined by $u_t(s) = u(t + s)$ for $s \in [-\rho, 0]$. We denote by $(\cdot, \cdot)_Y$ the inner product in a Hilbert space Y. The letter c is a generic positive constant which may change its values from line to line.

This work is organized as follows. Section 2 establishes the continuous cocycle for stochastic system (1.1). In Section 3, all necessary uniform estimates of the solutions are provided. Section 4 is dedicated to proving the existence and uniqueness of random attractors for system (1.1). Finally, in Section 5, we investigate the upper semicontinuity of random attractors when a family of two-dimensional unbounded thin domains collapses onto $\mathbb{R}$.

2 Existence of continuous cocycle

To transfer system (1.1) into boundary value problems on the fixed domain $\mathcal{O}$, we define a transformation $T_\varepsilon : \mathcal{O}_\varepsilon \to \mathcal{O}$ by $T_\varepsilon(x_1, x_2) = (x_1, \frac{x_2}{\varepsilon g(x_1)})$ for $x = (x_1, x_2) \in \mathcal{O}_\varepsilon$. Let $y = (y_1, y_2) = T_\varepsilon(x_1, x_2)$, then we obtain

$$x_1 = y_1, \quad x_2 = \varepsilon g(y_1)y_2,$$

and the Jacobian matrix of T_ε is given by

$$J = \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{pmatrix} 1 & 0 \\ -\frac{y_2}{g} g_{y_1} & \frac{1}{\varepsilon g(y_1)} \end{pmatrix}.$$

Let J^* be the transpose of J, then

$$JJ^* = \begin{pmatrix} 1 & -\frac{y_2}{g} g_{y_1} \\ -\frac{y_2}{g} g_{y_1} & \left(\frac{1}{\varepsilon g(y_1)}\right)^2 + \left(\frac{y_2}{g} g_{y_1}\right)^2 \end{pmatrix}.$$

By [13], for $x \in \mathcal{O}_\varepsilon$ and $y \in \mathcal{O}$, we obtain

$$\nabla_x \hat{u}(x) = J^* \nabla_y u(y)$$

and

$$\Delta_x \hat{u}(x) = |J| \operatorname{div}_y(|J|^{-1} JJ^* \nabla_y u(y)) = \frac{1}{g} \operatorname{div}_y(P_\varepsilon u(y)),$$

where $\hat{u}(x) = u(y)$, ∇_x and Δ_x are the gradient operator and the Laplace operator in $x \in \mathcal{O}_\varepsilon$, respectively, div_y and ∇_y are the divergence operator and the gradient operator in $y \in \mathcal{O}$, respectively, and P_ε is the operator defined by

$$P_\varepsilon u(y) = \begin{pmatrix} g u_{y_1} - g_{y_1} y_2 u_{y_2} \\ -y_2 g_{y_1} u_{y_1} + \frac{1}{\varepsilon^2 g}(1 + (\varepsilon y_2 g_{y_1})^2) u_{y_2} \end{pmatrix}.$$

For $y = (y_1, y_2) \in \mathcal{O}$, $\tau \in \mathbb{R}$, and $t > \tau$, system (1.1) can be rewritten as

$$\begin{cases} \mathrm{d}u^\varepsilon = (\lambda + i\alpha)\frac{1}{g} \operatorname{div}_y(P_\varepsilon u^\varepsilon)\mathrm{d}t - (\kappa + i\beta)|u^\varepsilon|^2 u^\varepsilon \mathrm{d}t - (\gamma + i\delta)u^\varepsilon \mathrm{d}t + h(u^\varepsilon(t - \tilde{\rho}(t)))\mathrm{d}t + f(t, y_1)\mathrm{d}t \\ \qquad + \sum\limits_{j=1}^m c_j u^\varepsilon \circ \mathrm{d}\omega_j, \\ P_\varepsilon u^\varepsilon \cdot \nu = 0, \quad y \in \partial\mathcal{O}, \\ u_\tau^\varepsilon(s, y) = \psi^\varepsilon(s, y) = \hat{\psi}^\varepsilon(s, x) \circ T_\varepsilon^{-1}(y), \quad s \in [-\rho, 0], \end{cases} \tag{2.1}$$

where ν is the unit outward normal vector to $\partial\mathcal{O}$.

To transfer system (2.1) to an abstract evolutionary system, we first define the inner product $(\cdot, \cdot)_{H_g(\mathcal{O})}$ on complex space $L^2(\mathcal{O})$ by

$$(u, v)_{H_g(\mathcal{O})} = \int_{\mathcal{O}} g u \bar{v} \mathrm{d}y, \quad u, v \in L^2(\mathcal{O}),$$

where $\bar{v}$ is the conjugate function of v. Denote by $H_g(\mathcal{O})$ the space equipped with $(\cdot, \cdot)_{H_g(\mathcal{O})}$. Since $g \in C_b^2(\mathbb{R}, (0, +\infty))$, there exist positive constants k_1 and k_2 such that

$$k_1 \le g(x_1) \le k_2, \quad x_1 \in \mathbb{R}.$$

It is easy to obtain that $H_g(\mathcal{O})$ is a Hilbert space with equivalent norm of $L^2(\mathcal{O})$.

For $0 < \varepsilon \le 1$, let $a_\varepsilon(\cdot, \cdot) : H^1(\mathcal{O}) \times H^1(\mathcal{O}) \to \mathbb{C}$ be a bilinear form, which is given by

$$a_\varepsilon(u, v) = (J^* \nabla_y u, J^* \nabla_y v)_{H_g(\mathcal{O})},$$

where

$$J^*\nabla_y u(y) = \left(u_{y_1} - \frac{g_{y_1}}{g} y_2 u_{y_2}, \frac{1}{\varepsilon g} u_{y_2}\right).$$

Denote

$$\|u\|_{H^1_\varepsilon(O)} = \left(\|u\|^2_{H^1(O)} + \frac{1}{\varepsilon^2}\|u_{y_2}\|^2_{L^2(O)}\right)^{\frac{1}{2}}.$$

There exist positive constants ε_0, l_1, and l_2 such that for all $0 < \varepsilon \le \varepsilon_0$ and $u \in H^1(O)$,

$$l_1\|u\|^2_{H^1_\varepsilon(O)} \le a_\varepsilon(u, u) + \|u\|^2_{L^2(O)} \le l_2\|u\|^2_{H^1_\varepsilon(O)}. \tag{2.2}$$

Define

$$A_\varepsilon u = -\frac{1}{g}\operatorname{div}_y(P_\varepsilon u), \quad u \in D(A_\varepsilon),$$

where $D(A_\varepsilon) = \{u \in H^2(O), P_\varepsilon u \cdot \nu = 0 \quad \text{on } \partial O\}$. Then, we have

$$a_\varepsilon(u, v) = (A_\varepsilon u, v)_{H_g(O)}, \quad \text{for all } u \in D(A_\varepsilon) \text{ and } v \in H^1(O).$$

Using the definition of A_ε, for $\tau \in \mathbb{R}$, $t > \tau$, and $y \in O$, system (2.1) can be rewritten as

$$\begin{cases} du^\varepsilon = -(\lambda + i\alpha)A_\varepsilon u^\varepsilon dt - (\kappa + i\beta)|u^\varepsilon|^2 u^\varepsilon dt - (\gamma + i\delta)u^\varepsilon dt + h(u^\varepsilon(t - \tilde{\rho}(t)))dt + f(t, y_1)dt \\ \qquad + \sum_{j=1}^m c_j u^\varepsilon \circ d\omega_j, \\ P_\varepsilon u^\varepsilon \cdot \nu = 0, \quad y \in \partial O, \\ u^\varepsilon_\tau(s, y) = \psi^\varepsilon(s, y) = \hat{\psi}^\varepsilon(s, x) \circ T_\varepsilon^{-1}(y), \quad s \in [-\rho, 0]. \end{cases} \tag{2.3}$$

Next we define an inner product $(\cdot, \cdot)_{H_g(\mathbb{R})}$ on $L^2(\mathbb{R})$ by

$$(u, v)_{H_g(\mathbb{R})} = \int_{\mathbb{R}} g u \bar{v} dy_1, \quad u, v \in L^2(\mathbb{R}),$$

and denote by $H_g(\mathbb{R})$ the space equipped with $(\cdot, \cdot)_{H_g(\mathbb{R})}$. Let $a_0(\cdot, \cdot) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ be a bilinear form, which is defined by

$$a_0(u, v) = \int_{\mathbb{R}} g \nabla u \cdot \nabla \bar{v} dy_1.$$

Define

$$A_0 u = -\frac{1}{g}(g u_{y_1})_{y_1}, \quad u \in D(A_0),$$

where $D(A_0) = H^2(\mathbb{R})$. Then, we obtain

$$a_0(u, v) = (A_0 u, v)_{H_g(\mathbb{R})}, \quad u \in D(A_0), \text{ and } v \in H^1(\mathbb{R}).$$

Using the definition of A_0, for $\tau \in \mathbb{R}$, $t > \tau$, and $y_1 \in \mathbb{R}$, system (1.2) can be written as

$$\begin{cases} du^0 = -(\lambda + i\alpha)A_0 u^0 dt - (\kappa + i\beta)|u^0|^2 u^0 dt - (\gamma + i\delta)u^0 dt + h(u^0(t - \tilde{\rho}(t)))dt + f(t, y_1)dt \\ \qquad + \sum_{j=1}^m c_j u^0 \circ d\omega_j, \\ \dfrac{\partial u^0}{\partial \nu_0} = 0, \quad y_1 \in \partial O, \\ u^0_\tau(s, y_1) = \psi^0(s, y_1) = \hat{\psi}^0(s, x_1) \circ T_\varepsilon^{-1}(y_1), \quad s \in [-\rho, 0]. \end{cases} \tag{2.4}$$

Now, we consider the probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot+t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}. \tag{2.5}$$

Then, $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system with the filtration

$$\mathcal{F} = \bigvee_{s\le t} \mathcal{F}_s^t, \quad t \in \mathbb{R},$$

where $\mathcal{F}_s^t = \sigma\{\omega(t_2) - \omega(t_1) : s \le t_1 \le t_2 \le t\}$ is the smallest σ-algebra generated by $\omega(t_2) - \omega(t_1)$ for all $s \le t_1 \le t_2 \le t$ (refer [33] for details).

On the other hand, let us consider the following stochastic equation:

$$\mathrm{d}z + \alpha z \mathrm{d}t = \mathrm{d}\omega(t) \tag{2.6}$$

for $\alpha > 0$. As a matter of fact, we can obtain the following lemma.

Lemma 2.1. *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\Omega' \in \mathcal{F}$ of full measure such that*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0 \quad \text{for all } \omega \in \Omega', \tag{2.7}$$

and the random variable given by

$$z^*(\omega) = -\alpha \int_{-\infty}^{0} e^{\alpha s}\omega(s)\mathrm{d}s$$

is well defined. Moreover, for $\omega \in \Omega'$, the mapping

$$(t, \omega) \to z^*(\theta_t\omega) = -\alpha \int_{-\infty}^{0} e^{\alpha s}\theta_t\omega(s)\mathrm{d}s = -\alpha \int_{-\infty}^{0} e^{\alpha s}\omega(t+s)\mathrm{d}s + \omega(t)$$

is a stationary solution of (2.6) *with continuous trajectories. In addition, for $\omega \in \Omega'$*

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{t} = 0, \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^*(\theta_s\omega)\mathrm{d}s = 0, \tag{2.8}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^*(\theta_s\omega)\mathrm{d}s = \mathbb{E}|z^*| < +\infty. \tag{2.9}$$

Let z_j^* be the associated Ornstein-Uhlenbeck process on Ω_j' for system (2.6) with $\alpha = 1$ and replaced ω by ω_j for $j = 1, \ldots, m$. Then, for any $j = 1, \ldots, m$, we obtain a stationary Ornstein-Uhlenbeck process generated by a random variable z_j^* which is defined on a metric dynamical system $(\Omega_j', \mathcal{F}_j, P_j, \{\theta_t\}_{t\in\mathbb{R}})$. Denote

$$\tilde{\Omega} = \Omega_1' \times \ldots \times \Omega_m', \quad P = P_1 \times \ldots \times P_m, \quad \text{and} \quad \mathcal{F} = \bigotimes_{j=1}^{m} \mathcal{F}_j.$$

Then, $(\tilde{\Omega}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system. Let

$$S_{C_j}(t)u = e^{c_j t}u, \quad \text{for } u \in L^2(O),$$

and

$$\mathcal{T}(\omega) := S_{C_1}(z_1^*(\omega)) \circ \ \ldots \ \circ S_{C_m}(z_m^*(\omega)) = e^{\sum_{j=1}^m c_j z_j^*(\omega)} Id_{L^2(O)}, \quad \omega \in \tilde{\Omega}.$$

Then, for every $\omega \in \tilde{\Omega}$, $\mathcal{T}(\omega)$ is a homeomorphism on $L^2(O)$, and its inverse operator is defined by

$$\mathcal{T}^{-1}(\omega) := S_{C_m}(-z_m^*(\omega)) \circ \ \ldots \ \circ S_{C_1}(-z_1^*(\omega)) = e^{-\sum_{j=1}^m c_j z_j^*(\omega)} Id_{L^2(O)}.$$

Indeed, we can extend these operators from $L^2(O)$ to Q. For any $\zeta \in Q$, denote

$$(\tilde{\mathcal{T}}(\omega)\zeta)(s) = \mathcal{T}(\theta_s\omega)\zeta(s), \quad (\tilde{\mathcal{T}}^{-1}(\omega)\zeta)(s) = \mathcal{T}^{-1}(\theta_s\omega)\zeta(s), \quad \text{for } s \in [-\rho, 0].$$

It is easy to see that $\|\mathcal{T}^{-1}(\omega)\|$ and $\|\mathcal{T}(\omega)\|$ are tempered.

Moreover, since $z_j^*, j = 1, \ldots, m$, are independent random variables, by the ergodic theorem, we obtain a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\hat{\Omega} \in \mathcal{F}$ of full measure such that

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \|\mathcal{T}(\theta_\tau\omega)\|^2 \mathrm{d}\tau = \mathbb{E}\|\mathcal{T}\|^2 = \prod_{j=1}^m \mathbb{E}(e^{2c_j z_j^*}) < +\infty$$

and

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \|\mathcal{T}^{-1}(\theta_\tau\omega)\|^2 \mathrm{d}\tau = \mathbb{E}\|\mathcal{T}^{-1}\|^2 = \prod_{j=1}^m \mathbb{E}(e^{-2c_j z_j^*}) < +\infty.$$

Next we investigate θ defined in (2.5) on $\tilde{\Omega} \cap \hat{\Omega}$ instead of Ω. This mapping possesses the same properties as the original one if we choose $\mathcal{F}$ as the trace σ-algebra with respect to $\tilde{\Omega} \cap \hat{\Omega}$. We still denote by $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ the corresponding metric dynamical system throughout this study.

To convert the stochastic system into a deterministic one, let $v^\varepsilon(t) = \mathcal{T}^{-1}(\theta_t\omega)u^\varepsilon(t)$ and $\sigma(\omega) = \sum_{j=1}^m c_j z_j^*(\omega)$, where u^ε is a solution of system (2.3). Then, for $\tau \in \mathbb{R}$, $t > \tau$, and $y \in O$, v^ε satisfies

$$\begin{cases} \dfrac{\mathrm{d}v^\varepsilon}{\mathrm{d}t} = -(\lambda + i\alpha)A_\varepsilon v^\varepsilon - (\kappa + i\beta)|v^\varepsilon|^2 v^\varepsilon\, |\mathcal{T}(\theta_t\omega)|^2 - (\gamma + i\delta)v^\varepsilon + \sigma(\theta_t\omega)v^\varepsilon \\ \qquad\qquad + \mathcal{T}^{-1}(\theta_t\omega)h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^\varepsilon(t - \tilde{\rho}(t))) + \mathcal{T}^{-1}(\theta_t\omega)f(t, y_1), \\ P_\varepsilon v^\varepsilon \cdot \nu = 0, \quad y \in \partial O, \\ v_\tau^\varepsilon(s, y) = \varphi^\varepsilon(s, y), \quad s \in [-\rho, 0], \end{cases} \tag{2.10}$$

where $\varphi^\varepsilon = \tilde{\mathcal{T}}^{-1}(\theta_\tau\omega)\psi^\varepsilon$.

By the Galerkin method, we know that for every $\omega \in \Omega$, $\tau \in \mathbb{R}$, and $\varphi^\varepsilon \in Q$, system (2.10) has a unique solution $v^\varepsilon(\cdot, \tau, \omega, \varphi^\varepsilon) \in C([\tau - \rho, \tau + T], L^2(O)) \cap L^2((\tau, \tau + T), H^1(O))$ for every $T > 0$ and $s \in [-\rho, 0]$ with $v_\tau^\varepsilon(s, \tau, \omega, \varphi^\varepsilon) = \varphi^\varepsilon(s)$. In addition, the solution $v_t^\varepsilon(\cdot, \tau, \omega, \varphi^\varepsilon)$ is $(\mathcal{F}, \mathcal{B}(Q))$-measurable in $\omega \in \Omega$ and continuous with respect to $\varphi^\varepsilon \in Q$ for all $t \geq \tau$. We now define a mapping $\Psi^\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times Q \to Q$ for system (2.3). Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $\psi^\varepsilon \in Q$, let

$$\Psi^\varepsilon(t, \tau, \omega, \psi^\varepsilon)(\cdot) = u_{t+\tau}^\varepsilon(\cdot, \tau, \theta_{-\tau}\omega, \psi^\varepsilon) = \tilde{\mathcal{T}}(\theta_{t+\tau}\omega)v_{t+\tau}^\varepsilon(\cdot, \tau, \theta_{-\tau}\omega, \varphi^\varepsilon),$$

where $\psi^\varepsilon = \tilde{\mathcal{T}}(\theta_\tau\omega)\varphi^\varepsilon$. We can check that Ψ^ε is a continuous cocycle for system (2.3) on Q.

Let $R_\varepsilon : C([-\rho, 0], L^2(O_\varepsilon)) \to C([-\rho, 0], L^2(O))$ be a mapping given by

$$(R_\varepsilon\hat{\psi}^\varepsilon)(y) = \hat{\psi}^\varepsilon(T_\varepsilon^{-1}(y)), \quad \hat{\psi}^\varepsilon \in C([-\rho, 0], L^2(O_\varepsilon)), \quad y \in O.$$

Define a mapping $\hat{\Psi}^\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C([-\rho, 0], L^2(O_\varepsilon)) \to C([-\rho, 0], L^2(O_\varepsilon))$ by

$$\hat{\Psi}^\varepsilon(t, \tau, \omega, \hat{\psi}^\varepsilon) = R_\varepsilon^{-1}\Psi^\varepsilon(t, \tau, \omega, R_\varepsilon\hat{\psi}^\varepsilon)$$

for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $\hat{\psi}^\varepsilon \in C([-\rho, 0], L^2(O))$. Then, $\hat{\Psi}^\varepsilon$ is a continuous cocycle for system (1.1).

Similarly, let $v^0(t) = \mathcal{T}^{-1}(\theta_t\omega)u^0(t)$, system (2.4) can be transformed into the following system on $\mathbb{R}$, for $t > \tau$:

$$\begin{cases} \dfrac{\mathrm{d}v^0}{\mathrm{d}t} = -(\lambda + i\alpha)A_0 v^0 - (\kappa + i\beta)|v^0|^2 v^0\, |\mathcal{T}(\theta_t\omega)|^2 - (\gamma + i\delta)v^0 + \sigma(\theta_t\omega)v^0 \\ \qquad\qquad + \mathcal{T}^{-1}(\theta_t\omega)h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^0(t - \tilde{\rho}(t))) + \mathcal{T}^{-1}(\theta_t\omega)f(t, y_1), \\ \dfrac{\partial v^0}{\partial \nu_0} = 0, \quad y_1 \in \partial O, \\ v_\tau^0(s, y_1) = \varphi^0(s, y_1), \quad s \in [-\rho, 0], \end{cases} \tag{2.11}$$

where $\varphi^0 = \tilde{\mathcal{T}}^{-1}(\theta_\tau\omega)\psi^0$.

By the above argument, we find that system (2.4) generates a continuous cocycle $\Psi^0(t,\tau,\omega,\psi^0)$ in the space $\mathcal{P}$.

Denote by $X_\varepsilon = C([-\rho,0],L^2(O_\varepsilon))$, $X_0=\mathcal{P}$, and $X_1=\mathcal{Q}$. For each $i=\varepsilon$, 0, or 1, let $D_i=\{D_i(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ be a family of nonempty subsets of X_i. Then, D_i is called tempered if for every $\zeta>0$, it holds that

$$\lim_{t\to+\infty} e^{-\zeta t}\|D_i(\tau-t,\theta_{-t}\omega)\|_{X_i}=0,$$

where $\|D_i\|_{X_i}=\sup_{x\in D_i}\|x\|_{X_i}$. Denote by $\mathcal{D}_i$ the collection of families of tempered nonempty subsets of X_i, i.e.,

$$\mathcal{D}_i=\{D_i=\{D_i(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}:D_i \text{ is tempered in } X_i\}.$$

Moreover, we assume that γ satisfies

$$\bar{\xi}\triangleq\gamma-2\mathbb{E}(|\sigma(\omega)|)-\frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\left(\prod_{j=1}^m\mathbb{E}(e^{2c_jz_j^*})+\prod_{j=1}^m\mathbb{E}(e^{-2c_jz_j^*})\right)>0. \tag{2.12}$$

Let

$$\xi(\omega)=\gamma-2|\sigma(\omega)|-\frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}(\|\mathcal{T}(\omega)\|^2+\|\mathcal{T}^{-1}(\omega)\|^2). \tag{2.13}$$

By the ergodic theory and (2.12), we have

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\xi(\theta_l\omega)\mathrm{d}l=\mathbb{E}\xi=\bar{\xi}>0. \tag{2.14}$$

In this study, we need the following conditions to derive uniform estimates of solutions, for every $\tau\in\mathbb{R}$:

$$\int_{-\infty}^{\tau}e^{\frac{1}{2}\bar{\xi}s}\|f(s,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}s<+\infty, \tag{2.15}$$

and for any $\zeta>0$,

$$\lim_{r\to-\infty}e^{\zeta r}\int_{-\infty}^{0}e^{\frac{1}{2}\bar{\xi}s}\|f(s+r,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}s=0. \tag{2.16}$$

In the sequel, we will use the Agmon inequality as follows:

$$\|u\|_{L^\infty(O)}\le c\|u\|^{\frac{1}{2}}_{L^2(O)}\|u\|^{\frac{1}{2}}_{H^2(O)},\quad \forall u\in H^2(O), \text{ and } O\subset\mathbb{R}^2. \tag{2.17}$$

3 Uniform estimates of solutions

In this section, we obtain uniform estimates of solutions to system (2.10) in $H_g(O)$, which is important for establishing the existence and uniqueness of $\mathcal{D}_1$-pullback random attractors for system (2.10).

Lemma 3.1. *Suppose* (2.12) *and* (2.15) *hold. For every* $0<\varepsilon\le\varepsilon_0$, $\tau\in\mathbb{R}$, $\omega\in\Omega$, *and* $D_1=\{D_1(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$, *there exists* $T=T(\tau,\omega,D_1)>\rho$, *independent of* ε, *such that for all* $t\ge T$, *the solution* v^ε *of system* (2.10) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\|v^\varepsilon_\tau(\cdot,\tau-t,\theta_{-\tau}\omega,\varphi^\varepsilon)\|^2_{C([-\rho,0],H_g(O))}\le R_1(\tau,\omega),$$

where $\varphi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$ *and* $R_1(\tau, \omega)$ *is defined by*

$$R_1(\tau, \omega) = c \int_{-\infty}^{0} e^{\int_0^r \xi(\theta_l\omega)\mathrm{d}l} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r, \tag{3.1}$$

where c *is a positive constant independent of* ε.

Proof. Taking the inner product of system (2.10) with v^ε in $H_g(O)$ and keeping real part, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^\varepsilon\|^2_{H_g(O)} &= -\lambda a_\varepsilon(v^\varepsilon, v^\varepsilon) + (-\gamma + \sigma(\theta_t\omega))\|v^\varepsilon\|^2_{H_g(O)} - \kappa \int_O g\, |v^\varepsilon|^4 |\mathcal{T}(\theta_t\omega)|^2 \mathrm{d}y \\
&\quad + \mathrm{Re}(\mathcal{T}^{-1}(\theta_t\omega) f(t, y_1), v^\varepsilon)_{H_g(O)} + \mathrm{Re}(\mathcal{T}^{-1}(\theta_t\omega) h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega) v^\varepsilon(t - \tilde{\rho}(t))), v^\varepsilon)_{H_g(O)} \\
&\leq -\lambda a_\varepsilon(v^\varepsilon, v^\varepsilon) + (-\gamma + \sigma(\theta_t\omega))\|v^\varepsilon\|^2_{H_g(O)} + c\|\mathcal{T}^{-1}(\theta_t\omega)\|^2 \|f(t, y_1)\|^2_{L^2(\mathbb{R})} \\
&\quad + \frac{\gamma}{4}\|v^\varepsilon\|^2_{H_g(O)} + \mathrm{Re}(\mathcal{T}^{-1}(\theta_t\omega) h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega) v^\varepsilon(t - \tilde{\rho}(t))), v^\varepsilon)_{H_g(O)}.
\end{aligned} \tag{3.2}$$

By (3.2), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int_\tau^t \xi(\theta_l\omega)\mathrm{d}l}\|v^\varepsilon\|^2_{H_g(O)}\right) + 2\lambda e^{\int_\tau^t \xi(\theta_l\omega)\mathrm{d}l} a_\varepsilon(v^\varepsilon, v^\varepsilon) + \frac{\gamma}{2} e^{\int_\tau^t \xi(\theta_l\omega)\mathrm{d}l}\|v^\varepsilon\|^2_{H_g(O)} \\
&\leq e^{\int_\tau^t \xi(\theta_l\omega)\mathrm{d}l}(-\gamma + 2\sigma(\theta_t\omega) + \xi(\theta_t\omega))\|v^\varepsilon\|^2_{H_g(O)} + c e^{\int_\tau^t \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}^{-1}(\theta_t\omega)\|^2 \|f(t, y_1)\|^2_{L^2(\mathbb{R})} \\
&\quad + 2 e^{\int_\tau^t \xi(\theta_l\omega)\mathrm{d}l}\, \mathrm{Re}(\mathcal{T}^{-1}(\theta_t\omega) h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega) v^\varepsilon(t - \tilde{\rho}(t))), v^\varepsilon)_{H_g(O)}.
\end{aligned} \tag{3.3}$$

For any $s \geq \tau$, we obtain

$$\begin{aligned}
&e^{\int_\tau^s \xi(\theta_l\omega)\mathrm{d}l}\|v^\varepsilon(s)\|^2_{H_g(O)} + 2\lambda \int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l} a_\varepsilon(v^\varepsilon(r), v^\varepsilon(r))\mathrm{d}r + \frac{\gamma}{2}\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}\|v^\varepsilon(r)\|^2_{H_g(O)}\mathrm{d}r \\
&\leq \|v^\varepsilon(\tau)\|^2_{H_g(O)} + \int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}(-\gamma + 2\sigma(\theta_r\omega) + \xi(\theta_r\omega))\|v^\varepsilon(r)\|^2_{H_g(O)}\mathrm{d}r \\
&\quad + c\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r, y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r \\
&\quad + 2\,\mathrm{Re}\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}(\mathcal{T}^{-1}(\theta_r\omega) h(\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega) v^\varepsilon(r - \tilde{\rho}(r))), v^\varepsilon(r))_{H_g(O)}\mathrm{d}r.
\end{aligned} \tag{3.4}$$

For the last term of (3.4), by the fact of $\xi(\omega) \leq \gamma$, we have

$$\begin{aligned}
&2\,\mathrm{Re}\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}(\mathcal{T}^{-1}(\theta_r\omega) h(\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega) v^\varepsilon(r - \tilde{\rho}(r))), v^\varepsilon(r))_{H_g(O)}\mathrm{d}r \\
&\leq \varepsilon\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}(\mathcal{T}^{-1}(\theta_r\omega))^2\|v^\varepsilon(r)\|^2_{H_g(O)}\mathrm{d}r + \varepsilon^{-1}\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}\|h(\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega) v^\varepsilon(r - \tilde{\rho}(r)))\|^2_{H_g(O)}\mathrm{d}r \\
&\leq \varepsilon\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}(\mathcal{T}^{-1}(\theta_r\omega))^2\|v^\varepsilon(r)\|^2_{H_g(O)}\mathrm{d}r \\
&\quad + \varepsilon^{-1}L^2 e^{\gamma\rho}\int_\tau^s e^{\int_\tau^{r-\tilde{\rho}(r)} \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega) v^\varepsilon(r - \tilde{\rho}(r))\|^2_{H_g(O)}\mathrm{d}r \\
&\leq \varepsilon\int_\tau^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}(\mathcal{T}^{-1}(\theta_r\omega))^2\|v^\varepsilon(r)\|^2_{H_g(O)}\mathrm{d}r + \frac{L^2 e^{\gamma\rho}}{\varepsilon(1 - \rho_0)}\int_{\tau-\rho}^s e^{\int_\tau^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}(\theta_r\omega) v^\varepsilon(r)\|^2_{H_g(O)}\mathrm{d}r.
\end{aligned} \tag{3.5}$$

Let $\varepsilon = \frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}$, then we obtain from (2.13), (3.4), and (3.5) that

$$\begin{aligned}
&\|v^\varepsilon(s)\|^2_{H_g(O)} + 2\lambda\int_\tau^s e^{\int_s^r \xi(\theta_l\omega)\mathrm{d}l} a_\varepsilon(v^\varepsilon(r), v^\varepsilon(r))\mathrm{d}r + \frac{\gamma}{2}\int_\tau^s e^{\int_s^r \xi(\theta_l\omega)\mathrm{d}l}\|v^\varepsilon(r)\|^2_{H_g(O)}\mathrm{d}r \\
&\le \|v^\varepsilon(\tau)\|^2_{H_g(O)} e^{-\int_\tau^s \xi(\theta_l\omega)\mathrm{d}l} + c\int_\tau^s e^{\int_s^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|f(r,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r \\
&\quad + \frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\|\varphi^\varepsilon\|^2_{C([-\rho,0],H_g(O))}\int_{\tau-\rho}^{\tau} e^{\int_s^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}(\theta_r\omega)\|^2\mathrm{d}r.
\end{aligned} \tag{3.6}$$

Considering time $\tau - t$ instead of τ with $t \ge 0$, and then replacing ω by $\theta_{-\tau}\omega$, we obtain

$$\begin{aligned}
&\|v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,\varphi^\varepsilon)\|^2_{H_g(O)} + \frac{\gamma}{2}\int_{\tau-t}^s e^{\int_s^r \xi(\theta_{l-\tau}\omega)\mathrm{d}l}\|v^\varepsilon(r,\tau-t,\theta_{-\tau}\omega,\varphi^\varepsilon)\|^2_{H_g(O)}\mathrm{d}r \\
&\quad + 2\lambda\int_{\tau-t}^s e^{\int_s^r \xi(\theta_{l-\tau}\omega)\mathrm{d}l} a_\varepsilon(v^\varepsilon(r,\tau-t,\theta_{-\tau}\omega,\varphi^\varepsilon), v^\varepsilon(r,\tau-t,\theta_{-\tau}\omega,\varphi^\varepsilon))\mathrm{d}r \\
&\le \left(e^{-\int_{\tau-t}^s \xi(\theta_{l-\tau}\omega)\mathrm{d}l} + \frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\int_{\tau-t-\rho}^{\tau-t} e^{\int_s^r \xi(\theta_{l-\tau}\omega)\mathrm{d}l}\|\mathcal{T}(\theta_{r-\tau}\omega)\|^2\mathrm{d}r\right)\|\varphi^\varepsilon\|^2_{C([-\rho,0],H_g(O))} \\
&\quad + c\int_{\tau-t}^s e^{\int_s^r \xi(\theta_{l-\tau}\omega)\mathrm{d}l}\|\mathcal{T}^{-1}(\theta_{r-\tau}\omega)\|^2\|f(r,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r.
\end{aligned} \tag{3.7}$$

Replacing s by $\tau + \iota$ in (3.7) and taking supremum when $\iota \in [-\rho, 0]$, then we obtain

$$\begin{aligned}
&\|v^\varepsilon_\tau(\cdot,\tau-t,\theta_{-\tau}\omega,\varphi^\varepsilon)\|^2_{C([-\rho,0],H_g(O))} \\
&\le \sup_{-\rho\le\iota\le0} e^{\int_\iota^0 \xi(\theta_l\omega)\mathrm{d}l}\left(\left(e^{-\int_{-t}^0 \xi(\theta_l\omega)\mathrm{d}l} + \frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\int_{-t-\rho}^{-t} e^{\int_0^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}(\theta_r\omega)\|^2\mathrm{d}r\right)\right. \\
&\quad \left. \times\|\varphi^\varepsilon\|^2_{C([-\rho,0],H_g(O))} + c\int_{-\infty}^0 e^{\int_0^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|f(r+\tau,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r\right).
\end{aligned} \tag{3.8}$$

Since $\xi(\omega) \le \gamma$, we have

$$\sup_{-\rho\le\iota\le0} e^{\int_\iota^0 \xi(\theta_l\omega)\mathrm{d}l} \le e^{\gamma\rho}. \tag{3.9}$$

By (2.14), for any $\varepsilon_1 > 0$ and $\omega \in \Omega$, there is a $T_1 = T_1(\varepsilon_1, \omega) > 0$ such that for all $|t| \ge T_1$, we obtain

$$\left|\int_0^t (\xi(\theta_r\omega) - \bar{\xi})\mathrm{d}r\right| \le \varepsilon_1|t|. \tag{3.10}$$

By the temperedness of $\|\mathcal{T}(\theta_t\omega)\|^2$ and $\|\mathcal{T}^{-1}(\theta_t\omega)\|^2$, for any $\varepsilon_2 > 0$ and $\omega \in \Omega$, there exists a $T_2 = T_2(\varepsilon_2, \omega) \ge T_1$, such that for $|t| \ge T_2$,

$$\|\mathcal{T}(\theta_t\omega)\|^2 \le e^{\varepsilon_2|t|} \quad \text{and} \quad \|\mathcal{T}^{-1}(\theta_t\omega)\|^2 \le e^{\varepsilon_2|t|}. \tag{3.11}$$

Taking $\varepsilon_1 = \varepsilon_2 = \frac{1}{4}\bar{\xi}$, for all $t \ge T_2$, we obtain from (3.10) and (3.11) that

$$e^{-\int_{-t}^0 \xi(\theta_l\omega)\mathrm{d}l} = e^{-\int_{-t}^0 (\xi(\theta_l\omega)-\bar{\xi})\mathrm{d}l - \bar{\xi}t} \le e^{-\frac{3}{4}\bar{\xi}t} \tag{3.12}$$

and

$$\int_{-t-\rho}^{-t} e^{\int_0^r \xi(\theta_l\omega)\mathrm{d}l} \|\mathcal{T}(\theta_r\omega)\|^2 \mathrm{d}r \le \int_{-t-\rho}^{-t} e^{\frac{1}{2}\bar{\xi}r} \mathrm{d}r \le \frac{2}{\bar{\xi}} e^{-\frac{1}{2}\bar{\xi}t}. \tag{3.13}$$

Since $\varphi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$ and $D_1 \in \mathcal{D}_1$, we find from (3.12) and (3.13) that

$$\limsup_{t\to+\infty}\left(e^{-\int_{-t}^0 \xi(\theta_l\omega)\mathrm{d}l} + \frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}} \int_{-t-\rho}^{-t} e^{\int_0^r \xi(\theta_l\omega)\mathrm{d}l} \|\mathcal{T}(\theta_r\omega)\|^2 \mathrm{d}r \right) \|\varphi^\varepsilon\|^2_{C([-\rho,0],H_g(O))} = 0. \tag{3.14}$$

By (3.8), (3.14), and (2.15), we can obtain the result. □

From Lemma 3.1 we can obtain the following inequality.

Lemma 3.2. *Suppose* (2.12) *and* (2.15) *hold. For all* $0 < \varepsilon \le \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, *there exists* $T = T(\tau, \omega, D_1) \ge 2\rho + 1$, *independent of* ε, *such that for all* $t \ge T$, *the solution* v^ε *of system* (2.10) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\int_{\tau-2\rho-1}^{\tau} (a_\varepsilon(v^\varepsilon(r, \tau-t, \theta_{-\tau}\omega, \varphi^\varepsilon), v^\varepsilon(r, \tau-t, \theta_{-\tau}\omega, \varphi^\varepsilon)) + \|v^\varepsilon(r, \tau-t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{H_g(O)})\mathrm{d}r \le cR_1(\tau, \omega),$$

where $\varphi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, c *is a positive constant independent of* ε, *and* $R_1(\tau,\omega)$ *is given by* (3.1).

Next we obtain uniform estimates of solutions to system (2.10) in $C([-\rho, 0], H^1_\varepsilon(O))$.

Lemma 3.3. *Suppose* (2.12) *and* (2.15) *hold. For all* $0 < \varepsilon \le \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, *there exists* $T = T(\tau, \omega, D_1) \ge 2\rho + 1$, *independent of* ε, *such that for all* $t \ge T$, *the solution* v^ε *of system* (2.10) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\|v^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{C([-\rho,0],H^1_\varepsilon(O))} \le R_2(\tau, \omega), \tag{3.15}$$

where $\varphi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, $R_2(\tau,\omega)$ *is determined by*

$$R_2(\tau,\omega) = r_1(\omega)R_1(\tau,\omega) + ce^{\bar{\xi}(\rho+1)} \int_{-\infty}^{0} e^{\bar{\xi}r} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r+\tau, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r, \tag{3.16}$$

where $r_1(\omega)$ *is tempered,* c *is a positive constant independent of* ε, *and* $R_1(\tau,\omega)$ *is given by* (3.1).

Proof. Taking the inner product of system (2.10) with $A_\varepsilon v^\varepsilon$ in $H_g(O)$ and keeping real part, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a_\varepsilon(v^\varepsilon, v^\varepsilon) + \lambda\|A_\varepsilon v^\varepsilon\|^2_{H_g(O)} + \gamma a_\varepsilon(v^\varepsilon, v^\varepsilon) - \sigma(\theta_t\omega) a_\varepsilon(v^\varepsilon, v^\varepsilon) \\ &= -\operatorname{Re}((\kappa + i\beta)|v^\varepsilon|^2 v^\varepsilon\, |\mathcal{T}(\theta_t\omega)|^2, A_\varepsilon v^\varepsilon)_{H_g(O)} + \operatorname{Re}(\mathcal{T}^{-1}(\theta_t\omega) f(t, y_1), A_\varepsilon v^\varepsilon)_{H_g(O)} \\ &\quad + \operatorname{Re}(\mathcal{T}^{-1}(\theta_t\omega) h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega) v^\varepsilon(t - \tilde{\rho}(t))), A_\varepsilon v^\varepsilon)_{H_g(O)}. \end{aligned} \tag{3.17}$$

According to the definition of $a_\varepsilon(u, v)$, we obtain

$$\begin{aligned} &(|v^\varepsilon|^2 v^\varepsilon\, |\mathcal{T}(\theta_t\omega)|^2, A_\varepsilon v^\varepsilon)_{H_g(O)} \\ &= \|\mathcal{T}(\theta_t\omega)\|^2 (J^*\nabla_y(|v^\varepsilon|^2 v^\varepsilon), J^*\nabla_y v^\varepsilon)_{H_g(O)} \\ &= \|\mathcal{T}(\theta_t\omega)\|^2 \int_O g\Bigg((|v^\varepsilon|^2 v^\varepsilon)_{y_1} \bar{v}^\varepsilon_{y_1} - \frac{g_{y_1}}{g} y_2 (|v^\varepsilon|^2 v^\varepsilon)_{y_1} \bar{v}^\varepsilon_{y_2} - \frac{g_{y_1}}{g} y_2 (|v^\varepsilon|^2 v^\varepsilon)_{y_2} \bar{v}^\varepsilon_{y_1} \\ &\quad + \left(\frac{g_{y_1}}{g} y_2\right)^2 (|v^\varepsilon|^2 v^\varepsilon)_{y_2} \bar{v}^\varepsilon_{y_2} + \frac{1}{\varepsilon^2 g^2} (|v^\varepsilon|^2 v^\varepsilon)_{y_2} \bar{v}^\varepsilon_{y_2} \Bigg) \mathrm{d}y. \end{aligned} \tag{3.18}$$

Note that

$$
\begin{aligned}
&-\mathrm{Re}(\kappa + i\beta)(|v^\varepsilon|^2 v^\varepsilon)_{y_j}\bar{v}^\varepsilon_{y_j} \\
&= -\kappa\,|v^\varepsilon|^2\left|\frac{\partial v^\varepsilon}{\partial y_j}\right|^2 - \mathrm{Re}(\kappa + i\beta)v^\varepsilon\frac{\partial|v^\varepsilon|^2}{\partial y_j}\frac{\partial\bar{v}^\varepsilon}{\partial y_j} \\
&= -\kappa\,|v^\varepsilon|^2\left|\frac{\partial v^\varepsilon}{\partial y_j}\right|^2 - \kappa\,|v^\varepsilon|^2\left|\frac{\partial v^\varepsilon}{\partial y_j}\right|^2 - \kappa\,\mathrm{Re}\left(v^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial y_j}\right)^2 + \beta\,\mathrm{Im}\left(v^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial y_j}\right)^2 \\
&= -\kappa|v^\varepsilon|\left|\frac{\partial v^\varepsilon}{\partial y_j}\right|^2 - \frac{\kappa}{2}\left(\frac{\partial|v^\varepsilon|^2}{\partial y_j}\right)^2 + \beta\,\mathrm{Im}\left(v^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial y_j}\right)^2 \\
&\le -(\kappa - |\beta|)|v^\varepsilon|^2\left|\frac{\partial v^\varepsilon}{\partial y_j}\right|^2 - \frac{\kappa}{2}\left(\frac{\partial|v^\varepsilon|^2}{\partial y_j}\right)^2, \quad \forall j = 1,2
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
&\mathrm{Re}(\kappa + i\beta)((|v^\varepsilon|^2 v^\varepsilon)_{y_1}\bar{v}^\varepsilon_{y_2} + (|v^\varepsilon|^2 v^\varepsilon)_{y_2}\bar{v}^\varepsilon_{y_1}) \\
&= \mathrm{Re}(\kappa + i\beta)\left(\frac{\partial|v^\varepsilon|^2}{\partial y_1}v^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial y_2} + \frac{\partial v^\varepsilon}{\partial y_1}|v^\varepsilon|^2\frac{\partial\bar{v}^\varepsilon}{\partial y_2} + \frac{\partial|v^\varepsilon|^2}{\partial y_2}v^\varepsilon\frac{\partial\bar{v}^\varepsilon}{\partial y_1} + \frac{\partial v^\varepsilon}{\partial y_2}|v^\varepsilon|^2\frac{\partial\bar{v}^\varepsilon}{\partial y_1}\right) \\
&\le \sqrt{\kappa^2 + \beta^2}\left|\frac{\partial|v^\varepsilon|^2}{\partial y_1}\right||v^\varepsilon|\left|\frac{\partial\bar{v}^\varepsilon}{\partial y_2}\right| + 2\kappa\,|v^\varepsilon|^2\left|\frac{\partial v^\varepsilon}{\partial y_1}\right|\left|\frac{\partial\bar{v}^\varepsilon}{\partial y_2}\right| + \sqrt{\kappa^2 + \beta^2}\left|\frac{\partial|v^\varepsilon|^2}{\partial y_2}\right||v^\varepsilon|\left|\frac{\partial\bar{v}^\varepsilon}{\partial y_1}\right|.
\end{aligned}
\tag{3.20}
$$

By (3.18)–(3.20), $\kappa > |\beta|$ and Young's inequality, there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \le \varepsilon_1$,

$$
-\mathrm{Re}(\kappa + i\beta)(|v^\varepsilon|^2 v^\varepsilon\,|\mathcal{T}(\theta_t\omega)|^2, A_\varepsilon v^\varepsilon)_{H_g(O)} \le 0. \tag{3.21}
$$

By Young's inequality again, we have

$$
\mathrm{Re}(\mathcal{T}^{-1}(\theta_t\omega)f(t,y_1), A_\varepsilon v^\varepsilon)_{H_g(O)} \le \frac{\lambda}{4}\|A_\varepsilon v^\varepsilon\|^2_{H_g(O)} + c\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|f(t,y_1)\|^2_{L^2(\mathbb{R})}. \tag{3.22}
$$

For the last term of (3.17), we obtain

$$
\begin{aligned}
&\mathrm{Re}(\mathcal{T}^{-1}(\theta_t\omega)h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^\varepsilon(t - \tilde{\rho}(t))), A_\varepsilon v^\varepsilon)_{H_g(O)} \\
&\le \frac{\lambda}{4}\|A_\varepsilon v^\varepsilon\|^2_{H_g(O)} + cL^2\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)\|^2\|v^\varepsilon(t - \tilde{\rho}(t))\|^2_{H_g(O)}.
\end{aligned}
\tag{3.23}
$$

Applying (3.17) and (3.21)–(3.23), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}a_\varepsilon(v^\varepsilon, v^\varepsilon) + \lambda\|A_\varepsilon v^\varepsilon\|^2_{H_g(O)} + (2\gamma - 2\sigma(\theta_t\omega))a_\varepsilon(v^\varepsilon, v^\varepsilon) \\
&\le c\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|f(t,y_1)\|^2_{L^2(\mathbb{R})} + cL^2\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)\|^2\|v^\varepsilon(t - \tilde{\rho}(t))\|^2_{H_g(O)},
\end{aligned}
\tag{3.24}
$$

which implies that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}a_\varepsilon(v^\varepsilon, v^\varepsilon) \le{}& (-2\gamma + 2\sigma(\theta_t\omega))a_\varepsilon(v^\varepsilon, v^\varepsilon) + c\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|f(t,y_1)\|^2_{L^2(\mathbb{R})} \\
&+ cL^2\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)\|^2\|v^\varepsilon(t - \tilde{\rho}(t))\|^2_{H_g(O)}.
\end{aligned}
\tag{3.25}
$$

Given $t \geq 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $s \in (\tau + \iota - 1, \tau + \iota)$, where $\iota \in [-\rho, 0]$, integrating (3.25) on $(s, \tau + \iota)$, it can be derived that

$$\begin{aligned}
&a_\varepsilon(v^\varepsilon(\tau+\iota, \tau - t, \omega, \varphi^\varepsilon), v^\varepsilon(\tau+\iota, \tau - t, \omega, \varphi^\varepsilon)) \\
&\quad \leq a_\varepsilon(v^\varepsilon(s, \tau - t, \omega, \varphi^\varepsilon), v^\varepsilon(s, \tau - t, \omega, \varphi^\varepsilon)) \\
&\qquad + \int_s^{\tau+\iota} (-2\gamma + 2\sigma(\theta_r\omega)) a_\varepsilon(v^\varepsilon(r, \tau - t, \omega, \varphi^\varepsilon), v^\varepsilon(r, \tau - t, \omega, \varphi^\varepsilon)) \mathrm{d}r \\
&\qquad + c \int_s^{\tau+\iota} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r \\
&\quad + cL^2 \int_s^{\tau+\iota} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega)\|^2 \|v^\varepsilon(r - \tilde{\rho}(r), \tau - t, \omega, \varphi^\varepsilon)\|^2_{H_g(O)} \mathrm{d}r.
\end{aligned} \tag{3.26}$$

Integrating (3.26) with respect to s over $(\tau + \iota - 1, \tau + \iota)$, and replacing ω by $\theta_{-\tau}\omega$, we have

$$\begin{aligned}
&a_\varepsilon(v^\varepsilon(\tau+\iota, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon), v^\varepsilon(\tau+\iota, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)) \\
&\leq \int_{\tau+\iota-1}^{\tau+\iota} (1 - 2\gamma + 2|\sigma(\theta_{r-\tau}\omega)|) a_\varepsilon(v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon), v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)) \mathrm{d}r \\
&\quad + c \int_{\tau+\iota-1}^{\tau+\iota} \|\mathcal{T}^{-1}(\theta_{r-\tau}\omega)\|^2 \|f(r, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r \\
&\quad + cL^2 \int_{\tau+\iota-1}^{\tau+\iota} \|\mathcal{T}^{-1}(\theta_{r-\tau}\omega)\|^2 \|\mathcal{T}(\theta_{r-\tilde{\rho}(r)-\tau}\omega)\|^2 \|v^\varepsilon(r - \tilde{\rho}(r), \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{H_g(O)} \mathrm{d}r \\
&\leq \left(1 - 2\gamma + 2 \max_{\nu \in [-\rho-1, 0]} |\sigma(\theta_\nu\omega)|\right) \int_{\tau-\rho-1}^{\tau} a_\varepsilon(v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon), v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)) \mathrm{d}r \\
&\quad + c \int_{-\rho-1}^{0} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r \\
&\quad + \frac{cL^2}{1 - \rho_0} \max_{\nu \in [-\rho-1, 0]} \|\mathcal{T}^{-1}(\theta_\nu\omega)\|^2 \max_{\nu \in [-2\rho-1, 0]} \|\mathcal{T}(\theta_\nu\omega)\|^2 \int_{\tau-2\rho-1}^{\tau} \|v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{H_g(O)} \mathrm{d}r,
\end{aligned}$$

which along with Lemma 3.2 implies that there is a $T = T(\tau, \omega, D_1) \geq 2\rho + 1$ such that for all $t \geq T$ and $\omega \in \Omega$, we obtain

$$\begin{aligned}
&a_\varepsilon(v^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon), v^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)) \\
&\quad \leq c(1 - 2\gamma + 2 \max_{\nu \in [-\rho-1, 0]} |\sigma(\theta_\nu\omega)|) R_1(\tau, \omega) + \frac{cL^2}{1 - \rho_0} \max_{\nu \in [-\rho-1, 0]} \|\mathcal{T}^{-1}(\theta_\nu\omega)\|^2 \max_{\nu \in [-2\rho-1, 0]} \|\mathcal{T}(\theta_\nu\omega)\|^2 R_1(\tau, \omega) \\
&\qquad + c \int_{-\rho-1}^{0} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r.
\end{aligned} \tag{3.27}$$

For the last term of (3.27), we have

$$c \int_{-\rho-1}^{0} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r \leq c e^{\bar{\xi}(\rho+1)} \int_{-\infty}^{0} e^{\bar{\xi} r} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r, \tag{3.28}$$

which together with (3.27) and Lemma 3.1 implies the desired result. □

The next step involves establishing the uniform smallness of solutions for system (2.10) with respect to large space and time variables, which is crucial in proving the asymptotic compactness of solutions on the unbounded domain O.

Lemma 3.4. *Suppose* (2.12) *and* (2.15) *hold. For all* $0 < \varepsilon \le \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ *and* $\eta > 0$, *there exist* $T = T(\tau,\omega,D_1,\eta) \ge 2\rho + 1$ *and* $N = N(\tau,\omega,\eta) \ge 1$, *independent of* ε, *such that for all* $t \ge T$ *and* $n \ge N$, *the solution* v^ε *of system* (2.10) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\sup_{-\rho\le s\le 0}\int_{|y_1|\ge N}\int_0^1 |v^\varepsilon(s+\tau,\tau-t,\theta_{-\tau}\omega,\varphi^\varepsilon)(y_1,y_2)|^2 \mathrm{d}y_1\,\mathrm{d}y_2 \le \eta, \tag{3.29}$$

where $\varphi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$.

Proof. We choose a smooth function ϑ such that $0 \le \vartheta \le 1$ for all $\mu \in \mathbb{R}^+$, and

$$\vartheta(\mu) = \begin{cases} 0, & 0 \le \mu \le 1; \\ 1, & \mu \ge 2. \end{cases} \tag{3.30}$$

Let $n \in \mathbb{Z}^+$, multiplying system (2.10) with $\vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon$ in $H_g(O)$ and keeping real part, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_O g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^\varepsilon|^2\mathrm{d}y + \gamma\int_O g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^\varepsilon|^2\mathrm{d}y - \sigma(\theta_t\omega)\int_O g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^\varepsilon|^2\mathrm{d}y \\
&\quad = -\operatorname{Re}(\lambda + i\alpha)\left(A_\varepsilon v^\varepsilon, \vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon\right)_{H_g(O)} - \operatorname{Re}(\kappa + i\beta)\left(|v^\varepsilon|^2 v^\varepsilon\,|\mathcal{T}(\theta_t\omega)|^2, \vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon\right)_{H_g(O)} \\
&\qquad + \operatorname{Re}\left(\mathcal{T}^{-1}(\theta_t\omega)f(t,y_1), \vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon\right)_{H_g(O)} + \operatorname{Re}\left(\mathcal{T}^{-1}(\theta_t\omega)h(\mathcal{T}(\theta_{t-\tilde\rho(t)}\omega)v^\varepsilon(t-\tilde\rho(t))), \vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon\right)_{H_g(O)}.
\end{aligned} \tag{3.31}$$

Note that

$$\begin{aligned}
&-\operatorname{Re}(\lambda + i\alpha)\left(A_\varepsilon v^\varepsilon, \vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon\right)_{H_g(O)} \\
&\quad = -\operatorname{Re}(\lambda + i\alpha)\int_O g\left(J^*\nabla_y\vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon\right)\cdot(J^*\nabla_y\bar v^\varepsilon)\mathrm{d}y \\
&\quad = -\operatorname{Re}(\lambda + i\alpha)\int_O g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|J^*\nabla_y v^\varepsilon|^2\mathrm{d}y - \operatorname{Re}(\lambda + i\alpha)\int_O g v^\varepsilon\left(J^*\nabla_y\vartheta\left(\frac{|y_1|^2}{n^2}\right)\right)\cdot(J^*\nabla_y\bar v^\varepsilon)\mathrm{d}y \\
&\quad \le \sqrt{\lambda^2+\alpha^2}\int_O g|v^\varepsilon|\left|J^*\nabla_y\vartheta\left(\frac{|y_1|^2}{n^2}\right)\right||J^*\nabla_y\bar v^\varepsilon|\mathrm{d}y \\
&\quad \le \sqrt{\lambda^2+\alpha^2}\int_{n\le|y_1|\le\sqrt{2}n}\int_0^1 g|v^\varepsilon|\left|\frac{2|y_1|}{n^2}\varphi'\left(\frac{|y_1|^2}{n^2}\right)\right||J^*\nabla_y\bar v^\varepsilon|\mathrm{d}y \\
&\quad \le \frac{c}{n}\int_O g\left(|v^\varepsilon|^2 + \left|\bar v^\varepsilon_{y_1} - \frac{g_{y_1}}{g}y_2\bar v^\varepsilon_{y_2}\right|^2\right)\mathrm{d}y \\
&\quad \le \frac{c}{n}(\|v^\varepsilon\|^2_{H_g(O)} + a_\varepsilon(v^\varepsilon,v^\varepsilon)) \\
&\quad \le \frac{c}{n}\|v^\varepsilon\|^2_{H^1_\varepsilon(O)}.
\end{aligned} \tag{3.32}$$

Observe that

$$\operatorname{Re}\left(\mathcal{T}^{-1}(\theta_t\omega)f(t,y_1), \vartheta\left(\frac{|y_1|^2}{n^2}\right)v^\varepsilon\right)_{H_g(O)}$$

$$\begin{aligned}
&= \mathrm{Re} \int_{O} g\mathcal{T}^{-1}(\theta_t\omega) f(t, y_1) \vartheta\left(\frac{|y_1|^2}{n^2}\right) \bar{v}^{\varepsilon} \mathrm{d}y \\
&\leq \frac{\gamma}{2} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y + c\int_{O} g\, |\mathcal{T}^{-1}(\theta_t\omega)|^2 \vartheta\left(\frac{|y_1|^2}{n^2}\right) |f(t, y_1)|^2 \mathrm{d}y.
\end{aligned} \tag{3.33}$$

For the last term of (3.31), we have

$$\begin{aligned}
&\mathrm{Re}\left(\mathcal{T}^{-1}(\theta_t\omega) h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega) v^{\varepsilon}(t-\tilde{\rho}(t))), \vartheta\left(\frac{|y_1|^2}{n^2}\right) v^{\varepsilon}\right)_{H_g(O)} \\
&= \mathrm{Re} \int_{O} g\mathcal{T}^{-1}(\theta_t\omega) h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega) v^{\varepsilon}(t-\tilde{\rho}(t))) \vartheta\left(\frac{|y_1|^2}{n^2}\right) \bar{v}^{\varepsilon} \mathrm{d}y \\
&\leq \frac{\varepsilon}{2} \int_{O} g\, |\mathcal{T}^{-1}(\theta_t\omega)|^2 \vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y + \frac{L^2}{2\varepsilon} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)|^2 |v^{\varepsilon}(t-\tilde{\rho}(t))|^2 \mathrm{d}y.
\end{aligned} \tag{3.34}$$

By using (3.31)–(3.34), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y + (\gamma - \sigma(\theta_t\omega)) \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y + \kappa \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^4 \mathcal{T}(\theta_t\omega)|^2 \mathrm{d}y \\
&\leq \frac{c}{n} \|v^{\varepsilon}\|^2_{H^1_{\varepsilon}(O)} + \frac{\gamma}{2} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y + c\int_{O} g|\mathcal{T}^{-1}(\theta_t\omega)|^2 \vartheta\left(\frac{|y_1|^2}{n^2}\right) |f(t, y_1)|^2 \mathrm{d}y \\
&\quad + \frac{\varepsilon}{2} \int_{O} g\, |\mathcal{T}^{-1}(\theta_t\omega)|^2 \vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y + \frac{L^2}{2\varepsilon} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)|^2 |v^{\varepsilon}(t-\tilde{\rho}(t))|^2 \mathrm{d}y,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int_{\tau}^{t} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y\right) \\
&\leq \frac{c}{n} \|v^{\varepsilon}\|^2_{H^1_{\varepsilon}(O)} e^{\int_{\tau}^{t} \xi(\theta_l\omega)\mathrm{d}l} + (-\gamma + \xi(\theta_t\omega) + 2\sigma(\theta_t\omega)) e^{\int_{\tau}^{t} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y \\
&\quad + ce^{\int_{\tau}^{t} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\, |\mathcal{T}^{-1}(\theta_t\omega)|^2 \vartheta\left(\frac{|y_1|^2}{n^2}\right) |f(t, y_1)|^2 \mathrm{d}y + \varepsilon e^{\int_{\tau}^{t} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\, |\mathcal{T}^{-1}(\theta_t\omega)|^2 \vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}|^2 \mathrm{d}y \\
&\quad + \frac{L^2}{\varepsilon} e^{\int_{\tau}^{t} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)|^2 |v^{\varepsilon}(t-\tilde{\rho}(t))|^2 \mathrm{d}y.
\end{aligned}$$

For any $s \geq \tau$, we obtain

$$\begin{aligned}
&e^{\int_{\tau}^{s} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}(s)|^2 \mathrm{d}y \\
&\leq \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}(\tau)|^2 \mathrm{d}y + \frac{c}{n} \|v^{\varepsilon}\|^2_{H^1_{\varepsilon}(O)} \int_{\tau}^{s} e^{\int_{\tau}^{r} \xi(\theta_l\omega)\mathrm{d}l} \mathrm{d}r \\
&\quad + \int_{\tau}^{s} (-\gamma + \xi(\theta_r\omega) + 2\sigma(\theta_r\omega)) e^{\int_{\tau}^{r} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\vartheta\left(\frac{|y_1|^2}{n^2}\right) |v^{\varepsilon}(r)|^2 \mathrm{d}y\mathrm{d}r \\
&\quad + c\int_{\tau}^{s} e^{\int_{\tau}^{r} \xi(\theta_l\omega)\mathrm{d}l} \int_{O} g\, |\mathcal{T}^{-1}(\theta_r\omega)|^2 \vartheta\left(\frac{|y_1|^2}{n^2}\right) |f(r, y_1)|^2 \mathrm{d}y\mathrm{d}r
\end{aligned} \tag{3.35}$$

$$
\begin{aligned}
&+\varepsilon\int_{\tau}^{s} e^{\int_{\tau}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\,|\mathcal{T}^{-1}(\theta_r\omega)|^2\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^{\varepsilon}(r)|^2\mathrm{d}y\mathrm{d}r\\
&+\frac{L^2}{\varepsilon}\int_{\tau}^{s} e^{\int_{\tau}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega)|^2|v^{\varepsilon}(r-\tilde{\rho}(r))|^2\mathrm{d}y\mathrm{d}r.
\end{aligned}
$$

For the last term of (3.35), by the fact of $\xi(\omega)\le\gamma$, we obtain

$$
\begin{aligned}
&\frac{L^2}{\varepsilon}\int_{\tau}^{s} e^{\int_{\tau}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega)|^2|v^{\varepsilon}(r-\tilde{\rho}(r))|^2\mathrm{d}y\mathrm{d}r\\
&\le\frac{L^2e^{\gamma\rho}}{\varepsilon(1-\rho_0)}\int_{\tau-\rho}^{s} e^{\int_{\tau}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|\mathcal{T}(\theta_r\omega)|^2|v^{\varepsilon}(r)|^2\mathrm{d}y\mathrm{d}r.
\end{aligned}
\tag{3.36}
$$

Let $\varepsilon=\frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}$, then we obtain from (2.13), (3.35) and (3.36) that

$$
\begin{aligned}
\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^{\varepsilon}(s)|^2\mathrm{d}y &\le e^{-\int_{\tau}^{s}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^{\varepsilon}(\tau)|^2\mathrm{d}y+\frac{c}{n}\|v^{\varepsilon}\|^2_{H^1_\varepsilon(\mathcal{O})}\int_{\tau}^{s} e^{\int_{s}^{r}\xi(\theta_l\omega)\mathrm{d}l}\mathrm{d}r\\
&+c\int_{\tau}^{s} e^{\int_{s}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\,|\mathcal{T}^{-1}(\theta_r\omega)|^2\vartheta\left(\frac{|y_1|^2}{n^2}\right)|f(r,y_1)|^2\mathrm{d}y\mathrm{d}r\\
&+\frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\|\varphi^{\varepsilon}\|^2_{C([-\rho,0],H_g(\mathcal{O}))}\int_{\tau-\rho}^{\tau} e^{\int_{s}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|\mathcal{T}(\theta_r\omega)|^2\mathrm{d}y\mathrm{d}r.
\end{aligned}
$$

Considering time $\tau-t$ instead of τ with $t\ge0$, and then replacing ω by $\theta_{-\tau}\omega$, we obtain

$$
\begin{aligned}
&\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^{\varepsilon}(s,\tau-t,\theta_{-\tau}\omega,\varphi^{\varepsilon})|^2\mathrm{d}y\\
&\le e^{-\int_{\tau-t}^{s}\xi(\theta_{l-\tau}\omega)\mathrm{d}l}\|\varphi^{\varepsilon}\|^2_{C([-\rho,0],H_g(\mathcal{O}))}+\frac{c}{n}\|v^{\varepsilon}\|^2_{H^1_\varepsilon(\mathcal{O})}\int_{\tau-t}^{s} e^{\int_{s}^{r}\xi(\theta_{l-\tau}\omega)\mathrm{d}l}\mathrm{d}r\\
&\quad+c\int_{\tau-t}^{s} e^{\int_{s}^{r}\xi(\theta_{l-\tau}\omega)\mathrm{d}l}\int_{\mathcal{O}} g\,|\mathcal{T}^{-1}(\theta_{r-\tau}\omega)|^2\vartheta\left(\frac{|y_1|^2}{n^2}\right)|f(r,y_1)|^2\mathrm{d}y\mathrm{d}r\\
&\quad+\frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\int_{\tau-t-\rho}^{\tau-t} e^{\int_{s}^{r}\xi(\theta_{l-\tau}\omega)\mathrm{d}l}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|\mathcal{T}(\theta_{r-\tau}\omega)|^2\mathrm{d}y\mathrm{d}r\|\varphi^{\varepsilon}\|^2_{C([-\rho,0],H_g(\mathcal{O}))}.
\end{aligned}
\tag{3.37}
$$

Replacing s by $\tau+\iota$ in (3.37) and taking supremum when $\iota\in[-\rho,0]$, we obtain

$$
\begin{aligned}
&\sup_{-\rho\le\iota\le0}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|v^{\varepsilon}(\tau+\iota,\tau-t,\theta_{-\tau}\omega,\varphi^{\varepsilon})|^2\mathrm{d}y\\
&\le e^{\gamma\rho}e^{-\int_{-t}^{0}\xi(\theta_l\omega)\mathrm{d}l}\|\varphi^{\varepsilon}\|^2_{C([-\rho,0],H_g(\mathcal{O}))}+e^{\gamma\rho}\frac{c}{n}\|v^{\varepsilon}\|^2_{H^1_\varepsilon(\mathcal{O})}\int_{-t}^{0} e^{\int_{0}^{r}\xi(\theta_l\omega)\mathrm{d}l}\mathrm{d}r\\
&\quad+ce^{\gamma\rho}\int_{-\infty}^{0} e^{\int_{0}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\,|\mathcal{T}^{-1}(\theta_r\omega)|^2\vartheta\left(\frac{|y_1|^2}{n^2}\right)|f(r+\tau,y_1)|^2\mathrm{d}y\mathrm{d}r\\
&\quad+e^{\gamma\rho}\frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\int_{-t-\rho}^{-t} e^{\int_{0}^{r}\xi(\theta_l\omega)\mathrm{d}l}\int_{\mathcal{O}} g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|\mathcal{T}(\theta_r\omega)|^2\mathrm{d}y\mathrm{d}r\|\varphi^{\varepsilon}\|^2_{C([-\rho,0],H_g(\mathcal{O}))}.
\end{aligned}
\tag{3.38}
$$

By (3.13), $\varphi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, and $D_1 \in \mathcal{D}_1$, we find that there exists $T_1 = T_1(\tau, \omega, D_1, \eta) \geq 2\rho + 1$ such that for all $t \geq T_1$,

$$\begin{aligned}&e^{\gamma\rho}e^{-\int_{-t}^{0}\xi(\theta_l\omega)\mathrm{d}l}\|\varphi^\varepsilon\|^2_{C([-\rho,0],H_g(O))}\\&\quad + e^{\gamma\rho}\frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\int_{-t-\rho}^{-t} e^{\int_0^r\xi(\theta_l\omega)\mathrm{d}l}\int_O g\vartheta\left(\frac{|y_1|^2}{n^2}\right)|\mathcal{T}(\theta_r\omega)|^2\mathrm{d}y\mathrm{d}r\|\psi^\varepsilon\|^2_{C([-\rho,0],H_g(O))}\\&\leq \frac{\eta}{3}.\end{aligned} \tag{3.39}$$

By Lemma 3.3 and (3.13), there exist $T_2 = T_2(\tau, \omega, D_1) \geq T_1$ and $N_1 = N_1(\tau, \omega, \eta) \geq 1$ such that for all $t \geq T_2$ and $n \geq N_1$,

$$e^{\gamma\rho}\frac{c}{n}\|v^\varepsilon\|^2_{H^1_\varepsilon(O)}\int_{-t}^{0} e^{\int_0^r\xi(\theta_l\omega)\mathrm{d}l}\mathrm{d}r \leq \frac{\eta}{3}. \tag{3.40}$$

By (2.15), we can obtain that there exists $N_2 = N_2(\tau, \omega, \eta) \geq N_1$ such that for all $n \geq N_2$,

$$\begin{aligned}&ce^{\gamma\rho}\int_{-\infty}^{0} e^{\int_0^r\xi(\theta_l\omega)\mathrm{d}l}\int_O g\,|\mathcal{T}^{-1}(\theta_r\omega)|^2\vartheta\left(\frac{|y_1|^2}{n^2}\right)|f(r+\tau, y_1)|^2\mathrm{d}y\mathrm{d}r\\&= ce^{\gamma\rho}\int_{-\infty}^{0} e^{\int_0^r\xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\int_{|y_1|>n}\vartheta\left(\frac{|y_1|^2}{n^2}\right)|f(r+\tau, y_1)|^2\mathrm{d}y\mathrm{d}r\\&\leq \frac{\eta}{3}.\end{aligned} \tag{3.41}$$

By using (3.38)–(3.41), we can obtain the result. □

The following estimate is needed to obtain the equicontinuity of the solution of system (2.10).

Lemma 3.5. *Suppose* (2.12) *and* (2.15) *hold. For all* $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, *there exists* $T = T(\tau, \omega, D_1) \geq 2\rho + 1$, *independent of* ε, *such that for all* $t \geq T$ *and* $-\rho \leq \iota_1 < \iota_2 \leq 0$, *the solution* v^ε *of system* (2.10) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\int_{\tau+\iota_1}^{\tau+\iota_2}\|A_\varepsilon v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{H_g(O)}\mathrm{d}r \leq R_2(\tau, \omega), \tag{3.42}$$

where $\varphi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, c *is a positive constant independent of* ε *and* $R_2(\tau, \omega)$ *is determined by* (3.16).

Proof. By (3.24), we have

$$\begin{aligned}&\lambda\int_{\tau+\iota_1}^{\tau+\iota_2}\|A_\varepsilon v^\varepsilon(r, \tau - t, \omega, \varphi^\varepsilon)\|^2_{H_g(O)}\mathrm{d}r\\&\leq a_\varepsilon(v^\varepsilon(\tau+\iota_1, \tau - t, \omega, \psi^\varepsilon), v^\varepsilon(\tau+\iota_1, \tau - t, \omega, \varphi^\varepsilon))\\&\quad + \int_{\tau+\iota_1}^{\tau+\iota_2}(-2\gamma + 2\sigma(\theta_t\omega))a_\varepsilon(v^\varepsilon(r, \tau - t, \omega, \varphi^\varepsilon), v^\varepsilon(r, \tau - t, \omega, \varphi^\varepsilon))\mathrm{d}r\\&\quad + c\int_{\tau+\iota_1}^{\tau+\iota_2}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|f(r, y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r + cL^2\int_{\tau+\iota_1}^{\tau+\iota_2}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|\mathcal{T}(\theta_{r-\tilde\rho(r)}\omega)\|^2\|v^\varepsilon(r-\tilde\rho(r))\|^2_{H_g(O)}\mathrm{d}r.\end{aligned} \tag{3.43}$$

Replacing ω with $\theta_{-\tau}\omega$, we find that

$$\begin{aligned}
&\lambda \int_{\tau+\iota_1}^{\tau+\iota_2} \|A_\varepsilon v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{H_g(O)} \mathrm{d}r \\
&\quad \le a_\varepsilon(v^\varepsilon(\tau + \iota_1, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon), v^\varepsilon(\tau + \iota_1, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)) \\
&\qquad + 2 \max_{\iota\in[-\rho,0]} |\sigma(\theta_\iota\omega)| \int_{\tau-\rho}^{\tau} a_\varepsilon(v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon), v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon))\mathrm{d}r \\
&\qquad + c\int_{-\rho}^{0} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r \\
&\qquad + cL^2 \frac{1}{1-\rho_0} \max_{\iota\in[-\rho,0]} \|\mathcal{T}^{-1}(\theta_\iota\omega)\|^2 \max_{\iota\in[-2\rho,0]} \|\mathcal{T}(\theta_\nu\omega)\|^2 \int_{\tau-2\rho}^{\tau} \|v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{H_g(O)}\mathrm{d}r.
\end{aligned} \tag{3.44}$$

Note that

$$c\int_{-\rho}^{0} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r \le ce^{\bar{\xi}\rho}\int_{-\infty}^{0} e^{\bar{\xi}r}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|f(r + \tau, y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r,$$

which together with (3.44) and Lemmas 3.2 and 3.3 completes the proof. □

Next we obtain the uniform estimates for the solution u^ε to the stochastic system (2.3).

Lemma 3.6. *Suppose* (2.12) *and* (2.15) *hold. For all* $0 < \varepsilon \le \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, *there exists* $T = T(\tau, \omega, D_1) \ge 2\rho + 1$, *independent of* ε, *such that for all* $t \ge T$, *the solution* u^ε *of system* (2.3) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\|u^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon)\|^2_{C([-\rho,0],H^1_\varepsilon(O))} \le L(\tau, \omega), \tag{3.45}$$

where $L(\tau, \omega) = r_3(\omega)R_2(\tau, \omega)$, $\psi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, *and* $r_3(\omega)$ *is tempered. Moreover,* $u^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon) : [-\rho, 0] \to L^2(O)$ *is equicontinuous.*

Proof. Given $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, denote

$$\hat{D}_1 = \{\hat{D}_1(\tau, \omega) = \{v \in Q : \|v\|_Q \le \|\tilde{\mathcal{T}}^{-1}(\omega)\|\|D_1(\tau, \omega)\|_Q\} : \tau \in \mathbb{R}, \omega \in \Omega\}. \tag{3.46}$$

Since $D_1 \in \mathcal{D}_1$ and $\|\tilde{\mathcal{T}}^{-1}\|$ is tempered, we know that $\hat{D}_1$ also belongs to $\mathcal{D}_1$. By $\psi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, we obtain that

$$\varphi^\varepsilon = \tilde{\mathcal{T}}^{-1}(\theta_{-t}\omega)\psi^\varepsilon \quad \text{and} \quad \|\varphi^\varepsilon\|_Q \le \|\tilde{\mathcal{T}}^{-1}(\theta_{-t}\omega)\|\|D_1(\tau - t, \theta_{-t}\omega)\|_Q. \tag{3.47}$$

By (3.46) and (3.47), we obtain $\psi^\varepsilon \in \hat{D}_1(\tau - t, \theta_{-t}\omega)$. By Lemma 3.3, there exists $T = T(\tau, \omega, D_1) \ge 2\rho + 1$ such that for all $t \ge T$,

$$\|v^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{C([-\rho,0],H^1_\varepsilon(O))} \le R_2(\tau, \omega).$$

Note that

$$u^\varepsilon(\iota, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon) = \mathcal{T}(\theta_{\iota-\tau}\omega)v^\varepsilon(\iota, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon). \tag{3.48}$$

For all $t \ge T$, we obtain

$$\begin{aligned}
\|u^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon)\|^2_{C([-\rho,0],H^1_\varepsilon(O))} &\le \max_{\iota\in[-\rho,0]} \|\mathcal{T}(\theta_\iota\omega)\|^2 \|v^\varepsilon_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|^2_{C([-\rho,0],H^1_\varepsilon(O))} \\
&\le \max_{\iota\in[-\rho,0]} \|\mathcal{T}(\theta_\iota\omega)\|^2 R_2(\tau, \omega).
\end{aligned}$$

Thus (3.45) is proved.

By Lemmas 3.3 and 3.5 and (2.10), we obtain that there exists $T = T(\tau, \omega, D_1) \geq 2\rho + 1$ such that for all $t \geq T$,

$$\int_{\tau-\rho}^{\tau} \left\| \frac{\mathrm{d}}{\mathrm{d}r} v_\tau^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon) \right\|_{L^2(O)}^2 \mathrm{d}r \leq c,$$

where $c = c(\tau, \omega) > 0$. For any $\iota_1, \iota_2 \in [-\rho, 0]$, we obtain

$$\begin{aligned}
&\|v_\tau^\varepsilon(\iota_2, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon) - v_\tau^\varepsilon(\iota_1, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|_{L^2(O)} \\
&= \left\| \int_{\tau+\iota_1}^{\tau+\iota_2} \frac{\mathrm{d}}{\mathrm{d}r} v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon) \mathrm{d}r \right\|_{L^2(O)} \\
&\leq |\iota_1 - \iota_2|^{\frac{1}{2}} \left(\int_{\tau+\iota_1}^{\tau+\iota_2} \left\| \frac{\mathrm{d}}{\mathrm{d}r} v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon) \right\|_{L^2(O)}^2 \mathrm{d}r \right)^{\frac{1}{2}} \\
&\leq |\iota_1 - \iota_2|^{\frac{1}{2}} \left(\int_{\tau-\rho}^{\tau} \left\| \frac{\mathrm{d}}{\mathrm{d}r} v^\varepsilon(r, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon) \right\|_{L^2(O)}^2 \mathrm{d}r \right)^{\frac{1}{2}} \\
&\leq c\, |\iota_1 - \iota_2|^{\frac{1}{2}},
\end{aligned} \tag{3.49}$$

which together with (3.48) implies that

$$\begin{aligned}
&\|u_\tau^\varepsilon(\iota_2, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon) - u_\tau^\varepsilon(\iota_1, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon)\|_{L^2(O)} \\
&\leq \|\mathcal{T}(\theta_{\iota_2}\omega)\| \|v_\tau^\varepsilon(\iota_2, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon) - v_\tau^\varepsilon(\iota_1, \tau - t, \theta_{-t}\omega, \varphi^\varepsilon)\|_{L^2(O)} \\
&+ \|\mathcal{T}(\theta_{\iota_2}\omega) - \mathcal{T}(\theta_{\iota_1}\omega)\| \|v_\tau^\varepsilon(\cdot, \tau - t, \theta_{-\tau}\omega, \varphi^\varepsilon)\|_Q \\
&\leq c\, |\iota_1 - \iota_2|^{\frac{1}{2}} + cR_1^{\frac{1}{2}}(\tau, \omega) \|\mathcal{T}(\theta_{\iota_2}\omega) - \mathcal{T}(\theta_{\iota_1}\omega)\|.
\end{aligned} \tag{3.50}$$

By the continuity of $\mathcal{T}(\theta_t\omega)$, we can obtain the result. □

4 Existence of random attractors

In this section, we will show the existence of $\mathcal{D}_1$-pullback attractors for system (2.3) and $\mathcal{D}_0$-pullback attractors for system (2.4), respectively. First, we prove that system (2.3) has a tempered $\mathcal{D}_1$-pullback absorbing set.

Lemma 4.1. *Suppose* (2.12) *and* (2.16) *hold. For all* $0 < \varepsilon \leq \varepsilon_0$, *the continuous cocycle* Ψ^ε *associated with system* (2.3) *has a closed measurable* $\mathcal{D}_1$*-pullback absorbing set* $K \in \mathcal{D}_1$, *which is defined by*

$$K(\tau, \omega) = \{u \in Q : \|u\|_Q^2 \leq L(\tau, \omega)\}, \tau \in \mathbb{R}, \omega \in \Omega,$$

where $L(\tau, \omega)$ *is given by* (3.45).

Proof. For each $\tau \in \mathbb{R}$, we obtain $L(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathfrak{B}(\mathbb{R}))$-measurable. On the other hand, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 \in \mathcal{D}_1$, there is a $T = T(\tau, \omega, D_1) \geq 2\rho + 1$, independent of ε, such that for all $t \geq T$,

$$\Psi^\varepsilon(t, \tau - t, \theta_{-t}\omega, D_1(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega).$$

Consequently, $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set for Φ_ε in $\mathcal{D}_1$. Next we prove that K is tempered. Given $\varsigma > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, we obtain

$$e^{\varsigma t} R_2(\tau + t, \theta_t\omega) = c e^{\varsigma t} r_1(\theta_t\omega) \int_{-\infty}^{0} e^{\int_t^{t+r} \xi(\theta_l\omega)\mathrm{d}l} \|\mathcal{T}^{-1}(\theta_{r+t}\omega)\|^2 \|f(r + \tau + t, y_1)\|_{L^2(\mathbb{R})}^2 \mathrm{d}r \tag{4.1}$$

$$+ ce^{\varsigma t}e^{\bar{\xi}(\rho+1)}\int_{-\infty}^{0} e^{\bar{\xi}r}\|\mathcal{T}^{-1}(\theta_{r+t}\omega)\|^2\|f(r+\tau+t,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r.$$

Let $0<\varepsilon<\min\{\frac{\varsigma}{4},\frac{\bar{\xi}}{4}\}$ and $\varepsilon=\varepsilon_1=\varepsilon_2$ in (3.10) and (3.11), by the temperedness of $r_1(\theta_t\omega)$, for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$, we have

$$\begin{aligned}
&\limsup_{t\to-\infty} e^{\varsigma t}R_2(\tau+t,\theta_t\omega)\\
&\le \limsup_{t\to-\infty} ce^{(\varsigma-\varepsilon)t}\int_{-\infty}^{0} e^{\int_0^{t+r}(\xi(\theta_l\omega)-\bar{\xi})\mathrm{d}l+\bar{\xi}(r+t)-\int_0^{t}(\xi(\theta_l\omega)-\bar{\xi})\mathrm{d}l-\bar{\xi}t}e^{\varepsilon|r+t|}\|f(r+\tau+t,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r\\
&\quad+\limsup_{t\to-\infty} ce^{\bar{\xi}(\rho+1)}e^{\varsigma t}\int_{-\infty}^{0} e^{\bar{\xi}r}e^{\varepsilon|r+t|}\|f(r+\tau+t,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r\\
&\le ce^{-(\varsigma-4\varepsilon)\tau}\limsup_{t\to-\infty} e^{(\varsigma-4\varepsilon)(t+\tau)}\int_{-\infty}^{0} e^{(\bar{\xi}-2\varepsilon)r}\|f(r+\tau+t,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r\\
&\quad+ce^{-(\varsigma-\varepsilon)\tau}\limsup_{t\to-\infty} e^{\bar{\xi}(\rho+1)}e^{(\varsigma-\varepsilon)(t+\tau)}\int_{-\infty}^{0} e^{(\bar{\xi}-\varepsilon)r}\|f(r+\tau+t,y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r.
\end{aligned}\tag{4.2}$$

By (2.16), we obtain

$$\lim_{t\to-\infty} e^{\varsigma r}R_2(\tau+t,\theta_t\omega)=0,$$

which together with the temperedness of $r_3(\omega)$ completes the proof. □

Lemma 4.2. *Suppose* (2.12) *and* (2.16) *hold. For all* $0<\varepsilon\le\varepsilon_0$, $\tau\in\mathbb{R}$, and $\omega\in\Omega$ *and* $D_1=\{D_1(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$, *the sequence* $\{\Psi^\varepsilon(t_n,\tau-t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^\infty$ has a *convergent subsequence in* Q, *provided* $t_n\to\infty$ *and* $x_n\in D_1(\tau-t_n,\theta_{-t_n}\omega)$.

Proof. For $t_n\to\infty$, by Lemma 3.6, there is a $N=N(\tau,\omega,D_1)$ such that for all $n\ge N$ and $\iota\in[-\rho,0]$,

$$\|u^\varepsilon_\tau(\iota,\tau-t_n,\theta_{-t_n}\omega,x_n)\|^2_{H^1_\varepsilon(O)}\le L(\tau,\omega).\tag{4.3}$$

By the compactness of imbedding $H^1(O)\hookrightarrow L^2(O)$ and (4.3), for each $\iota\in[-\rho,0]$, the sequence $\{u^\varepsilon_\tau(\iota,\tau-t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^\infty$ is relatively compact in $L^2(O)$. In addition, by Lemma 3.6, the sequence $\{u^\varepsilon_\tau(\iota,\tau-t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^\infty$ is equicontinuous. By using the Ascoli-Arzelà theorem, the sequence $\{u^\varepsilon_\tau(\iota,\tau-t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^\infty$ is relatively compact in Q. This completes the proof. □

Theorem 4.1. *Suppose* (2.12) *and* (2.16) *hold. For all,* $0<\varepsilon\le\varepsilon_0$, *the cocycle* Ψ^ε *associated with system* (2.3) *has a unique* $\mathcal{D}_1$*-pullback attractor* $\mathcal{A}_\varepsilon=\{\mathcal{A}_\varepsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$ *in* Q.

Proof. By Lemma 4.1, Ψ^ε has a closed measurable $\mathcal{D}_1$-pullback absorbing set $K\in\mathcal{D}_1$, and Lemma 4.2 shows that Ψ^ε is asymptotically compact in Q with respect to $\mathcal{D}_1$. Hence, the existence and uniqueness of $\mathcal{D}_1$-pullback random attractor $\mathcal{A}_\varepsilon$ of Ψ^ε follows from Proposition 3.5 of [34] immediately. □

Similar to Theorem 4.1, we have the following theorem.

Theorem 4.2. *Suppose* (2.12) *and* (2.16) *hold. Then, the cocycle* Ψ^0 *associated with system* (2.4) *has a unique* $\mathcal{D}_0$*-pullback attractor* $\mathcal{A}_0=\{\mathcal{A}_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_0$ in $\mathcal{P}$.

5 Upper semicontinuity of random attractors

In this section, we need the following estimates to establish the convergence of random attractors.

Lemma 5.1. *Suppose* (2.12) *holds. For all* $0 < \varepsilon \le \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, *and* $\varphi^\varepsilon \in Q$, *then the solution* v^ε *of system* (2.10) *satisfies, for all* $t \in [\tau, \tau + T]$,

$$\int_\tau^t \|v^\varepsilon(r, \tau, \omega, \varphi^\varepsilon)\|^2_{H^1_\varepsilon(O)} \mathrm{d}r \le c\|\varphi^\varepsilon\|^2_Q + c\int_\tau^{\tau+T} \|f(r, y_1)\|^2_{L^2(\mathbb{R})} \mathrm{d}r, \tag{5.1}$$

where c is a positive constant depending on τ, ω, *and T, but independent of* ε.

Proof. By (3.6), for $t \in [\tau, \tau + T]$, we obtain

$$\begin{aligned} &2\lambda\int_\tau^t e^{\int_t^r \xi(\theta_l\omega)\mathrm{d}l} a_\varepsilon(v^\varepsilon(r, \tau, \omega, \varphi^\varepsilon), v^\varepsilon(r, \tau, \omega, \varphi^\varepsilon))\mathrm{d}r + \frac{\gamma}{2}\int_\tau^t e^{\int_t^r \xi(\theta_l\omega)\mathrm{d}l}\|v^\varepsilon(r, \tau, \omega, \varphi^\varepsilon)\|^2_{H_g(O)}\mathrm{d}r \\ &\le \|v^\varepsilon(\tau)\|^2_{H_g(O)} e^{-\int_\tau^t \xi(\theta_l\omega)\mathrm{d}l} + c\int_\tau^t e^{\int_t^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|f(r, y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r \\ &\quad + \frac{Le^{\frac{1}{2}\gamma\rho}}{\sqrt{1-\rho_0}}\|\psi^\varepsilon\|^2_{C([-\rho,0],H_g(O))}\int_{\tau-\rho}^{\tau} e^{\int_t^r \xi(\theta_l\omega)\mathrm{d}l}\|\mathcal{T}(\theta_r\omega)\|^2\mathrm{d}r, \end{aligned} \tag{5.2}$$

which along with (2.2) and the continuity of $\xi(\theta_t\omega)$ and $\mathcal{T}(\theta_t\omega)$ with respect to t implies (5.1). □

Similar to Lemma 5.1, we obtain the following lemma.

Lemma 5.2. *Suppose* (2.12) *holds, there exists* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, *and* $\psi^0 \in \mathcal{P}$, *the solution* v^0 *of* (2.11) *satisfies, for all* $t \in [\tau, \tau + T]$,

$$\int_\tau^t \|v^0(r, \tau, \omega, \psi^0)\|^2_{L^2(\mathbb{R})}\mathrm{d}r + \int_\tau^t \|v^0(r, \tau, \omega, \psi^0)\|^2_{H^2(\mathbb{R})}\mathrm{d}r \le c\|\psi^0\|^2_{\mathcal{P}} + c\int_\tau^{\tau+T} \|f(r, y_1)\|^2_{L^2(\mathbb{R})}\mathrm{d}r, \tag{5.3}$$

where c is a positive constant depending on τ, ω, *and T, but independent of* ε.

Next we introduce average function defined by

$$\mathcal{G}u = \int_0^1 u(y_1, y_2)\mathrm{d}y_2 \quad \text{for } u \in L^2(O).$$

From Lemma 3.1 of [13], we can obtain the average function to the case of the unbounded domains.

Lemma 5.3. *If* $u \in H^1(O)$, *then* $\mathcal{G}u \in H^1(\mathbb{R})$ *and*

$$\|u - \mathcal{G}u\|_{H_g(O)} \le c\varepsilon\|u\|_{H^1_\varepsilon(O)},$$

where c is a positive constant independent of ε.

Since $\mathcal{P}$ can be embedded into Q as the subspace of functions independent of y_2, we can investigate the cocycle Ψ^0 as a mapping from $\mathcal{P}$ into Q, then we obtain the convergence of Ψ^ε.

Theorem 5.1. *Suppose* (2.15) *and* (2.16) *hold. Let* $\tau \in \mathbb{R}$, $t \ge \tau$, $\omega \in \Omega$, *and* $\mu(\tau, \omega) > 0$, *if* $\psi^\varepsilon \in C([-\rho, 0], H^1_\varepsilon(O))$ *such that* $\|\psi^\varepsilon\|_{C([-\rho,0],H^1_\varepsilon(O))} \le \mu(\tau, \omega)$, *then we obtain*

$$\lim_{\varepsilon\to 0}\|\Psi^\varepsilon(t, \tau, \omega, \psi^\varepsilon) - \Psi^0(t, \tau, \omega, \mathcal{G}\psi^\varepsilon)\|_Q = 0.$$

Proof. For $l \in H^1(\mathbb{R})$, multiplying (2.11) with l and then integrating over $H_g(\mathbb{R})$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} g\frac{\mathrm{d}v^0}{\mathrm{d}t}\bar{l}\,\mathrm{d}y_1 + (\lambda + i\alpha)\int_{\mathbb{R}} g v^0_{y_1}\bar{l}_{y_1}\mathrm{d}y_1 + (\gamma + i\delta - \sigma(\theta_t\omega))\int_{\mathbb{R}} g v^0\bar{l}\,\mathrm{d}y_1 \\
&\quad = -(k + i\beta)\int_{\mathbb{R}} g\,|v^0|^2 v^0\,|\mathcal{T}(\theta_t\omega)|^2\bar{l}\,\mathrm{d}y_1 + \int_{\mathbb{R}} g\mathcal{T}^{-1}(\theta_t\omega)f(t, y_1)\bar{l}\,\mathrm{d}y_1 \\
&\qquad + \int_{\mathbb{R}} g\mathcal{T}^{-1}(\theta_t\omega)h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^0(t - \tilde{\rho}(t)))\bar{l}\,\mathrm{d}y_1.
\end{aligned} \tag{5.4}$$

If ϱ is in $H^1(\mathcal{O})$, $\int_0^1 \varrho(y_1, y_2)\mathrm{d}y_2$ belongs to $H^1(\mathbb{R})$. For any $\varrho \in H^1(\mathcal{O})$, (5.4) can be written as

$$\begin{aligned}
&\left(\frac{\mathrm{d}v^0}{\mathrm{d}t}, \varrho\right)_{H_g(\mathcal{O})} + (\lambda + i\alpha)(v^0_{y_1}, \varrho_{y_1})_{H_g(\mathcal{O})} + (\gamma + i\delta - \sigma(\theta_t\omega))(v^0, \varrho)_{H_g(\mathcal{O})} \\
&\quad = -(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2(|v^0|^2 v^0, \varrho)_{H_g(\mathcal{O})} + \mathcal{T}^{-1}(\theta_t\omega)(f(t, y_1), \varrho)_{H_g(\mathcal{O})} \\
&\qquad + \mathcal{T}^{-1}(\theta_t\omega)(h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^0(t - \tilde{\rho}(t))), \varrho)_{H_g(\mathcal{O})}.
\end{aligned} \tag{5.5}$$

Since v^0 is independent of y_2, we obtain from (5.5), the following equation:

$$\begin{aligned}
&\left(\frac{\mathrm{d}v^0}{\mathrm{d}t}, \varrho\right)_{H_g(\mathcal{O})} + (\lambda + i\alpha)a_\varepsilon(v^0, \varrho) + (\gamma + i\delta - \sigma(\theta_t\omega))(v^0, \varrho)_{H_g(\mathcal{O})} \\
&\quad = -(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2(|v^0|^2 v^0, \varrho)_{H_g(\mathcal{O})} + \mathcal{T}^{-1}(\theta_t\omega)(f(t, y_1), \varrho)_{H_g(\mathcal{O})} \\
&\qquad + \mathcal{T}^{-1}(\theta_t\omega)(h(\mathcal{T}(\theta_{r-\tilde{\rho}(r)}\omega)v^0(r - \tilde{\rho}(r))), \varrho)_{H_g(\mathcal{O})} - (\lambda + i\alpha)\left(\frac{g_{y_1}}{g}v^0_{y_1}, y_2\varrho_{y_2}\right)_{H_g(\mathcal{O})}.
\end{aligned} \tag{5.6}$$

By (2.10) and (5.6), we have that for any $\varrho \in H^1(\mathcal{O})$,

$$\begin{aligned}
&\left(\frac{\mathrm{d}(v^\varepsilon - v^0)}{\mathrm{d}t}, \varrho\right)_{H_g(\mathcal{O})} + (\lambda + i\alpha)a_\varepsilon(v^\varepsilon - v^0, \varrho) + (\gamma + i\delta - \sigma(\theta_t\omega))(v^\varepsilon - v^0, \varrho)_{H_g(\mathcal{O})} \\
&\quad = -(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2((|v^\varepsilon|^2 v^\varepsilon - |v^0|^2 v^0), \varrho)_{H_g(\mathcal{O})} + (\lambda + i\alpha)\left(\frac{g_{y_1}}{g}v^0_{y_1}, y_2\varrho_{y_2}\right)_{H_g(\mathcal{O})} \\
&\qquad + \mathcal{T}^{-1}(\theta_t\omega)(h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^\varepsilon(t - \tilde{\rho}(t))) - h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^0(t - \tilde{\rho}(t))), \varrho)_{H_g(\mathcal{O})}.
\end{aligned} \tag{5.7}$$

Setting $\varrho = v^\varepsilon - v^0$ and keeping the real part in (5.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^\varepsilon - v^0\|^2_{H_g(\mathcal{O})} + \lambda a_\varepsilon(v^\varepsilon - v^0, v^\varepsilon - v^0) + (\gamma - \sigma(\theta_t\omega))\|v^\varepsilon - v^0\|_{H_g(\mathcal{O})} \\
&\quad = -\operatorname{Re}(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2(|v^\varepsilon|^2 v^\varepsilon - |v^0|^2 v^0, v^\varepsilon - v^0)_{H_g(\mathcal{O})} \\
&\qquad + \operatorname{Re}\mathcal{T}^{-1}(\theta_t\omega)(h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^\varepsilon(t - \tilde{\rho}(t))) - h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^0(t - \tilde{\rho}(t))), v^\varepsilon - v^0)_{H_g(\mathcal{O})} \\
&\qquad + \operatorname{Re}(\lambda + i\alpha)\left(\frac{g_{y_1}}{g}v^0_{y_1}, y_2(v^\varepsilon - v^0)_{y_2}\right)_{H_g(\mathcal{O})}.
\end{aligned} \tag{5.8}$$

By using the Agmon inequality, we obtain

$$\begin{aligned}
&-\operatorname{Re}(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2(|v^\varepsilon|^2 v^\varepsilon - |v^0|^2 v^0, v^\varepsilon - v^0)_{H_g(\mathcal{O})} \\
&\quad = -\operatorname{Re}(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2\int_{\mathcal{O}} g\,|v^\varepsilon|^2(v^\varepsilon - v^0)\overline{(v^\varepsilon - v^0)}\mathrm{d}y \\
&\qquad - \operatorname{Re}(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2\int_{\mathcal{O}} g(|v^\varepsilon|^2 - |v^0|^2)v^0\overline{(v^\varepsilon - v^0)}\mathrm{d}y \\
&\quad \le -\operatorname{Re}(k + i\beta)|\mathcal{T}(\theta_t\omega)|^2\int_{\mathcal{O}} g(|v^\varepsilon|^2 - |v^0|^2)v^0\overline{(v^\varepsilon - v^0)}\mathrm{d}y
\end{aligned} \tag{5.9}$$

$$
\begin{aligned}
&= -\operatorname{Re}(k+i\beta)|\mathcal{T}(\theta_t\omega)|^2\int_O g v^0(\bar{v}^\varepsilon(v^\varepsilon - v^0) + v^0(\bar{v}^\varepsilon - \bar{v}^0))\overline{(v^\varepsilon - v^0)}\mathrm{d}y \\
&\le c\,|\mathcal{T}(\theta_t\omega)|^2\int_O g(|v^0||v^\varepsilon||v^\varepsilon - v^0|^2 + |v^0|^2|v^\varepsilon - v^0|^2)\mathrm{d}y \\
&\le c\,|\mathcal{T}(\theta_t\omega)|^2(\|v^0\|_{L^\infty(\mathbb{R})}\|v^\varepsilon\|_{L^\infty(O)} + \|v^0\|^2_{L^\infty(\mathbb{R})})\int_O g\,|v^\varepsilon - v^0|^2\mathrm{d}y \\
&\le c\|\mathcal{T}(\theta_t\omega)\|^2(\|v^\varepsilon\|^2_{H^2(O)} + \|v^\varepsilon\|^2_{L^2(O)} + \|v^0\|^2_{H^2(\mathbb{R})} + \|v^0\|^2_{L^2(\mathbb{R})})\|v^\varepsilon - v^0\|^2_{H_g(O)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\operatorname{Re}\mathcal{T}^{-1}(\theta_t\omega)(h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^\varepsilon(t-\tilde{\rho}(t))) - h(\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)v^0(t-\tilde{\rho}(t))), v^\varepsilon - v^0)_{H_g(O)} \\
&\le L\|\mathcal{T}^{-1}(\theta_t\omega)\|\|\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)\|\|v^\varepsilon(t-\tilde{\rho}(t)) - v^0(t-\tilde{\rho}(t))\|_{H_g(O)}\|v^\varepsilon - v^0\|_{H_g(O)} \\
&\le \frac{1}{4}L^2\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)\|^2\|v^\varepsilon - v^0\|^2_{H_g(O)} + \|v^\varepsilon(t-\tilde{\rho}(t)) - v^0(t-\tilde{\rho}(t))\|^2_{H_g(O)}.
\end{aligned}
\tag{5.10}
$$

By (2.2), we have

$$
\begin{aligned}
\operatorname{Re}(\lambda+i\alpha)\left(\frac{g_{y_1}}{g}v^0_{y_1}, y_2(v^\varepsilon - v^0)_{y_2}\right)_{H_g(O)} &= \operatorname{Re}(\lambda+i\alpha)(g_{y_1}v^0_{y_1}, y_2(v^\varepsilon_{y_2} - v^0_{y_2}))_{L^2(O)} \\
&\le c\varepsilon\|v^0\|_{H^1(\mathbb{R})}\|v^\varepsilon - v^0\|_{H^1_\varepsilon(O)} \\
&\le c\varepsilon(\|v^\varepsilon\|^2_{H^1_\varepsilon(O)} + \|v^0\|^2_{H^1(\mathbb{R})}).
\end{aligned}
\tag{5.11}
$$

From (5.8)–(5.11), for $t \ge \tau$, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|v^\varepsilon - v^0\|^2_{H_g(O)} \\
&\le 2|\sigma(\theta_t\omega)|\|v^\varepsilon - v^0\|^2_{H_g(O)} + c\varepsilon(\|v^\varepsilon\|^2_{H^1_\varepsilon(O)} + \|v^0\|^2_{H^1(\mathbb{R})}) \\
&\quad + c\|\mathcal{T}(\theta_t\omega)\|^2(\|v^\varepsilon\|^2_{H^2(O)} + \|v^\varepsilon\|^2_{L^2(O)} + \|v^0\|^2_{H^2(\mathbb{R})} + \|v^0\|^2_{L^2(\mathbb{R})})\|v^\varepsilon - v^0\|^2_{H_g(O)} \\
&\quad + \frac{1}{2}L^2\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)\|^2\|v^\varepsilon - v^0\|^2_{H_g(O)} + 2\|v^\varepsilon(t-\tilde{\rho}(t)) - v^0(t-\tilde{\rho}(t))\|^2_{H_g(O)}.
\end{aligned}
\tag{5.12}
$$

Integrating (5.12) on (τ, t), for all $t \in [\tau, \tau + T]$ with $T > 0$, we have

$$
\begin{aligned}
\|v^\varepsilon(t) - v^0(t)\|^2_{H_g(O)} \le \|v^\varepsilon(\tau) - v^0(\tau)\|^2_{H_g(O)} &+ \int_\tau^t \varpi(r,w)\|v^\varepsilon(r) - v^0(r)\|^2_{H_g(O)}\mathrm{d}r \\
&+ c\varepsilon\int_\tau^t(\|v^\varepsilon(r)\|^2_{H^1_\varepsilon(O)} + \|v^0(r)\|^2_{H^1(\mathbb{R})})\mathrm{d}r \\
&+ 2\int_\tau^t\|v^\varepsilon(r-\tilde{\rho}(r)) - v^0(r-\tilde{\rho}(r))\|^2_{H_g(O)}\mathrm{d}r,
\end{aligned}
\tag{5.13}
$$

where

$$
\begin{aligned}
\varpi(t,w) = c\|\mathcal{T}(\theta_t\omega)\|^2(\|v^\varepsilon(t)\|^2_{H^2(O)} + \|v^\varepsilon(t)\|^2_{L^2(O)} + \|v^0(t)\|^2_{H^2(\mathbb{R})} + \|v^0(t)\|^2_{L^2(\mathbb{R})}) \\
+ 2|\sigma(\theta_t\omega)| + \frac{1}{2}L^2\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|\mathcal{T}(\theta_{t-\tilde{\rho}(t)}\omega)\|^2.
\end{aligned}
\tag{5.14}
$$

Then, we have for $t \in [\tau, \tau + T]$,

$$\begin{aligned}\|v_t^\varepsilon(\cdot) - v_t^0(\cdot)\|_Q^2 \le c\|v_\tau^\varepsilon(\cdot) - v_\tau^0(\cdot)\|_Q^2 &+ c\max_{\iota\in[\tau,t]}\varpi(\iota, w)\int_\tau^t \|v_r^\varepsilon(\cdot) - v_r^0(\cdot)\|_Q^2 \mathrm{d}r \\ &+ c\varepsilon\int_\tau^{\tau+T}(\|v^\varepsilon(r)\|_{H_\varepsilon^1(O)}^2 + \|v^0(r)\|_{H^1(\mathbb{R})}^2)\mathrm{d}r + c\int_\tau^t \|v_r^\varepsilon(\cdot) - v_r^0(\cdot)\|_Q^2 \mathrm{d}r.\end{aligned} \tag{5.15}$$

By (5.15) and Lemmas 5.1 and 5.2, we find that there exists a positive constant $\chi = \chi(\tau, \omega, T)$, independent of ε, such that for all $t \in [\tau, \tau + T]$,

$$\begin{aligned}\|v_t^\varepsilon(\cdot) - v_t^0(\cdot)\|_Q^2 &\le e^{c(1+\max_{\iota\in[\tau,\tau+T]}\varpi(\iota,w))T}\|v_\tau^\varepsilon(\cdot) - v_\tau^0(\cdot)\|_Q^2 \\ &+ \varepsilon\chi e^{c(1+\max_{\iota\in[\tau,\tau+T]}\varpi(\iota,w))T}(\|\psi^0\|_{\mathcal{P}}^2 + \|\psi^\varepsilon\|_Q^2 + \int_\tau^{\tau+T}\|f(r,y_1)\|_{L^2(\mathbb{R})}^2 \mathrm{d}r).\end{aligned} \tag{5.16}$$

Then, by Lemma 5.3, for all $t \in [\tau, \tau + T]$, we obtain

$$\begin{aligned}&\|u_t^\varepsilon(\cdot, \tau, \omega, \psi^\varepsilon) - u_t^0(\cdot, \tau, \omega, \mathcal{G}\psi^\varepsilon)\|_Q^2 \\ &\le \max_{\iota\in[\tau-\rho,\tau+T]}\|\mathcal{T}(\theta_\iota\omega)\|^2\|v_t^\varepsilon(\cdot, \tau, \omega, \tilde{\mathcal{T}}^{-1}(\theta_\tau\omega)\psi^\varepsilon) - v_t^0(\cdot, \tau, \omega, \tilde{\mathcal{T}}^{-1}(\theta_\tau\omega)\mathcal{G}\psi^\varepsilon)\|_Q^2 \\ &\le \max_{\iota\in[\tau-\rho,\tau+T]}\|\mathcal{T}(\theta_\iota\omega)\|^2 e^{c(1+\max_{\iota\in[\tau,\tau+T]}\varpi(\iota,w))T}\|\tilde{\mathcal{T}}^{-1}(\theta_\tau\omega)\psi^\varepsilon - \tilde{\mathcal{T}}^{-1}(\theta_\tau\omega)\mathcal{G}\psi^\varepsilon\|_Q^2 \\ &\quad + \max_{\iota\in[\tau-\rho,\tau+T]}\|\mathcal{T}(\theta_\iota\omega)\|^2\varepsilon\chi e^{c(1+\max_{\iota\in[\tau,\tau+T]}\varpi(\iota,w))T}(\|\psi^0\|_{\mathcal{P}}^2 + \|\psi^\varepsilon\|_Q^2 + \int_\tau^{\tau+T}\|f(r,y_1)\|_{L^2(\mathbb{R})}^2 \mathrm{d}r) \\ &\le \max_{\iota\in[\tau-\rho,\tau+T]}\|\mathcal{T}(\theta_\iota\omega)\|^2 c\varepsilon e^{c(1+\max_{\iota\in[\tau,\tau+T]}\varpi(\iota,w)T)}\|\tilde{\mathcal{T}}^{-1}(\theta_\tau\omega)\|\|\psi^\varepsilon\|_{C([-\rho,0],H_\varepsilon^1(O))} \\ &\quad + \max_{\iota\in[\tau-\rho,\tau+T]}\|\mathcal{T}(\theta_\iota\omega)\|^2\varepsilon\chi e^{c(1+\max_{\iota\in[\tau,\tau+T]}\varpi(\iota,w)T)}(\|\psi^0\|_{\mathcal{P}}^2 + \|\psi^\varepsilon\|_Q^2 + \int_\tau^{\tau+T}\|f(r,y_1)\|_{L^2(\mathbb{R})}^2 \mathrm{d}r).\end{aligned}$$

Since $\|\psi^\varepsilon\|_{C([-\rho,0],H_\varepsilon^1(O))} \le \mu(\tau, \omega)$, we can derive the desired result. □

Now we establish the upper semicontinuity of random attractors as $\varepsilon \to 0$.

Theorem 5.2. *Suppose* (2.12) *and* (2.16) *hold, then for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0}\mathrm{dist}_Q(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0.$$

Proof. For the proof, please see Theorem 5.5 in [28]. □

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