Research Article

Hong Yan Xu and Goutam Haldar*

Characterizations of transcendental entire solutions of trinomial partial differential-difference equations in $\mathbb{C}^{2\#}$

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Abstract: This study is devoted to exploring the existence and the precise form of finite-order transcendental entire solutions of second-order trinomial partial differential-difference equations

$$L(f)^2 + 2hL(f)f(z_1 + c_1, z_2 + c_2) + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1,z_2)}$$

and

$$\tilde{L}(f)^2 + 2h\tilde{L}(f)(f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2)) + (f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2))^2 = e^{g(z_1,z_2)},$$

where $L(f)$ and $\tilde{L}(f)$ are defined in (2.1) and (2.2), respectively, and $g(z)$ is a polynomial in $\mathbb{C}^2$. Our results are the extensions of some of the previous results of Liu et al. Also, we exhibit a series of examples to explain that the forms of transcendental entire solutions of finite-order in our results are precise.

Keywords: functions of several complex variables, Fermat-type equations, entire solutions, Nevanlinna theory

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1 Introduction

Fermat's last theorem [1] says that the Fermat equation $x^m + y^m = 1$ does not admit nontrivial solutions in rational numbers when $m \geqslant 3$ and does admit nontrivial rational solutions when $m = 2$. For a positive integer m, the equation

$$f^m + g^m = 1 \tag{1.1}$$

is known as Fermat-type equation over function fields. With the help of the Nevanlinna theory [2,3], Montel [4], Iyer [5], and Gross [6] studied the existence and form of the solutions of the functional equation (1.1) and pointed out the following:

(i) For $m = 2$, the entire solutions of (1.1) are $f(z) = \cos(\xi(z))$ and $g(z) = \sin(\xi(z))$, where ξ is a non-constant entire function.

(ii) For $m > 2$, there are no non-constant entire solutions of (1.1).

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* **Corresponding author: Goutam Haldar**, Department of Mathematics, Ghani Khan Choudhury Institute of Engineering and Technology, Narayanpur, Malda, PIN 732141, West Bengal, India, e-mail: goutamiit1986@gmail.com, goutamiitm@gmail.com
Hong Yan Xu: School of Arts and Sciences, Suqian University, Suqian, Jiangsu 223800, P. R. China, e-mail: xhyhhh@126.com

(iii) For $m = 2$, the meromorphic solutions of (1.1) are of the form

$$f(z) = \frac{2\phi(z)}{1-\phi^2(z)} \quad \text{and} \quad g(z) = \frac{1-\phi^2(z)}{1+\phi^2(z)},$$

where ϕ is a non-constant meromorphic function.

(iv) For $m = 3$, the meromorphic solutions of (1.1) are of the form

$$f(z) = \frac{1}{2\wp(h)}\left(1 + \frac{\wp'(h)}{\sqrt{3}}\right) \quad \text{and} \quad g(z) = \frac{\omega}{2\wp(h)}\left(1 - \frac{\wp'(h)}{\sqrt{3}}\right),$$

where $\omega^3 = 1$ and $\wp$ satisfies $(\wp')^2 = 4\wp^3 - 1$.

(v) For $m > 3$, there are no non-constant meromorphic solutions.

In 2004, Yang and Li [7] investigated (1.1) by replacing g with f' when $m = 2$, and proved that the transcendental entire solution of $f(z)^2 + f'(z)^2 = 1$ has the form $f(z) = Ae^{\alpha z}/2 + e^{-\alpha z}/2A$, where A and α are non-zero complex constants.

The advent of the difference analog lemma of the logarithmic derivative [8,9] expedites the research activity to characterize the entire or meromorphic solutions of Fermat-type difference and differential-difference equations [10–13]. In 2012, Liu et al. [14] proved that the transcendental entire solutions with finite-order of the Fermat-type difference equation $f(z)^2 + f(z+c)^2 = 1$ must satisfy $f(z) = \sin(Az + B)$, where B is a constant and $A = (4k+1)\pi/2c$, where k is an integer. In 2016, Liu and Yang [15] studied the existence and the form of solutions of some quadratic trinomial functional equations and obtained some important results as follows. The equation $f(z)^2 + 2\alpha f(z)f'(z) + f'^2(z) = 1$ has no transcendental meromorphic solutions, and the finite-order transcendental entire solution of $f(z)^2 + 2\alpha f(z)f(z+c) + f(z+c)^2 = 1$ must be of order equal to one, where in both the equations, $\alpha \neq 0, \pm 1$.

The study of several characteristics of the solutions to partial differential equations in several complex variables [16–24] is an important topic. The Fermat-type equation like appears in particle mechanics; in particular, it appears as the Lagrange function in the Lagrangian functional describing the "action" of a system. As far as the author's knowledge is concerned, Saleebly, in 1999, first started investigation about the existence and form of entire and meromorphic solutions of Fermat-type partial differential equations [25,26]. Most noticeably, Khavinson [27] proved that any entire solution of the partial differential equation $f_{z_1}^2 + f_{z_2}^2 = 1$ must be linear, i.e., $f(z_1, z_2) = az_1 + bz_2 + c$, where $a, b, c \in \mathbb{C}$, and $a^2 + b^2 = 1$. Later, Li [28,29] investigated the partial differential equations with more general forms such as $f_{z_1}^2 + f_{z_2}^2 = p$ and $f_{z_1}^2 + f_{z_2}^2 = e^q$, where p and q are the polynomials in $\mathbb{C}^2$.

Hereinafter, we denote by $z + w = (z_1 + w_1, z_2 + w_2)$ for any $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{C}^2$.

Recently, Xu and Cao [30] extended several results from one complex variable to several complex variables. In fact, they considered two Fermat-type equations $f(z)^2 + f(z+c)^2 = 1$ and $f(z+c)^2 + f_{z_1}^2 = 1$ and proved that any finite-order transcendental entire solution $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ has the form $f(z) = \cos(L(z) + B)$, where L is a linear function of the form $L(z) = a_1 z_1 + \ldots + a_n z_n$ on $\mathbb{C}^n$ such that $L(c) = -\pi/2 - 2k\pi$ $(k \in \mathbb{Z})$, and $B \in \mathbb{C}$ and $f(z) = \cos(A_1 z_1 + A_2 z_2 + \text{Constant})$, where $A_1 c_1 + A_2 c_2 = -\pi/2 - 2k\pi$, $k \in \mathbb{Z}$, respectively.

In 2022, Xu et al. [31] explored the precise form of entire and meromorphic solutions of the following partial differential difference equations:

$$\left(\alpha\frac{\partial f}{\partial z_1} + \beta\frac{\partial f}{\partial z_2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1, z_2)} \tag{1.2}$$

and

$$\left(\alpha\frac{\partial f}{\partial z_1} + \beta\frac{\partial f}{\partial z_2}\right)^2 + (f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2))^2 = e^{g(z_1, z_2)}, \tag{1.3}$$

where $g(z)$ is a polynomial in $\mathbb{C}^2$ and α and β are the constants in $\mathbb{C}$. For detail study, we insist the readers to go through [31].

As far as we know, it was Saleeby [32], who first considered another quadratic functional equation

$$f^2 + 2\alpha fg + g^2 = 1, \tag{1.4}$$

and investigated the existence and form of transcendental entire and meromorphic solutions in $\mathbb{C}^n$. We recall the result.

Theorem A. [32] *The transcendental entire solutions of* (1.4) *must be of the form* $f(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} + \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right)$ *and* $\frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} - \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right)$, *where* h *is entire* $\mathbb{C}^n$. *The meromorphic solutions of* (1.4) *must be of the form* $f(z) = \frac{\alpha_1 - \alpha_2\beta(z)^2}{(\alpha_1-\alpha_2)\beta(z)}$ *and* $g(z) = \frac{1-\beta(z)^2}{(\alpha_1-\alpha_2)\beta(z)}$, *where* $\beta(z)$ *is meromorphic in* $\mathbb{C}^n$ *and* $\alpha_1 = -\alpha + \sqrt{\alpha^2-1}$, $\alpha_2 = -\alpha - \sqrt{\alpha^2-1}$, *and* $\alpha^2 \neq 0, 1$.

In 2021, Li and Xu [33] considered the quadratic trinomial partial differential difference equations

$$f(z_1 + c_1, z_2 + c_2)^2 + 2\alpha f(z_1 + c_1, z_2 + c_2)\frac{\partial f}{\partial z_1} + \left(\frac{\partial f}{\partial z_1}\right)^2 = e^{g(z_1+c_1, z_2+c_2)} \tag{1.5}$$

and obtained the following result.

Theorem B. [33] *Let* $\alpha^2 \neq 0, 1$, $c_2 \neq 0$, *and* g *be a non-constant polynomial in* $\mathbb{C}^2$, *and not the form of* $\phi(z_2)$. *If* (1.5) *admits a finite-order transcendental entire solution f, then g must be of the form* $g(z_1, z_2) = a_1z_1 + a_2z_2 + b$, *where* $a_1(\neq 0), a_2, b_2 \in \mathbb{c}$. *Furthermore,* $f(z)$ *must satisfy one of the following cases*:

(*i*)

$$f(z_1, z_2) = \frac{\sqrt{2}}{a_1}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}(a_1z_1+a_2z_2+b)},$$

where ξ ($\neq 0$), $a_1, a_2, b, c_1, c_2, A_1, A_2 \in \mathbb{C}$ *satisfying*

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{a_1(A_1\xi + A_2\xi^{-1})}{2(A_2\xi + A_1\xi^{-1})}.$$

(*ii*)

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}\left(\frac{A_2}{a_{11}}e^{a_{11}z_1+a_{12}z_2+b_1} + \frac{A_1}{a_{21}}e^{a_{21}z_1+a_{22}z_2+b_2}\right),$$

where a_j ($\neq 0$), $b_j \in \mathbb{C}$, ($j = 1, 2$) satisfy $a_{11}z_1 + a_{12}z_2 \neq a_{21}z_1 + a_{22}z_2$,

$$g(z_1, z_2) = (a_{11} + a_{21})z_1 + (a_{12} + a_{22})z_2 + b_1 + b_2, \quad \textit{and}$$

$$e^{a_{11}c_1+a_{12}c_2} = \frac{A_2}{A_1}a_{11}, \quad e^{a_{21}c_1+a_{2}c_2} = \frac{A_1}{A_2}a_{21}, \quad e^{a_1c_1+a_2c_2} = a_{11}a_{21}.$$

2 Main results

Inspired by Theorems A and B, and utilizing difference analogs of the Nevanlinna theory of several complex variables [34,35], we investigate the existence and forms of transcendental entire solutions of the following trinomial second-order partial differential-difference equations:

$$L(f)^2 + 2hL(f)f(z_1 + c_1, z_2 + c_2) + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1,z_1)} \tag{2.1}$$

and

$$\tilde{L}(f)^2 + 2h\tilde{L}(f)f(z_1 + c_1, z_2 + c_2) + (f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2))^2 = e^{g(z_1,z_2)}, \tag{2.2}$$

where $\tilde{L}(f) = \gamma\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \delta\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2} + \eta\frac{\partial^2 f(z_1,z_2)}{\partial z_1\partial z_2}$, and $L(f) = \tilde{L} + \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}$, $\alpha,\beta,\gamma,\delta$, and η are the constants in $\mathbb{C}$, $c = (c_1, c_2) \in \mathbb{C}^2$, and $g(z)$ is a polynomial in $\mathbb{C}^2$.

Before we state our main results, let us first set that

$$A_1 := \frac{1}{2\sqrt{1+h}} - \frac{1}{2i\sqrt{1-h}} \quad \text{and} \quad A_2 := \frac{1}{2\sqrt{1+h}} + \frac{1}{2i\sqrt{1-h}}, \tag{2.3}$$

where $h \neq 0, \pm 1, 2$ is a constant in $\mathbb{C}$,

$$\begin{cases} R_1 = \alpha + \gamma a_1 + \frac{1}{2}\eta a_2, \quad R_2 = \beta + \delta a_2 + \frac{1}{2}\eta a_1, \\ R_3 = \gamma c_2^2 + \delta c_1^2 - \eta c_1 c_2, \quad R_4 = \alpha a_1 + \beta a_2 + \frac{1}{2}(\gamma a_1^2 + \delta a_2^2 + \eta a_1 a_2), \\ R_{j5} = \alpha + 2\gamma a_{j1} + \eta a_{j2}, \quad R_{j6} = \beta + 2\delta a_{j2} + \eta a_{j1}, \\ R_{j7} = \alpha a_{j1} + \beta a_{j2} + \gamma a_{j1}^2 + \delta a_{j2}^2 + \eta a_{j1}a_{j2}, \quad j = 1, 2. \end{cases} \tag{2.4}$$

Now, we state our results as follows.

Theorem 2.1. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $h \in \mathbb{C}$ such that $h \neq 0, \pm 1$ and $g(z_1, z_2)$ is a polynomial in $\mathbb{C}^2$. If $f(z)$ be a finite-order transcendental entire solution of equation* (2.1)*, then one of the following cases occurs.*

(*i*) *The form of the solution f is of the form $f(z_1, z_2) = \sum A e^{l_1 z_1 + l_2 z_2}$, where A, l_1, and l_2 are the arbitrary constants in $\mathbb{C}$ with $\alpha l_1 + \beta l_2 + \gamma l_1^2 + \delta l_2^2 + \eta l_1 l_2 = 0$ and*

$$g(z_1, z_2) = 2Ln\left(\pm\sum A' e^{l_1 z_1 + l_2 z_2}\right), \text{ where } A' = Ae^{(l_1c_1 + l_2c_2)}.$$

(*ii*) *g must be of the form $g(z_1, z_2) = L(z_1, z_2) + H(s_1) + B$, where $L(z_1, z_2) = a_1 z_1 + a_2 z_2$, $H(s_1)$ is a polynomial in $s_1 = c_2 z_1 - c_1 z_2$, a_1, a_2, B are constants in $\mathbb{C}$, and the form of the solution is*

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}[L(z_1,z_2)+H(s_1)-L(c)+B]}.$$

$L(z_1, z_2)$ satisfies relation

$$e^{\frac{1}{2}L(c_1,c_2)} = \frac{A_2\xi + A_1\xi^{-1}}{2(A_1\xi + A_2\xi^{-1})}\left[R_4 + (R_1c_2 - R_2c_1)a_0 + \frac{1}{2}R_3 a_0^2\right],$$

where a_0 is the coefficient of s_1 of the polynomial $H(s_1)$ and R_j's are defined in (2.4)*. In particular, if $R_1c_2 - R_2c_1 \neq 0$ or $R_3 \neq 0$, then $H(s_1)$ becomes linear in s_1.*

(*iii*) *$g(z_1, z_2)$ must be of the form $g(z_1, z_2) = L(z_1, z_2) + H(s_1) + B$, where $L(z_1, z_2) = L_1(z_1, z_2) + L_2(z_1, z_2)$ and $H(s_1) = H_1(s_1) + H_2(s_1)$ with $L_1(z_1, z_2) + H_1(s_1) \neq L_2(z_1, z_2) + H_2(s_1)$ and $L_j(z_1, z_2) = a_{j1}z_1 + a_{j2}z_2$, $B = B_1 + B_2$; $H_j(s_1)$ is a polynomial in $s_1 = c_2 z_1 - c_1 z_2$ for $j = 1, 2$, and B_1, B_2, and a_{ji} are the constants in $\mathbb{C}$, and the form of the solution is*

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}[A_2 e^{(L_1(z_1,z_2)+H_1(s_1)-L_1(c)+B_1)} + A_1 e^{(L_2(z_1,z_2)+H_2(s_1)-L_2(c)+B_2)}],$$

where $L_1(z_1, z_2)$ and $L_2(z_1, z_2)$, respectively, satisfy the relations

$$e^{L_1(c)} = \frac{A_2}{A_1}[R_{17} + (R_{15}c_2 - R_{16}c_1)a_0 + R_3 a_0^2] \quad \text{and} \quad e^{L_2(c)} = \frac{A_1}{A_2}[R_{27} + (R_{25}c_2 - R_{26}c_1)a_{00} + R_3 a_{00}^2],$$

a_0 and a_{00}, respectively, the coefficients of the linear term of the polynomials $H_1(s_1)$ and $H_2(s_1)$, and R_{ij}'s are defined in (2.4)*. In particular, if $R_{15}c_2 - R_{16}c_1 \neq 0$ or $R_3 \neq 0$, then H_1 becomes linear in s_1. Similarly, if $R_{25}c_2 - R_{26}c_1 \neq 0$ or $A_3 \neq 0$, then H_2 becomes linear in s_1.*

Next, we exhibit some examples in support of Theorem 2.1.

Example 2.2. Let $\alpha = \beta = 0, \gamma = \delta = 1, \eta = -2, h = -5/4, c_1 = c_2 = \pi i$, and $g(z) = 2(z_1 + z_2)$. Then, it can be easily seen that $f(z_1, z_2) = e^{z_1+z_2}$ is a solution of (2.1), which is the conclusion (i) of Theorem 2.1.

Example 2.3. $\alpha = \beta = 0,\ \xi = c_1 = c_2 = \gamma = \delta = 1,\ \eta = 2$, and $g(z_1, z_2) = z_1 + z_2 + (z_1 - z_2)^2 + 1$. Then, in view of conclusion (ii) of Theorem 2.1, one can easily verify that $f(z_1, z_2) = \frac{A_1 + A_2}{\sqrt{2}} e^{[z_1+z_2+(z_1-z_2)^2+1]/2}$ is a solution of

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2} + 2\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + 2h\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2} + 2\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right) f(z_1 + c_1, z_2 + c_2)$$
$$+ f(z_1 + c_1, z_2 + c_2)^2 = e^{[z_1+z_2+(z_1-z_2)^2+1]},$$

where A_1 and A_2 are given by (2.3).

Example 2.4. Let $\alpha = \beta = 0,\ \gamma = \delta = 1,\ \eta = -2,\ L_1(z) = z_1 + z_2,\ L_2(z) = 2z_1 + z_2,\ H_1(s_1) = H_2(s_2) = 0$, and $B_1 = B_2 = 1$. Then, in view of conclusion (iii) of Theorem 2.1, it can be easily verified that $f(z_1, z_2) = \frac{1}{2}[A_1 e^{z_1+2z_2+1} + A_2 e^{2z_1+z_2+1}]$ is a solution of

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2} - 2\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + 2h\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2} - 2\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right) f(z_1 + c_1, z_2 + c_2)$$
$$+ f(z_1 + c_1, z_2 + c_2)^2 = e^{[3z_1+3z_2+2]}, \quad \text{where } A_1 \text{ and } A_2 \text{ are given by (2.3).}$$

Theorem 2.5. *Let* $c = (c_1, c_2) \in \mathbb{C}^2, \gamma, \delta, \eta, h \in \mathbb{C}$ *such that* $\gamma \neq 0, h \neq 0, \pm 1$ *and* $g(z_1, z_2)$ *is a polynomial in* $\mathbb{C}^2$. *Let* $f(z)$ *be a finite-order transcendental entire solution of* (2.2). *Then, one of the following cases must occur.*

(i) $f(z_1, z_2) = \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2)$, *where* ϕ_1 *and* ϕ_2 *are finite-order transcendental entire functions in* $\mathbb{C}^2$ *satisfying*

$$\phi_1(\beta_1 z_1 - \alpha_1 z_2 + \beta_1 c_1 - \alpha_1 c_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2 + \beta_2 c_1 - \alpha_2 c_2) - \phi_1(\beta_1 z_1 - \alpha_1 z_2) - \phi_2(\beta_2 z_1 - \alpha_2 z_2) = \pm e^{g(z)/2},$$

where $\alpha_1 \alpha_2 = \gamma, \beta_1 \beta_2 = \delta$ *and* $\alpha_1 \beta_2 + \beta_1 \alpha_2 = \eta$.

(ii) $g(z_1, z_2) = a_1 z_1 + a_2 z_2 + H(c_2 z_1 - c_1 z_2) + B$, *where* H *is a polynomial in* $c_2 z_1 - c_1 z_2$ *and* $a_1 c_1 + a_2 c_2 = 4k\pi i$, $k \in \mathbb{Z}$,

$$f(z_1, z_2) = \pm \frac{1}{\gamma} \int_0^{z_1/\alpha_1} \int_0^{z_1/\alpha_2} e^{\frac{1}{2}[a_1 z_1 + a_2 z_2 + H(c_2 z_1 - c_1 z_2) + B]} dz_1 dz_1$$
$$+ \frac{1}{\alpha_2} \int_0^{z_1/\alpha_2} G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1}\right) dz_1 + G_1\left(\frac{\alpha_2 z_2 - \beta_2 z_1}{\alpha_2}\right),$$

where $\alpha_1, \alpha_2, \beta_1$, *and* β_2 *are defined in* (i) *and* G_0 *and* G_1 *are finite-order transcendental entire functions in* $\mathbb{C}^2$ *satisfying*

$$\frac{1}{\alpha_2} \int_0^{z_1/\alpha_2} \left[G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1} + \frac{\alpha_1 c_2 - \beta_1 c_1}{\alpha_1}\right) - G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1}\right)\right] dz_1$$
$$+ G_1\left(\frac{\alpha_2 z_2 - \beta_2 z_1}{\alpha_2} + \frac{\alpha_2 c_2 - \beta_2 c_1}{\alpha_2}\right) - G_1\left(\frac{\alpha_2 z_2 - \beta_2 z_1}{\alpha_2}\right) = 0.$$

(iii) *If* $\gamma c_2^2 + \delta c_1^2 \neq \eta c_1 c_2$, *then* g *must be of the form* $g(z_1, z_2) = a_1 z_1 + a_2 z_2 + B$, $a_1, a_2, B \in \mathbb{C}$, *and the form of the solution is*

$$f(z_1, z_2) = \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2) + \frac{2\sqrt{2}(A_1 \xi + A_2 \xi^{-1})}{\gamma a_1^2 + \delta a_2^2 + \eta a_1 a_2} e^{\frac{1}{2}[a_1 z_1 + a_2 z_2 + B]},$$

where $\xi(\neq 0) \in \mathbb{C}$, $\gamma a_1^2 + \delta a_2^2 + \eta a_1 a_2 \neq 0$, $\alpha_1, \alpha_2, \beta_1$, *and* β_2 *are the same as in* (i), A_1 *and* A_2 *are defined in* (2.3), ϕ_1 *and* ϕ_2 *are finite-order transcendental entire functions in* $\mathbb{C}^2$ *such that*

$$\phi_1(\beta_1 z_1 - \alpha_1 z_2 + \beta_1 c_1 - \alpha_1 c_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2 + \beta_2 c_1 - \alpha_2 c_2) = \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2) \quad \text{and}$$

$$e^{\frac{1}{2}[a_1c_1 + a_2c_2]} = \frac{(A_2\xi + A_1\xi^{-1})(\gamma a_1^2 + \delta a_2^2 + \eta a_1 a_2)}{4(A_1\xi + A_2\xi^{-1})} + 1.$$

(*iv*) *If* $\gamma c_2^2 + \delta c_1^2 \neq \eta c_1 c_2$, *then g must be of the form* $g(z_1, z_2) = L_1(z_1, z_2) + L_2(z_1, z_2) + B_1 + B_2$, *where* $L_j(z_1, z_2) = a_{j1}z_1 + a_{j2}z_2$ *with* $L_1(z_1, z_2) \neq L_2(z_1, z_2)$, $a_{ij}, B_j \in \mathbb{C}$ *and the form of the solution is*

$$f(z_1, z_2) = \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2) + \frac{A_1 e^{L_1(z_1,z_2)+B_1}}{\sqrt{2}(\gamma a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12})} + \frac{A_2 e^{L_2(z_1,z_2)+B_2}}{\sqrt{2}(\gamma a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22})},$$

where $\gamma a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22} \neq 0$, $\gamma a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12} \neq 0$, $\alpha_1, \alpha_2, \beta_1, \beta_2$ *are the same as in* (i), A_1 *and* A_2 *are defined in* (2.3), ϕ_1 *and* ϕ_2 *are finite-order transcendental entire functions in* $\mathbb{C}^2$ *such that*

$$\phi_1(\beta_1 z_1 - \alpha_1 z_2 + \beta_1 c_1 - \alpha_1 c_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2 + \beta_2 c_1 - \alpha_2 c_2) = \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2) \quad \text{and}$$

$$\begin{cases} e^{L_1(c)} = \dfrac{A_2}{A_1}(\gamma a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12}) + 1, \\ e^{L_2(c)} = \dfrac{A_1}{A_2}(\gamma a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22}) + 1. \end{cases}$$

The following examples show that the forms of solutions are precise.

Example 2.6. Let $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$. Choose $c = (c_1, c_2) \in \mathbb{C}^2$ such that $c_1 - c_2 = 2\log 2$. Then, in view of conclusion (i) of Theorem 2.5, we can easily deduce that $f(z_1, z_2) = 2e^{(z_1+z_2)/2} + e^{z_2/2}$ is a solution of (2.2) with $g(z_1, z_2) = z_1 - z_2$.

Example 2.7. Let $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$, $a_1 = a_2 = 1$, and $c_1 = c_2 = 4\pi i$. Let $g(z_1, z_2) = z_1 + z_2$, $G_0(z_2) = 2e^{z_2/2}$ and $G_1(z_2) = 4e^{z_2/2}$. Then, in view of conclusion (ii) of Theorem 2.5, we can easily deduce that $f(z_1, z_2) = 4e^{(z_1+z_2)/2}$ is a solution of (2.2).

Example 2.8. Let $\xi = \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $c_1 = \log 2 + \pi i$, $c_2 = \log 2 - \pi i$, and $\psi_1(z_1 - z_2) = \phi_1(z_1 - z_2) + \phi_2(z_1 - z_2) = e^{z_1 - z_2}$. Then, in view of conclusion (iii) of Theorem 2.5, we can easily see that $f(z_1, z_2) = e^{z_1 - z_2} + \frac{A_1 + A_2}{\sqrt{2}}e^{(z_1+z_2+1)/2}$ is a solution of (2.2), where A_1 and A_2 are given by (2.3).

Example 2.9. Let $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\phi_1(z_1 - z_2) = e^{z_1 - z_2}$, and $\phi_2(z_1 - z_2) = 0$. Choose $c = (c_1, c_2) \in \mathbb{C}^2$ such that $c_1 - c_2 = 4\pi i$. Let $L_1(z) = z_1 + 2z_2$ and $L_2(z) = 2z_1 + z_2$ such that $e^{L_1(c)} = (9A_2 + A_1)/A_1$ and $e^{L_2(c)} = (9A_1 + A_2)/A_2$, where A_1 and A_2 are given by (2.3). Then, we can easily verify that $f(z_1, z_2) = e^{z_1 - z_2} + \frac{1}{9\sqrt{2}}(A_1 e^{z_1 + 2z_2 + 5} + A_2 e^{2z_1 + z_2 + 6})$ is a solution of (2.2) with $g(z_1, z_2) = 2z_1 + 2z_2 + 11$.

3 Proofs of the main results

Before proving the main results, we present here some necessary lemmas that will play a key role to prove the main results of this article.

Lemma 3.1. [36] *Let* $f_j \not\equiv 0 \ \ (j = 1, 2, 3)$ *be meromorphic functions on* $\mathbb{C}^n$ *such that* f_1 *is not constant,* $f_1 + f_2 + f_3 = 1$, *and such that*

$$\sum_{j=1}^{3}\left\{N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j)\right\} < \lambda T(r, f_j) + O(\log^+ T(r, f_j))$$

holds for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ *is a positive number. Then, either* $f_2 = 1$ *or* $f_3 = 1$.

Lemma 3.2. [36] *Let* $a_0(z), a_1(z), \dots, a_n(z)$ $(n \geqslant 1)$ *be meromorphic functions on* $\mathbb{C}^m$ *and* $g_0(z), g_1(z), \dots, g_n(z)$ *are entire functions on* $\mathbb{C}^m$ *such that* $g_j(z) - g_k(z)$ *are not constants for* $0 \leqslant j < k \leqslant n$. *If* $\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$, *and* $\|T(r, a_j) = o(T(r))$, *where* $T(r) = \min_{0\leqslant j<k\leqslant n} T(r, e^{g_j - g_k})$ *for* $j = 0, 1, \dots, n$, *then* $a_j(z) \equiv 0$ *for each* $j = 0, 1, \dots, n$.

Lemma 3.3. [37–39] *For an entire function* F *on* $\mathbb{C}^n$, $F(0) \not\equiv 0$ *and put* $\rho(n_F) = \rho < \infty$. *Then, there exist a canonical function* f_F *and a function* $g_F \in \mathbb{C}^n$ *such that* $F(z) = f_F(z)e^{g_F(z)}$. *For the special case* $n = 1$, f_F *is the canonical product of Weierstrass.*

Lemma 3.4. [40] *If* g *and* h *are entire functions on the complex plane* $\mathbb{C}$ *and* $g(h)$ *is an entire function of finite-order, then there are only two possible cases: either*

(*i*) *the internal function* h *is a polynomial and the external function* g *is of finite-order; or else*
(*ii*) *the internal function* h *is not a polynomial but a function of finite-order, and the external function* g *is of zero order.*

Proof of Theorem 2.1. Let $f(z_1, z_2)$ be a transcendental entire solution of equation (2.1). Let us make a transformation

$$L(f) = \frac{1}{\sqrt{2}}(U - V), \quad f(z_1 + c_1, z_2 + c_2) = \frac{1}{\sqrt{2}}(U + V). \tag{3.1}$$

Then, equation (2.1) becomes

$$(1 + h)U^2 + (1 - h)V^2 = e^{g(z_1,z_2)},$$

which can be rewritten as

$$\left(\frac{\sqrt{1+h}\,U}{e^{g(z_1,z_2)/2}} + i\frac{\sqrt{1-h}\,V}{e^{g(z_1,z_2)/2}}\right)\left(\frac{\sqrt{1+h}\,U}{e^{g(z_1,z_2)/2}} - i\frac{\sqrt{1-h}\,V}{e^{g(z_1,z_2)/2}}\right) = 1. \tag{3.2}$$

Since f is finite-order transcendental entire, in view of Lemmas 3.3 and 3.4, it follows from (3.2) that

$$\begin{cases} \dfrac{\sqrt{1+h}\,U}{e^{g(z_1,z_2)/2}} + i\dfrac{\sqrt{1-h}\,V}{e^{g(z_1,z_2)/2}} = e^{h_1(z_1,z_2)} \\ \dfrac{\sqrt{1+h}\,U}{e^{g(z_1,z_2)/2}} - i\dfrac{\sqrt{1-h}\,V}{e^{g(z_1,z_2)/2}} = e^{-h_1(z_1,z_2)}, \end{cases}$$

where $h_1(z_1, z_2)$ is a polynomial in $\mathbb{C}^2$, from which we obtain

$$U = \frac{e^{h_1(z_1,z_2)} + e^{-h_1(z_1,z_2)}}{2\sqrt{1+h}} e^{g(z_1,z_2)/2} \quad \text{and} \quad V = \frac{e^{h_1(z_1,z_2)} - e^{-h_1(z_1,z_2)}}{2i\sqrt{1-h}} e^{g(z_1,z_2)/2}. \tag{3.3}$$

Set

$$\gamma_1(z_1, z_2) = \frac{g(z_1, z_2)}{2} + h_1(z_1, z_2) \quad \text{and} \quad \gamma_2(z_1, z_2) = \frac{g(z_1, z_2)}{2} - h_1(z_1, z_2). \tag{3.4}$$

Therefore, it follows from (3.3) and (3.4) that

$$U = \frac{e^{\gamma_1(z_1,z_2)} + e^{\gamma_2(z_1,z_2)}}{2\sqrt{1+h}} \quad \text{and} \quad V = \frac{e^{\gamma_1(z_1,z_2)} - e^{\gamma_2(z_1,z_2)}}{2i\sqrt{1-h}}. \tag{3.5}$$

In view of (3.1) and (3.5), we obtain that

$$\begin{cases} L(f) = \dfrac{1}{\sqrt{2}}[A_1 e^{\gamma_1(z_1,z_2)} + A_2 e^{\gamma_2(z_1,z_2)}], \\ f(z_1 + c_1, z_2 + c_2) = \dfrac{1}{\sqrt{2}}[A_2 e^{\gamma_1(z_1,z_2)} + A_1 e^{\gamma_2(z_1,z_2)}]. \end{cases} \tag{3.6}$$

After simple calculation, it follows from the two equations of (3.6) that

$$\frac{A_2}{A_1}Q_1(z_1,z_2)e^{\gamma_1(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}+Q_2(z_1,z_2)e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}-\frac{A_2}{A_1}e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}=1, \tag{3.7}$$

where

$$\begin{aligned}Q_j(z_1,z_2)&=\alpha\frac{\partial\gamma_j}{\partial z_1}+\beta\frac{\partial\gamma_j}{\partial z_2}+\gamma\left(\left(\frac{\partial\gamma_j}{\partial z_1}\right)^2+\frac{\partial^2\gamma_j}{\partial z_1^2}\right)+\delta\left(\left(\frac{\partial\gamma_j}{\partial z_2}\right)^2+\frac{\partial^2\gamma_j}{\partial z_2^2}\right)\\&\quad+\eta\left(\frac{\partial\gamma_j}{\partial z_1}\frac{\partial\gamma_j}{\partial z_2}+\frac{\partial^2\gamma_j}{\partial z_1\partial z_2}\right),\quad j=1,2.\end{aligned} \tag{3.8}$$

Now, we discuss two possible cases.

Case 1. Let $\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)=k$, a constant in $\mathbb{C}$. In view of (3.4), we conclude that $h_1(z_1,z_2)$ is constant. Let $\xi=e^{h_1(z_1,z_2)}\in\mathbb{C}$. Then, (3.6) yields that

$$L(f)=C_1e^{g(z_1,z_2)/2},\quad f(z_1+c_1,z_2+c_2)=C_2e^{g(z_1,z_2)/2}, \tag{3.9}$$

where $C_1=\frac{1}{\sqrt{2}}(A_1\xi+A_2\xi^{-1})$, $C_2=\frac{1}{\sqrt{2}}(A_2\xi+A_1\xi^{-1})$.

Note that

$$C_2\neq0\quad\text{and}\quad C_1^2+C_2^2=\frac{1}{2}[(A_1^2+A_2^2)(\xi^2+\xi^{-2})+4A_1A_2]. \tag{3.10}$$

If $C_1=0$, in view of (3.9) and (3.10), it follows that

$$\begin{cases}\alpha\frac{\partial f}{\partial z_1}+\beta\frac{\partial f}{\partial z_2}+\gamma\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1\partial z_2}=0,\\ f(z_1+c_1,z_2+c_2)=\pm e^{g(z_1,z_2)/2}.\end{cases} \tag{3.11}$$

From the first equation of (3.11), we obtain that

$$f(z_1,z_2)=\sum Ae^{l_1z_1+l_2z_2},$$

where A,l_1, and l_2 are the arbitrary constants in $\mathbb{C}$ satisfying the relation $\alpha l_1+\beta l_2+\gamma l_1^2+\delta l_2^2+\eta l_1l_2=0$. From the second equation of (3.11), we obtain that

$$g(z_1,z_2)=2Ln\left(\pm\sum A'e^{l_1z_1+l_2z_2}\right),$$

where $A'=Ae^{l_1c_1+l_2c_2}$. This is conclusion (i).

If $C_1\neq0$, then from (3.9), we obtain that

$$\begin{aligned}&\alpha\frac{\partial g}{\partial z_1}+\beta\frac{\partial g}{\partial z_2}+\gamma\left(\frac{1}{2}\left(\frac{\partial g}{\partial z_1}\right)^2+\frac{\partial^2 g}{\partial z_1^2}\right)+\delta\left(\frac{1}{2}\left(\frac{\partial g}{\partial z_2}\right)^2+\frac{\partial^2 g}{\partial z_2^2}\right)\\&\quad+\eta\left(\frac{1}{2}\frac{\partial g}{\partial z_1}\frac{\partial g}{\partial z_2}+\frac{\partial^2 g}{\partial z_1\partial z_2}\right)=\frac{2C_1}{C_2}e^{\frac{1}{2}[g(z_1+c_1,z_2+c_2)-g(z_1,z_2)]}.\end{aligned} \tag{3.12}$$

Since g is a polynomial in $\mathbb{C}^2$, it follows from (3.12) that $g(z_1+c_1,z_2+c_2)-g(z_1,z_2)$ must be constant, say $\zeta\in\mathbb{C}$.

Then, g can be written as $g(z_1,z_2)=L(z_1,z_2)+H(s_1)+B$, where $L(z_1,z_2)=a_1z_1+a_2z_2$, $H(s_1)$ is a polynomial in $s_1=c_2z_1-c_1z_2$, a_1,a_2, and B are the constants in $\mathbb{C}$. Hence, it follows from (3.12) that

$$(R_1c_2-R_2c_1)H'+R_3\left(\frac{1}{2}H'^2+H''\right)=\frac{2(A_1\xi+A_2\xi^{-1})}{A_2\xi+A_1\xi^{-1}}e^{\frac{1}{2}L(c)}-R_4, \tag{3.13}$$

where $R's$ are defined in (2.4).

If $R_1c_2 - R_2c_1 = 0 = R_3$, then from (3.13), we obtain

$$e^{\frac{1}{2}L(c)} = \frac{A_2\xi + A_1\xi^{-1}}{2(A_1\xi + A_2\xi^{-1})}R_4.$$

If $R_1c_2 - R_2c_1 \neq 0$ or (here “or” is in inclusive sense) $R_3 \neq 0$, then it follows from (3.13) that H' must be constant, say a_0, which is the coefficient of s_1 in the polynomial $H(s_1)$.

Therefore, from (3.13), we obtain that

$$e^{\frac{1}{2}L(c)} = \frac{A_2\xi + A_1\xi^{-1}}{2(A_1\xi + A_2\xi^{-1})}\left[R_4 + (R_1c_2 - R_2c_1)a_0 + \frac{1}{2}R_3a_0^2\right]. \tag{3.14}$$

Hence, in either case $L(z)$ satisfies relation (3.14).

Therefore, in view of the second equation of (3.9), we obtain the form of the solution as

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}[L(z_1,z_2)+H(s_1)-L(c)+B]}.$$

This is conclusion (ii).

Case 2 Let $\gamma_2(z_1 + c_1, z_2 + c_2) - \gamma_1(z_1 + c_1, z_2 + c_2)$ be non-constant. Then, in view of (3.7), it follows that $Q_1(z_1, z_2)$ and $Q_2(z_1, z_2)$ both cannot be zero at the same time.

If $Q_1(z_1, z_2) \equiv 0$ and $Q_2(z_1, z_2) \not\equiv 0$, then (3.7) yields that

$$Q_2(z_1, z_2)e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)} - \frac{A_2}{A_1}e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)} = 1.$$

In view of the aforementioned equation, it follows that

$$\begin{aligned} N\left(r, \frac{1}{A_2e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)} + A_1}\right) &= N\left(r, \frac{1}{Q_2(z_1, z_2)e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}}\right) \\ &= S(r, e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}). \end{aligned}$$

Also, note that

$$\begin{aligned} &N(r, e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}) = S(r, e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}),\\ &N\left(r, \frac{1}{e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}}\right) = S(r, e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}). \end{aligned}$$

By the second main theorem of Nevanlinna for several complex variables, we obtain

$$\begin{aligned} &T(r, e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)})\\ &\leqslant \overline{N}(r, e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}) + \overline{N}\left(r, \frac{1}{e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}}\right)\\ &\quad + \overline{N}\left(r, \frac{1}{e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)} - A_1/A_2}\right) + S(r, e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)})\\ &\leqslant S(r, e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}) + S(r, e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}). \end{aligned}$$

This implies that $\gamma_2(z_1 + c_1, z_2 + c_2) - \gamma_1(z_1 + c_1, z_2 + c_2)$ is constant, which is a contradiction. Similarly, we can obtain a contradiction for the case $Q_1(z_1, z_2) \not\equiv 0$ and $Q_1(z_1, z_2) \equiv 0$. Hence, $Q_1(z_1, z_2) \not\equiv 0$ and $Q_2(z_1, z_2) \not\equiv 0$.

Since $\gamma_1(z_1, z_2)$ and $\gamma_2(z_1, z_2)$ are the polynomials in $\mathbb{C}^2$ and $\gamma_2(z_1 + c_1, z_2 + c_2) - \gamma_1(z_1 + c_1, z_2 + c_2)$ is non-constant, applying Lemma 3.1 to equation (3.7), we obtain that either

$$\frac{A_2}{A_1}Q_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} = 1, \quad \text{or} \quad Q_2(z_1, z_2)e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)} = 1.$$

If

$$\frac{A_2}{A_1}Q_1(z_1, z_2)e^{\gamma_1(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)} = 1, \tag{3.15}$$

then from (3.7), it follows that

$$Q_2(z_1, z_2)e^{\gamma_2(z_1,z_2)-\gamma_2(z_1+c_1,z_2+c_2)} = \frac{A_1}{A_2}. \tag{3.16}$$

As $\gamma_1(z_1, z_2)$ and $\gamma_2(z_1, z_2)$ are polynomials, in view of (3.15) and (3.16), we conclude that $\gamma_1(z_1, z_2) - \gamma_1(z_1 + c_1, z_2 + c_2)$ and $\gamma_2(z_1, z_2) - \gamma_2(z_1 + c_1, z_2 + c_2)$ both are constants in $\mathbb{C}$, and hence, we can obtain that $\gamma_1(z_1, z_2) = L_1(z_1, z_2) + H_1(s_1) + B_1$ and $\gamma_2(z_1, z_2) = L_2(z_1, z_2) + H_2(s_1) + B_2$, where $L_j(z_1, z_2) = a_{j1}z_1 + a_{j2}z_2$, $H_j(s_1)$ is a polynomial in $s_1 = c_2z_1 - c_1z_2$, and a_{j1}, a_{j2}, B_1, and B_2 are the constants in $\mathbb{C}$ for $j = 1, 2$. Note that $L_1(z_1, z_2) + H_1(s_1) \neq L_2(z_1, z_2) + H_2(s_1)$. Otherwise, $\gamma_2(z_1 + c_1, z_2 + c_2) - \gamma_1(z_1 + c_1, z_2 + c_2)$ would become constant, a contradiction to our assumption. Hence, the form of the polynomial g is $g(z_1, z_2) = L(z_1, z_2) + H(s_1) + B$, where $L(z_1, z_2) = L_1(z_1, z_2) + L_2(z_1, z_2)$, $H(s_1) = H_1(s_1) + H_2(s_1)$, and $B = B_1 + B_2$.

Therefore, in view of (3.15) and (3.16), we obtain that

$$\begin{cases} (R_{15}c_2 - R_{16}c_1)H_1' + R_3(H_1'^2 + H_1'') = \dfrac{A_1}{A_2}e^{L_1(c)} - R_{17}, \\ (R_{25}c_2 - R_{26}c_1)H_2' + R_3(H_2'^2 + H_2'') = \dfrac{A_2}{A_1}e^{L_2(c)} - R_{27}, \end{cases} \tag{3.17}$$

where $R's$ are defined in (2.4).

Then, by similar arguments as in Case 1, we obtain from (3.17) that

$$\begin{cases} e^{L_1(c)} = \dfrac{A_2}{A_1}(R_{17} + (R_{15}c_2 - R_{16}c_1)a_0 + R_3a_0^2), \\ e^{L_2(c)} = \dfrac{A_1}{A_2}(R_{27} + (R_{25}c_2 - R_{26}c_1)a_{00} + R_3a_{00}^2), \end{cases} \tag{3.18}$$

where a_0 and a_{00}, respectively, the coefficients of the linear term of the polynomials $H_1(s_1)$ and $H_2(s_1)$.

Therefore, in view of the second equation of (3.6), we obtain

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}(A_2e^{L_1(z_1,z_2)+H_1(s_1)-L_1(c)+B_1} + A_1e^{L_2(z_1,z_2)+H_2(s_1)-L_2(c)+B_2}),$$

where $L_1(c)$ and $L_2(c)$ can be found from (3.18).

This is conclusion (iii).

If $Q_2(z_1, z_2)e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)} = 1$, then it follows from equation (3.7) that $Q_1(z_1, z_2)e^{\gamma_1(z_1,z_2)-\gamma_2(z_1+c_1,z_2+c_2)} = 1$. Since $\gamma_1(z_1, z_2)$ and $\gamma_2(z_1, z_2)$ are both polynomials in $\mathbb{C}^2$, it follows that $\gamma_2(z_1, z_2) - \gamma_1(z_1 + c_1, z_2 + c_2) = \eta_1$ and $\gamma_1(z_1, z_2) - \gamma_2(z_1 + c_1, z_2 + c_2) = \eta_2$, where $\eta_1, \eta_2 \in \mathbb{C}$. This implies that $\gamma_1(z_1, z_2) - \gamma_1(z_1 + c_1, z_2 + c_2) = \gamma_2(z_1, z_2) - \gamma_2(z_1 + c_1, z_2 + c_2) = \eta_1 + \eta_2$. Therefore, we can write $\gamma_1(z_1, z_2) = L(z_1, z_2) + H(s_1) + \zeta_1$ and $\gamma_2(z_1, z_2) = L(z_1, z_2) + H(s_1) + \zeta_2$. But, then we obtain $\gamma_2(z_1 + c_1, z_2 + c_2) - \gamma_1(z_1 + c_1, z_2 + c_2) = \zeta_2 - \zeta_1$, constants, which is a contradiction. □

Proof of Theorem 2.5. Let $f(z_1, z_2)$ be a finite-order transcendental entire solution of equation (2.2). Let us make a transformation

$$\tilde{L}(f) = \frac{1}{\sqrt{2}}(U - V), \quad f(z + c) - f(z) = \frac{1}{\sqrt{2}}(U + V). \tag{3.19}$$

Then, by similar arguments as in Theorem 2.1, we can obtain that

$$\begin{cases} \tilde{L}(f) = \dfrac{1}{\sqrt{2}}[A_1e^{\gamma_1(z_1,z_2)} + A_2e^{\gamma_2(z_1,z_2)}] \\ f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) = \dfrac{1}{\sqrt{2}}[A_2e^{\gamma_1(z_1,z_2)} + A_1e^{\gamma_2(z_1,z_2)}], \end{cases} \tag{3.20}$$

where γ_1 and γ_2 are defined in (3.4) and A_1 and A_2 are defined in (2.3).

In view of equations in (3.20), we obtain that

$$\frac{A_2Q_1(z_1,z_2)+A_1}{A_1}e^{\gamma_1(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}+\frac{A_1Q_2(z_1,z_2)+A_2}{A_1}e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}-\frac{A_2}{A_1}e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}=1, \tag{3.21}$$

where

$$Q_j(z_1,z_2)=\gamma\left(\left(\frac{\partial\gamma_j}{\partial z_1}\right)^2+\frac{\partial^2\gamma_j}{\partial z_1^2}\right)+\delta\left(\left(\frac{\partial\gamma_j}{\partial z_2}\right)^2+\frac{\partial^2\gamma_j}{\partial z_2^2}\right)+\eta\left(\frac{\partial\gamma_j}{\partial z_1}\frac{\partial\gamma_j}{\partial z_2}+\frac{\partial^2\gamma_j}{\partial z_1\partial z_2}\right),\quad j=1,2. \tag{3.22}$$

Now, we consider two possible cases.

Case 1. Let $\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)=k\in\mathbb{C}$. In view of (3.4), we conclude that $h_1(z_1,z_2)$ is constant. Set $e^{h_1}=\xi\in\mathbb{C}$. Then, (3.20) yields that

$$\tilde{L}(f)=C_1e^{g(z_1,z_2)/2},\quad f(z_1+c_1,z_2+c_2)-f(z_1,z_2)=C_2e^{g(z_1,z_2)/2}, \tag{3.23}$$

where $C_1=\frac{1}{\sqrt{2}}(A_1\xi+A_2\xi^{-1})$, $C_2=\frac{1}{\sqrt{2}}(A_2\xi+A_1\xi^{-1})$.

Note that

$$C_1^2+C_2^2=\frac{1}{2}[(A_1^2+A_2^2)(\xi^2+\xi^{-2})+4A_1A_2]. \tag{3.24}$$

Subcase 1.1. Let $C_1=0$. Then, in view of (3.24), we obtain that $C_2=\pm1$. Therefore, it follows from (3.23) that

$$\begin{cases}\gamma\dfrac{\partial^2f(z_1,z_2)}{\partial z_1^2}+\delta\dfrac{\partial^2f(z_1,z_2)}{\partial z_2^2}+\eta\dfrac{\partial^2f(z_1,z_2)}{\partial z_1\partial z_2}=0,\\ f(z_1+c_1,z_2+c_2)-f(z_1,z_2)=\pm e^{g(z_1,z_2)/2}.\end{cases} \tag{3.25}$$

Now, in view of the results in [35, page 2178, line 21], the first equation of (3.25) can be written as

$$(\alpha_1D+\beta_1D')(\alpha_2D+\beta_2D')f(z)=0, \tag{3.26}$$

where $D\equiv\frac{\partial}{\partial z_1}$, $D'\equiv\frac{\partial}{\partial z_2}$, $\alpha_1\alpha_2=\gamma$, $\beta_1\beta_2=\delta$, and $\alpha_1\beta_2+\alpha_2\beta_1=\eta$.

Solving (3.26), we obtain that

$$f(z_1,z_2)=\phi_1(\beta_1z_1-\alpha_1z_2)+\phi_2(\beta_2z_1-\alpha_2z_2),$$

where ϕ_1 and ϕ_2 are finite-order transcendental entire functions in $\mathbb{C}^2$.

In view of the second equation of (3.25), we obtain that

$$\phi_1(\beta_1z_1-\alpha_1z_2+\beta_1c_1-\alpha_1c_2)+\phi_2(\beta_2z_1-\alpha_2z_2+\beta_2c_1-\alpha_2c_2)-\phi_1(\beta_1z_1-\alpha_1z_2)-\phi_2(\beta_2z_1-\alpha_2z_2)=\pm e^{\frac{1}{2}g(z_1,z_2)}.$$

This is conclusion (i).

Subcase 1.2. Let $C_2=0$. Then, in view of (3.24), it yields that $C_1=\pm1$.

Therefore, it follows from (3.23) that

$$\begin{cases}\gamma\dfrac{\partial^2f(z_1,z_2)}{\partial z_1^2}+\delta\dfrac{\partial^2f(z_1,z_2)}{\partial z_2^2}+\eta\dfrac{\partial^2f(z_1,z_2)}{\partial z_1\partial z_2}=\pm e^{\frac{1}{2}g(z_1,z_2)},\\ f(z_1+c_1,z_2+c_2)-f(z_1,z_2)=0.\end{cases} \tag{3.27}$$

Clearly, second equation of (3.27) shows that f is a periodic function of period c. In view of the two equations in (3.27), it follows that $e^{\frac{1}{2}(g(z_1+c_1,z_2+c_2)-g(z_1,z_2))}=1$. This implies that $g(z_1,z_2)=a_1z_1+a_2z_2+H(c_2z_1-c_1z_2)+B$, where H is a polynomial in $c_2z_1-c_1z_2$ and $a_1c_1+a_2c_2=4k\pi i$, $k\in\mathbb{Z}$.

Now, in view of the results in [35, page 2178, line 21], the first equation of (3.27) can be written as

$$(\alpha_1D+\beta_1D')(\alpha_2D+\beta_2D')f(z_1,z_2)=\pm e^{\frac{1}{2}g(z_1,z_2)}, \tag{3.28}$$

where $D, D', \alpha_1, \alpha_2, \beta_1$, and β_2 are defined in (3.26).

Let $(\alpha_2 D + \beta_2 D')f(z_1, z_2) = u(z_1, z_2)$. Then, in view of the results in [[31], page 2182, line 22], we obtain

$$u(z_1, z_2) = \pm \frac{1}{\alpha_1} \int_0^{z_1/\alpha_1} e^{\frac{1}{2}g(z_1,z_2)} dz_1 + G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1}\right),$$

where G_0 is a finite-order transcendental entire function in $\mathbb{C}^2$.

Since we have assumed that $(\alpha_2 D + \beta_2 D')f(z_1, z_2) = u(z_1, z_2)$, we obtain that

$$\alpha_2 \frac{\partial f}{\partial z_1} + \beta_2 \frac{\partial f}{\partial z_2} = \pm \frac{1}{\alpha_1} \int_0^{z_1/\alpha_1} e^{\frac{1}{2}g(z_1,z_2)} \mathrm{d}z_1 + G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1}\right). \tag{3.29}$$

By similar argument, we obtain from (3.29) that

$$f(z_1, z_2) = \pm \frac{1}{\gamma} \int_0^{z_1/\alpha_1} \int_0^{z_1/\alpha_2} e^{\frac{1}{2}[a_1 z_1 + a_2 z_2 + H(c_2 z_1 - c_1 z_2) + B]} dz_1 dz_1 + \frac{1}{\alpha_2} \int_0^{z_1/\alpha_2} G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1}\right) dz_1 + G_1\left(\frac{\alpha_2 z_2 - \beta_2 z_1}{\alpha_2}\right),$$

where G_1 is a finite-order transcendental entire function in $\mathbb{C}^2$.

In view of the fact that $a_1 c_1 + a_2 c_2 = 4k\pi i$, $k \in \mathbb{Z}$, it follows from the second equation of (3.27) that

$$\frac{1}{\alpha_2} \int_0^{z_1/\alpha_2} \left[G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1} + \frac{\alpha_1 c_2 - \beta_1 c_1}{\alpha_1}\right) - G_0\left(\frac{\alpha_1 z_2 - \beta_1 z_1}{\alpha_1}\right)\right] dz_1 + G_1\left(\frac{\alpha_2 z_2 - \beta_2 z_1}{\alpha_2} + \frac{\alpha_2 c_2 - \beta_2 c_1}{\alpha_2}\right) - G_1\left(\frac{\alpha_2 z_2 - \beta_2 z_1}{\alpha_2}\right) = 0.$$

This is conclusion (ii).

Subcase 1.3. Let $C_1 \neq 0$ and $C_2 \neq 0$. Then, after simple calculations, (3.23) yields that

$$\begin{aligned} &\gamma\left(\frac{\partial^2 g}{\partial z_1^2} + \frac{1}{2}\left(\frac{\partial g}{\partial z_1}\right)^2\right) + \delta\left(\frac{\partial^2 g}{\partial z_2^2} + \frac{1}{2}\left(\frac{\partial g}{\partial z_2}\right)^2\right) + \eta\left(\frac{\partial^2 g}{\partial z_1 \partial z_2} + \frac{1}{2}\frac{\partial g}{\partial z_1}\frac{\partial g}{\partial z_2}\right) \\ &= \frac{2C_1}{C_2}\left[e^{\frac{1}{2}[g(z_1+c_1, z_2+c_2) - g(z_1,z_2)]} - 1\right]. \end{aligned} \tag{3.30}$$

Since g is a polynomial in $\mathbb{C}^2$, in view of (3.30), we conclude that $g(z_1 + c_1, z_2 + c_2) - g(z_1, z_2) = \xi$, $\xi \in \mathbb{C}$. This implies that $g(z_1, z_2) = L_1(z_1, z_2) + H(s_1) + B_1$, where $L_1(z_1, z_2) = a_{11} z_1 + a_{12} z_2$, $H(s_1)$ is a polynomial in $s_1 \coloneqq c_2 z_1 - c_1 z_2$, $a_{11}, a_{12}, B_1 \in \mathbb{C}$. Hence, we obtain from (3.30) that

$$\left[\left(\gamma a_{11} + \frac{1}{2}\eta a_{12}\right)c_2 - \left(\delta a_{12} + \frac{1}{2}\eta a_{11}\right)c_1\right]H' + (\gamma c_2^2 + \delta c_1^2 - \eta c_1 c_2)\left(\frac{1}{2}H'^2 + H''\right) = \frac{2C_1}{C_2}\left[e^{\frac{1}{2}L_1(c)} - 1\right]. \tag{3.31}$$

Since $\gamma c_2^2 + \delta c_1^2 \neq \eta c_1 c_2$, in view of (3.31), we conclude that H' is constant. This implies that $H(s_1) = a_0 s_1 + b_0$. Hence, $g(z)$ reduces to the form

$$g(z_1, z_2) = L(z_1, z_2) + B = a_1 z_1 + a_2 z_2 + B, \tag{3.32}$$

where $a_1 = a_{11} + a_0 c_2$, $a_2 = a_{12} - a_0 c_1$ and $B = B_1 + b_0$.

Therefore, in view of (3.30) and (3.32), we obtain that

$$e^{\frac{1}{2}[a_1 c_1 + a_2 c_2]} = \frac{C_2}{4C_1}(\gamma a_1^2 + \delta a_2^2 - \eta a_1 a_2) + 1. \tag{3.33}$$

Now, the first equation of (3.23) can be rewritten as

$$(\gamma D^2 + \delta D'^2 + \eta DD')f(z_1, z_2) = C_1 e^{\frac{1}{2}[a_1 z_1 + a_2 z_2 + B]}, \tag{3.34}$$

where $D \equiv \frac{\partial}{\partial z_1}$ and $D' \equiv \frac{\partial}{\partial z_2}$.

Therefore, complementary function of (3.34) is C.F. $= \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2)$, where $\alpha_1, \alpha_2, \beta_1$, and β_2 are same as in (i), ϕ_1 and ϕ_2 are finite-order transcendental entire functions in $\mathbb{C}^2$, and the particular integral is

$$\text{P.I.} = \frac{4C_1 e^{B/2}}{\gamma a_1^2 + \delta a_2^2 + \eta a_1 a_2} \iint e^{v} dv dv = \frac{4C_1}{\gamma a_1^2 + \delta a_2^2 + \eta a_1 a_2} e^{\frac{1}{2}[a_1 z_1 + a_2 z_2 + B]},$$

where $v = a_1 z_1 + a_2 z_2$.

Hence, from (3.23), we obtain

$$f(z_1,z_2)=\phi_1(\beta_1 z_1-\alpha_1 z_2)+\phi_2(\beta_2 z_1-\alpha_2 z_2)+\frac{2\sqrt{2}(A_1\xi+A_2\xi^{-1})}{\gamma a_1^2+\delta a_2^2+\eta a_1 a_2}e^{\frac{1}{2}[a_1z_1+a_2z_2+B]}.$$

Substituting $f(z_1,z_2)$ into the second equation of (3.23) and combining with (3.33), we obtain that

$$\phi_1(\beta_1z_1-\alpha_1z_2+\beta_1c_1-\alpha_1c_2)+\phi_2(\beta_2z_1-\alpha_2z_2+\beta_2c_1-\alpha_2c_2)=\phi_1(\beta_1z_1-\alpha_1z_2)+\phi_2(\beta_2z_1-\alpha_2z_2).$$

This is conclusion (iii).

Case 2. Let $\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)$ be non-constant. Then, obviously, $A_2Q_1(z_1,z_2)+A_1$ and $A_1Q_2(z_1,z_2)+A_2$ cannot be identically zero at the same time. Otherwise, in view of (3.21), it follows that $e^{\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)}$ is a constant, which implies that $\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)$ is a constant. This is a contradiction to our assumption.

If $A_2Q_1(z_1,z_2)+A_1\equiv 0$ and $A_1Q_2(z_1,z_2)+A_2\not\equiv 0$, then (3.21) it yields

$$(A_1Q_2(z_1,z_2)+A_2)e^{\gamma_2(z_1,z_2)}-A_2e^{\gamma_2(z_1+c_1,z_2+c_2)}-A_1e^{\gamma_1(z_1+c_1,z_2+c_2)}\equiv 0. \tag{3.35}$$

In view of Lemma 3.2 and (3.35), we can easily obtain a contradiction. Similarly, we can obtain a contradiction for the case $A_2Q_1(z_1,z_2)+A_1\not\equiv 0$ and $A_1Q_2(z_1,z_2)+A_2\equiv 0$. Hence, we must have $A_2Q_1(z_1,z_2)+A_1\not\equiv 0$ and $A_1Q_2(z_1,z_2)+A_2\not\equiv 0$.

Now, in view of Lemma 3.1, we obtain from (3.21) that either

$$\frac{A_2Q_1(z_1,z_2)+A_1}{A_1}e^{\gamma_1(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}\equiv 1$$

or

$$\frac{A_1Q_2(z_1,z_2)+A_2}{A_1}e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}\equiv 1.$$

If $\frac{A_1Q_2(z_1,z_2)+A_2}{A_1}e^{\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}\equiv 1$, then in view of (3.21), it follows that $\frac{A_2Q_1(z_1,z_2)+A_1}{A_2}e^{\gamma_1(z_1,z_2)-\gamma_2(z_1+c_1,z_2+c_2)}\equiv 1$. Therefore, we must obtain that $\gamma_2(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)=\xi_1$ and $\gamma_1(z_1,z_2)-\gamma_2(z_1+c_1,z_2+c_2)=\xi_2$, $\xi_1,\xi_2\in\mathbb{C}$. Thus, it follows that $\gamma_1(z_1,z_2)-\gamma_1(z_1+2c_1,z_2+2c_2)=\gamma_2(z_1,z_2)-\gamma_2(z_1+2c_1,z_2+2c_2)=\xi_1+\xi_2$. This implies that $\gamma_1(z_1,z_2)=L(z_1,z_2)+H(s_1)+B_1$ and $\gamma_2(z_1,z_2)=L(z_1,z_2)+H(s_1)+B_2$, where $L(z_1,z_2)=a_1z_1+a_2z_2$ and $H(s_1)$ is a polynomial in $s_1\coloneqq c_2z_1-c_1z_2$, $a_1,a_2,B_1,B_2\in\mathbb{C}$. Hence, we must obtain that $\gamma_2(z_1+c_1,z_2+c_2)-\gamma_1(z_1+c_1,z_2+c_2)=B_2-B_1$, a constant in $\mathbb{C}$, which is a contradiction to the assumption.

Therefore, we must have

$$\frac{A_2Q_1(z_1,z_2)+A_1}{A_1}e^{\gamma_1(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)}\equiv 1. \tag{3.36}$$

In view of (3.21) and (3.36), we obtain that

$$\frac{A_1Q_2(z_1,z_2)+A_2}{A_2}e^{\gamma_2(z_1,z_2)-\gamma_2(z_1+c_1,z_2+c_2)}\equiv 1. \tag{3.37}$$

Since $\gamma_1(z_1,z_2)$ and $\gamma_2(z_1,z_2)$ are the polynomials in $\mathbb{C}^2$, from (3.36) and (3.37), we can conclude that $\gamma_1(z_1,z_2)-\gamma_1(z_1+c_1,z_2+c_2)=\eta_1$ and $\gamma_2(z_1,z_2)-\gamma_2(z_1+c_1,z_2+c_2)=\eta_2$, $\eta_1,\eta_2\in\mathbb{C}$. Thus, we have $\gamma_1(z_1,z_2)=L_1(z_1,z_2)+H_1(s_1)+B_1$ and $\gamma_2(z_1,z_2)=L_2(z_1,z_2)+H_2(s_1)+B_2$, where $L_j(z_1,z_2)=a_{j1}z_1+a_{j2}z_2$ and $H_j(s_1)$ is a polynomial in $s_1\coloneqq c_2z_1-c_1z_2$, $a_{j1},a_{j2},B_j\in\mathbb{C}$ for $j=1,2$.

Therefore, in view of (3.22) and (3.36), we obtain that

$$[(2\gamma a_{11}+\eta a_{12})c_2-(2\delta a_{12}+\eta a_{11})c_1]H_1'+(\gamma c_2^2+\delta c_1^2-\eta c_1c_2)(H_1'^2+H_1'')=\frac{A_1}{A_1}[e^{L_1(c)}-1]-(\gamma a_{11}^2+\delta a_{12}^2+\eta a_{11}a_{12}).$$

Since $\gamma c_2^2 + \delta c_1^2 - \eta c_1 c_2 \neq 0$, we conclude that $H_1(s_1)$ is a linear polynomial in s_1, and thus, $L_1(z) + H_1(s_1)$ becomes linear in $\mathbb{C}$. For the sake of convenience, we still denote that $\gamma_1(z) = L_1(z) + B_1$. In a similar manner, from (3.22) and (3.37), we can conclude that $\gamma_2(z) = L_2(z) + B_2$.

Therefore, it follows from (3.36) and (3.37) that

$$\begin{cases} e^{L_1(c)} = \dfrac{A_2}{A_1}(\gamma a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12}) + 1, \\ e^{L_2(c)} = \dfrac{A_1}{A_2}(\gamma a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22}) + 1. \end{cases} \tag{3.38}$$

Now, in view of the first equation of (3.20), it follows that

$$(\gamma D^2 + \delta D'^2 + \eta DD')f(z_1, z_2) = \frac{1}{\sqrt{2}}[A_1 e^{L_1(z_1,z_2)+B_1} + A_2 e^{L_2(z_1,z_2)+B_2}], \tag{3.39}$$

where $D \equiv \frac{\partial}{\partial z_1}$ and $D' \equiv \frac{\partial}{\partial z_2}$.

Now, in view of the results in [[31], page 2178, line 21], the complementary function of (3.39) is $\phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2)$, where α_1, α_2, β_1, and β_2 are same as in (i), ϕ_1 and ϕ_2 are finite-order transcendental entire functions in $\mathbb{C}^2$, and the particular integral is

$$\text{P.I.} = \frac{A_1 e^{L_1(z_1,z_2)+B_1}}{\sqrt{2}(\gamma a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12})} + \frac{A_2 e^{L_2(z_1,z_2)+B_2}}{\sqrt{2}(\gamma a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22})}.$$

Hence, the form of the solution of (3.39) is

$$\begin{aligned} f(z_1, z_2) = {} & \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2) \\ & + \frac{A_1 e^{L_1(z_1,z_2)+B_1}}{\sqrt{2}(\gamma a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12})} + \frac{A_2 e^{L_2(z_1,z_2)+B_2}}{\sqrt{2}(\gamma a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22})}. \end{aligned} \tag{3.40}$$

Substituting (3.40) into the second equation of (3.20) and combining with (3.38), we obtain that

$$\phi_1(\beta_1 z_1 - \alpha_1 z_2 + \beta_1 c_1 - \alpha_1 c_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2 + \beta_2 c_1 - \alpha_2 c_2) = \phi_1(\beta_1 z_1 - \alpha_1 z_2) + \phi_2(\beta_2 z_1 - \alpha_2 z_2).$$

This is conclusion (iv). □

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