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Review Article

Norah Saud Almutairi*, Awatef Shahen, and Hanan Darwish

Differential sandwich theorems for *p*-valent analytic functions associated with a generalization of the integral operator

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Abstract: In this study, subordination, superordination, and sandwich theorems are established for a class of p-valent analytic functions involving a generalized integral operator that has as a special case p-valent Sălăgean integral operator. Relevant connections of the new results with several well-known ones are given as a conclusion for this investigation.

Keywords: multivalent function, differintegral operator, differential subordination, differential superordination, sandwich theorem, p-valent Sălăgean integral operator

MSC 2020: 30C45

1 Introduction and definitions

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk

$$\mathbb{U} := \{\mathfrak{z} : \mathfrak{z} \in \mathbb{C} \quad \text{and} \quad |\mathfrak{z}| < 1\}$$

and $\mathcal{H}[a, p]$ $(a \in \mathbb{C}, p \in \mathbb{N} := 1, 2, 3, \ldots)$ be the subclass of $\mathcal{H}$ consisting of functions of the following form:

$$\mathfrak{f}(\mathfrak{z}) = a + a_p\mathfrak{z}^p + a_{p+1}\mathfrak{z}^{p+1} + \ldots \quad (\mathfrak{z} \in \mathbb{U}).$$

Suppose that $\mathfrak{f}$ and g are in $\mathcal{H}$. We say that $\mathfrak{f}$ is subordinate to g (or g is superordinate to $\mathfrak{f}$), which can be written as

$$\mathfrak{f} \prec g \quad \text{in} \quad \mathbb{U} \quad \text{or} \quad \mathfrak{f}(\mathfrak{z}) \prec g(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}),$$

if there exists a function $\omega \in \mathcal{H}$, satisfying the conditions of the Schwarz lemma (i.e., $\omega(0) = 0$ and $|\omega(\mathfrak{z})| < 1$) such that

$$\mathfrak{f}(\mathfrak{z}) = g(\omega(\mathfrak{z})) \quad (\mathfrak{z} \in \mathbb{U}).$$

It follows that

$$\mathfrak{f}(\mathfrak{z}) \prec g(\mathfrak{z}) \ (\mathfrak{z} \in \mathbb{U}) \Rightarrow \mathfrak{f}(0) = g(0) \quad \text{and} \quad \mathfrak{f}(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if g is univalent in $\mathbb{U}$, then the reverse implication also holds (cf. [1]).

We recall here some more definitions and terminologies from the theory of *differential subordination and differential superordination* developed by Miller and Mocanu (cf. [1,2]).

*** Corresponding author: Norah Saud Almutairi,** Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, e-mail: norah.s.almutairi@gmail.com
Awatef Shahen, Hanan Darwish: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
ORCID: Norah Saud Almutairi 0009-0009-0444-9423

Let $\phi(r, s;\, \mathfrak{z}) : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $\mathfrak{L}(\mathfrak{z})$ be univalent in $\mathbb{U}$. If $p \in \mathcal{H}$ satisfies

$$\phi(p(\mathfrak{z}), \mathfrak{z}p'(\mathfrak{z});\, \mathfrak{z}) \prec \mathfrak{L}(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}), \tag{1}$$

then $p(\mathfrak{z})$ is called a solution of the first-order differential subordination (1). A univalent function q is called a dominant of the solutions of the differential subordination, or more precisely a dominant if $p \prec q$, for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$, for all dominant q of (1) is called the best dominant of (1). The best dominant is unique up to rotations of $\mathbb{U}$.

Similarly, let $\varphi(r, s;\, \mathfrak{z}) : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $\mathfrak{L} \in \mathcal{H}$. Let $p \in \mathcal{H}$ be such that $p(\mathfrak{z})$ and $\varphi(p(\mathfrak{z}), \mathfrak{z}p'(\mathfrak{z});\, \mathfrak{z})$ are univalent in $\mathbb{U}$. If $p(\mathfrak{z})$ satisfies

$$\mathfrak{L}(\mathfrak{z}) \prec \varphi(p(\mathfrak{z}), \mathfrak{z}p'(\mathfrak{z});\, \mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}), \tag{2}$$

then $p(\mathfrak{z})$ is called a solution of the first-order differential superordination (2).

An analytic function q is called a subordinant of the solutions of the differential superordination, or more precisely a subordinant, if $q \prec p$, for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$, for all subordinants q of (2) is said to be the best subordinant. The best subordinant is unique up to rotations of $\mathbb{U}$. The well-known monograph of Miller and Mocanu [1] and the more recent work of Bulboacă [3] provide detailed expositions on the theory of differential subordination and superordination.

Miller and Mocanu [1,2] obtained sufficient conditions on certain broad class of functions $\mathfrak{L}_1, q_1, \mathfrak{L}_2, q_2, \varphi_1$ and φ_2 for which the following implications hold true:

$$\varphi_1(p(\mathfrak{z}), \mathfrak{z}p'(\mathfrak{z}), \mathfrak{z}^2p''(\mathfrak{z});\, \mathfrak{z}) \prec \mathfrak{L}_1(\mathfrak{z}) \Rightarrow p(\mathfrak{z}) \prec q_1(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U})$$

and

$$\mathfrak{L}_2(\mathfrak{z}) \prec \varphi_2(p(\mathfrak{z}), \mathfrak{z}p'(\mathfrak{z}), \mathfrak{z}^2p''(\mathfrak{z});\, \mathfrak{z}) \Rightarrow q_2(\mathfrak{z}) \prec p(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}).$$

Bulboacă [4,5], Ali et al. [6], and Shanmugam et al. [7,8] found adequate conditions on the normalized analytic function $\mathfrak{f}$ in a series of follow-up studies such that sandwich subordinations of the following kind are true:

$$q_1(\mathfrak{z}) \prec \frac{z\mathfrak{f}'(\mathfrak{z})}{\mathfrak{f}(\mathfrak{z})} \prec q_2(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}),$$

where q_1, q_2 are univalent in $\mathbb{U}$ and I is a suitable operator. Refer [9–18] for sandwich results from more recent studies.

Studies with intriguing results were recently inspired by p-valent analytic classes of functions. Recent publications have provided information on the following topics: the properties of p-valent analytic functions related to cosine and exponential functions [19], results of subordination and superordination obtained by applying operators on p-valent analytic functions [20,21], and the introduction of new classes through the application of operators on p-valent analytic functions [22].

The following studies, also recently published, served as further inspiration and motivation for the study's findings. Two new classes of p-valent functions were introduced using generalized differential operators [23,24]. Geometric features of a newly developed operator involving p-valent functions were studied and a subclass of multivalent functions was introduced in [25]. A new generalized integral operator is presented and analyzed considering numerous subordination and coefficient properties in [26].

In view of the recent investigation listed above, the subclass $\mathcal{H}_p$ of $\mathcal{H}[0, p]$ consists of functions of the following form:

$$\mathfrak{f}(\mathfrak{z}) = \mathfrak{z}^p + \sum_{k=p+1}^{\infty} a_k \mathfrak{z}^k \quad (\mathfrak{z} \in \mathbb{U}), \tag{3}$$

which will be investigated using a new generalized integral operator [27] defined for $p \in N, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda > 0$ and $\mathfrak{f} \in \mathcal{H}_p$, defined as follows:

$$I^0_{p,\lambda}\mathfrak{f}(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z})$$

$$I^1_{p,\lambda}\mathfrak{f}(\mathfrak{z}) = \frac{p}{\lambda}\mathfrak{z}^{p-\frac{p}{\lambda}}\int_0^{\mathfrak{z}} t^{\frac{p}{\lambda}-p-1}\mathfrak{f}(t)\mathrm{d}t = \mathfrak{z}^p + \sum_{k=p+1}^{\infty}\left[\frac{p}{p+\lambda(k-p)}\right]a_k\mathfrak{z}^k$$

$$I^2_{p,\lambda}\mathfrak{f}(\mathfrak{z}) = \frac{p}{\lambda}\mathfrak{z}^{p-\frac{p}{\lambda}}\int_0^{\mathfrak{z}} t^{\frac{p}{\lambda}-p-1}I^1_{p,\lambda}\mathfrak{f}(t)\mathrm{d}t = \mathfrak{z}^p + \sum_{k=p+1}^{\infty}\left[\frac{p}{p+\lambda(k-p)}\right]^2 a_k\mathfrak{z}^k$$

and (in general)

$$\begin{aligned} I^n_{p,\lambda}\mathfrak{f}(\mathfrak{z}) &= \frac{p}{\lambda}\mathfrak{z}^{p-\frac{p}{\lambda}}\int_0^{\mathfrak{z}} t^{\frac{p}{\lambda}-p-1}I^{n-1}_{p,\lambda}\mathfrak{f}(t)\mathrm{d}t = \mathfrak{z}^p + \sum_{k=p+1}^{\infty}\left[\frac{p}{p+\lambda(k-p)}\right]^n a_k\mathfrak{z}^k \\ &= \underbrace{I^1_{p,\lambda}\left(\frac{\mathfrak{z}^p}{1-\mathfrak{z}}\right) * I^1_{p,\lambda}\left(\frac{\mathfrak{z}^p}{1-\mathfrak{z}}\right) * \ldots * I^1_{p,\lambda}\left(\frac{\mathfrak{z}^p}{1-\mathfrak{z}}\right)}_{n\text{-times}} * \mathfrak{f}(\mathfrak{z}), \end{aligned} \tag{4}$$

then from (4), we can easily deduce that

$$\frac{\lambda}{p}\mathfrak{z}(I^n_{p,\lambda}\mathfrak{f}(\mathfrak{z}))' = I^{n-1}_{p,\lambda}\mathfrak{f}(\mathfrak{z}) - (1-\lambda)I^n_{p,\lambda}\mathfrak{f}(\mathfrak{z}) \quad (p, n \in \mathbb{N}; \lambda > 0). \tag{5}$$

We note that

$$\begin{aligned} (i) \quad I^n_{1,\lambda}\mathfrak{f}(\mathfrak{z}) &= I^{-n}_{\lambda}\mathfrak{f}(\mathfrak{z}) \quad \text{(see [28])} \\ &= \left\{\mathfrak{f}(\mathfrak{z}) \in \mathcal{H} : I^{-n}_{\lambda}\mathfrak{f}(\mathfrak{z}) = \mathfrak{z} + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^{-n}a_k\mathfrak{z}^k \quad (n \in \mathbb{N}_0)\right\}, \\ (ii) \quad I^n_{1,1}\mathfrak{f}(\mathfrak{z}) &= I^n\mathfrak{f}(\mathfrak{z}) \quad \text{(see [28])} \\ &= \left\{\mathfrak{f}(\mathfrak{z}) \in \mathcal{H} : I^n\mathfrak{f}(\mathfrak{z}) = \mathfrak{z} + \sum_{k=2}^{\infty}k^{-n}a_k\mathfrak{z}^k \quad (n \in \mathbb{N}_0)\right\}. \end{aligned}$$

Also, we note that $I^n_{p,1}\mathfrak{f}(\mathfrak{z}) = I^n_p\mathfrak{f}(\mathfrak{z})$, where I^n_p is p-valent Sălăgean integral operator

$$I^n_p\mathfrak{f}(\mathfrak{z}) = \left\{\mathfrak{f}(\mathfrak{z}) \in \mathcal{H}_p : I^n_p\mathfrak{f}(\mathfrak{z}) = \mathfrak{z}^p + \sum_{k=p+1}^{\infty}\left(\frac{p}{k}\right)^n a_k\mathfrak{z}^k \quad (p \in \mathbb{N}, n \in \mathbb{N}_0)\right\}. \tag{6}$$

In the sequel to earlier investigations, in the present study, we find interesting sufficient conditions on the functions $\mathfrak{f} \in \mathcal{H}_p$ and $q_1, q_2 \in \mathcal{H}$ such that sandwich relation of the form [29]

$$q_1(\mathfrak{z}) \prec \frac{I^n_{p,\lambda}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \prec q_2(\mathfrak{z})$$

or

$$q_1(\mathfrak{z}) \prec \left(\frac{(1-\rho)I^{n-1}_{p,\lambda}\mathfrak{f}(\mathfrak{z}) + \rho I^n_{p,\lambda}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right)^{\eta} \prec q_2(\mathfrak{z})$$

holds. For particular values of the parameters λ and p, our results obtained here include several classical as well as recent results. We will derive several subordination results, superordination results, and sandwich results involving the operator $I^n_{p,\lambda}$.

2 Preliminaries

To establish our results, we need the following:

Definition 1. ([2], Definition 2, p. 817; also see [1], Definition 2.2b, p. 21) Let Q be the set of functions $\mathfrak{f}$ that are analytic and injective on $\overline{\mathbb{U}}\backslash\mathbb{E}(\mathfrak{f})$, where

$$\mathbb{E}(\mathfrak{f}) := \left\{\zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{\mathfrak{z}\to\zeta} \mathfrak{f}(\mathfrak{z}) = \infty\right\}$$

and such that $\mathfrak{f}'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash\mathbb{E}(\mathfrak{f})$.

Lemma 1. ([1], Theorem 3.4h, p. 132) *Let q be univalent in the open unit disk $\mathbb{U}$ and θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $\Phi(\mathfrak{z}) = \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z}))$ and $\mathfrak{L}(\mathfrak{z}) = \theta(q(\mathfrak{z})) + \Phi(\mathfrak{z})$. Suppose that*
(1) *Φ is starlike in $\mathbb{U}$*
and
(2) $\Re\left(\frac{z\mathfrak{L}'(\mathfrak{z})}{\Phi(\mathfrak{z})}\right) > 0 \quad (\mathfrak{z} \in \mathbb{U})$.

If $p \in \mathcal{H}[q(0), n]$ for some $n \in \mathbb{N}$ with $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z})) \prec \theta(q(\mathfrak{z})) + \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})), \tag{7}$$

then $p \prec q$ and q is the best dominant.

Lemma 2. [7] *Let q be univalent convex in the open unit disk $\mathbb{U}$ and $\psi, \gamma \in \mathbb{C}$ with $\Re\left(1 + \frac{\mathfrak{z}q''(\mathfrak{z})}{q'(\mathfrak{z})}\right) > \max\{0, -\Re(\psi\backslash\gamma)\}$. If $p(\mathfrak{z})$ is analytic and*

$$\psi(p(\mathfrak{z})) + \gamma\mathfrak{z}p'(\mathfrak{z}) \prec \psi(q(\mathfrak{z})) + \gamma\mathfrak{z}q'(\mathfrak{z}),$$

then $p \prec q$ and q is the best dominant.

Lemma 3. [30] *Let q be univalent in the open unit disk $\mathbb{U}$ and θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$. Set $\Phi(\mathfrak{z}) = \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z}))$. Suppose that*
(1) *Φ is univalent starlike in $\mathbb{U}$*
and
(2) $\Re\left(\frac{\theta'(q(\mathfrak{z}))}{\phi(q(\mathfrak{z}))}\right) > 0 \quad (\mathfrak{z} \in \mathbb{U})$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$ with $p(\mathbb{U}) \subseteq \mathbb{D}$; $\theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z}))$ is univalent in $\mathbb{U}$ and

$$\theta(q(\mathfrak{z})) + \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})) \prec \theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z})) \quad (\mathfrak{z} \in \mathbb{U}),$$

then $q \prec p$ and q is the best dominant.

Lemma 4. ([2], Theorem 8, p. 822) *Let q be univalent convex in the open unit disk $\mathbb{U}$ and $\gamma \in \mathbb{C}$, with $\Re(\gamma) > 0$. If $p \in \mathcal{H}[q(0), 1] \cap Q$, $p(\mathfrak{z}) + \gamma\mathfrak{z}p'(\mathfrak{z})$ is univalent in $\mathbb{U}$ and*

$$q(\mathfrak{z}) + \gamma\mathfrak{z}q'(\mathfrak{z}) \prec p(\mathfrak{z}) + \gamma\mathfrak{z}p'(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}),$$

then $q \prec p$ and q is the best subordinant.

3 Subordination and superordination results

We state and prove the following subordination and superordination results.

Theorem 1. *Let $q \in \mathcal{H}$ be a convex univalent function in $\mathbb{U}$ with $q(0) = 1$. Let the function $\mathfrak{f} \in \mathcal{H}_p$ satisfy the following subordination condition*:

$$\tau \frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^P} + (1-\tau)\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \prec q(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}q'(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U};\ p, n \in \mathbb{N}, \tau, \lambda > 0), \tag{8}$$

where $I_{p,\lambda}^{n}$ is defined by (4). *Then,*

$$\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \prec q(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}) \tag{9}$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(\mathfrak{z}) = \frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \quad (\mathfrak{z} \in \mathbb{U}) \tag{10}$$

$$\begin{aligned} p'(\mathfrak{z}) &= \frac{\mathfrak{z}^p(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))' - p\mathfrak{z}^{p-1}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))}{(\mathfrak{z}^P)^2} \\ \mathfrak{z}^{p+1}p'(\mathfrak{z}) &= \mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))' - p(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})) \\ \mathfrak{z}^{p+1}p'(\mathfrak{z}) + p(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})) &= \mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))', \end{aligned}$$

which, upon differentiation followed by multiplication by $\mathfrak{z}$, gives

$$\mathfrak{z}^{p+1}p'(\mathfrak{z}) + p\mathfrak{z}^p p(\mathfrak{z}) = \mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'. \tag{11}$$

By using (5) we obtain the following, after a routine simplification:

$$\begin{aligned} \mathfrak{z}^{p+1}p'(\mathfrak{z}) + p\mathfrak{z}^p p(\mathfrak{z}) &= \frac{p}{\lambda}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) - (1-\lambda)I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})) \\ \mathfrak{z}^{p+1}p'(\mathfrak{z}) + p\mathfrak{z}^p p(\mathfrak{z}) + \frac{p}{\lambda}(1-\lambda)I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}) &= \frac{p}{\lambda}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})) \\ \mathfrak{z}^{p+1}p'(\mathfrak{z}) + \left(p + \frac{p}{\lambda}(1-\lambda)\right)\mathfrak{z}^p p(\mathfrak{z}) &= \frac{p}{\lambda}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})) \\ \mathfrak{z}^{p+1}p'(\mathfrak{z}) + \left(p + \frac{p}{\lambda} - p\right)\mathfrak{z}^p p(\mathfrak{z}) &= \frac{p}{\lambda}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})), \\ \frac{\lambda}{p}\mathfrak{z}p'(\mathfrak{z}) + p(\mathfrak{z}) &= \frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}. \end{aligned}$$

This further gives that

$$\begin{aligned} \frac{\lambda}{p}\mathfrak{z}p'(\mathfrak{z}) &= \frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} - p(\mathfrak{z}) \\ \frac{\lambda}{p}\mathfrak{z}p'(\mathfrak{z}) &= \frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) - \mathfrak{z}^p p(\mathfrak{z})}{\mathfrak{z}^P} \\ \tau\left(\frac{\lambda}{p}\right)\mathfrak{z}p'(\mathfrak{z}) &= \frac{\tau(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) - I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))}{\mathfrak{z}^P} \\ p(\mathfrak{z}) + \tau\frac{\lambda}{p}\mathfrak{z}p'(\mathfrak{z}) &= \frac{\tau(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) - I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))}{\mathfrak{z}^P} + \frac{I_{p\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \\ p(\mathfrak{z}) + \tau\frac{\lambda}{p}\mathfrak{z}p'(\mathfrak{z}) &= \frac{\tau I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^P} + (1-\tau)\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}. \end{aligned}$$

Therefore, in the light of the hypothesis (9), we have

$$p(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}p'(\mathfrak{z}) \prec q(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}q'(\mathfrak{z}).$$

Now, an application of Lemma 2 with

$$\gamma = \frac{\tau\lambda}{p} \quad \text{and} \quad \psi = 1$$

gives the assertion in (10). This completes the proof of Theorem 1. □

Theorem 2. *Let $q \in \mathcal{H}$ be a univalent convex function in $\mathbb{U}$ with $q(0) = 1$. Also, let the function $\mathfrak{f} \in \mathcal{H}_p$, be such that*

$$\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \in H[1,1] \cap Q$$

and for $\tau > 0$, the function $\tau\frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} + (1-\tau)\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}$ be univalent in $\mathbb{U}$, where $I_{p,\lambda}^{n}$ is defined by (4). If

$$q(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}q'(\mathfrak{z}) \prec \tau\frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} + (1-\tau)\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \quad (\mathfrak{z} \in \mathbb{U};\ p, n \in \mathbb{N}, \tau, \lambda > 0), \tag{12}$$

then

$$q(\mathfrak{z}) \prec \frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \quad (\mathfrak{z} \in \mathbb{U})$$

and q is the best subordinant.

Proof. As in the proof of our Theorem 1, let the function $p(\mathfrak{z})$ be defined by (10). Then,

$$\tau\frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} + (1-\tau)\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} = p(\mathfrak{z}) + \tau\frac{\lambda}{p}\mathfrak{z}p'(\mathfrak{z}).$$

Therefore, the hypothesis (12) is equivalent to

$$q(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}q'(\mathfrak{z}) \prec p(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}p'(\mathfrak{z}).$$

Now, an application of Lemma 4 yields

$$q(\mathfrak{z}) \prec p(\mathfrak{z}) = \frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p},$$

and q is the best subordinant. The proof of Theorem 2 is completed. □

Theorem 3. *Let the function $q \in \mathcal{H}$ be nonzero univalent in $\mathbb{U}$ with $q(0) = 1$ and*

$$\Re\left\{1 + \frac{\mathfrak{z}q''(\mathfrak{z})}{q'(\mathfrak{z})} - \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}\right\} > 0 \quad (\mathfrak{z} \in \mathbb{U}). \tag{13}$$

Let $0 \le \rho \le 1$, $\lambda, p \in \mathbb{N}$, $p > 0$, and $\eta \in \mathbb{C}$. If $\mathfrak{f} \in \mathcal{H}_p$ *satisfies the following*:

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right] \neq 0 \quad (\mathfrak{z} \in \mathbb{U})$$

and

$$\eta\left[\frac{(1-\rho)\mathfrak{z}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}))' + \rho\mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'}{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})} - p\right] \prec \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}, \tag{14}$$

then

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]^{\eta} \prec q(\mathfrak{z}) \tag{15}$$

and q is the best dominant in (15).

Proof. Let the function $p(\mathfrak{z})$ be defined on $\mathbb{U}$ by

$$p(\mathfrak{z})=\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]^{\eta}. \tag{16}$$

Then, p is analytic in $\mathbb{U}$. The logarithmic differentiation of (16) yields

$$\frac{\mathfrak{z}p'(\mathfrak{z})}{p(\mathfrak{z})}=\eta\left[\frac{(1-\rho)\mathfrak{z}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}))'+\rho\mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'}{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}-p\right]. \tag{17}$$

In order to apply Lemma 1, we set

$$\theta(\mathfrak{z})=1,\ \phi(w)=\frac{1}{w}\quad (w\in\mathbb{C}\backslash\{0\}),$$

$$\Phi(\mathfrak{z})=\mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z}))=\frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}\quad (\mathfrak{z}\in\mathbb{U}),$$

and

$$\mathfrak{L}(\mathfrak{z})=\theta(q(\mathfrak{z}))+\Phi(\mathfrak{z})=1+\frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}.$$

By making use of hypothesis (13), we see that $\Phi(\mathfrak{z})$ is univalent starlike in U. Since $\mathfrak{L}(\mathfrak{z})=1+\Phi(\mathfrak{z})$, we further obtain that

$$\Re\left(\frac{\mathfrak{z}\mathfrak{L}'(\mathfrak{z})}{\Phi(\mathfrak{z})}\right)>0.$$

By a routine calculation using (16) and (17), we have

$$\theta(p(\mathfrak{z}))+\mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z}))=1+\eta\left[\frac{(1-\rho)\mathfrak{z}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}))'+\rho\mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'}{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}-p\right].$$

Therefore, hypothesis (14) is equivalently written as follows:

$$\theta(p(\mathfrak{z}))+\mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z}))\prec 1+\frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}=\theta(q(\mathfrak{z}))+\mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})).$$

We see that condition (7) is also satisfied. Now, by an application of Lemma 1, we have

$$p(\mathfrak{z})\prec q(\mathfrak{z}).$$

We, thus, obtain the assertions in (15). This completes the proof of Theorem 3. □

Theorem 4. *Let $q\in\mathcal{H}$ be a univalent mapping of U into the right half plane with $q(0)=1$ and*

$$\Re\left\{1+\frac{\mathfrak{z}q''(\mathfrak{z})}{q'(\mathfrak{z})}-\frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}\right\}>0\quad (\mathfrak{z}\in\mathbb{U}). \tag{18}$$

Let $0\le\rho\le 1$, $\lambda, p\in\mathbb{N}$, $p>0$, *and* $\eta\in\mathbb{C}$. *Suppose that the function* $\mathfrak{f}\in\mathcal{H}_p$ *satisfies the following*:

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]\neq 0\quad (\mathfrak{z}\in\mathbb{U}).$$

Set

$$\Delta(\mathfrak{z}) = \left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right]^{\eta} + \eta\left[\frac{(1-\rho)\mathfrak{z}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}))' + \rho\mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'}{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})} - p\right] \quad (\mathfrak{z} \in \mathbb{U}). \tag{19}$$

If

$$\Delta(\mathfrak{z}) \prec q(\mathfrak{z}) + \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}, \tag{20}$$

then

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right]^{\eta} \prec q(\mathfrak{z}) \tag{21}$$

and q is the best dominant in (21).

Proof. We follow the lines of proof of Theorem 3. Let the function $p(\mathfrak{z})$ be defined as in (16). We set

$$\theta(w) = w, \quad \phi(w) = \frac{1}{w} \quad (w \in \mathbb{C}\backslash\{0\}),$$

$$\Phi(\mathfrak{z}) = \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})) = \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})} \quad (\mathfrak{z} \in \mathbb{U}),$$

and

$$\mathfrak{L}(\mathfrak{z}) = \theta(q(\mathfrak{z})) + \Phi(\mathfrak{z}) = q(\mathfrak{z}) + \Phi(\mathfrak{z}).$$

In this case,

$$\Re\left\{\frac{\mathfrak{z}\mathfrak{L}'(\mathfrak{z})}{\Phi(\mathfrak{z})}\right\} = \Re\left\{q(\mathfrak{z}) + 1 + \frac{\mathfrak{z}q''(\mathfrak{z})}{q'(\mathfrak{z})} - \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}\right\} > 0 \quad (\mathfrak{z} \in \mathbb{U}).$$

By making use of (17), hypothesis (20) can be equivalently written as

$$\theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(\mathfrak{z}) \prec \theta(q(\mathfrak{z})) + \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})).$$

Therefore, by applying Lemma 1, we obtain

$$p(\mathfrak{z}) \prec q(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}).$$

We obtain the assertion in (21). The proof of Theorem 4 is completed. □

Theorem 5. *Let $q \in \mathcal{H}$ be a univalent mapping of $\mathbb{U}$ into the right half plane with $q(0) = 1$ and satisfy*

$$\Re\left\{1 + \frac{\mathfrak{z}q''(\mathfrak{z})}{q'(\mathfrak{z})} - \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})}\right\} > 0 \quad (\mathfrak{z} \in \mathbb{U}). \tag{22}$$

Let $0 \le \rho \le 1$ and $\eta \in \mathbb{C}$. Let the function $\mathfrak{f} \in \mathcal{H}_p$ be such that

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right]^{\eta} \in \mathcal{H}[1,1] \cap Q.$$

Suppose that the function $\Delta(\mathfrak{z})$ is also univalent in $\mathbb{U}$, where $\Delta(\mathfrak{z})$ is defined by (19). *If*

$$q(\mathfrak{z}) + \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})} \prec \Delta(\mathfrak{z}), \tag{23}$$

then

$$q(\mathfrak{z}) \prec \left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right]^{\eta} \tag{24}$$

and q is the best subordinant in (24).

Proof. In order to apply Lemma 3, we set

$$\theta(w) = w, \phi(w) = \frac{1}{w} \quad (w \in \mathbb{C}\backslash\{0\})$$

and

$$\Phi(\mathfrak{z}) = \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})) = \frac{\mathfrak{z}q'(\mathfrak{z})}{q(\mathfrak{z})} \quad (\mathfrak{z} \in \mathbb{U}).$$

We first observe that Φ is starlike in $\mathbb{U}$. Furthermore,

$$\Re\left\{\frac{\theta'(q(\mathfrak{z}))}{\phi(\mathfrak{z})}\right\} = \Re\{q(\mathfrak{z})\} > 0 \quad (\mathfrak{z} \in \mathbb{U}).$$

Now, let the function p be defined on $\mathbb{U}$ as in (16). By a routine calculation using (17), we have

$$\theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z})) = \Delta(\mathfrak{z}).$$

Hence, condition (23) is equivalent to the following:

$$\theta(q(\mathfrak{z})) + \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})) \prec \theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z})).$$

Therefore, by using Lemma 3, we have

$$q(\mathfrak{z}) \prec p(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}),$$

and q is the best subordinant. This is precisely the assertion of (24). The proof of Theorem 5 is completed. □

Theorem 6. *Let $0 \le \rho \le 1$ and $\alpha, \eta \in \mathbb{C}$. Let the function $q \in \mathcal{H}$ be univalent in $\mathbb{U}$ and*

$$\Re\left(1 + \frac{\mathfrak{z}q''(\mathfrak{z})}{q'(\mathfrak{z})}\right) > \max\{0, -\Re(\alpha)\}. \tag{25}$$

Suppose that $\mathfrak{f} \in \mathcal{H}_p$ satisfies the following:

$$\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \neq 0 \quad (\mathfrak{z} \in \mathbb{U}).$$

Set

$$\Omega(\mathfrak{z}) = \left\{\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right\}^{\eta}\left\{\alpha + \eta\left(\frac{(1-\rho)\mathfrak{z}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}))' + \rho\mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'}{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})} - p\right)\right\} \quad (\mathfrak{z} \in \mathbb{U}) \tag{26}$$

if

$$\Omega(\mathfrak{z}) \prec \alpha q(\mathfrak{z}) + \mathfrak{z}q'(\mathfrak{z}), \tag{27}$$

then

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right]^{\eta} \prec q(\mathfrak{z}) \tag{28}$$

and q is the best dominant.

Proof. The proof of this theorem is similar to the proof of Theorem 4. Therefore, we sketch only the main steps. Let the function $p(\mathfrak{z})$ be defined on $\mathbb{U}$ by (16). By using (17), we write:

$$\frac{\mathfrak{z}p'(\mathfrak{z})}{p(\mathfrak{z})} = \eta\left[\frac{(1-\rho)\mathfrak{z}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}))' + \rho\mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'}{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})} - p\right]$$

$$\mathfrak{z}p'(\mathfrak{z}) = \eta p(\mathfrak{z})\left(\frac{(1-\rho)\mathfrak{z}(I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}))' + \rho\mathfrak{z}(I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z}))'}{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})} - p\right). \tag{29}$$

In this case setting,

$$\theta(w) = \alpha w, \phi(w) = 1 \quad (w \in \mathbb{C}),$$

$$\Phi(\mathfrak{z}) = \mathfrak{z}q'(\mathfrak{z})\phi(q(\mathfrak{z})) = \mathfrak{z}q'(\mathfrak{z}),$$

and

$$\mathfrak{L}(\mathfrak{z}) = \theta(q(\mathfrak{z})) + \Phi(\mathfrak{z}) = \alpha q(\mathfrak{z}) + \mathfrak{z}q'(\mathfrak{z}),$$

we see that, by (25), Φ is starlike in $\mathbb{U}$ and

$$\Re\left(\frac{\mathfrak{z}\mathfrak{L}'(\mathfrak{z})}{\Phi(\mathfrak{z})}\right) = \Re\left\{\alpha + 1 + \frac{\mathfrak{z}q''(\mathfrak{z})}{q'(\mathfrak{z})}\right\} > 0.$$

Furthermore, by substituting the expression for $p(\mathfrak{z})$ from (16) and the expression for $\mathfrak{z}p'(\mathfrak{z})$ from (29), we have

$$\theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z})) = \alpha p(\mathfrak{z}) + \mathfrak{z}p'(\mathfrak{z}) = \Omega(\mathfrak{z}),$$

where $\Omega(\mathfrak{z})$ is defined by (26). The hypothesis (27) is now equivalently written as

$$\theta(p(\mathfrak{z})) + \mathfrak{z}p'(\mathfrak{z})\phi(p(\mathfrak{z})) \prec \theta(q(\mathfrak{z})) + \mathfrak{z}q'\phi(q(\mathfrak{z})).$$

An application of Lemma 1 yields

$$p(\mathfrak{z}) \prec q(\mathfrak{z}).$$

This last statement gives the assertion in (28). The proof of Theorem 6 is completed. □

Theorem 7. *Let* $0 \le \rho \le 1$, $\eta \in \mathbb{C}, \alpha \in \mathbb{C}\backslash\{0\}$, *and* $\Re(\alpha) > 0$. *Let the function* q *be univalent convex in* $\mathbb{U}$ *with* $q(0) = 1$. *Suppose that the function* $\mathfrak{f} \in \mathcal{H}_p$ *is such that*

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right] \neq 0 \quad (\mathfrak{z} \in \mathbb{U})$$

and

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right]^{\eta} \in \mathcal{H}[1,1] \cap Q.$$

If $\Omega(\mathfrak{z})$ *defined by* (26) *is univalent and satisfies the following*:

$$\alpha q(\mathfrak{z}) + \mathfrak{z}q'(\mathfrak{z}) \prec \Omega(\mathfrak{z}), \tag{30}$$

then

$$q(\mathfrak{z}) \prec \left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right]^{\eta}. \tag{31}$$

The function q *is the best subordinant in* (31).

Proof. Let the function $p(\mathfrak{z})$ be defined as in (17). Then, by making use of (18), we write

$$\Omega(\mathfrak{z}) = \alpha p(\mathfrak{z}) + \mathfrak{z}p'(\mathfrak{z}).$$

The hypothesis (31) is now equivalently written as

$$q(\mathfrak{z}) + \left(\frac{1}{\alpha}\right)\mathfrak{z}q'(\mathfrak{z}) \prec p(\mathfrak{z}) + \left(\frac{1}{\alpha}\right)\mathfrak{z}p'(\mathfrak{z}).$$

Therefore, an application of Lemma 4 with $\gamma = \frac{1}{\alpha}$ yields (32). The proof of Theorem 7 is completed. □

4 Sandwich theorems

By combining Theorem 1 with Theorem 2, we obtain the following differential sandwich theorem:

Theorem 8. *Let the functions q_1 and q_2 be univalent convex in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$. Let $\mathfrak{f} \in \mathcal{H}_p$ be such that*

$$\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \in \mathcal{H}[1,1] \cap Q$$

and for $\tau > 0$, the function

$$\tau\frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} + (1-\tau)\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}$$

is univalent in $\mathbb{U}$, where $I_{p,\lambda}^{n}$ is defined by (4). If

$$q_1(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}q_1'(\mathfrak{z}) \prec \tau\frac{I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} + (1-\tau)\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \prec q_2(\mathfrak{z}) + \frac{\tau\lambda}{p}\mathfrak{z}q_2'(\mathfrak{z}),$$

then

$$q_1(\mathfrak{z}) \prec \tau\frac{I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p} \prec q_2(\mathfrak{z}). \tag{32}$$

The functions q_1 and q_2 are, respectively, the best subordinant and the best dominant in (32).

By combining Theorems 4 and 5, we obtain following.

Theorem 9. *Let the functions $q_1, q_2 \in \mathcal{H}$ be univalent mappings of $\mathbb{U}$ into the right half plane and further satisfy the following conditions:*

$$q_1(0) = q_2(0) = 1$$

and

$$\Re\left\{1 + \frac{\mathfrak{z}q_j''(\mathfrak{z})}{q_j'(\mathfrak{z})} - \frac{\mathfrak{z}q_j'(\mathfrak{z})}{q_j(\mathfrak{z})}\right\} > 0 \quad (j = 1,2;\ \mathfrak{z} \in \mathbb{U}).$$

Let $0 \le \rho \le 1$ and $\eta \in \mathbb{C}$. Let $\mathfrak{f} \in \mathcal{H}_p$ be such that the following conditions hold true:

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z}) + \rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^p}\right] \neq 0 \quad (\mathfrak{z} \in \mathbb{U})$$

and

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]^{\eta}\in\mathcal{H}[1,1]\cap Q.$$

Let the function $\Delta(\mathfrak{z})$ be defined on $\mathbb{U}$ as in (19). *If*

$$q_1(\mathfrak{z})+\frac{\mathfrak{z}q_1'(\mathfrak{z})}{q_1(\mathfrak{z})}\prec\Delta(\mathfrak{z})\prec q_2+\frac{\mathfrak{z}q_2'(\mathfrak{z})}{q_2(\mathfrak{z})},$$

then

$$q_1(\mathfrak{z})\prec\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]^{\eta}\prec q_2(\mathfrak{z}), \tag{33}$$

where q_1 and q_2 are, respectively, the best subordinant and the best dominant in (33).

By combining Theorems 6 and 7, we obtain following.

Theorem 10. *Let $0\le\rho\le1$, $\eta\in\mathbb{C}$, and $\alpha\in\mathbb{C}\backslash\{0\}$ with $\Re(\alpha)>0$. Let the functions q_1 and q_2 be univalent convex in $\mathbb{U}$ with $q_1(0)=q_2(0)=1$. Suppose that $\mathfrak{f}\in\mathcal{H}_p$ is such that*

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]\neq0\quad(\mathfrak{z}\in\mathbb{U})$$

and

$$\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]^{\eta}\in\mathcal{H}[1,1]\cap Q.$$

Let the function $\Omega(\mathfrak{z})$ be defined by (27). *If*

$$\alpha q_1(\mathfrak{z})+\mathfrak{z}q_1'(\mathfrak{z})\prec\Omega(\mathfrak{z})\prec\alpha q_2(\mathfrak{z})+\mathfrak{z}q_2'(\mathfrak{z}),$$

then

$$q_1(\mathfrak{z})\prec\left[\frac{(1-\rho)I_{p,\lambda}^{n-1}\mathfrak{f}(\mathfrak{z})+\rho I_{p,\lambda}^{n}\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}^{p}}\right]^{\eta}\prec q_2(\mathfrak{z}), \tag{34}$$

where q_1 and q_2 are, respectively, the best subordinant and the best dominant in (34).

5 Concluding remarks

By taking particular values for the parameters λ, p, n, and choosing different dominant functions $q(\mathfrak{z})$ in our results of Section 3, we obtain several interesting consequences. As the first example, let Ω_k $(0,\le k<\infty)$ be the convex conic region in the w-plane defined by the following:

$$\Omega_k\coloneqq\{w=u+iv\in\mathbb{C}:u^2>k^2(u-1)^2+k^2v^2,u>0\}.$$

Also, let R_k be the Riemann map of $\mathbb{U}$ onto Ω_k satisfying $R_k(0)=1$ and $R_k'(0)>0$. Let the function q_k be defined by

$$q_k(\mathfrak{z})=\exp\int_0^{\mathfrak{z}}\frac{R_k(s)-1}{s}\mathrm{d}s\quad(\mathfrak{z}\in\mathbb{U}). \tag{35}$$

The region Ω_k; the functions $R_k(z)$ and $q_k(z)$ are widely discussed in the literature in the context of *k-uniformly convex* functions. (See e.g. [31], also see [32].) Moreover, we can readily verify that

$$\Re\left\{1+\frac{z q_k''(z)}{q_k'(z)}-\frac{z q_k'(z)}{q_k(z)}\right\}=\Re\left\{\frac{R_k'(z)}{R_k(z)-1}\right\}>\frac{1}{2}\quad(z\in\mathbb{U}).$$

Therefore, condition (13) is satisfied. Now, by choosing $p=1,\ \rho=1,\ n=1,\ \lambda=1,\ \eta$ real, and $q(z)=q_k(z)$ in Theorem 3, where $q_k(z)$ is defined by (35), we obtain

If the function $f\in\mathcal{H}_1$ satisfies the following:

$$\frac{f(z)}{z}\neq 0\quad(z\in\mathbb{U})$$

and

$$\eta\left(\frac{z f'(z)}{f(z)}-1\right)\prec(R_k(z)-1)\quad(\eta\in\mathbb{C},z\in\mathbb{U}),$$

then

$$\left(\frac{f(z)}{z}\right)^{\eta}\prec q_k(z) \tag{36}$$

and $q_k(z)$ is the best dominant in (36).

For $\eta=1$, this result is due to Kanas and Wisniowska [33]. (Also see [32,34,35] for generalizations.)

In the second example, we choose $q(z)=\frac{1+Az}{1+Bz}(-1\le B<A\le 1),\rho=1$ in Theorem 3, obtain the following:

If the function $f\in\mathcal{H}_p$ satisfies

$$\eta\left[\frac{z(I_{p,\lambda}^n f(z))'}{I_{p,\lambda}^n f(z)}-p\right]\prec\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\eta\left[\frac{(I_{p,\lambda}^n f(z))}{z^p}\right]^{\eta}\prec\frac{1+Az}{(1+Bz)}\quad(z\in\mathbb{U})$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Similarly setting $p=1,\rho=1,n=1,\eta$ real, and $q(z)=(1+Bz)^{\eta(A-B)/B}$, which is univalent if and only if $|(\eta(A-B)/B)-1|\le 1$ or $|(\eta(A-B)/B)+1|\le 1$ [36], Theorem 3 reduces to the following.

Let the real numbers A,B be such that $-1\le B<A\le 1$ and suppose that the real number η satisfies $1\le\frac{\eta(A-B)}{B}\le 2$. For $f\in\mathcal{H}_1$, if

$$\frac{z f'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}),$$

then

$$\left[\frac{f(z)}{z}\right]^{\eta}\prec(1+Bz)^{\eta(A-B)/B}$$

and $(1+Bz)^{\eta(A-B)/B}$ is the best dominant.

By further specializing $A=1-2\alpha,(0\le\alpha<1),B=-1$, and $\eta=1$, here, we obtain the following well-known result on univalent starlike functions (see [28], also see [38]):

If $f\in\mathcal{H}_1$ is univalent starlike of order $\alpha(0\le\alpha<1)$ in $\mathbb{U}$, then

$$\frac{f(z)}{z}\prec\frac{1}{(1-z)^{2(1-\alpha)}} \tag{37}$$

and $\frac{1}{(1-z)^{2(1-\alpha)}}$ is the best dominant.

Again, setting $\rho = 0, p = 1, \eta = 1$, and $q(\mathfrak{z}) = \frac{1}{(1-\mathfrak{z})^{2(1-\alpha)}}$ $(0 \le \alpha < 1)$ in Theorem 3, we obtain the following well-known result for univalent convex functions (see [28], also see [37,38]).

If $\mathfrak{f} \in \mathcal{H}_1$ is univalent convex of order $\alpha (0 \le \alpha < 1)$ in $\mathbb{U}$, then

$$\mathfrak{f}'(\mathfrak{z}) \prec \frac{1}{(1-\mathfrak{z})^{2(1-\alpha)}}$$

and $\frac{1}{(1-\mathfrak{z})^{2(1-\alpha)}}$ is the best dominant.

Particular cases of our Theorems 1, 4, and 6 also yield interesting consequences. However, we omit the details for the sake of brevity. Finally, we address the following problem:

For $0 \le \alpha < 1$, let the function q be defined on $\mathbb{U}$ by

$$q(\mathfrak{z}) = \begin{cases} \frac{1}{2-\alpha}\left[\frac{1}{(1-\mathfrak{z})^{(1-2\alpha)}} - 1\right]; & \alpha \neq \frac{1}{2}, \\ -\log(1-\mathfrak{z}); & \alpha = \frac{1}{2}, \end{cases} \tag{38}$$

A result analogous to (37) for univalent convex functions of order $\alpha(1/2 \le \alpha < 1)$ is well known, i.e.,

$$\frac{\mathfrak{f}(\mathfrak{z})}{\mathfrak{z}} \prec q(\mathfrak{z}) \quad (\mathfrak{z} \in \mathbb{U}),$$

where $q(\mathfrak{z})$ is defined by (38). However, a similar result in the range $0 \le \alpha < 1/2$ seems to be an open problem [39].

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