DE GRUYTER

Demonstratio Mathematica 2024; 57: 20240050

Review Article

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Penalty method for unilateral contact problem with Coulomb's friction in time-fractional derivatives

https://doi.org/10.1515/dema-2024-0050
received February 13, 2023; accepted August 6, 2024

Abstract: The purpose of this work is to study a mathematical model that describes a contact between a deformable body and a rigid foundation. A linear viscoelastic Kelvin-Voigt constitutive law with time-fractional derivatives describes the material's behavior. The contact is modeled with Signorini's condition coupled with Coulomb's friction law. We derive a variational formulation of the model, and we prove the existence of a weak solution using the theory of monotone operators and Caputo derivative and the Rothe method. We also introduce the penalized problem and prove its solvability using the Galerkin method. Furthermore, we study the convergence of its solution to the solution of the original problem as the penalization parameter tends to zero.

Keywords: fractional viscoelastic constitutive law, Caputo derivative, contact with friction, Rothe method, Faedo-Galerkin method, compactness method, penalty method

MSC 2020: 35R11, 74M10, 74M15, 65N30, 65M60, 46B50

1 Introduction

The study of the behavior of many materials, such as polymers, reveals a low-frequency dependence of their damping properties over a wide frequency range. This weak frequency dependence is challenging to describe within the framework of classical viscoelastic models (integer-order derivatives) and without an excessive number of parameters. The challenge is overcome by employing fractional-order operators instead of integer-order operators in the constitutive laws. This substitution leads to a decreased number of parameters required to characterize the material's properties.

Time-fractional-order viscoelastic models find extensive applications in various fields such as mechanics, chemistry, and engineering [1–3]. For the specific viscoelastic materials, we can find the fractional constitutive model in previous studies [4–6].

The theory of fractional differential equations has been studied in several works, including [7–10] and more recently [11,12]. Models utilizing fractional derivatives and their corresponding modeling can be found in [5,13–15]. The investigation of differential hemivariational inequalities in Banach spaces was initiated [16,17]. In previous studies [18,19], the authors explore another application of partial differential equations (PDEs) involving fractional-order derivatives, specifically the synchronization of fractional-order stochastic systems in finite-dimensional spaces. Numerical illustrations are provided to validate the theoretical findings.

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In 2018, Zeng et al. [17,20] addressed a novel category of generalized differential hemivariational inequalities. Their study incorporated the temporal fractional-order derivative operator while considering a frictional contact problem. They employed Rothe's method to establish the existence of a weak solution to the contact problem. More recently, in the investigation by Bouallala and Essoufi [21], a fractional contact problem with normal compliance and Coulomb's friction was examined. This research concentrated on the interplay between a thermo-viscoelastic body and a thermally conductive foundation. A more recent contribution in [22] initiated the exploration of a novel frictionless dynamic contact problem model for a viscoelastic body with normal compliance, incorporating the Kelvin-Voigt constitutive law with a time-fractional component.

The study of a contact problem with friction, considering fractional-order derivatives using the penalization method, presents several motivations. Theoretically, it allows for the generalization of classical models and stimulates research and the development of new methods and theories in applied mathematics and numerical analysis. Practically, important motivations include applications in engineering, more accurate predictions of material behavior, and the optimization of industrial processes.

Furthermore, given that the fractional-order derivative generalizes the constitutive laws of rheological models commonly used in linear viscoelasticity, including springs and dampers. The objective of this document is to analyze a frictional contact problem involving a viscoelastic material and a rigid foundation. In this framework, we express the constitutive equation using the fractional Kelvin-Voigt law. The process is quasistatic, the contact is unilateral, and friction is modeled by a version of Coulomb's law. We introduce a weak formulation of the model, involving a variational inequality for the displacement field with a fractional derivative component in time.

The novelties of this study are as follows: first, the utilization of a time-fractional Kelvin-Voigt constitutive law for the viscosity tensor is as follows:

$$\sigma(t) = \mathcal{V}\varepsilon({}_0^C D_t^\alpha u(t)) + \mathcal{B}\varepsilon(u(t)),$$

where $\alpha \in (0, 1]$ and $t \in [0, T]$.

The second novelty lies in tackling a novel problem by penalizing the contact law and regularizing friction. This approach transforms the problem into a variational equation that includes a fractional-order derivative in time. Consequently, this penalized formulation manifests as a nonlinear quasi-variational equation involving fractional-order derivatives.

The third innovation of this study is the combination of two major methods, Rothe and finite difference-Galerkin, which are highly effective in the resolution of PDEs, particularly those involving fractional derivatives. Additionally, the former is used for a quasi-variational inequality and the latter for a variational equation, both of which are nonlinear and involve fractional-order derivatives.

We demonstrate the existence of a penalized solution using the Galerkin method for such equations. Among the difficulties encountered in this study are the estimation of certain non-linear terms and the convergence of the two problems. These challenges stem from the combination of fractional-order derivatives and non-linear boundary conditions.

The remaining sections of this article are organized as follows: In Section 2, we present the model of the contact quasistatic process of a viscoelastic body with the fractional Kelvin-Voigt law. We introduce some preliminary material and list assumptions on the problem data. Additionally, we derive the variational formulation of the problem and present the main results concerning the existence of a weak solution. In Section 3, we prove the existence of a weak solution by leveraging the Rothe method and a surjectivity theorem for multivalued pseudomonotone operators. In Section 4, we discuss the existence of a penalty problem and investigate the convergence of the solution as the penalty parameter approaches zero. The proof is based on utilizing the fractional Caputo derivative, implementing the Galerkin method, and employing the compactness method for the Caputo derivative. Finally, in Appendix, we provide a summary of relevant results, including known definitions and properties in nonlinear analysis and fractional calculus.

2 Problem statement and variational formulation

We assume that a viscoelastic body occupies a regular domain Ω of $\mathbb{R}^d$, $d = 2, 3$, which will be supposed bounded with a smooth boundary $\Gamma = \partial\Omega$. This boundary is divided into three open disjoint parts Γ_D, Γ_N, and Γ_C, such that meas$(\Gamma_D) > 0$. The interval is denoted as $[0, T]$, where T represents a fixed positive value that defines the upper bound of the time interval.

The body is assumed to be clamped in $\Gamma_D \times (0, T)$ and is subjected to a volume force f_1 in $\Omega \times (0, T)$. A density of traction force f_N acts on $\Gamma_N \times (0, T)$. The normalized gap between Γ_C and a rigid foundation is denoted by g (Figure 1).

In the following, we use $\mathbb{S}^d$ to denote the space of second-order symmetric tensors on $\mathbb{R}^d$, while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively.

$$u \cdot v = u_i \cdot v_i, \quad \|v\| = (v, v)^{\frac{1}{2}}, \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}, \quad \|\tau\| = (\tau, \tau)^{\frac{1}{2}}.$$

We denote by $u : \Omega \times (0, T) \to \mathbb{R}^d$ the displacement field, $\sigma = (\sigma_{ij}) : \Omega \times (0, T) \to \mathbb{S}^d$ the stress tensor, $\varepsilon(u) = (\varepsilon_{ij}(u))$ the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, and div and Div denote the divergence operator for vector-valued and tensor-valued functions, respectively. Specifically, $Div\ \sigma = (\sigma_{ij,j})$ and div $\xi = (\xi_{j,j})$.

We represent the normal and tangential components of the displacement field u on Γ as follows:

$$u_\nu = u.\nu, \quad \text{and} \quad u_\tau = u - u_\nu \nu.$$

The normal and tangential components of the stress field σ on the boundary are defined as follows:

$$\sigma_\nu = (\sigma\nu).\nu, \quad \text{and} \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

respectively, where ν denote the outward normal vector on Γ.

The classical formulation of the fractional contact problem can be expressed in the following manner:

- **Problem (P)**: Find a displacement field $u : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ and a stress field $\sigma : \Omega \times (0, T) \longrightarrow \mathbb{S}^d$ such that

$$\sigma(t) = \mathcal{V}\varepsilon({}_0^C D_t^\alpha u(t)) + \mathcal{B}\varepsilon(u(t)), \qquad \text{in } \Omega \times (0, T), \tag{1}$$

$$\text{Div } \sigma(t) + f_1(t) = 0, \qquad \text{in } \Omega \times (0, T), \tag{2}$$

$$u = 0, \qquad \text{on } \Gamma_D \times (0, T), \tag{3}$$

$$\sigma(t)\nu = f_N(t), \qquad \text{on } \Gamma_N \times (0, T), \tag{4}$$

$$u(0, x) = u_0, \qquad \text{in } \Omega \times (0, T), \tag{5}$$

$$\sigma_\nu(u(t)) \le 0, \quad u_\nu(t) \le g, \quad \sigma_\nu(u(t))(u_\nu(t) - g) = 0, \qquad \text{on } \Gamma_C \times (0, T), \tag{6}$$

$$\left.\begin{array}{l} \|\sigma_\tau(t)\| \le \mu(\|u_\tau(t)\|)\|\sigma_\nu(t)\|, \\ \|\sigma_\tau(t)\| < \mu(\|u_\tau(t)\|)\|\sigma_\nu(t)\| \Rightarrow u_\tau(t) = 0, \\ \|\sigma_\tau(t)\| = \mu(\|u_\tau(t)\|)\|\sigma_\nu(t)\| \Rightarrow \exists \lambda \ne 0 \text{ such that } \sigma_\tau(t) = -\lambda u_\tau(t) \end{array}\right\}, \quad \text{on } \Gamma_C \times (0, T) \tag{7}$$

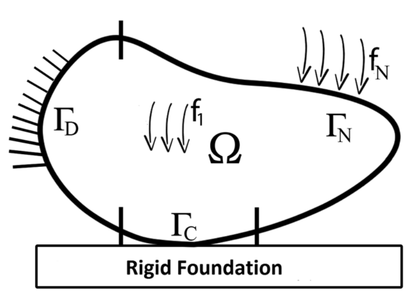

Figure 1: Domain in the initial configuration.

Equation (1) corresponds to the Caputo-type time-fractional Kelvin-Voigt viscoelastic constitutive law, as described in [23]. Here, $\mathcal{B} = (\mathcal{B}_{ijkl})$ and $\mathcal{V} = (\mathcal{V}_{ijkl})$ denote the elastic tensor and viscosity tensor, respectively, both of which are fourth-order tensors. Equation (2) represents the stress equilibrium condition. The relations (3) and (4) represent the mechanical boundary conditions. Additionally, the initial condition is described by equation (5). Relation (6) captures the frictional contact on Γ_C with Signorini's conditions. Furthermore, equation (7) represents Coulomb's friction, where μ denotes the coefficient of friction.

In the context of a real Banach space X and $1 \leq p \leq \infty$, we adopt the conventional notation to represent the spaces $L^p(0, T; X)$, $C(0, T; X)$, and $W^{k,p}(0, T; X)$, where $k = 1, 2, \dots$.

To establish the variational formulation of **Problem (P)**, we will utilize the function spaces:

$$\begin{aligned} H &= L^2(\Omega)^d = \{v = (v_i) \mid v_i \in L^2(\Omega),\ i = 1, \dots, d\},\\ H_1 &= H^1(\Omega)^d = \{v = (v_i) \mid v_i \in H^1(\Omega),\ i = 1, \dots, d\},\\ \mathcal{H} &= \{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ i, j = 1, \dots, d\}, \end{aligned}$$

and

$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} \mid \operatorname{Div} \sigma \in H\}.$$

Endowed with the following inner products:

$$\begin{aligned} (u, v)_H &= \int_\Omega u_i v_i \, dx, \quad (u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},\\ (\sigma, \tau)_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \quad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\operatorname{Div} \sigma, \operatorname{Div} \tau)_{\mathcal{H}}, \end{aligned}$$

with the associated norm $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$, and $\|\cdot\|_{\mathcal{H}_1}$.

Taking into account (3), we introduce the following space:

$$V = \{v \in H_1 \ : \ v = 0 \ \text{on} \ \Gamma_D\},$$

endowed with the inner products and norm given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \ \ \|v\|_V = (v, v)_V^{\frac{1}{2}}.$$

The set of admissible displacements is defined as follows:

$$V_{ad} = \{v \in V \ : \ v_\nu \leq g \ \text{on} \ \Gamma_C\}.$$

Given that $\operatorname{meas}(\Gamma_D) > 0$, Korn's inequality holds.

$$\|\varepsilon(u)\|_{\mathcal{H}} \geq c_K \|v\|_{H_1}, \ \text{for all} \ v \in V,$$

where $c_K > 0$ is a constant, which depends only on Ω and Γ_D.

Furthermore, according to Sobolev's trace theorem, there exists a positive constant c_d that depends solely on Ω and Γ_C, such that

$$\|v\|_{L^2(\Gamma_C)^d} \leq c_0 \|v\|_V, \ \text{for all} \ v \in V. \tag{8}$$

For simplicity, let us denote the following bilinear and symmetric operators:

$$\mathfrak{a}(u, v) \coloneqq (\mathcal{V}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \mathfrak{b}(u, v) \coloneqq (\mathcal{B}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}. \tag{9}$$

Applying Riesz's representation theorem, we define the element $f(t) \in V$ as follows:

$$(f(t), v)_V \coloneqq \int_\Omega f_1(t).v \, dx + \int_{\Gamma_N} f_N(t).v \, d\Gamma, \ \text{for all} \ v \in V.$$

Also, we define the mapping $j : V \times V \to \mathbb{R}$ by

$$j(u(t), v) := \int_{\Gamma_C} \mu(\|u_\tau(t)\|)|\sigma(t)|.\|v_\tau\| d\Gamma, \quad \text{for all } v \in V. \tag{10}$$

In the study of mechanical **Problem (P)**, we impose the following assumptions:

(HP_1)

(i) The viscosity tensor $\mathcal{V} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ and the elasticity $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ exhibit the standard property of symmetry:

$$\mathcal{V}_{ijkl} = \mathcal{V}_{jikl} = \mathcal{V}_{lkij} \in L^\infty(\Omega), \quad \mathcal{B}_{ijkl} = \mathcal{B}_{jikl} = \mathcal{B}_{lkij} \in L^\infty(\Omega).$$

(ii) The forms a, b satisfy the property of ellipticity

$$\mathfrak{a}(u, u) \geq m_a\|u\|_V^2, \text{ and } \mathfrak{b}(u, u) \geq m_b\|u\|_V^2,$$

where $m_a, m_b > 0$ for all $u \in V$.

(iii) The operators a and b adhere to the conventional property of boundedness.

$$|\mathfrak{a}(u, v)| \leq M_a\|u\|_V.\|v\|_V, \text{ and } |\mathfrak{b}(u, v)| \leq M_b\|u\|_V.\|v\|_V,$$

where $M_a, M_b > 0$, for all $u, v \in V$.

(HP_2)

(i) The forces and tractions satisfy the following conditions:

$$f_1 \in L^2(0, T; L^2(\Omega)), \text{ and } f_N \in L^2(0, T; L^2(\Gamma_N)^d).$$

(ii) The gap function and the initial condition fulfill the following conditions:

$$g > 0, \ g \in L^\infty(\Gamma_C), \text{ and } u_0 \in V_{ad}.$$

(HP_3) The coefficient of friction $\mu : \Gamma_C \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

(i) There exists $L_\mu > 0$, for all $x, y \in \mathbb{R}^+$

$$|\mu(\cdot, x_1) - \mu(\cdot, x_2)| < L_\mu|x_1 - x_2| \text{ a.e. on } \Gamma_C.$$

(ii) The mapping $z \mapsto \mu(z, x)$ is measurable on Γ_C, for all $x \in \mathbb{R}^+$.

(iii) The mapping $z \mapsto \mu(z, x)$ is μ^*-bounded a.e. on Γ_C, where

$$\mu^* = \sup_{t \in [0,T]} \|\mu(t)\|_{L^\infty(\Gamma_C)}.$$

(HP_4) The mapping j satisfies

(i) j is measurable on Γ_C.

(ii) j is locally Lipschitz on Γ_C.

(iii) j is a proper convex and l.s.c on V.

(iv) There exists $c_j > 0$ such that

$$\|\partial j(v)\|_{V^*} \leq c_j(1 + \|v\|_V).$$

As V is dense in H, the inclusion mapping from $(V, \|\cdot\|_V)$ to $(H, \|\cdot\|_H)$ is continuous and dense. Consequently, we identify H with its dual space H^*, and we express $V \subset H \equiv H^* \subset V^*$, with V^* being the dual space of V.

By employing a standard procedure relying on Green's formula, we derive the subsequent variational formulation of (1)–(7):

• **Variational Problem (PV)** : Find a displacement field $u : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ and a stress field $\sigma : \Omega \times (0, T) \longrightarrow \mathbb{S}^d$ for all $v \in V$ such that

$$\sigma(t) = \mathcal{V}\varepsilon({}_0^C D_t^\alpha u(t)) + \mathcal{B}\varepsilon(u(t)), \tag{11}$$

$$\mathfrak{a}({}_0^C D_t^\alpha u(t), v - u(t)) + \mathfrak{b}(u(t), v - u(t)) + j(u(t), v) - j(u(t), u(t)) \geq (f(t), v - u(t)), \tag{12}$$

$$u(0) = u_0. \tag{13}$$

3 Existence result

In this section, we present and demonstrate the existence of the result

Theorem 3.1. *Assuming that hypotheses* (HP_1)–(HP_4) *and* (9)–(10) *are satisfied, it follows that* **Problem (PV)** *possesses at least one solution.*

$$(u, \sigma) \in W^{1,2}(0, T; V) \times L^2(0, T; L^2(\Omega, \mathbb{S}^d)).$$

The demonstration of Theorem 3.1 hinges upon the application of logic involving the nonlinear operator, the Caputo derivative, and the Rothe method.

Given $y(t) = {}_0^C D_t^\alpha u(t)$ and $u(t) = {}_0 I_t^\alpha y(t) + u_0$, inequality (12) can be restated as follows:

$$\mathfrak{a}(y(t)) + \mathfrak{b}({}_0 I_t^\alpha y(t) + u_0) + \partial j({}_0 I_t^\alpha y(t) + u_0) \ni f(t). \tag{14}$$

Now, let $N \in \mathbb{N}$ be a fixed integer, and $\Delta t = \delta = \frac{T}{N}$. We proceed to examine the following approximation of the fractional integral operator ${}_0 I_{t_n}^\alpha y(t)$ by:

$${}_0\tilde{I}_{t_n}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} \int_{t_{j-1}}^{t_j} (t_n - s)^{\alpha-1} y(t_i) \mathrm{d}s = \frac{\delta^\alpha}{\Gamma(1+\alpha)} \sum_{i=1}^{n} y(t_i)[(n-i+1)^\alpha - (n-i)^\alpha],$$

where $t_k = k\delta$. Additionally, we define the functional f_δ^k as follows:

$$f_\delta^i = \frac{1}{\delta} \int_{t_{i_1}}^{t_i} f(s) \mathrm{d}s,$$

for $i = 1, \ldots, N$.

By applying the Rothe method to equation (14), we derive the ensuing fractional Rothe problem:

Fractional Rothe Problem (FRP): Find $\{y_\delta^k\} \subset V$ for $k = 1, \ldots, N$ such that

$$\mathfrak{a}(y_\delta^k) + \mathfrak{b}(u_\delta^k) + \partial j(u_\delta^k) \ni f_\delta^k, \tag{15}$$

where

$$u_\delta^k = u_0 + \frac{\delta}{\Gamma(1+\alpha)} \sum_{i=1}^{k} y_\delta^i [(k-i+1)^\alpha - (k-i)^\alpha]. \tag{16}$$

The subsequent result is as follows:

Lemma 3.1. *There exists a positive constant* $\bar{\delta}$ *such that if* δ *belongs to the interval* $(0, \bar{\delta})$, *the* **Problem (FRP)** *possesses at least one solution.*

Proof. We assume that $\{y_\delta^k\}_{k=0}^{n-1}$ are provided, and we will select $y_\delta^n \in V$ satisfying (15)–(16). To accomplish this, we introduce the following multivalued operators: $\Pi : V \to V$ and $\Pi_0 : V \to V^*$:

$$\Pi(y) \coloneqq \Pi_0 + \partial j(y)$$

and

$$\Pi_0(y) \coloneqq \mathfrak{a}(y) + \mathfrak{b}\left(\frac{\delta^\alpha}{\Gamma(1+\alpha)} \sum_{i=1}^{n-1} [(n-i+1)^\alpha - (n-i)^\alpha] \frac{\delta^\alpha}{y} + u_0\right).$$

Now, we aim to demonstrate that Π is a surjection operator.

First, we establish the coercivity of the operator Π. To this end, let $c_t > 0$ denote the constant defined by

$$c_t = \frac{\delta^\alpha}{\Gamma(1+\alpha)} \sum_{i=1}^{n-1} \|y_\delta^i\| [(n-i+1)^\alpha - (n-i)^\alpha]. \tag{17}$$

Utilizing hypotheses (HP_4) and equation (17), we obtain

$$\left\|\partial j\left(\frac{\delta^\alpha}{\Gamma(1+\alpha)}\sum_{i=1}^{n-1} y_\delta^i[(n-i+1)^\alpha-(n-i)^\alpha]+\frac{\delta^\alpha}{\Gamma(1+\alpha)}+u_0\right)\right\|_{V^*}$$
$$\leq c_j\left(\frac{\delta^\alpha}{\Gamma(1+\alpha)}\sum_{i=1}^{n-1}\|y_\delta^i\|_V[(n-i+1)^\alpha-(n-i)^\alpha]+\frac{\delta^\alpha}{\Gamma(1+\alpha)}\|y\|_V+1+\|u_0\|_V\right)$$
$$\leq c_j\left(1+c_t+\frac{\delta^\alpha}{\Gamma(1+\alpha)}\|y\|_V\right).$$

Based on hypothesis (HP_1) and equation (8), we conclude that

$$\Pi(y,y)\geq m_a\|y\|_V^2-M_b\left(c_t+\frac{\delta^\alpha}{\Gamma(1+\alpha)}\|y\|_V\right)\|y\|_V-c_j\left(1+c_t+\frac{\delta^\alpha}{\Gamma(1+\alpha)}\|y\|_V\right)$$
$$\geq\left(m_a-\frac{\delta^\alpha(M_b+\nu_j)}{\Gamma(1+\alpha)}\right)\|y\|_V^2-\left(M_bc_t+\frac{c_j\delta^\alpha}{\Gamma(1+\alpha)}\right)\|y\|_V-c_j(1+c_t).$$

We define $\bar{\delta}=\frac{\Gamma(1+\alpha)m_a}{\delta^\alpha(M_b+c_j)}$ to establish the coercivity of the operator Π.

Subsequently, we utilize the assumptions concerning $\mathfrak{a}$ and $\mathfrak{b}$ to deduce

$$\Pi_0(y,y)\geq\left(m_a-\frac{\delta^\alpha M_b}{\Gamma(1+\alpha)}\right)\|y\|_V^2.$$

Therefore, Π_0 exhibits pseudomonotonicity.

Now, it is necessary to demonstrate that the operator $\Psi: V\to V^*$, which is defined as

$$\Psi(y)\coloneqq\partial j\left(\frac{\delta^\alpha}{\Gamma(1+\alpha)}\sum_{i=1}^{n-1}[(n-i+1)^\alpha-(n-i)^\alpha]+\frac{\delta^\alpha}{\Gamma(1+\alpha)}y+u_0\right),$$

for $y\in V$ is pseudomonotone.

It follows from the properties of j and the reflexivity of V that $\Pi(y)$ is nonempty, convex, and weakly compact for all $y\in V$. Moreover, according to (HP_4), Π is bounded.

In other words, suppose $\{y_m\}\subset V$ such that $y_m\to y$ weakly in V, as $m\to\infty$, and

$$\beta_m\in\partial j\left(\frac{\delta^\alpha}{\Gamma(1+\alpha)}\sum_{i=1}^{n-1}[(n-i+1)^\alpha-(n-i)^\alpha]+\frac{\delta^\alpha}{\Gamma(1+\alpha)}y_m+u_0\right).$$

As the operator ∂_j is bounded, the sequence $\{\beta_m\}$ is bounded in V^*.

Consequently, by considering a subsequence if necessary, we observe that $y_m\to y$ weakly in V^* as $m\to\infty$.

Given that the graph of the multivalued mapping

$$y\mapsto\partial_j\left(\frac{\delta^\alpha}{\Gamma(1+\alpha)}\sum_{i=1}^{n-1}[(n-i+1)^\alpha-(n-i)^\alpha]+\frac{\delta^\alpha}{\Gamma(1+\alpha)}+u_0\right),$$

is closed with respect to the $V\times V^*$ topology (see [15, Proposition 3.23(v)]), we conclude that

$$\beta\in\partial_j\left(\frac{\delta^\alpha}{\Gamma(1+\alpha)}\sum_{i=1}^{n-1}y_\delta^\alpha[(n-i+1)^\alpha-(n-i)^\alpha]+\frac{\delta^\alpha}{\Gamma(1+\alpha)}y+u_0\right).$$

Subsequently, it becomes evident that $\beta\in\Pi(y)$, and we define

$$\langle\beta_m,y_m\rangle\to\langle\beta,y\rangle_{V^*\times V},\quad\text{as } m\to\infty.$$

Utilizing Lemma A.1, we infer the pseudomonotonicity of the operator Π, thus establishing that it is pseudomonotone. Consequently, the operator Π is pseudomonotone. Consequently, **Problem (FRP)** has at least one solution.

Next, we will present the sequence of solutions for the fractional Rothe problem (15). □

Lemma 3.2. *Given assumptions* (HP_1)–(HP_4) *and equation* (17), *there exists* $\delta > 0$ *and a positive constant* c *independent of* δ *such that* $0 < \delta < \bar{\delta}$; *the solution of* (15) *satisfies*

$$\max_k \|y_\tau^k\| + \max_k \|u_\tau^k\| + \max_k \|\beta_\tau^k\| \leq c, \tag{18}$$

for $k = 1, \dots, N$ *and* $\beta_\delta^k \in \partial j(y_\delta^k)$, *and*

$$\mathfrak{a}(y_\delta^k) + \mathfrak{b}(u_\delta^k) + \beta_\delta^k = f_\delta^k. \tag{19}$$

Proof. For all $1 \leq n \leq N$, multiplying equation (19) by y_δ^n, we obtain

$$\mathfrak{a}(y_\delta^n, y_\delta^n) + \mathfrak{a}(u_\delta^n, y_\delta^n) + \langle \beta_\delta^n, y_\delta^n \rangle_{V^* \times V} = \langle f_\delta^n, y_\delta^n \rangle_{V^* \times V}.$$

With reference to (16), (HP_1), and (HP_4), we deduce

$$\begin{aligned}
\langle f_\delta^n, y_\tau^n \rangle &\geq m_a \|y_\delta^n\|_V^2 - M_b \|u_\delta^n\|_V \|y_\delta^n\|_V - c_j(1 + \|u_\delta^n\|_V)\|y_\delta^n\|_V \\
&\geq m_a \|y_\delta^n\|_V - M_b \left(\|u_0\|_V + \frac{\delta^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \|y_\delta^i\|_V [(n-i+1)^\alpha - (n-i)^\alpha] \right) \|y_\delta^n\|_V \\
&\quad - \left(1 + \|u_0\|_V + \frac{\delta^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \|y_\delta^i\|_V [(n-i+1)^\alpha - (n-i)^\alpha] \right) \|y_\delta^n\|_V \\
&\geq m_a \|y_\tau^n\|_V^2 - \frac{M_b \delta^\alpha}{\Gamma(\alpha+1)} \|y_\delta^n\|_V^2 - M_b \|u_0\|_V \|y_\delta^n\|_V \\
&\quad - c_j \|y_\delta^n\|_V - c_j \|u_0\|_V \|y_\delta^n\|_V - \frac{c_j \delta^\alpha}{\Gamma(\alpha+1)} \|y_\delta^n\|_V^2 \\
&\quad - \frac{M_b \delta^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n-1} [(n-i+1)^\alpha - (n-i)^\alpha] \|y_\delta^i\|_V \|y_\delta^n\|_V \\
&\quad - \frac{c_j \delta^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n-1} [(n-i+1)^\alpha - (n-i)^\alpha] \|y_\delta^i\|_V \|y_\delta^n\|_V .
\end{aligned}$$

Hence, drawing from the preceding analysis, we conclude that

$$\begin{aligned}
&\|f_\delta^n\|_V + \frac{\delta^\alpha (M_b + c_j)}{\Gamma(\alpha+1)} \|y_\delta^i\|_V \sum_{i=1}^{n-1} [(n-i+1)^\alpha - (n-i)^\alpha] + c_j + (c_j + M_b)\|u_0\|_V \\
&\geq \left(m_a - \frac{\delta^\alpha (M_b + c_j)}{\Gamma(\alpha+1)} \right) \|y_\delta^n\|_V .
\end{aligned}$$

Selecting $\bar{\delta} = \left(\frac{m_1 \Gamma(1+\alpha)}{2(M_b + c_j)} \right)^{1/\alpha}$, we deduce that $m_a - \frac{\delta^\alpha (M_b + c_j)}{\Gamma(1+\alpha)} \geq \frac{m_a}{2}$ for all $0 < \delta < \bar{\delta}$. Thus,

$$2\frac{\|f_\delta^n\|_V}{m_a} + 2\frac{c_j + c_j \|u_0\|_V + M_b \|u_0\|_V}{m_a} + 2\frac{\delta^\alpha (M_b + c_j)}{m_a \Gamma(\alpha+1)} \sum_{i=1}^{n-1} \|y_\delta^i\|_V [(n-i+1)^\alpha - (n-i)^\alpha] \geq \|y_\delta^n\|_V .$$

Utilizing hypotheses (HP_1) (i), for every $\delta > 0$ and $n \in \mathbb{N}$, there exists a positive constant $c_f > 0$ such that $|f_\delta^n|_V \leq c_f$.

For brevity, let us denote $\tilde{c} = \frac{2}{m_a}(c_f + M_b \|u_0\|_V + c_j \|u\|_V + c_j)$.

By applying the generalized discrete Gronwall's inequality from Lemma A.1, we observe that

$$\|f_\delta^n\|_V \leq c_0 \exp\left(\frac{2\delta^\alpha (M_b + c_j)}{m_a \Gamma(\alpha+1)} \sum_{i=1}^{n-1} [(n-i+1)^\alpha - (n-i)^\alpha] \right) \leq c_0 \exp\left(\frac{2\delta^\alpha (M_b + c_j)}{m_a \Gamma(\alpha+1)} \right) \leq c. \tag{20}$$

By combining equations (20) and (16), we derive the following result:

$$\|u_\delta^n\|_V = \left\| u_0 + \frac{\delta^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} y_\delta^i [(n-i+1)^\alpha - (n-i)^\alpha] \right\|_V$$
$$\leq \|u_0\|_V + \sum_{i=1}^{n} (t_{n-i+1}^\alpha - t_{n-i}^\alpha) \leq \|u_0\|_V + \frac{cT^\alpha}{\Gamma(\alpha+1)} \leq c.$$

Finally, according to (HP_1), we obtain the following estimate for β_δ^n:

$$\|\beta_\delta^n\|_{V^*} \leq c_j(1 + \|u_\delta^n\|_V) \leq c_j(1+c). \tag{21}$$

Thus, Lemma A.1 is established.

The solvability of **Problem (PV)** ensues from the subsequent result. □

Proof of Theorem 3.1. Consider a sequence $\{\delta_n\}$ such that $\delta_n \to 0$, as $n \to \infty$.

Based on the estimate (18), the sequences $\{\overline{y}_\delta\}$, $\{\overline{u}_\delta\}$, and $\{\overline{\beta}_\delta\}$, which interpolate to $\{y_\delta\}$, $\{u_\delta\}$, and $\{\beta_\delta\}$ respectively, are bounded for $k = 1, \ldots, N$.

Therefore, there exists $y \in V$, $u \in V$ and $\beta \in V^*$ such that

$$\overline{y}_\delta \to y \text{ weakly in } V, \text{ as } \delta \to 0,$$
$$\overline{u}_\delta \to u \text{ weakly in } V, \text{ as } \tau \to 0, \tag{22}$$
$$\overline{\beta}_\delta \to \beta \text{ weakly in } V^*, \text{ as } \tau \to 0. \tag{23}$$

Utilizing [24, Lemma 4 (a)], we derive that

$${}_0I_t^\alpha \overline{y}_\delta \to {}_0I_t^\alpha y \text{ weakly in } V, \text{ as } \delta \to 0.$$

Employing equation (18), and for all $t \in (0,T)$, it follows that

$$\|\overline{u}_\delta(t) - u_0 - {}_0I_t^\alpha \overline{y}_\delta(t)\| = \left\| \frac{\delta^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} y_\delta^i [(n-i+1)^\alpha - (n-i)^\alpha] - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{y}_\delta(s) \mathrm{d}s \right\|$$
$$\leq \frac{c}{\Gamma(\alpha)} \int_t^{t_n} (t_n - s)^{\alpha-1} \mathrm{d}s + \int_0^t |(t-s)^{\alpha-1} - (t_n - s)^{\alpha-1}| \mathrm{d}s \tag{24}$$
$$\leq \frac{c}{\Gamma(\alpha)} [(t_n - t)^\alpha + t^\alpha + (t_n - t)^\alpha - t_n^\alpha],$$

for $t \in [t_{n-1}, t_n]$. Then,

$$\overline{u}_\delta(t) - u_0 - {}_0I_t^\alpha \overline{y}_\delta(t) \to 0 \text{ strongly in } V, \text{ as } \tau \to 0.$$

This, in conjunction with equation (24), results in

$$\overline{u}_\delta(t) \to u_0 + {}_0I_t^\alpha y(t) \text{ weakly in } V, \text{ as } \delta \to 0. \tag{25}$$

Considering the mapping $u \mapsto \partial j(u)$, which is upper semi-continuous from V to V^*, and relying on (23) and [25, Theorem 3.13], we have

$$\beta(t) \in \partial j(u_0 + {}_0I_t^\alpha y_\delta), \text{ for a.e., } t \in (0,T).$$

We now define the Nemytskii operators $\overline{a}$ and $\overline{b}$ corresponding to a and b as follows:

$$(\overline{a}y)(t) = ay(t), \text{ and } (\overline{b}y)(t) = b(u_0 + {}_0I_t^\alpha y(t)),$$

for all $w \in V$ and almost every $t \in (0,T)$.

Given assumption (HP_1) (iii), as well as equations (21) and (22), we have for $t \in (0,T)$ that

$$\overline{a}\, \overline{y}_\delta \to \overline{a}y \text{ weakly in } V, \text{ as } \tau \to 0, \tag{26}$$
$$b(u_0 + {}_0I_t^\alpha \overline{y}_\delta(t)) \to b(u_0 + {}_0I_t^\alpha y(t)) \text{ weakly in } V, \text{ as } \tau \to 0.$$

Derived from (HP_1) and equation (18), we have

$$\int_0^T \|b({}_0I_t^\alpha \overline{y}_\delta(t) + u_0)\|_V \, dt \leq \frac{M_b C}{\Gamma(\alpha+1)} \int_0^T t^\alpha dt + TM_b \, \|u_0\|_V = \frac{M_b C T^{\alpha+1}}{\Gamma(\alpha+2)} + TM_b \, \|u_0\|_V . \tag{27}$$

By applying the Lebesgue dominated convergence theorem, we can express

$$\lim_{\delta\to 0} b(\overline{y}_\delta, v) = \lim_{\tau\to 0} \int_0^T b({}_0I_t^\alpha \overline{y}_\delta(t) + u_0, v(t))dt = \int_0^T \lim_{\tau\to 0} b({}_0I_t^\alpha \overline{y}_\delta(t) + u_0, v(t))dt$$
$$= \int_0^T b({}_0I_t^\alpha y(t) + u_0, v(t))dt = b(y, v).$$

On the contrary, according to [25, Lemma 3.3], we are aware that

$$f_\tau \to f \ \text{ strongly in } \ V, \ \text{ as } \ \tau \to 0.$$

Finally, by utilizing equations (25), (26), and (27), we can take the limit in equation (15), which implies that $y \in L^2(0, T; V)$ is a solution to **Problem (FRP)**.

Hence, we deduce that $u \in W^{1,2}(0, T; V)$, given by $u(t) = u_0 + {}_0I_t^\alpha y(t)$ for almost every $t \in (0, T)$, is a solution to **Problem (PV)**. □

4 Convergence analysis of the penalty method

In this section, we employ the penalty approach by replacing the Signorini's condition (6) with a modified condition:

$$\sigma_\nu(u_\varepsilon(t) - g) := -\frac{1}{\varepsilon}[u_{\varepsilon_\nu}(t) - g]^+,$$

where $\varepsilon > 0$ represents the penalty parameter.

For any $u, v \in V$ and $t \in (0, T)$, we define the function $\Phi : V \times V \to \mathbb{R}$ as follows:

$$\Phi(u(t), v) := \int_{\Gamma_C} [u_\nu(t)]^+ v_\nu d\Gamma = \langle [u_\nu(t)]^+, v_\nu \rangle_{\Gamma_C}. \tag{28}$$

As j is not differentiable, we introduce a regularization using j_ε defined as follows:

$$j_\varepsilon(u(t), v) := \int_{\Gamma_C} \mu(\|u_\tau(t)\|).|\sigma_\nu(u(t))|.\, \Psi_\varepsilon(v)d\Gamma,$$

for all $v \in V$, where $\Psi_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ is the family of convex and differentiable function and satisfy the following property:

$$0 < \Psi_\varepsilon(v) - \|v\| \leq \varepsilon.$$

The functionals j_ε are Gateaux-differentiable and serve as approximations of j. Specifically, there exists a constant $c > 0$ satisfying the following inequality:

$$|j_\varepsilon(u(t), v) - j(u(t), v)| \leq c \,.\, \varepsilon, \ \text{ for all } \ v \in V. \tag{29}$$

We denote by $j'_\varepsilon : V \times V \to \mathbb{R}$ the derivative of j_ε given by

$$\langle j'_\varepsilon(u_\varepsilon(t), v), w\rangle = \int_{\Gamma_C} \mu(\|u_{\varepsilon_\tau}(t)\|)|\sigma_\nu(t)|.\Psi'_\varepsilon(v_\tau)w_\tau d\Gamma, \tag{30}$$

where $\Psi'_\varepsilon(v)w = \frac{v.w}{\sqrt{\|v\|^2 + \varepsilon^2}}$, for all $u, v, w \in V$.

Finally, the regularized problem associated with equations (11) to (13) is given by:

• **Problem (PVP)** : Find a displacement field $u_\varepsilon : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ and a stress field $\sigma_\varepsilon : \Omega \times (0,T) \longrightarrow \mathbb{S}^d$ for all $v \in V$ and $\varepsilon > 0$ such that

$$\sigma_\varepsilon(t) = \mathcal{V}\varepsilon({}_0^C D_t^\alpha u_\varepsilon(t)) + \mathcal{B}\varepsilon(u_\varepsilon(t)), \tag{31}$$

$$\mathfrak{a}({}_0^C D_t^\alpha u_\varepsilon(t), v) + \mathfrak{b}(u_\varepsilon(t), v) + \frac{1}{\varepsilon}\Phi(u_\varepsilon(t), v) + \langle j_\varepsilon'(u_\varepsilon(t), u_\varepsilon(t)), v\rangle = (f(t), v), \tag{32}$$

$$u_\varepsilon(0) = u_0. \tag{33}$$

We have the following existence and convergence of penalized problem.

Theorem 4.1. *Assuming the condition stated in* Theorem 3.1, *for any* $\varepsilon > 0$, *we have the following*:

(*a*) **Problem (PVP)** *possesses at least one solution*

$$(u_\varepsilon, \sigma_\varepsilon) \in W^{1,2}(0,T;\, V) \times L^2(0,T;\, L^2(\Omega, \mathbb{S}^d)).$$

(*b*) *The solution* $(u_\varepsilon, \sigma_\varepsilon)$ *of* **Problem (PVP)** *converges to a solution of* **Problem (PV)**, *i.e.*,

$$\|u - u_\varepsilon\|_V \to 0, \quad \|\sigma - \sigma_\varepsilon\|_V \to 0, \quad as \quad \varepsilon \to 0.$$

In this paragraph, we establish the existence result of the penalized problem by employing the Faedo-Galerkin approximation method.

Proof of (a) in Theorem 4.1. We will utilize the Faedo-Galerkin approximation method. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ represent a complete orthonormal basis of $L^2(\Omega)^d$.

Consisting of eigenfunction of the operator $-\Delta$. For a positive integer m, we are to find a function

$$u_{\varepsilon_m} \coloneqq \sum_{i=1}^{m} d_i(t)\varphi_i. \tag{34}$$

The components $d_i = (d_1, d_2, \ldots, d_n)$ are chosen to be continuous in time and vector-valued functions. They are selected in such a way that satisfies the following conditions:

$$\mathfrak{a}({}_0^C D_t^\alpha u_{\varepsilon_m}(t), \varphi_i) + \mathfrak{b}(u_{\varepsilon_m}(t), \varphi_i) + \frac{1}{\varepsilon}\Phi(u_{\varepsilon_m}(t), \varphi_i) + \langle j_\varepsilon'(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), \varphi_i\rangle = (f(t), \varphi_i)_V,$$

$$u_{\varepsilon_m}(0) = u_0. \tag{35}$$

We denote by F_m the vector space generated by $\varphi_1, \varphi_2, \ldots, \varphi_n$, such that $u_{\varepsilon_m} \in F_m$ and u_{ε_m} converges to u_ε in V.

We consider the following approximation problem:

Find $u_{\varepsilon_m} \in L^2(0,T;\, F_m)$ such that ${}_0^C D_t^\alpha u_{\varepsilon_m} \in L^2(0,T;\, F_m)$ and

$$\mathfrak{a}({}_0^C D_t^\alpha u_{\varepsilon_m}(t), \varphi_k) + \mathfrak{b}(u_{\varepsilon_m}(t), \varphi_k) + \frac{1}{\varepsilon}\Phi(u_{\varepsilon_m}(t), \varphi_k) + \langle j_\varepsilon'(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), \varphi_k\rangle = (f(t), \varphi_k)_V, \tag{36}$$

$$u_{\varepsilon_m}(0) = u_0. \tag{37}$$

Taking into account (34) for $k = 1, 2, \ldots, m$, we obtain

$$\begin{cases} \mathfrak{a}({}_0^C D_t^\alpha u_{\varepsilon_m}(t), \varphi_k) = \mathcal{V}{}_0^C D_t^\alpha d_i(t), \\ \mathfrak{b}(u_{\varepsilon_m}(t), \varphi_k) = \mathcal{B} d_i(t), \\ \Phi(u_{\varepsilon_m}(t), \varphi_k) = \Phi\left(\sum_{i=1}^{m} d_i(t)\varphi_i, \varphi_k\right), \\ \langle j_\varepsilon'(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), \varphi_k\rangle = \left\langle j_\varepsilon'\left(\sum_{i=1}^{m} d_i(t)\varphi_i, \sum_{i=1}^{m} d_i(t)\varphi_i\right), \varphi_k \right\rangle, \\ (f(t), \varphi_k) = f_k(t). \end{cases}$$

Thus, the vector function u_{ε_m} is a solution of (36)–(37) if the vector $d(t) = (d_1(t), \dots, d_n(t))$ satisfies the fractional ordinary differential equation:

$$\begin{cases} {}_0^C D_t^\alpha d_i(t) = h(t, d_i(t)), \\ d_i(0) = (u_{\varepsilon_0}, \varphi_i), \end{cases} \tag{38}$$

for $i = 1, \dots, m$, where

$$h(t, d_i(t)) = \mathcal{V}^{-1}\left(f_k(t) - \mathcal{B} d_i(t) - \Phi\left(\sum_{i=1}^m d_i(t)\varphi_i, \varphi_k \right) + \left\langle j'_\varepsilon \left(\sum_{i=1}^m d_i(t)\varphi_i, \varphi_k \right) \right\rangle \right).$$

Let d_i^1 and d_i^2 be two functions that satisfy (34). By employing (HP_1) (iii), we obtain

$$|\mathcal{B} d_i^1(t) - \mathcal{B} d_i^2(t)| \le M_b \| d_i^1(t) - d_i^2(t) \|_V . \tag{39}$$

Combining (28) and (8) with the given inequality

$$|[x]^+ - [y]^+| \le |x - y|,$$

we have

$$\begin{aligned} &\left| \Phi\left(\sum_{i=1}^m d_i^1(t)\varphi_i, \varphi_k \right) - \Phi\left(\sum_{i=1}^m d_i^2(t)\varphi_i, \varphi_k \right) \right| \\ &\le \left| \int_{\Gamma_C} [d_{i_\nu}^1(t)\varphi_i]^+ \varphi_k \, d\Gamma - \int_{\Gamma_C} [d_{i_\nu}^2 \varphi_i(t)]^+ \varphi_k \, d\Gamma \right| \\ &\le \int_{\Gamma_C} |d_{i_\nu}^1(t) - d_{i_\nu}^2(t)| |\varphi_i \varphi_k| \, d\Gamma \\ &\le \| d_{i_\nu}^1(t) - d_{i_\nu}^2(t) \|_{L^2(\Gamma_C)} \le c_d \, \| d_i^1(t) - d_i^2(t) \|_V . \end{aligned}$$

Using (8), (30), and (HP_3), we can establish the existence of a positive constant c dependent on c_0, ε, L_μ, and μ^*, satisfying

$$\begin{aligned} &|\langle j'_\varepsilon(u_{\varepsilon_m}^1(t), u_{\varepsilon_m}^1(t)), \varphi_k \rangle - \langle j'_\varepsilon(u_{\varepsilon_m}^2(t), u_{\varepsilon_m}^2(t)), \varphi_k \rangle| \\ &= \left| \int_{\Gamma_C} \mu(\|u_{\varepsilon_m}^1(t)\|) |\sigma_\nu(u_{\varepsilon_m}^1)| \frac{u_{\varepsilon_{m_\nu}}^1 \varphi_{k_\nu}}{\sqrt{\varepsilon^2 + \|u_{\varepsilon_{m_\nu}}^1\|^2}} \, d\Gamma - \int_{\Gamma_C} \mu(\|u_{\varepsilon_m}^2(t)\|) |\sigma_\nu(u_{\varepsilon_m}^2)| \frac{u_{\varepsilon_{m_\nu}}^2 \varphi_{k_\nu}}{\sqrt{\varepsilon^2 + \|u_{\varepsilon_{m_\nu}}^2\|^2}} \, d\Gamma \right| \\ &\le c \, \| d_i^1(t) - d_i^2(t) \|_V . \end{aligned} \tag{40}$$

Combining relations (39) to (40), we can deduce the existence of a positive constant c such that

$$|h(t, d_i^1(t)) - h(t, d_i^2(t))| \le c \| d_i^1(t) - d_i^2(t) \|.$$

By applying a standard method for fractional calculus as described in [26, Proposition 4.6], we can conclude that the system of fractional ordinary differential equations (38) possesses a unique solution d_m on the interval $[0, T_*)$.

In the following analysis, we establish *a priori* estimates to ensure that the function d is well defined on the interval $[0, T]$ for all $T > 0$. This enables us to consider the limit as $n \to \infty$ and find a global weak solution to **Problem (PVP)**.

Estimate for u_{ε_m}:

To begin, multiply equation (36) by $d_i(t)$, summing over $i = 1, \dots, m$. Utilizing the fact that $u_{\varepsilon_m} \mapsto \frac{1}{2} \|u_{\varepsilon_m}\|_V^2$ is a convex function, we obtain

$$\begin{aligned} {}_0^C D_t^\alpha \left(\frac{1}{2} \mathfrak{a}(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)) \right) &\le \mathfrak{a}({}_0^C D_t^\alpha u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)) \\ &= (f(t), u_{\varepsilon_m}(t))_V - b(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)) - \frac{1}{\varepsilon} \Phi(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)) - \langle j'_\varepsilon(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), u_{\varepsilon_m}(t) \rangle. \end{aligned}$$

For any $\lambda > 0$ and the following several calculations, we can deduce that

$$\begin{cases} |(f(t), u_{\varepsilon_m}(t))_V| \le \dfrac{1}{2\lambda}\|f(t)\|_V^2 + \dfrac{\lambda}{2}\|u_{\varepsilon_m}(t)\|_V^2, \\ m_b\|u_{\varepsilon_m}(t)\|_V^2 \le |b(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t))|, \\ \left|\dfrac{1}{\varepsilon}\Phi(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t))\right| \le \dfrac{c_d^2}{\varepsilon}\|u_{\varepsilon_m}(t)\|_V^2, \\ |\langle j_\varepsilon'(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), u_{\varepsilon_m}(t)\rangle| \le \mu^* \mathrm{mes}(\Gamma_C) c_0^2 \|u_{\varepsilon_m}(t)\|_V^2. \end{cases}$$

Then,

$$ {}_0^C D_t^\alpha\left(\frac{1}{2}\mathfrak{a}(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t))\right) + c_1\|u_{\varepsilon_m}(t)\|_V^2 \le c_2\|f(t)\|_V^2. \tag{41}$$

We use Proposition A.1 (c) to (41) and the coercivity of operator a, we have

$$\|u_{\varepsilon_m}(t)\|_V^2 + \frac{2c_1}{m_a\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_{\varepsilon_m}(s)\|_V^2\,ds \le c_3(\|f\|_V^2 + \|u_0\|_V^2). \tag{42}$$

Therefore, we obtain $T_* = +\infty$.

Estimate for ${}_0^C D_t^\alpha u_{\varepsilon_m}$:

Taking $v = {}_0^C D_t^\alpha u_{\varepsilon_m}(t)$ in (36), we obtain

$$\begin{aligned} &\mathfrak{a}({}_0^C D_t^\alpha u_{\varepsilon_m}(t), {}_0^C D_t^\alpha u_{\varepsilon_m}(t)) + \mathfrak{b}(u_{\varepsilon_m}(t), {}_0^C D_t^\alpha u_{\varepsilon_m}(t)) + \frac{1}{\varepsilon}\Phi(u_{\varepsilon_m}(t), {}_0^C D_t^\alpha u_{\varepsilon_m}(t)) + \langle j_\varepsilon'(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), {}_0^C D_t^\alpha u_{\varepsilon_m}(t)\rangle \\ &\quad = (f(t), {}_0^C D_t^\alpha u_{\varepsilon_m}(t))_V. \end{aligned}$$

However, since we have $\Psi(u, v) \ge 0$ and $\langle j_\varepsilon(u, v), v\rangle \ge 0$, and based on assumption (HP_1), we can conclude that

$$m_a\|{}_0^C D_t^\alpha u_{\varepsilon_m}(t)\|_{V^*}^2 \le \frac{1}{2\lambda}\|f(t)\|_V^2 + \lambda\|u_{\varepsilon_m}(t)\|_V^2 + \frac{1}{2\lambda}\|{}_0^C D_t^\alpha u_{\varepsilon_m}(t)\|_{V^*}^2,$$

where $\lambda > 0$. Based on the estimate for u_{ε_m}, there exists a positive constant c such that

$$\sup_{t\in[0,T]}\|{}_0^C D_t^\alpha u_{\varepsilon_m}(t)\|_{V^*} \le c. \tag{43}$$

Passage to the limit

By combining the previous estimates with the compactness result from [27, Theorem 4.2] for the Caputo derivative, we can conclude that there exists a subsequence $u_{\varepsilon_{\tau_m}}$ and $u_\varepsilon \in L^2(0, T; V)$ such that

$$\begin{aligned} &u_{\varepsilon_{\tau_m}} \to u_\varepsilon, \ \text{ strongly in } L^2(0, T; V), \\ &{}_0^C D_t^\alpha u_{\varepsilon_{\tau_m}} \rightharpoonup {}_0^C D_t^\alpha u_\varepsilon, \ \text{ weakly in } L^2(0, T; V^*), \end{aligned}$$

where τ_m is a sequence such that $\tau_m \to 0$, as $m \to \infty$.

Furthermore, we have

$$\begin{cases} \mathfrak{a}({}_0^C D_t^\alpha u_{\varepsilon_{\tau_m}}(t), v) \to \mathfrak{a}({}_0^C D_t^\alpha u_\varepsilon(t), v), \ \text{ in } \mathbb{R}, \\ \mathfrak{b}(u_{\varepsilon_{\tau_m}}(t), v) \to \mathfrak{b}(u_\varepsilon(t), v), \ \text{ in } \mathbb{R}, \\ \sigma(u_{\varepsilon_{\tau_m}}(t)) \to \sigma(u_\varepsilon(t)), \ \text{ in } \mathbb{S}^d. \end{cases}$$

By utilizing the definitions of the operators Ψ and j_ε, it can be observed that

$$\begin{cases} |\Phi(u_{\varepsilon_m}(t), v)| \le \dfrac{c_0}{\varepsilon}\|u_{\varepsilon_m}(t)\|_V \cdot \|v\|_{L^2(\Gamma_C)^d}, \\ |\langle j_\varepsilon'(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), v\rangle| \le \mu^*\mathrm{mes}(\Gamma_C) c_0 \dfrac{c_0}{\varepsilon}\|u_{\varepsilon_m}(t)\|_V \cdot \|v\|_{L^2(\Gamma_C)^d}. \end{cases}$$

Then, $\{\Psi(u_{\varepsilon_m}(t))\}_{m=1}^{\infty}$ and $\{\langle j_\varepsilon'(u_{\varepsilon_m}(t), u_{\varepsilon_m}(t)), v\rangle\}_{m=1}^{\infty}$ are bounded in $\mathbb{R}$, and we may pass to a subsequence if it is necessary.

For $v = u_\varepsilon - u_{\varepsilon_{\tau m}}$, we obtain

$$|\Phi(u_\varepsilon(t), u_\varepsilon(t) - u_{\varepsilon_{\tau m}}(t)) - \Phi(u_{\varepsilon_{\tau m}}(t), u_\varepsilon(t) - u_{\varepsilon_{\tau m}}(t))| \leq c_0^2 \, \|u_\varepsilon(t) - u_{\varepsilon_{\tau m}}(t)\|_V^2$$

and

$$|\langle j_\varepsilon'(u_\varepsilon(t), u_\varepsilon(t)), u_\varepsilon(t) - u_{\varepsilon_{\tau m}}(t)\rangle - \langle j_\varepsilon'(u_{\varepsilon_{\tau m}}(t), u_{\varepsilon_{\tau m}}(t)), u_\varepsilon(t) - u_{\varepsilon_{\tau m}}(t)\rangle| \leq c \, \|u_\varepsilon(t) - u_{\varepsilon_{\tau m}}(t)\|_V^2 .$$

Due to the compactness of the trace operator $\gamma : V \to L^2(\Gamma_C)^d$, it follows from the weak convergence of $u_{\varepsilon_{\tau m}}$ that

$$u_{\varepsilon_{\tau m}} \to u_\varepsilon, \ \text{ strongly in } L^2(0, T; L^2(\Gamma_C)^d). \tag{44}$$

Then,

$$\begin{cases} \Phi(u_{\varepsilon_{\tau m}}, v) \to \Phi(u_\varepsilon, v), \ \text{ in } \mathbb{R}, \\ \langle j_\varepsilon'(u_{\varepsilon_{\tau m}}, u_{\varepsilon_{\tau m}}), v\rangle \to \langle j_\varepsilon'(u_\varepsilon, u_\varepsilon), v\rangle, \ \text{ in } \mathbb{R}. \end{cases}$$

Thus, we have successfully demonstrated the existence of a weak penalized solution to (31)–(33). □

Proof of (b) in Theorem 4.1. In this paragraph, we present a convergence result that involves the sequence u_ε, ${}_0^C D_t^\alpha u_\varepsilon$, and $[u_{\varepsilon_\nu}]^+$. Based on equations (42) and (43), we can deduce that

$$\begin{aligned} &\{u_\varepsilon\} \text{ is bounded in } L^2(0, T; V), \\ &\{\sigma_\varepsilon\} \text{ is bounded in } L^2(0, T; L^2(\Omega, \mathbb{S}^d)), \\ &\{{}_0^C D_t^\alpha u_\varepsilon\} \text{ is bounded in } L^2(0, T; V^*). \end{aligned} \tag{45}$$

Estimate for $[u_{\varepsilon_\nu}]^+$**:**

Taking into account (28) and (42), we obtain

$$\Phi([u_{\varepsilon_\nu}]^+, u_{\varepsilon_{nu}}) = \int_{\Gamma_C} [u_{\varepsilon_\nu}]^+ . \, u_{\varepsilon_\nu} d\Gamma = \|u_{\varepsilon_\nu}\|_{L^2(\Gamma_C)}^2 \leq c.$$

For almost every $t \in [0, T]$, integrating from 0 to t, we find

$$\{[u_{\varepsilon_\nu}]^+\} \text{ is bounded in } L^2(0, T; L^2(\Gamma_C)^d).$$

Passage to the limit in ε**:**

The result (45) ensures the existence of subsequences of u_ε that converge again to u_ε such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup \tilde{u}, \ \text{ weakly in } L^2(0, T; V), \\ &\sigma_\varepsilon \rightharpoonup \tilde{\sigma}, \ \text{ weakly in } L^2(0, T; L^2(\Omega, \mathbb{S}^d)), \\ &{}_0^C D_t^\alpha u_\varepsilon \rightharpoonup {}_0^C D_t^\alpha \tilde{u}, \ \text{ weakly in } L^2(0, T; V^*). \end{aligned} \tag{46}$$

Similar to (44), we can derive the following

$$\begin{aligned} &u_\varepsilon \to \tilde{u}, \ \text{ strongly in } L^2(0, T; L^2(\Gamma_C)^d), \\ &\sigma_\varepsilon \to \tilde{\sigma}, \ \text{ strongly in } L^2(0, T; L^2(\Gamma_C)^d), \\ &{}_0^C D_t^\alpha u_\varepsilon \to {}_0^C D_t^\alpha \tilde{u}, \ \text{ strongly in } L^2(0, T; L^2(\Gamma_C)^d). \end{aligned} \tag{47}$$

According to boundedness of $\{[u_{\varepsilon_\nu}]^+\}$, we obtain

$$\lim_{\varepsilon \to 0} \|[u_{\varepsilon_\nu}]^+ - g\|_{L^2(0,T;\, L^2(\Gamma_C)^d)} = \|[u_\nu]^+ - g\|_{L^2(0,T;\, L^2(\Gamma_C)^d)} = 0.$$

This implies that

$$[u_\nu]^+ = g, \quad \text{a.e. on } \Gamma_C, \quad \text{and } \tilde{u}_\nu \leq g, \quad \text{on } \Gamma_C.$$

Then,

$$\tilde{u} \in V_{ad}.$$

For any $v \in V$, utilizing (29), (32), and the fact that $\Phi(u(t), v - u(t)) \geq 0$, we can deduce that

$$\begin{aligned}
&\sigma_\varepsilon(t) = \mathcal{V}\varepsilon({}_0^C D_t^\alpha u_\varepsilon(t)) + \mathcal{B}\varepsilon(u_\varepsilon(t)),\\
&\mathfrak{a}({}_0^C D_t^\alpha u_\varepsilon(t), v - u_\varepsilon(t)) + \mathfrak{b}(u_\varepsilon(t), v - u_\varepsilon(t)) + j_\varepsilon(u_\varepsilon(t), v) - j_\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) = (f(t), v - u_\varepsilon(t)).
\end{aligned}$$

Considering (47), it is evident that

$$j_\varepsilon(u_\varepsilon(t), v) - j_\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) \to j(u(t), v) - j(u(t), u(t)), \quad \text{in } \mathbb{R}.$$

For any $v, w \in V$, applying the coercivity of j_ε and (29), we can conclude that

$$\langle j_\varepsilon'(u(t), v), w - v\rangle_{V^*,V} \leq j_\varepsilon(u(t), w) - j_\varepsilon(u(t), v) \leq j(u(t), w) - j(u(t), v) + 2c\varepsilon. \tag{48}$$

By combining (46) and (48), and taking the limit as $\varepsilon \to 0$, we deduce that

$$\begin{aligned}
&\tilde{\sigma}(t) = \mathcal{V}\varepsilon({}_0^C D_t^\alpha \tilde{u}(t)) + \mathcal{B}\varepsilon(\tilde{u}(t)),\\
&\mathfrak{a}({}_0^C D_t^\alpha \tilde{u}(t), v - \tilde{u}(t)) + \mathfrak{b}(\tilde{u}(t), v - \tilde{u}(t)) + j(\tilde{u}(t), v) - j(\tilde{u}(t), \tilde{u}(t)) = (f(t), v - \tilde{u}(t)).
\end{aligned}$$

Finally, based on (11)–(13), we can conclude that $(\tilde{u}, \tilde{\sigma}) = (u, \sigma)$. □

5 Conclusion and future directions

In this article, we have examined parabolic problems that incorporate the fractional time-derivative operator. We also employed the Rothe method and the Banch method for mathematical purposes. Additionally, we developed an optimization problem by penalizing contact conditions and regularizing friction conditions. This study can be regarded as foundational for further research into other issues related to fractional viscoelastic contact with friction.

This work can serve as a foundation for studying other problems involving piezoelectric, thermo-viscoelastic, and thermo-piezo-viscoelastic behaviors, considering various types of contacts and frictions.

The work presented here encompasses various extensions and perspectives, including the following notable aspects:

(1) Exploring new contact problems inspired by current industrial projects, such as energy production.
(2) The mathematical and numerical analysis of problems, considering additional mechanical and physical properties such as magnetism.
(3) Studying contact models using optimization tools and associated optimal control problems to yield physically applicable results.
(4) Utilize numerical methods based on convex optimization, such as the projected conjugate gradient, alternating direction method of multipliers, and augmented Lagrangian, as well as deep learning techniques.
(5) Investigate contact models using optimization tools and address the associated optimal control problems.

Acknowledgement: The authors would like to express their gratitude to the anonymous referees for their valuable comments that have contributed to the improvement of this article.

Funding information: The authors state that no funding is involved.

Author contributions: All authors have made equal and significant contributions to this article. They have all read and approved the final version of the manuscript.

Conflict of interest: The authors state no conflict interest.

Ethical approval: The research conducted in this study is not associated with either human or animal use.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during this study.

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Appendix

In this section, we provide a brief overview of some relevant results concerning fractional calculus and nonlinear analysis. More detailed information can be found in the following references: [25,28–30].

Definition A.1. (Riemann-Liouville fractional integral) Let X be a Banach space and $(0,T)$ be a finite time interval. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a given function $f \in L^1(0,T;X)$ is defined by

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad \forall t \in (0,T), $$

where $\Gamma(.)$ stands for the Gamma function defined by

$$ \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\mathrm{d}t. $$

To complement the definition, we set ${}_0I_t^0 = I$, where I is the identity operator, which means that ${}_0I_t^0 f(t) = f(t)$ for a.e. $t \in (0,T)$.

Definition A.2. (Caputo derivative of order $0 < \alpha \leq 1$) Let X be a Banach space, $0 < \alpha \leq 1$ and $(0,T)$ be a finite time interval. For a given function $f \in AC(0,T;W)$, the Caputo fractional derivative of f is defined by

$$ {}_0^CD_t^{\alpha} f(t) = {}_0I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\mathrm{d}s, \quad \forall t \in (0,T). $$

The notation $AC(0,T;X)$ refers to the space of all absolutely continuous functions from $(0,T)$ into X.

It is obvious that if $\alpha = 1$, the Caputo derivative reduces to the classical first-order derivative, i.e., we have

$$ {}_0^CD_t^1 f(t) = If'(t) = f'(t), \quad \text{for a.e. } t \in (0,T). $$

Proposition A.1. *Let X be a Banach space and* $\alpha, \beta > 0$. *Then, the following statements hold*:

(a) *For* $y \in L^1(0,T;X)$, *we have* ${}_0I_t^{\alpha}\,{}_0I_t^{\beta} y(t) = {}_0I_t^{\alpha+\beta} y(t)$, *for a.e.* $t \in (0,T)$.

(b) *For* $y \in AC(0,T;X)$ *and* $\alpha \in (0,\alpha]$, *we have*

$$ {}_0I_t^{\alpha}\,{}_0^CD_t^{\alpha} y(t) = y(t) - y(0), \text{ for a.e. } t \in (0,T), $$

(c) *For* $y \in L^1(0,T;X)$, *we have* ${}_0^CD_t^{\alpha}\,{}_0I_t^{\alpha} y(t) = y(t)$, *for a.e.* $t \in (0,T)$.

Now, we introduce the generalized Caputo derivative based on the modified Riemnn-Liouville operator:

Definition A.3. Let $0 < \gamma < 1$ and $\varphi \in L^1_{\mathrm{loc}}[0,T)$. For any $\varphi_0 \in \mathbb{R}$, we define the generalized Caputo derivative associated with φ_0 to be

$$ D_c^{\gamma} : \varphi \mapsto D_c^{\gamma}\varphi = J_{-\gamma}\varphi - \varphi_0 g_{1-\gamma} = J_{-\gamma}(\varphi - \varphi_0) \in \mathcal{E}. \tag{A1} $$

If $\varphi \in X$, we impose $\varphi_0 = \varphi(0+)$ unless explicitly mentioned, and in this case, we call D_c^{γ} the Caputo derivative of order γ, where

$$ \mathcal{E} := \{v \in \mathcal{D}'(\mathbb{R}) : \exists M_v \in \mathbb{R},\ \operatorname{supp}(v) \subset [-M_v, +\infty)\}, $$

and $\mathcal{D}'(\mathbb{R})$ is the space of distribution, which is the dual of $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$. Clearly, $\mathcal{E}$ is a linear vector space.

Proposition A.2. *By the definition, we have the following claims*:

(1) *For any constant* C, $D_c^\gamma C = 0$.

(2) $D_c^\gamma : X \to \mathcal{E}$ *is a linear operator.*

(3) *For all* $\varphi \in X$, $0 < \gamma_1 < 1$, and $\gamma_2 > \gamma_1 - 1$, *we have*

$$J_{\gamma_2} D_c^{\gamma_1}\varphi = \begin{cases} D_c^{\gamma_1-\gamma_2}\varphi, & \gamma_2 < \gamma_1, \\ J_{\gamma_2-\gamma_1}(\varphi - \varphi(0+)), & \gamma_2 \geq \gamma_1. \end{cases}$$

(4) *Suppose* $0 < \gamma_1 < 1$. *If* $f \in Y_{\gamma_1}$, *then* $D_c^{\gamma_2} J_{\gamma_1} f = J_{\gamma_1-\gamma_2} f$ *for* $0 < \gamma_2 < 1$.

(5) *If* $D_c^{\gamma_1}\varphi \in X$, *then for* $0 < \gamma_2 < 1$, $0 < \gamma_1 + \gamma_2 < 1$,

$$D_c^{\gamma_2} D_c^{\gamma_1}\varphi = D_c^{\gamma_1+\gamma_2} - D_c^{\gamma_1}\varphi(0+) g_{1-\gamma_2}. \tag{A2}$$

(6) $J_{\gamma-1} D_c^\gamma \varphi = J_{-1}\varphi - \varphi(0+)\delta(t)$. *If we define this to be* D_c^1, *then for* $\varphi \in C^1[0, T)$, $D_c^1\varphi = \varphi'$,

where

$$X \coloneqq \left\{ \varphi \in L^1_{\mathrm{loc}}[0, T) \;:\; \exists \varphi_0 \in \mathbb{R},\ \lim_{t\to 0^+} \frac{1}{t}\int_0^t |\varphi - \varphi_0| \mathrm{d}t = 0 \right\},$$

and

$$Y_\gamma \coloneqq \left\{ f \in L^1_{\mathrm{loc}}[0, T) \;:\; \lim_{T\to 0^+} \frac{1}{T}\int_0^T \left| \int_0^t (t-s)^{\gamma-1} f(s) \mathrm{d}s \right| \mathrm{d}t = 0 \right\}.$$

Definition A.4. (Clarke generalized directional derivative and generalized gradient) Let $J : X \to \mathbb{R}$ be a locally Lipschitz function. We denote by $J^0(u, v)$ the Clarke generalized directional derivative of J at the point $x \in X$ in the direction $y \in X$ is defined by

$$J^0(x, y) = \limsup_{\lambda\to 0^+,\ z\to x} \frac{J(z + \lambda y) - J(z)}{\lambda}.$$

The generalized gradient of $J : X \to \mathbb{R}$ at $x \in X$ is defined by

$$\partial J(x) = \{\xi \in X^*, /J^0(x, y) \geq \langle \xi, y\rangle_{X^*,X}, \text{ for all } y \in X\}.$$

Lemma A.1. *Let* $\{u_n\}$, $\{v_n\}$, *and* $\{w_n\}$ *be nonnegative sequences satisfying*

$$u_n \leq v_n + \sum_{k=1}^{n-1} w_k u_k, \quad \textit{for all } n \geq 1.$$

Then, we have

$$u_n \leq v_n + \sum_{k=1}^{n-1} v_k w_k \exp\left(\sum_{j=k+1}^{n-1} w_j \right), \quad \textit{for all } n \geq 1.$$

Moreover, if $\{u_n\}$ *and* $\{w_n\}$ *are such that*

$$u_n \leq \alpha = \sum_{k=1}^{n-1} w_k u_k, \quad \textit{for all } n \geq 1,$$

where $\alpha > 0$ *is a constant, then for all* $n \geq 1$, *it holds*

$$u_n \leq \alpha \exp\left(\sum_{k=1}^{n-1} w_k \right).$$