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Research Article

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A memory-type thermoelastic laminated beam with structural damping and microtemperature effects: Well-posedness and general decay

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Abstract: In previous work, Fayssal considered a thermoelastic laminated beam with structural damping, where the heat conduction is given by the classical Fourier's law and acting on both the rotation angle and the transverse displacements established an exponential stability result for the considered problem in case of equal wave speeds and a polynomial stability for the opposite case. This article deals with a laminated beam system along with structural damping, past history, and the presence of both temperatures and microtemperature effects. Employing the semigroup approach, we establish the existence and uniqueness of the solution. With the help of convenient assumptions on the kernel, we demonstrate a general decay result for the solution of the considered system, with no constraints regarding the speeds of wave propagations. The result obtained is new and substantially improves earlier results in the literature.

Keywords: laminated beam, stability, well-posedness, microtemperature effects, structural damping, past history, nonlinear equations

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1 Introduction

Mathematical modeling is indispensable in engineering, natural science, and applied mathematics to capture the effects of memory ingrained in the studied actualities. To this end, the inclusion of memory is often simplified for presentation purposes, as a specific description of basic operations can be intricate for mathematical manipulation. A key question to address is how certain behaviors are related to memory? In this study, we investigate the joint impact of an infinite memory and microtemperature effects on the system under consideration.

In 1921, Timoshenko made a whole study focusing on rectifying a differential equation for transverse vibrations of prismatic bars to account for shear effects. Furthermore, he is credited with pioneering a beams study characterizing the transverse vibration, problems (4) and (5) are relevant to the Timoshenko laminated beam.

The laminated beam model has become quite popular that both scientists and engineers are interested in it. This model is a pertinent study topic, because of the wide industry applicability of such materials. Hansen

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and Spies [1] were the first to introduce the following beam with two layers by developing this mathematical model:

$$\begin{cases}\rho_1\varpi_{tt} + G(\phi - \varpi_x)_x = 0,\\ \rho_2(3\psi - \phi)_{tt} - G(\phi - \varpi_x) - D(3\psi - \phi)_{xx} = 0,\\ \rho_3\psi_{tt} + G(\phi - \varpi_x) + \dfrac{4}{3}\gamma\psi + \dfrac{4}{3}\beta\psi_t - D\psi_{xx} = 0.\end{cases} \tag{1}$$

The laminated beam equations have produced some results so far, most of which are focused on the system's stability and existence. Provided that the assumption of equal wave speeds holds, it was demonstrated that system (1) is exponentially stable, when linear damping terms are incorporated in two of the three equations. However, if they are included in the three equations, then the system decays exponentially with no restriction on the speeds of wave propagations, see, for instance [2,3].

Lately, a renewed focus on investigating the asymptotic behavior of the solution of several thermoelastic laminated beams has grown. For more details about this topic, the reader may consult [4–10].

Apalara [5] studied the following thermoelastic laminated beam model:

$$\begin{cases}\rho\varpi_{tt} + G(\phi - \varpi_x)_x = 0,\\ I_\rho(3\psi_{tt} - \phi_{tt}) - G(\phi - \varpi_x) - D(3\psi_{xx} - \phi_{xx}) = 0,\\ 3I_\rho\psi_{tt} - 3D\psi_{xx} + 3G(\phi - \varpi_x) + 4\gamma\psi + \delta\theta_x = 0,\\ \rho_3\theta_t - \alpha\theta_{xx} + \delta\psi_{xt} = 0,\end{cases} \tag{2}$$

provided that the wave speeds are equal, the author managed to establish an exponential decay result without the need to add damping terms.

The thermoelastic laminated beam problem together with nonlinear weights and time-varying delay was the study topic of Nonato et al. [7], who considered two cases (with and without the structural damping) and proved an exponential decay result for both of them.

The infinite memory is a critical aspect in addressing problems; moreover, incorporating this term can lead to distinct solutions that differ from those found in previous studies. The infinite memory has been explored in various contexts such as the work of Liu and Zhao [11], in which they considered a thermoelastic laminated beam model with past history. The authors managed to establish both exponential and polynomial stabilities, depending on the kernel function for the system involving structural damping, and with no restriction on the wave speeds, moreover, for the system in the absence of structural damping, they were able to establish both exponential and polynomial stabilities, in case of equal wave speeds, and lack of exponential stability in the opposite case.

In [12], Fayssal investigated the following thermoelastic laminated beam given by this system:

$$\begin{cases}\varrho\varpi_{tt} + G(\phi - \varpi_x)_x + \gamma\theta_x = 0,\\ I_\varrho(3\psi - \phi)_{tt} - D(3\psi - \phi)_{xx} - G(\phi - \varpi_x) - \gamma\theta = 0,\\ 3I_\varrho\psi_{tt} - 3D\psi_{xx} + 3G(\phi - \varpi_x) + 4\delta\psi + 4\beta\psi_t = 0,\\ \varrho_3\theta_t - k\theta_{xx} + \gamma(3\psi - \phi)_t + \gamma\varpi_{tx} = 0,\end{cases} \tag{3}$$

provided that the condition of equal wave speeds holds, he obtained an exponential stability result.

In the current work, we study the following thermoelastic laminated beam, together with structural damping, infinite memory, and microtemperature effects:

$$\begin{cases}\varrho\varpi_{tt} + G(\phi - \varpi_x)_x + \gamma\theta_x = 0,\\ I_\varrho(3\psi - \phi)_{tt} - D(3\psi - \phi)_{xx} - G(\phi - \varpi_x) - m\theta + dr_x + \displaystyle\int_0^\infty g(s)(3\psi - \phi)_{xx}(x, t - s)\mathrm{d}s = 0,\\ 3I_\varrho\psi_{tt} - 3D\psi_{xx} + 3G(\phi - \varpi_x) + 4\delta\psi + 4\beta\psi_t = 0,\\ c\theta_t - k_0\theta_{xx} + m(3\psi - \phi)_t + \gamma\varpi_{tx} + k_1r_x = 0,\\ \alpha r_t - k_2r_{xx} + k_3r + k_1\theta_x + d(3\psi - \phi)_{tx} = 0,\end{cases} \tag{4}$$

where

$$(x,t)\in(0,1)\times(0,+\infty),$$

and the initial and boundary conditions are given by

$$\begin{cases}\varpi(x,0)=\varpi_0,\psi(x,0)=\psi_0,\phi(x,0)=\phi_0,\theta(x,0)=\theta_0,r(x,0)=r_0, & x\in(0,1),\\ \varpi_t(x,0)=\varpi_1,\psi_t(x,0)=\psi_1,\phi_t(x,0)=\phi_1, & x\in(0,1),\\ \varpi_x(0,t)=\phi(0,t)=\psi(0,t)=\theta(0,t)=r(0,t)=0, & t>0,\\ \varpi(1,t)=\phi_x(1,t)=\psi_x(1,t)=\theta_x(1,t)=r(1,t)=0, & t>0,\end{cases} \tag{5}$$

where ϖ denotes the transverse displacement, ϕ represents the rotation angle, ψ is relative to the amount of slip occurring along the interface, θ is the temperature difference, and r is the microtemperature vector. The coefficients $\delta,\beta,\varrho,I_\varrho,G$, and D are positive constants representing the adhesive stiffness, the adhesive damping parameter, the density, the shear stiffness, the flexural rigidity, and the mass moment of inertia, respectively. We denote by the constants $c,k_0,k_1,k_2,k_3,\mathrm{d}\gamma,m,\alpha>0$ the physical parameters describing the coupling between the various constituents of the materials.

The rest of this article is structured this way: in Section 2, we provide some resources required for our research and then highlight our major results. In Section 3, we establish the well-posedness of the system, and in Section 4, we introduce some fundamental lemmas required in the proof later. In Section 5, we demonstrate our general decay result.

2 Preliminaries and main results

In this section, we provide some practical materials required in the proof later and then state our major results.

– **(A$_1$)** Let $g:\mathbb{R}_+\to\mathbb{R}_+$ be a C^1 function that satisfies

$$g(0)>0, D-g_0=\bar{l}>0,\quad \text{where } g_0:=\int_0^\infty g(s)\mathrm{d}s. \tag{6}$$

– **(A$_2$)** There exists a strictly increasing convex function $\mathcal{G}:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2([0,+\infty])$ that satisfies

$$\begin{cases}\mathcal{G}(0)=\mathcal{G}'(0)=0,\\ \lim\limits_{t\to+\infty}\mathcal{G}'(t)=+\infty,\end{cases}$$

such that

$$\sup_{s\in\mathbb{R}_+}\int_0^\infty\frac{g(s)}{\mathcal{G}^{-1}(-g'(s))}\mathrm{d}s+\int_0^\infty\frac{g(s)}{\mathcal{G}^{-1}(-g'(s))}\mathrm{d}s<+\infty.$$

Now, we present the following useful inequalities.

Lemma 1. [13] *The following inequalities are valid*:

$$\int_0^1\left[\int_0^\infty g(s)((3\psi-\psi)(t)-(3\psi-\phi)(t-s))\mathrm{d}s\right]^2\mathrm{d}x\le c_1(g\diamond(3\psi-\phi)_x)(t), \tag{7}$$

$$\int_0^1\left[\int_0^\infty g'(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))\mathrm{d}s\right]^2\mathrm{d}x\le -g(0)(g'\diamond(3\psi-\phi)_x)(t), \tag{8}$$

$$\int_0^1 \left[\int_0^\infty g(s)((3\psi - \phi)_x(t) - (3\psi - \phi)_x(t-s))\mathrm{d}s \right]^2 \mathrm{d}x \le g_0(g \diamond (3\psi - \phi)_x)(t), \tag{9}$$

$$\int_0^1 \left[\int_0^\infty g'(s)((3\psi - \phi)(t) - (3\psi - \phi)(t-s))\mathrm{d}s \right]^2 \mathrm{d}x \le -c_2(g' \diamond (3\psi - \phi)_x)(t), \tag{10}$$

where $c_1, c_2 > 0$, *and*

$$(g \diamond v)(t) = \int_0^1 \int_0^\infty g(s)(v(x,t) - v(x,t-s))^2 \mathrm{d}s\mathrm{d}x.$$

Let us start by introducing

$$\eta^t(x,s) = (3\psi - \phi)(x,t) - (3\psi - \phi)(x,t-s), \tag{11}$$

where

$$(x,s,t) \in (0,1) \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Then, the variable η^t surely satisfies

$$\eta_t^t + \eta_s^t = (3\psi - \phi)_t. \tag{12}$$

Hence, system (4) can be rewritten as

$$\begin{cases} \varrho\varpi_{tt} + G(\phi - \varpi_x)_x + \gamma\theta_x = 0, \\ I_\varrho(3\psi - \phi)_{tt} - D(3\psi - \phi)_{xx} - G(\phi - \varpi_x) - m\theta + \mathrm{d}r_x + \int_0^\infty g(s)(3\psi - \phi)_{xx}(t-s)\mathrm{d}s = 0, \\ 3I_\varrho\psi_{tt} - 3Ds_{xx} + 3G(\phi - \varpi_x) + 4\delta\psi + 4\beta\psi_t = 0, \\ c\theta_t - k_0\theta_{xx} + m(3\psi - \phi)_t + \gamma\varpi_{tx} + k_1 r_x = 0, \\ \alpha r_t - k_2 r_{xx} + k_3 r + k_1\theta_x + d(3\psi - \phi)_{tx} = 0, \\ \eta_t^t + \eta_s^t = (3\psi - \phi)_t. \end{cases} \tag{13}$$

Certainly, system (13) depends on the following initial and boundary conditions:

$$\begin{cases} \varpi(x,0) = \varpi_0, \psi(x,0) = \psi_0, \phi(x,0) = \phi_0, \theta(x,0) = \theta_0, r(x,0) = r_0, \quad x \in (0,1), \\ \varpi_t(x,0) = \varpi_1, \psi_t(x,0) = \psi_1, \phi_t(x,0) = \phi_1, \quad x \in (0,1), \\ \varpi_x(0,t) = \phi(0,t) = \psi(0,t) = \theta(0,t) = r(0,t) = 0, \quad t > 0, \\ \varpi(1,t) = \phi_x(1,t) = \psi_x(1,t) = \theta_x(1,t) = r(1,t) = 0, \quad t > 0 \\ \eta^t(0,s) = \eta_x^t(1,s) = \eta^t(x,0) = 0, \eta^0(x,s) = \eta_0(x,s), \quad x \in (0,1), t, s > 0. \end{cases} \tag{14}$$

Now, let

$$\begin{cases} \zeta = 3\psi - \phi, \\ \zeta(0,t) = \zeta_x(1,t) = 0, \zeta(x,0) = \zeta_0, \zeta_t(x,0) = \zeta_1, \quad (x,t) \in (0,1) \times \mathbb{R}_+. \end{cases}$$

Then, system (13) is equivalent to

$$\begin{cases} \varrho\varpi_{tt} + G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x = 0, \\ I_\varrho\zeta_{tt} - D\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + \mathrm{d}r_x + \int_0^\infty g(s)\zeta_{xx}(t-s)\mathrm{d}s = 0, \\ 3I_\varrho\psi_{tt} - 3D\psi_{xx} + 3G(3\psi - \zeta - \varpi_x) + 4\delta\psi + 4\beta\psi_t = 0, \\ c\theta_t - k_0\theta_{xx} + m\zeta_t + \gamma\varpi_{tx} + k_1 r_x = 0, \\ \alpha r_t - k_2 r_{xx} + k_3 r + k_1\theta_x + \mathrm{d}\zeta_{tx} = 0, \\ \eta_t^t + \eta_s^t = \zeta_t. \end{cases} \tag{15}$$

Manipulating (11), one can rewrite the second equation of (15) as

$$I_\varrho \zeta_{tt} - \bar{l}\,\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + \mathrm{d}r_x - \int_0^\infty g(s)\eta^t_{xx}(x,s)\mathrm{d}s = 0,$$

and then, once introducing the vector function $U = (\varpi, u, \zeta, v, \psi, y, \theta, r, \eta^t)^T$, with

$$u = \varpi_t,$$
$$v = \zeta_t,$$
$$y = \psi_t,$$

system (15) becomes

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}U(t) = \mathfrak{A}U(t), \quad t > 0, \\ U(0) = U_0 = (\varpi_0, \varpi_1, \zeta_0, \zeta_1, \psi_0, \psi_1, \theta_0, r_0, \eta_0)^T, \end{cases} \tag{16}$$

here, $\mathfrak{A} : D(\mathfrak{A}) \subset \mathcal{H} \to \mathcal{H}$ stands for a linear operator indicated by

$$\mathfrak{A}U = \begin{pmatrix} u \\ -\dfrac{1}{\varrho}(G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x) \\ v \\ \dfrac{1}{I_\varrho}\left(\bar{l}\,\zeta_{xx} + G(3\psi - \zeta - \varpi_x) + m\theta - \mathrm{d}r_x + \displaystyle\int_0^\infty g(s)\eta^t_{xx}(x,s)\mathrm{d}s\right) \\ y \\ \dfrac{1}{I_\varrho}\left(D\psi_{xx} - G(3\psi - \zeta - \varpi_x) - \dfrac{4}{3}\delta\psi - \dfrac{4}{3}\beta y\right) \\ \dfrac{1}{c}(k_0\theta_{xx} - mv - \gamma u_x - k_1 r_x) \\ \dfrac{1}{\alpha}(k_2 r_{xx} - k_3 r - k_1\theta_x - \mathrm{d}v_x) \\ v - \eta^t_s \end{pmatrix}.$$

Now, we shall consider the ensuing energy space

$$\mathcal{H} = \tilde{\mathbb{J}}^1_*(0,1) \times L^2(0,1) \times \mathbb{J}^1_*(0,1) \times L^2(0,1) \times \mathbb{J}^1_*(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1) \times \mathbb{L}_g,$$

where

$$\mathbb{J}^1_*(0,1) = \{\varphi \in H^1(0,1) : \ \varphi(0) = 0\},$$
$$\tilde{\mathbb{J}}^1_*(0,1) = \{\varphi \in H^1(0,1) : \ \varphi(1) = 0\},$$
$$\mathbb{J}^2_*(0,1) = H^2(0,1) \cap \mathbb{J}^1_*(0,1),$$
$$\tilde{\mathbb{J}}^2_*(0,1) = H^2(0,1) \cap \tilde{\mathbb{J}}^1_*(0,1),$$

and

$$\mathbb{L}_g = \left\{\varphi : \mathbb{R}_+ \to \mathbb{J}^1_*(0,1), \int_0^1\int_0^\infty g(s)\varphi_x^2\mathrm{d}s\mathrm{d}x < \infty\right\}.$$

For the space $\mathbb{L}_g$, we take the following inner product:

$$\langle\varphi_1, \varphi_2\rangle_{\mathbb{L}_g} = \int_0^1\int_0^\infty g(s)\varphi_{1x}\varphi_{2x}\mathrm{d}s\mathrm{d}x.$$

To continue, we set the domain

$$\mathcal{L}_g(\mathbb{R}_+, \mathbb{J}^1_*(0,1)) = \{\eta^t \in \mathbb{L}_g, \eta^t_s \in \mathbb{L}_g, \eta^t(x,0) = 0\}.$$

Then, we introduce

$$\begin{aligned}\langle U, \bar{U}\rangle_{\mathcal{H}} &= \varrho\int_0^1 u\bar{u}\mathrm{d}x + I_\varrho\int_0^1 v\bar{v}\mathrm{d}x + 3I_\varrho\int_0^1 y\bar{y}\,\mathrm{d}x + c\int_0^1 \theta\bar{\theta}\mathrm{d}x + \alpha\int_0^1 r\bar{r}\mathrm{d}x\\ &+ \bar{I}\int_0^1 \zeta_x\bar{\zeta}_x\mathrm{d}x + G\int_0^1 (3\psi - \zeta - \varpi_x)(3\bar{\psi} - \bar{\zeta} - \bar{\varpi}_x)\mathrm{d}x + 4\delta\int_0^1 \psi\bar{\psi}\mathrm{d}x\\ &+ 3D\int_0^1 \psi_x\bar{\psi}_x\mathrm{d}x + \int_0^1\int_0^1 g(s)\eta^t_x(x,t)\bar{\eta}^t_x(x,s)\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{17}$$

We deduce that $\mathcal{H}$ along with (17) is a Hilbert space, once we do that, we define $D(\mathfrak{A})$ by

$$D(\mathfrak{A}) = \left\{\begin{array}{c} U \in \mathcal{H}:\ \varpi \in \tilde{\mathbb{J}}^2_*(0,1);\ \zeta, \psi \in \mathbb{J}^2_*(0,1);\\ u \in \tilde{\mathbb{J}}^1_*(0,1);\ v, y, \theta \in \mathbb{J}^1_*(0,1), \theta_t \in L^2(0,1);\\ r \in H^2(0,1) \cap H^1_0(0,1), \eta^t \in \mathcal{L}_g(\mathbb{R}_+, \mathbb{J}^1_*(0,1));\\ \varpi_x(0,t) = \zeta_x(1,t) = \psi_x(1,t) = \theta_x(1,t) = \eta^t_x(1,s) = 0. \end{array}\right\}$$

Obviously, $D(\mathfrak{A})$ is dense in $\mathcal{H}$.

Now, we are prepared to state our results.

Theorem 1. *Let $U_0 \in D(\mathfrak{A})$, then problems* (14) *and* (15) *admit a unique solution*

$$U \in C(\mathbb{R}_+, D(\mathfrak{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

In addition, if $U_0 \in \mathcal{H}$, then

$$U \in C(\mathbb{R}_+, \mathcal{H}).$$

We give the energy of the solution of problems (13) and (14) by

$$\begin{aligned}\mathscr{E}(t) = \frac{1}{2}\int_0^1 \{&\varrho\varpi_t^2 + G(\phi - \varpi_x)^2 + I_\varrho(3\psi_t - \phi_t)^2 + \bar{I}(3\psi_x - \phi_x)^2 + 3I_\varrho\psi_t^2\\ &+3D\psi_x^2 + 4\delta\psi^2 + c\theta^2 + ar^2\}\mathrm{d}x + \frac{1}{2}(g \diamond (3\psi - \phi)_x)(t).\end{aligned} \tag{18}$$

Then, we have the following stability result.

Theorem 2. *Let $(\varpi, \phi, \psi, \theta, r, \eta^t)$ be the solution of* (13) *and* (14), *suppose that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold. Then, for any initial data $U_0 \in D(\mathfrak{A})$ satisfying, for some $q_0 \geq 0$,*

$$\int_0^1 \eta^2_{0x}(x,s)\mathrm{d}x \leq q_0, \quad \text{for all } s > 0, \tag{19}$$

there exist positive constants α_1, α_2, and α_3, such that

$$\mathscr{E}(t) \leq \alpha_1\mathscr{G}^{-1}_*(\alpha_2 t + \alpha_3), \tag{20}$$

where

$$\mathscr{G}^{-1}_*(t) = \int_t^\infty \frac{\mathrm{d}s}{\mathscr{G}_0(s)}, \mathscr{G}_0(t) = t\mathscr{G}'(\varepsilon_0 t), \quad \text{for all } \varepsilon_0 \geq 0.$$

Many examples regarding our stability result have been presented in previous studies; for details, the reader may consult [14].

3 Existence and uniqueness

In this section, we utilize the semigroup approach to prove our well-posedness result [15,16].

Proof of Theorem 1. Let us establish the dissipativity of $\mathfrak{A}$. By (17) and for any $U \in D(\mathfrak{A})$, we have

$$\langle \mathfrak{A}U, U\rangle_{\mathcal{H}} = -4\beta\int_0^1 y^2\,\mathrm{d}x - k_3\int_0^1 r^2\mathrm{d}x - k_2\int_0^1 r_x^2\mathrm{d}x - k_0\int_0^1 \theta_x^2\mathrm{d}x + \frac{1}{2}(g' \diamond \zeta_x)(t) \le 0.$$

Thereby, $\mathfrak{A}$ is dissipative.

Thereafter, we establish the surjectivity of $(I - \mathfrak{A})$, that is, we show that

$$\forall H = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9)^T \in \mathcal{H}, \exists U \in D(\mathfrak{A}) \ : \ (I - \mathfrak{A})U = H. \tag{21}$$

We have

$$\begin{cases} \varpi - u = h_1, \\ \varrho u + G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x = \varrho h_2, \\ \zeta - v = h_3, \\ I_\varrho v - \bar{l}\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + \mathrm{d}r_x - \int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s = I_\varrho h_4, \\ \psi - y = h_5, \\ 3I_\varrho y - 3D\psi_{xx} + 3G(3\psi - \zeta - \varpi_x) + 4\delta\psi + 4\beta y = 3I_\varrho h_6, \\ c\theta - k_0\theta_{xx} + mv + \gamma u_x + k_1 r_x = ch_7, \\ (\alpha + k_3)r - k_2 r_{xx} + k_1\theta_x + \mathrm{d}v_x = \alpha h_8, \\ \eta^t - v + \eta_s^t = h_9. \end{cases} \tag{22}$$

Solving the last equation of (22), we find

$$\eta^t = e^{-s}\int_0^s e^{\sigma}(v + h_9(\sigma))\mathrm{d}\sigma. \tag{23}$$

Inserting (23) and

$$\begin{cases} u = \varpi - h_1, \\ v = \zeta - h_3, \\ y = \psi - h_5, \end{cases}$$

into $(22)_2$, $(22)_4$, $(22)_6$, $(22)_7$, and $(22)_8$, we obtain

$$\begin{cases} \varrho\varpi + G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x = \varrho(h_1 + h_2), \\ I_\varrho\zeta - \left(\bar{l} + \int_0^\infty (1 - e^{-s})g(s)\mathrm{d}s\right)\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + \mathrm{d}r_x = I_\varrho(h_3 + h_4) + \tilde{h}, \\ \mu_1\psi - 3D\psi_{xx} + 3G(3\psi - \zeta - \varpi_x) = (3I_\varrho + 4\beta)h_5 + 3I_\varrho h_6, \\ c\theta - k_0\theta_{xx} + m\zeta + \gamma\varpi_x + k_1 r_x = \gamma h_{1x} + ch_7 + mh_3, \\ (\alpha + k_3)r - k_2 r_{xx} + k_1\theta_x + d\zeta_x = \alpha h_8 + dh_{3x}, \end{cases} \tag{24}$$

where

$$\tilde{h} = \int_0^\infty g(s) \int_0^s e^{\sigma - s}(h_9 - h_3)_{xx} \mathrm{d}\sigma \mathrm{d}s$$

and

$$\mu_1 = 3I_\varrho + 4\delta + 4\beta.$$

We take the following variational formulation, to solve (24)

$$\mathcal{Q}((\varpi, \zeta, \psi, \theta, r), (\bar{\varpi}, \bar{\zeta}, \bar{\psi}, \bar{\theta}, \bar{r})) = L(\bar{\varpi}, \bar{\zeta}, \bar{\psi}, \bar{\theta}), \quad \forall (\bar{\varpi}, \bar{\zeta}, \bar{\psi}, \bar{\theta}, \bar{r}) \in X, \tag{25}$$

where $X = \tilde{\mathbb{J}}^1_*(0,1) \times \mathbb{J}^1_*(0,1) \times \mathbb{J}^1_*(0,1) \times L^2(0,1) \times H^1_0(0,1)$ is a Hilbert space endowed with

$$\|(\varpi, \zeta, \psi, \theta, r)\|^2_X = \|3\psi - \zeta - \varpi_x\|^2_2 + \|\varpi\|^2_2 + \|\zeta_x\|^2_2 + \|\psi_x\|^2_2 + \|\theta_x\|^2_2 + \|r\|^2_2 + \|r_x\|^2_2.$$

As a part of this step, we provide definitions for both the bilinear form $\mathcal{Q} : X \times X \to \mathbb{R}$ and the linear form $L : X \to \mathbb{R}$, as follows:

$$\begin{aligned}
&\mathcal{Q}((\varpi, \zeta, \psi, \theta, r), (\bar{\varpi}, \bar{\zeta}, \bar{\psi}, \bar{\theta}, \bar{r})) \\
&\quad = \varrho \int_0^1 \varpi\bar{\varpi} \mathrm{d}x + I_\varrho \int_0^1 \zeta\bar{\zeta} \mathrm{d}x + \mu_1 \int_0^1 \psi\bar{\psi} \mathrm{d}x + c\int_0^1 \theta\bar{\theta} \mathrm{d}x + (\alpha + k_3)\int_0^1 r\bar{r} \mathrm{d}x + k_2 \int_0^1 r_x\bar{r}_x \mathrm{d}x + \gamma \int_0^1 (\theta_x\bar{\varpi} + \varpi_x\bar{\theta}) \mathrm{d}x \\
&\qquad + k_0 \int_0^1 \theta_x\bar{\theta}_x \mathrm{d}x + G\int_0^1 (3\psi - \zeta - \varpi_x)(3\bar{\psi} - \bar{\zeta} - \bar{\varpi}_x) \mathrm{d}x + \left(\bar{l} + \int_0^\infty (1 - e^{-s}) g(s) \mathrm{d}s\right) \int_0^1 \zeta_x\bar{\zeta}_x \mathrm{d}x + 3D\int_0^1 \psi_x\bar{\psi}_x \mathrm{d}x \\
&\qquad + d\int_0^1 (r_x\bar{\zeta} + \zeta_x\bar{r}) \mathrm{d}x + k_1 \int_0^1 (r_x\bar{\theta} + \bar{r}\theta_x) \mathrm{d}x + m\int_0^1 (\zeta\bar{\theta} - \bar{\zeta}\theta) \mathrm{d}x
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
L(\bar{\varpi}, \bar{\zeta}, \bar{\psi}, \bar{\theta}, \bar{r}) &= \varrho \int_0^1 \bar{\varpi}(h_1 + h_2) \mathrm{d}x + I_\varrho \int_0^1 \bar{\zeta}(h_3 + h_4) \mathrm{d}x + \int_0^1 \bar{\zeta}\tilde{h} \mathrm{d}x \\
&\quad + \int_0^1 \bar{\theta}(\gamma h_{1x} + mh_3 + ch_7) \mathrm{d}x + \int_0^1 \bar{\psi}[(3I_\varrho + 4\beta)h_5 + 3I_\varrho h_6] \mathrm{d}x + \int_0^1 \bar{r}(\alpha h_8 + dh_{3x}) \mathrm{d}x.
\end{aligned}$$

We can easily prove the continuity of $\mathcal{Q}$ and L. Moreover, from (26) together with integration by parts, we arrive at

$$\begin{aligned}
&\mathcal{Q}((\varpi, \zeta, \psi, \theta, r), (\varpi, \zeta, \psi, \theta, r)) \\
&\quad = \varrho \int_0^1 \varpi^2 \mathrm{d}x + I_\varrho \int_0^1 \zeta^2 \mathrm{d}x + \mu_1 \int_0^1 \psi^2 \mathrm{d}x + c\int_0^1 \theta^2 \mathrm{d}x + (\alpha + k_3)\int_0^1 r^2 \mathrm{d}x + k_2 \int_0^1 r_x^2 \mathrm{d}x + k_0 \int_0^1 \theta_x^2 \mathrm{d}x \\
&\qquad + G\int_0^1 (3\psi - \zeta - \varpi_x)^2 \mathrm{d}x + 3D\int_0^1 \psi_x^2 \mathrm{d}x + \left(\bar{l} + \int_0^\infty (1 - e^{-s}) g(s) \mathrm{d}s\right) \int_0^1 \zeta_x^2 \mathrm{d}x \\
&\quad \geq M\|(\varpi, \zeta, \psi, \theta, r)\|^2_X,
\end{aligned}$$

where M is a positive constant. From which, we conclude the coercivity of $\mathcal{Q}$. It follows from the Lax-Milgram lemma that (24) admits a unique solution satisfying

$$\begin{aligned}
\varpi &\in \tilde{\mathbb{J}}^1_*(0,1), \\
\zeta, \psi &\in \mathbb{J}^1_*(0,1), \\
\theta &\in L^2(0,1),
\end{aligned}$$

and

$$r \in H_0^1(0,1).$$

If we substitute ϖ, ζ, and ψ into $(22)_1$, $(22)_3$ and $(22)_5$, we find

$$u \in \tilde{\mathbb{J}}_*^1(0,1)$$

and

$$v, y \in \mathbb{J}_*^1(0,1).$$

Besides, taking $(\bar{\zeta}, \bar{\psi}, \bar{\theta}, \bar{r}) \equiv (0,0,0,0) \in (\mathbb{J}_*^1(0,1))^2 \times L^2(0,1) \times H_0^1(0,1)$, then (25) becomes

$$G\int_0^1 \bar{\varpi}\varpi_{xx}\mathrm{d}x = \int_0^1 \bar{\varpi}(\varrho\varpi + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1 + h_2))\mathrm{d}x, \tag{27}$$

for all $\bar{\varpi} \in \tilde{\mathbb{J}}_*^1(0,1)$, which indicates that

$$G\varpi_{xx} = \varrho\varpi + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1 + h_2) \in L^2(0,1). \tag{28}$$

The standard elliptic regularity implies that

$$\varpi \in \tilde{\mathbb{J}}_*^2(0,1).$$

We note that (27) remains valid for $\bar{\varphi} \in C^1([0,1]) \subset \tilde{\mathbb{J}}_*^1(0,1)$, that is $\bar{\varphi}(1) = 0$. Then, we obtain

$$G\int_0^1 \bar{\varphi}_x\varpi_x\mathrm{d}x = \int_0^1 \bar{\varphi}(-\varrho\varpi - 3G\psi_x + G\zeta_x - \gamma\theta_x + \varrho(h_1 + h_2))\mathrm{d}x.$$

Integrating by parts, it follows that

$$\varpi_x(0)\bar{\varphi}(0) = 0, \quad \text{for all } \bar{\varphi} \in C^1([0,1]).$$

Hence,

$$\varpi_x(0) = 0.$$

Likewise, we show that

$$(\zeta, \psi) \in (\mathbb{J}_*^2(0,1))^2, \quad \theta \in \mathbb{J}_*^1(0,1), \quad r \in H^2(0,1) \cap H_0^1(0,1), \quad \text{and} \quad \zeta_x(1) = \psi_x(1) = \theta_x(1) = 0.$$

The standard elliptic regularity guarantees the existence of a unique $U \in D(\mathfrak{A})$ which fulfils (21). Thereby, $\mathfrak{A}$ is surjective.

As a consequence, we infer that $\mathfrak{A}$ is a maximal dissipative operator. Then, the well-posedness result follows using the Lumer-Philips theorem [15]. □

4 Technical lemmas

The main purpose of this section is to establish the essential practical lemmas required to prove our stability results. To attain this goal, we apply a specific approach known as the multiplier technique, which enables us to prove the stability results of problem (13). Nevertheless, this method necessitates creating an appropriate Lyapunov functional equivalent to the energy, and we will clarify this in the next section. To simplify matters, we will employ $\chi_* > 0$ to represent a generic constant.

Lemma 2. *Let $(\varpi, \phi, \psi, \theta, r, \eta^t)$ be the solution of* (13) *and* (14), *then, the energy functional satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t) = -4\beta\int_0^1 \psi_t^2\mathrm{d}x - k_0\int_0^1 \theta_x^2\mathrm{d}x - k_2\int_0^1 r_x^2\mathrm{d}x - k_3\int_0^1 r^2\mathrm{d}x + \frac{1}{2}(g' \diamond (3\psi - \phi)_x)(t) \le 0. \tag{29}$$

Proof. As a start, we multiply $(13)_1$, $(13)_2$, $(13)_3$, $(13)_4$, and $(13)_5$ by ϖ_t, $(3\psi_t - \phi_t)$, ψ_t, θ, and r, respectively, then, we integrate over $(0,1)$, use integration by parts together with boundary conditions (14) and (11), to find

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\{\varrho\varpi_t^2 + G(\phi - \varpi_x)^2 + I_\varrho(3\psi_t - \phi_t)^2 + \bar{l}(3\psi_x - \phi_x)^2 + 3I_\varrho\psi_t^2 + 3D\psi_x^2 + 4\delta\psi^2 + c\theta^2 + \alpha r^2\}\mathrm{d}x\\&\quad + 4\beta\int_0^1\psi_t^2\mathrm{d}x + k_0\int_0^1\theta_x^2\mathrm{d}x + k_2\int_0^1 r_x^2\mathrm{d}x + k_3\int_0^1 r^2\mathrm{d}x - \int_0^1(3\psi - \phi)_t\int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s\mathrm{d}x = 0.\end{aligned}\tag{30}$$

It follows from the sixth equation in (13) and the integration by parts that

$$\begin{aligned}\int_0^1(3\psi - \phi)_t\int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s\mathrm{d}x &= \int_0^\infty g(s)\left(\int_0^1\eta_t^t\eta_{xx}^t(x,s)\mathrm{d}x\right)\mathrm{d}s + \int_0^\infty g(s)\left(\int_0^1\eta_s^t\eta_{xx}^t(x,s)\mathrm{d}x\right)\mathrm{d}s\\&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g\diamond(3\psi - \phi)_x)(t) + \frac{1}{2}(g'\diamond(3\psi - \phi)_x)(t),\end{aligned}\tag{31}$$

which together with (30) gives us the desired result. □

Lemma 3. *Consider the functional*

$$\mathscr{I}_1(t) := -I_\varrho\int_0^1(3\psi_t - \phi_t)\int_0^\infty g(s)((3\psi - \phi)(t) - (3\psi - \phi)(t-s))\mathrm{d}s\mathrm{d}x,\tag{32}$$

then, it satisfies

$$\begin{aligned}\mathscr{I}_1'(t) \le \frac{-I_\varrho g_0}{2}\int_0^1(3\psi_t - \phi_t)^2\mathrm{d}x + \varepsilon_1\int_0^1(3\psi_x - \phi_x)^2\mathrm{d}x + \varepsilon_1\int_0^1(\phi - \varpi_x)^2\mathrm{d}x + \varepsilon_1\int_0^1\theta_x^2\mathrm{d}x + \chi_*\int_0^1 r^2\mathrm{d}x\\ + \chi_*\left(1 + \frac{1}{\varepsilon_1}\right)(g\diamond(3\psi - \phi)_x)(t) - \chi_*(g'\diamond(3\psi - \phi)_x)(t), \quad \forall\varepsilon_1 > 0.\end{aligned}\tag{33}$$

Proof. First, we note that

$$\begin{aligned}&\partial_t\left(\int_0^\infty g(s)((3\psi - \phi)(t) - (3\psi - \phi)(t-s))\mathrm{d}s\right)\\&= \partial_t\left(\int_{-\infty}^t g(t-s)((3\psi - \phi)(t) - (3\psi - \phi)(s))\mathrm{d}s\right)\\&= \int_{-\infty}^t g'(t-s)((3\psi - \phi)(t) - (3\psi - \phi)(s))\mathrm{d}s + \int_{-\infty}^t g(t-s)(3\psi - \phi)_t(t)\mathrm{d}s\\&= \int_0^\infty g'(s)((3\psi - \phi)(t) - (3\psi - \phi)(t-s))\mathrm{d}s + g_0(3\psi - \phi)_t(t).\end{aligned}\tag{34}$$

Next, we proceed by deriving $\mathscr{I}_1(t)$, using both $(13)_2$ and relation (34), then, integrating by parts, we obtain

$$\begin{aligned}\mathscr{F}_1'(t) = &-I_\varrho\int_0^1(3\psi_{tt} - \phi_{tt})\int_0^\infty g(s)((3\psi - \phi)(t) - (3\psi - \phi)(t-s))\mathrm{d}s\mathrm{d}x\\&- I_\varrho\int_0^1(3\psi_t - \phi_t)\frac{\partial}{\partial t}\left(\int_0^\infty g(s)((3\psi - \phi)(t) - (3\psi - \phi)(t-s))\mathrm{d}s\mathrm{d}x\right)\end{aligned}$$

$$\begin{aligned}
&= D\int_0^1 (3\psi_x - \phi_x)\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad - G\int_0^1 (\phi - \varpi_x)\int_0^\infty g(s)((3\psi-\phi)(t) - (3\psi-\phi)(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad - m\int_0^1 \theta\int_0^\infty g(s)((3\psi-\phi)(t) - (3\psi-\phi)(t-s))\mathrm{d}s\mathrm{d}x - I_\varrho g_0\int_0^1 (3\psi_t - \phi_t)^2\mathrm{d}x \\
&\quad - I_\varrho\int_0^1 (3\psi_t - \phi_t)\int_0^\infty g'(s)((3\psi-\phi)(t) - (3\psi-\phi)(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad - d\int_0^1 r\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad - \int_0^1\left(\int_0^\infty g(s)(3\psi-\phi)_x(x,t-s)\mathrm{d}s\right)\left(\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\right)\mathrm{d}x.
\end{aligned} \tag{35}$$

The last term in (35) can be rewritten as

$$\begin{aligned}
&-\int_0^1\left(\int_0^\infty g(s)(3\psi-\phi)_x(x,t-s)\mathrm{d}s\right)\left(\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\right)\mathrm{d}x \\
&\quad = \int_0^1\left(\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\right)^2\mathrm{d}x \\
&\quad - g_0\int_0^1 (3\psi-\phi)_x\left(\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\right)\mathrm{d}x.
\end{aligned} \tag{36}$$

Now, replacing (36) into (35) leads to

$$\begin{aligned}
\mathscr{F}_1'(t) &= \bar{l}\int_0^1 (3\psi_x - \phi_x)\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad - G\int_0^1 (\phi - \varpi_x)\int_0^\infty g(s)((3\psi-\phi)(t) - (3\psi-\phi)(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad - m\int_0^1 \theta\int_0^\infty g(s)((3\psi-\phi)(t) - (3\psi-\phi)(t-s))\mathrm{d}s\mathrm{d}x - I_\varrho g_0\int_0^1 (3\psi_t - \phi_t)^2\mathrm{d}x \\
&\quad - I_\varrho\int_0^1 (3\psi_t - \phi_t)\int_0^\infty g'(s)((3\psi-\phi)(t) - (3\psi-\phi)(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad - d\int_0^1 r\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\mathrm{d}x \\
&\quad + \int_0^1\left(\int_0^\infty g(s)((3\psi-\phi)_x(t) - (3\psi-\phi)_x(t-s))\mathrm{d}s\right)^2\mathrm{d}x.
\end{aligned}$$

Finally, applying Young's inequality and making use of Lemma 1, we obtain estimate (33). □

Lemma 4. *Consider the functional*

$$\mathscr{I}_2(t) \coloneqq -c\varrho\int_0^1 \varpi_t\left(\int_x^1 \theta(y)\mathrm{d}y\right)\mathrm{d}x,$$

then it satisfies

$$\mathscr{I}_2'(t) \le \frac{-\gamma\varrho}{2}\int_0^1 \varpi_t^2\mathrm{d}x + \chi_*\int_0^1 (3\psi_t - \phi_t)^2\mathrm{d}x + \varepsilon_2\int_0^1 (\phi - \varpi_x)^2\mathrm{d}x + \chi_*\int_0^1 r^2\mathrm{d}x + \chi_*\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \theta_x^2\mathrm{d}x, \quad \forall \varepsilon_2 > 0. \tag{37}$$

Proof. Simple calculations using $(13)_1$, $(13)_4$, and integration by parts indicate that

$$\begin{aligned}
\mathscr{I}_2'(t) &= -c\varrho\int_0^1 \varpi_{tt}\left(\int_x^1 \theta(y)\mathrm{d}y\right)\mathrm{d}x - c\varrho\int_0^1 \varpi_t\left(\int_x^1 \theta_t(y)\mathrm{d}y\right)\mathrm{d}x \\
&= cG\int_0^1 (\phi - \varpi_x)\theta\mathrm{d}x + k_0\varrho\int_0^1 \theta_x\varpi_t\mathrm{d}x + \gamma c\int_0^1 \theta^2\mathrm{d}x \\
&\quad - \gamma\varrho\int_0^1 \varpi_t^2\mathrm{d}x - k_1\varrho\int_0^1 r\varpi_t\mathrm{d}x + m\varrho\int_0^1 \varpi_t\int_x^1 (3\psi_t - \phi_t)(y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Now, thanks to Young's, Poincaré's and Cauchy-Schwarz's inequalities [17], we obtain, for any $\varepsilon_2 > 0$,

$$\mathscr{I}_2'(t) \le \frac{-\gamma\varrho}{2}\int_0^1 \varpi_t^2\mathrm{d}x + \chi_*\int_0^1 (3\psi_t - \phi_t)^2\mathrm{d}x + \varepsilon_2\int_0^1 (\phi - \varpi_x)^2\mathrm{d}x + \chi_*\int_0^1 r^2\mathrm{d}x + \chi_*\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \theta_x^2\mathrm{d}x.$$

The proof is then completed. □

Lemma 5. *Consider the functional*

$$\mathscr{I}_3(t) \coloneqq \varrho\int_0^1 \varpi_t\varpi\mathrm{d}x + \varrho\int_0^1 \phi\left(\int_0^x \varpi_t(y)\mathrm{d}y\right)\mathrm{d}x, \tag{38}$$

then it satisfies

$$\mathscr{I}_3'(t) \le -\frac{G}{2}\int_0^1 (\phi - \varpi_x)^2\mathrm{d}x + \varrho\int_0^1 (3\psi_t - \phi_t)^2\mathrm{d}x + \frac{3\varrho}{2}\int_0^1 \varpi_t^2\mathrm{d}x + \chi_*\int_0^1 \theta_x^2\mathrm{d}x + 9\varrho\int_0^1 \psi_t^2\mathrm{d}x. \tag{39}$$

Proof. We derive $\mathscr{I}_3$, use the first equation in system (13) together with integration by parts, to establish

$$\begin{aligned}
\mathscr{I}_3'(t) &= \varrho\int_0^1 \varpi_t^2\mathrm{d}x + \varrho\int_0^1 \varpi_{tt}\varpi\mathrm{d}x + \varrho\int_0^1 \phi_t\left(\int_0^x \varpi_t(y)\mathrm{d}y\right)\mathrm{d}x + \varrho\int_0^1 \phi\left(\int_0^x \varpi_{tt}(y)\mathrm{d}y\right)\mathrm{d}x \\
&= \varrho\int_0^1 \varpi_t^2\mathrm{d}x - G\int_0^1 (\phi - \varpi_x)_x\varpi\mathrm{d}x - \gamma\int_0^1 \varpi\theta_x\mathrm{d}x + \varrho\int_0^1 \phi_t\left(\int_0^x \varpi_t(y)\mathrm{d}y\right)\mathrm{d}x - G\int_0^1 (\phi - \varpi_x)\phi\mathrm{d}x - \gamma\int_0^1 \theta\phi\mathrm{d}x \\
&= \varrho\int_0^1 \varpi_t^2\mathrm{d}x - G\int_0^1 (\phi - \varpi_x)^2\mathrm{d}x - \gamma\int_0^1 (\phi - \varpi_x)\theta\mathrm{d}x + \varrho\int_0^1 \phi_t\left(\int_0^x \varpi_t(y)\mathrm{d}y\right)\mathrm{d}x.
\end{aligned}$$

Noting that

$$\int_0^1 \phi_t^2 \mathrm{d}x \leq 2\int_0^1 (3\psi_t - \phi_t)^2 \mathrm{d}x + 18\int_0^1 \psi_t^2 \mathrm{d}x,$$

and taking advantage of Young's, Poincaré's and Cauchy-Schwarz inequalities, we easily prove (39). □

Lemma 6. *Consider the functional*

$$\mathscr{I}_4(t) \coloneqq I_\varrho \int_0^1 (3\psi - \phi)_t (3\psi - \phi)\mathrm{d}x, \tag{40}$$

then it satisfies

$$\begin{aligned}\mathscr{I}_4'(t) \leq & -\frac{\bar{l}}{2}\int_0^1 (3\psi_x - \phi_x)^2 \mathrm{d}x + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2 \mathrm{d}x + \chi_* \int_0^1 (r^2 + \theta_x^2)\mathrm{d}x + \chi_* \int_0^1 (\phi - \varpi_x)^2 \mathrm{d}x \\ & + \chi_* (g \diamond (3\psi - \phi)_x)(t).\end{aligned} \tag{41}$$

Proof. We proceed by differentiating the functional $\mathscr{I}_4$, using equation $(13)_2$ together with integration by parts, this leads to

$$\begin{aligned}\mathscr{I}_4'(t) &= I_\varrho \int_0^1 (3\psi - \phi)_{tt}(3\psi - \phi)\mathrm{d}x + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2 \mathrm{d}x \\ &= I_\varrho \int_0^1 (3\psi_t - \phi_t)^2 \mathrm{d}x - \bar{l}\int_0^1 (3\psi_x - \phi_x)^2 \mathrm{d}x + G\int_0^1 (3\psi - \phi)(\phi - \varpi_x)\mathrm{d}x \\ &\quad + m\int_0^1 (3\psi - \phi)\theta \mathrm{d}x + d\int_0^1 (3\psi - \phi)_x r \mathrm{d}x - \int_0^1 (3\psi - \phi)_x \int_0^\infty g(s)((3\psi - \phi)_x(t) - (3\psi - \phi)_x(t - s))\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{42}$$

By virtue of Young's inequality and (9), we have

$$\begin{aligned}\mathscr{I}_4'(t) &\leq -\frac{\bar{l}}{2}\int_0^1 (3\psi_x - \phi_x)^2 \mathrm{d}x + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2 \mathrm{d}x + \chi_* \int_0^1 (r^2 + \theta_x^2)\mathrm{d}x \\ &\quad + \chi_* \int_0^1 (\phi - \varpi_x)^2 \mathrm{d}x + C^1 \int_0^1 \left[\int_0^\infty g(s)((3\psi - \phi)_x(t) - (3\psi - \phi)_x(t - s))\mathrm{d}s\right]^2 \mathrm{d}x \\ &\leq -\frac{\bar{l}}{2}\int_0^1 (3\psi_x - \phi_x)^2 \mathrm{d}x + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2 \mathrm{d}x + \chi_* \int_0^1 (r^2 + \theta_x^2)\mathrm{d}x + \chi_* \int_0^1 (\phi - \varpi_x)^2 \mathrm{d}x + \chi_* (g \diamond (3\psi - \phi)_x)(t).\end{aligned}$$

This makes the proof of (41) complete. □

Lemma 7. *Consider the functional*

$$\mathscr{I}_5(t) \coloneqq 3I_\varrho \int_0^1 \psi_t \psi \mathrm{d}x + 2\beta \int_0^1 \psi^2 \mathrm{d}x, \tag{43}$$

then it satisfies the estimate

$$\mathscr{I}_5'(t) \leq -3D\int_0^1 \psi_x^2 \mathrm{d}x - 3\delta \int_0^1 \psi^2 \mathrm{d}x + 3I_\varrho \int_0^1 \psi_t^2 \mathrm{d}x + \chi_* \int_0^1 (\phi - \varpi_x)^2 \mathrm{d}x. \tag{44}$$

Proof. Simple computations, using equation $(13)_3$ integration by parts, yield

$$\mathscr{I}_5'(t) = 3I_\varrho\int_0^1 \psi_t^2 \mathrm{d}x - 3D\int_0^1 \psi_x^2 \mathrm{d}x - 4\delta\int_0^1 \psi^2 \mathrm{d}x - 3G\int_0^1 (\phi - \varpi_x)\psi \mathrm{d}x. \tag{45}$$

Employing Young's inequality, we conclude (44). □

5 Stability results

Let us here prove our stability result by using the lemmas already mentioned in Section 4.

Proof of Theorem 2. We proceed by introducing a Lyapunov functional

$$\mathscr{L}(t) = N\mathscr{E}(t) + \sum_{j=1}^{5} N_j \mathscr{I}_j(t), \tag{46}$$

where constants $N, N_j > 0,\ j = 1, \dots, 5$, will be chosen later.

From (46), we are in liberty to write

$$\begin{aligned}
|\mathscr{L}(t) - N\mathscr{E}(t)| \leq I_\varrho N_1 \int_0^1 \left| (3\psi - \phi)_t \int_0^\infty g(s)((3\psi - \phi)(t) - (3\psi - \phi)(t - s))\mathrm{d}s \right| \mathrm{d}x \\
+ c\varrho N_2 \int_0^1 \left| \varpi_t \int_x^1 \theta(y)\mathrm{d}y \right| \mathrm{d}x + \varrho N_3 \int_0^1 |\varpi_t \varpi| \mathrm{d}x + \varrho N_3 \int_0^1 \left| \phi \int_0^x \varpi_t(y)\mathrm{d}y \right| \mathrm{d}x \\
+ I_\varrho N_4 \int_0^1 |(3\psi - \phi)_t (3\psi - \phi)| \mathrm{d}x + 3I_\varrho N_5 \int_0^1 |\psi_t \psi| \mathrm{d}x + 2\beta N_5 \int_0^1 \psi^2 \mathrm{d}x.
\end{aligned}$$

Thanks to Young's, Cauchy-Schwarz's and Poincaré's inequalities, we come to

$$|\mathscr{L}(t) - N\mathscr{E}(t)| \leq \vartheta_1 \mathscr{E}(t) \quad \text{where } \vartheta_1 > 0,$$

i.e.,

$$(N - \vartheta_1)\mathscr{E}(t) \leq \mathscr{L}(t) \leq (N + \vartheta_1)\mathscr{E}(t). \tag{47}$$

Now, differentiating the Lyapunov functional $\mathscr{L}(t)$, making benefit of (29), (33), (37), (39), (41), (44), and fixing

$$N_4 = N_5 = 1,\ \varepsilon_1 = \frac{\bar{l}}{4N_1},\ \varepsilon_2 = \frac{GN_3}{4N_2},$$

we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \leq &-\left(\frac{\gamma\varrho}{2}N_2 - \frac{3\varrho}{2}N_3\right)\int_0^1 \varpi_t^2 \mathrm{d}x - \left(\frac{I_\varrho g_0}{2}N_1 - \chi_* N_2 - \varrho N_3 - I_\varrho\right)\int_0^1 (3\psi_t - \phi_t)^2 \mathrm{d}x \\
&- (4\beta N - 9\varrho N_3 - 3I_\varrho)\int_0^1 \psi_t^2 \mathrm{d}x - \frac{\bar{l}}{4}\int_0^1 (3\psi_x - \phi_x)^2 \mathrm{d}x - \left(\frac{G}{4}N_3 - \left(\frac{\bar{l}}{4} + 2\chi_*\right)\right)\int_0^1 (\phi - \varpi_x)^2 \mathrm{d}x \\
&- 3\delta\int_0^1 \psi^2 \mathrm{d}x - 3D\int_0^1 \psi_x^2 \mathrm{d}x - \left(k_0 N - \chi_*\left(1 + \frac{N_2}{N_3}\right)N_2 - \chi_* N_3 - \chi_* - \frac{\bar{l}}{4}\right)\int_0^1 \theta_x^2 \mathrm{d}x \\
&- k_2 N\int_0^1 r_x^2 \mathrm{d}x - (k_3 N - \chi_* N_1 - \chi_* N_2 - \chi_*)\int_0^1 r^2 \mathrm{d}x + \left(\frac{N}{2} - \chi_* N_1\right)(g' \diamond (3\psi - \phi)_x)(t) \\
&+ \left(\chi_*\left(1 + \frac{4N_1}{\bar{l}}\right)N_1 + \chi_*\right)(g \diamond (3\psi - \phi)_x)(t).
\end{aligned} \tag{48}$$

Next, we choose our coefficients in (48) in a way that they all except the last two become negative. We start by selecting N_3 big enough so that

$$\frac{G}{4}N_3 - \left(\frac{\bar{l}}{4} + 2\chi_*\right) > 0,$$

then we take N_2 fairly wide such that

$$\frac{\gamma\varrho}{2}N_2 - \frac{3\varrho}{2}N_3 > 0,$$

now, we select N_1 sufficiently large such that

$$\frac{I_\varrho g_0}{2}N_1 - \chi_* N_2 - \varrho N_3 - I_\varrho > 0.$$

Lastly, we select N huge enough in a way that we have (47) and

$$\begin{cases} \frac{1}{2}N - \chi_* N_1 > 0, \\ 4\beta N - 9\varrho N_3 - 3I_\varrho > 0, \\ k_3 N - \chi_* N_1 - \chi_* N_2 - \chi_* > 0, \\ k_0 N - \chi_*\left(1 + \frac{N_2}{N_3}\right)N_2 - \chi_* N_3 - \chi_* - \frac{\bar{l}}{4} > 0. \end{cases}$$

Hence, relation (48) and Poincaré's inequality give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \le -\vartheta_2\int_0^1 \{\varpi_t^2 + (\phi - \varpi_x)^2 + (3\psi_t - \phi_t)^2 + \psi^2 + (3\psi_x - \phi_x)^2 + \psi_x^2 + \psi_t^2 + \theta^2 + r^2\}\mathrm{d}x \\ + \vartheta_3(g \diamond (3\psi - \phi)_x)(t), \quad \vartheta_2, \vartheta_3 > 0. \end{aligned} \tag{49}$$

Now, by exploiting (18), we obtain

$$\begin{aligned} \mathscr{E}(t) \le \vartheta_4\int_0^1 \{\varpi_t^2 + (\phi - \varpi_x)^2 + (3\psi_t - \phi_t)^2 + \psi^2 + (3\psi_x - \phi_x)^2 + \psi_x^2 + \psi_t^2 + \theta^2 + r^2\}\mathrm{d}x \\ + \vartheta_4(g \diamond (3\psi - \phi)_x)(t) \quad \text{where } \vartheta_4 > 0. \end{aligned}$$

From which

$$\begin{aligned} -\int_0^1 \{\varpi_t^2 + (\phi - \varpi_x)^2 + (3\psi_t - \phi_t)^2 + \psi^2 + (3\psi_x - \phi_x)^2 + \psi_x^2 + \psi_t^2 + \theta^2 + r^2\}\mathrm{d}x - (g \diamond (3\psi - \phi)_x)(t) \\ \le -\vartheta_5\mathscr{E}(t), \end{aligned} \tag{50}$$

where $\vartheta_5 > 0$. Thereby, if we combine (50) and (49), we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \le -\vartheta_6\mathscr{E}(t) + \vartheta_7(g \diamond (3\psi - \phi)_x)(t) \quad \text{where } \vartheta_6, \vartheta_7 > 0. \tag{51}$$

Next, we multiply (51) by

$$\mathscr{G}'\left(\frac{\varepsilon_0\mathscr{E}(t)}{\mathscr{E}(0)}\right),$$

we find

$$\mathscr{G}'\left(\frac{\varepsilon_0\mathscr{E}(t)}{\mathscr{E}(0)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \le -\vartheta_6\mathscr{G}'\left(\frac{\varepsilon_0\mathscr{E}(t)}{\mathscr{E}(0)}\right)\mathscr{E}(t) + \vartheta_7\mathscr{G}'\left(\frac{\varepsilon_0\mathscr{E}(t)}{\mathscr{E}(0)}\right)(g \diamond (3\psi - \phi)_x)(t). \tag{52}$$

Now, we estimate the last term in (52), as in [8], and use both ($\mathbf{A}_2$) and (19) to find

$$\vartheta_7\mathscr{G}'\left(\frac{\varepsilon_0\mathscr{E}(t)}{\mathscr{E}(0)}\right)(g\diamond(3\psi-\phi)_x)(t)\le-\vartheta_7'\mathscr{E}'(t)+\vartheta_7'\varepsilon_0\mathscr{G}_0\left(\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right),\quad \vartheta_7'>0, \tag{53}$$

we insert (53) in (52), and set $\varepsilon_0=\frac{\vartheta_6\mathscr{E}(0)}{2\vartheta_7'}$, we obtain

$$\mathscr{G}'\left(\frac{\varepsilon_0\mathscr{E}(t)}{\mathscr{E}(0)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t)+\vartheta_7'\mathscr{E}'(t)\le-\Gamma\mathscr{G}_0\left(\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right),\quad \Gamma>0. \tag{54}$$

We consider now the functional

$$\mathscr{L}_1(t)\coloneqq\mathscr{G}'\left(\frac{\varepsilon_0\mathscr{E}(t)}{\mathscr{E}(0)}\right)\mathscr{L}(t)+\vartheta_7'\mathscr{E}(t).$$

It is clear that

$$\mathscr{L}_1(t)\sim\mathscr{E}(t),$$

moreover, noting that $\mathscr{E}'(t)\le0$, $\mathscr{G}''(t)>0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_1(t)\le-\Gamma\mathscr{G}_0\left(\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right). \tag{55}$$

Next, we present the functional

$$\mathscr{L}_2(t)\coloneqq b_1\frac{\mathscr{L}_1(t)}{\mathscr{E}(0)}\sim\mathscr{E}(t),\quad\text{such that}$$

$$\begin{cases}\mathscr{L}_2(t)\le1,\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_2(t)\le-\alpha_2\mathscr{G}_0(\mathscr{L}_2(t)),\end{cases}$$

where α_2 is a positive constant, therefore,

$$\mathscr{G}_*'(\mathscr{L}_2(t))\ge\alpha_2,$$

we integrate over $(0,t)$ to find

$$\mathscr{L}_2(t)\le\mathscr{G}_*^{-1}(\alpha_2t+\alpha_3).$$

From which, we deduce that

$$\mathscr{E}(t)\le\alpha_1\mathscr{G}_*^{-1}(\alpha_2t+\alpha_3),$$

where α_1 and α_3 are positive constants. The proof is then completed. □

6 Conclusion

The existence, uniqueness, and stability to solutions of mathematical problems always attracts the attention of researchers. This article considered a memory-type thermoelastic laminated beam with structural damping and microtemperature effects, in which the exponential decay of solutions was established using the energy method. We propose in the forthcoming work the study of the global existence and uniqueness of solutions of the problem considered in this article by searching the sufficient conditions that help us derive the continuation theorems and solve numerically this type of problem.

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