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Research Article

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Absence of global solutions to wave equations with structural damping and nonlinear memory

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Abstract: We prove the nonexistence of global solutions for the following wave equations with structural damping and nonlinear memory source term

$$u_{tt} + (-\Delta)^{\frac{\alpha}{2}}u + (-\Delta)^{\frac{\beta}{2}}u_t = \int_0^t (t-s)^{\delta-1}|u(s)|^p \mathrm{d}s$$

and

$$u_{tt} + (-\Delta)^{\frac{\alpha}{2}}u + (-\Delta)^{\frac{\beta}{2}}u_t = \int_0^t (t-s)^{\delta-1}|u_s(s)|^p \mathrm{d}s,$$

posed in $(x,t) \in \mathbb{R}^N \times [0,\infty)$, where $u = u(x,t)$ is the real-valued unknown function, $p > 1$, $\alpha, \beta \in (0,2)$, $\delta \in (0,1)$, by using the test function method under suitable sign assumptions on the initial data. Furthermore, we give an upper bound estimate of the life span of solutions.

Keywords: damped wave equation, nonexistence of global solution, life span

MSC 2020: 35L05, 35A01, 26A33

1 Introduction

In this article, we study the nonexistence of global solutions for the following problems:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\alpha}{2}}u + (-\Delta)^{\frac{\beta}{2}}u_t = \int_0^t (t-s)^{\delta-1}|u(s)|^p \mathrm{d}s, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p > 1$, $\alpha, \beta \in (0,2)$, $\delta \in (0,1)$ and to

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\alpha}{2}}u + (-\Delta)^{\frac{\beta}{2}}u_t = \int_0^t (t-s)^{\delta-1}|u_s(s)|^p \mathrm{d}s, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

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where $p > 1$, $\alpha, \beta \in (0, 2)$, $\delta \in (0, 1)$, $(-\Delta)^{\frac{\nu}{2}}$ is the fractional Laplacian operator of order $\nu \in (0, 2)$, $(\nu = \alpha \text{ or} \beta)$, which stands for propagation in media with impurities; it is defined, for a regular function, by

$$(-\Delta)^{\frac{\nu}{2}} v(x) = \mathcal{F}^{-1}(|\xi|^{\nu} \mathcal{F}(v)(\xi))(x), \tag{1.3}$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

Fractional differential equations of various types have been introduced in physics, control systems, life sciences, economical sciences, and engineering as fractional derivatives account of qualitative properties of various phenomena.

Before we state and prove our results, we dwell a while on the existing literature. There are many results about the nonexistence of global solutions of equations with fractional derivatives, see [1–7], to cite but a few.

In [8], D'Abbicco obtained the critical exponent for the following wave equation with structural damping and nonlinear memory:

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}} u_t = \int\limits_0^t (t-s)^{-\delta} |u(s)|^p \mathrm{d}s, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $\mu > 0$, $\delta \in (0, 1)$ and $p > 1$. In the supercritical case, the author has shown the existence of small data global solutions of problem (1.4) , whereas, in the sub-critical case, he proved the nonexistence of global solutions for suitable arbitrarily small data, in the special case when $\mu = 2$.

After that, D'Abbicco and Ebert [9] considered the following semilinear evolution equations

$$\begin{cases} u_{tt} - (\Delta)^{\sigma} u + (-\Delta)^{\delta} u_t = |u|^p, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $\sigma \in \mathbb{N}\backslash\{0\}$, $\delta \in \mathbb{N}$; and

$$\begin{cases} u_{tt} - (\Delta)^{\sigma} u + (-\Delta)^{\delta} u_t = |u_t|^p, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $\sigma, \delta \in \mathbb{N}\backslash\{0\}$; when $2\sigma < \delta$ and under suitable sign assumptions on the initial data, they obtained the critical exponents $p_0 \coloneqq 1 + \frac{2\sigma}{(N-2\sigma)_+}$ for problems (1.5) and $p_1 \coloneqq 1 + \frac{2\delta}{N}$ for problems (1.6) for global small data solutions. Moreover, they gave an upper estimate of the life span of solutions.

On the other hand, Dao and Fino [10] considered the semilinear structurally damped wave equation with nonlinear memory term

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = \int\limits_0^t (t-s)^{-\delta} |u|^p \mathrm{d}s, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $\mu > 0$, $\sigma \in (0, 2)$, $\delta \in (0, 1)$ and $p > 1$. By applying the method of duality, they proved that, under some restrictions on initial data $u_0(x)$ and $u_1(x)$, no global weak solution of problem (1.7) exists.

The method we use in this article has been successfully applied by Mitidieri and Pohozahev [4], Pohozahev and Tesei [11], Pohozahev and Véron [12], Zhang [2], Kirane et al. [13], and many others after them. It consists of a judicious choice of the test function in the weak formulation of the sought solution of problems (1.1) and (1.2).

The remainder of this article is organized as follows: In Section 2, some preliminaries and main results are presented. Section 3 is devoted to the proof of the main results.

2 Preliminaries and main results

The left-hand side and right-hand side Riemann-Liouville fractional integrals of order $\delta \in (0,1)$ for an integrable function f are, respectively, defined by

$$\mathbb{I}^{\delta}_{0|t} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\,\mathrm{d}s, \quad t > 0,$$

and

$$\mathbb{I}^{\delta}_{t|T} f(t) = \frac{1}{\Gamma(\delta)} \int_t^T (s-t)^{\delta-1} f(s)\,\mathrm{d}s, \quad 0 < t < T,$$

where Γ is the Euler gamma function. Let $AC([0,T])$ be the space of all functions which are absolutely continuous on $[0,T]$ with $0 < T < \infty$. Then, for $f \in AC([0,T])$, the left-hand side and right-hand side Riemann-Liouville fractional derivatives $\mathbb{D}^{\delta}_{0|t} f(t)$ and $\mathbb{D}^{\delta}_{t|T} f(t)$ of order $\delta \in (0,1)$ are, respectively, defined by

$$\mathbb{D}^{\delta}_{0|t} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{I}^{1-\delta}_{0|t} f(t) = \frac{1}{\Gamma(1-\delta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\delta} f(s)\,\mathrm{d}s, \quad t > 0,$$

and

$$\mathbb{D}^{\delta}_{t|T} f(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{I}^{1-\delta}_{t|T} f(t) = -\frac{1}{\Gamma(1-\delta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^T (s-t)^{-\delta} f(s)\,\mathrm{d}s, \quad t < T.$$

Note that for every $f, g \in C([0,T])$ such that $\mathbb{D}^{\delta}_{0|t} f(t), \mathbb{D}^{\delta}_{t|T} g(t)$ exist and are continuous, for all $t \in [0,T]$, and $0 < \delta < 1$, we have the formula of integration by parts

$$\int_0^T (\mathbb{D}^{\delta}_{0|t} f)(t) g(t)\,\mathrm{d}t = \int_0^T (\mathbb{D}^{\delta}_{t|T} g)(t) f(t)\,\mathrm{d}t. \tag{2.1}$$

Further, if $f \in L^p(0,T)$, $g \in L^q(0,T)$, and $p, q \geq 1$, $q = p/(p-1)$, then

$$\int_0^T (\mathbb{I}^{\delta}_{0|t} f)(t) g(t)\,\mathrm{d}t = \int_0^T (\mathbb{I}^{\delta}_{t|T} g)(t) f(t)\,\mathrm{d}t. \tag{2.2}$$

Note also that, for all $f \in AC^{k+1}[0,T]$, $k \geq 0$, we have

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathbb{D}^{\delta}_{t|T} f(t) = \mathbb{D}^{\delta+k}_{t|T} f(t), \tag{2.3}$$

where

$$AC^{k+1}[0,T] = \{ f : [0,T] \to \mathbb{R} \quad \text{and} \quad \frac{\mathrm{d}^k f}{\mathrm{d}t^k} \in AC[0,T] \}.$$

Moreover, for all $f \in L^q(0,T)$, $1 \leq q \leq \infty$, the following formula holds true

$$\mathbb{D}^{\delta}_{0|t} \mathbb{I}^{\delta}_{0|t} f(t) = f(t), \quad \text{for all } t \in (0,T). \tag{2.4}$$

In addition, let

$$\tilde{\psi}(t) = \begin{cases} \left(1 - \frac{t}{T}\right)^m, & 0 < t \leq T, \\ 0, & t > T, \end{cases} \tag{2.5}$$

then, for $n \geq 0$, we have

$$\mathbb{D}_{t|T}^{\delta+n}\tilde{\psi}(t) = \frac{\Gamma(m+1)}{\Gamma(m+1-n-\delta)} T^{-(\delta+n)}\left(1-\frac{t}{T}\right)^{n-m-\delta}, \tag{2.6}$$

and

$$\int_0^T [\tilde{\psi}(t)]^{-\frac{1}{p-1}} |\mathbb{D}_{t|T}^{\delta+n}\tilde{\psi}(t)|^{\frac{p}{p-1}} \mathrm{d}t = CT^{1-(n+\delta)\frac{p}{p-1}}. \tag{2.7}$$

On the other hand, for a function $v : \mathbb{R}^N \to \mathbb{R}$ in the Schwartz space of rapidly decaying functions,

$$(-\Delta)^{\frac{\nu}{2}} v(x) \coloneqq C_{N,\nu}\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{v(x)-v(\xi)}{|x-\xi|^{N+\nu}} \mathrm{d}\xi, \tag{2.8}$$

where P.V. denotes the principal value and $C_{N,\nu}$ is a suitable normalization constant.

Remark. Note that the two definitions of fractional Laplacian (1.3) and (2.8) are equivalent [14].

Let $\nu \in (0,2)$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \langle x\rangle^{-N-\nu} & \text{if } |x| \geq 1, \end{cases} \tag{2.9}$$

with $\langle x\rangle = (1+(|x|-1)^4)^{\frac{1}{4}}$ for all $x \in \mathbb{R}^N$; then, $\varphi \in C^2(\mathbb{R}^N)$, and the following estimate holds true

$$|(-\Delta)^{\frac{\nu}{2}}\varphi(x)| \leq C\varphi(x), \quad \text{for all } x \in \mathbb{R}^N. \tag{2.10}$$

Furthermore, let φ be a smooth function, and let

$$\varphi_T(x) = \varphi(x/T), \quad \text{for all } x \in \mathbb{R}^N.$$

Then,

$$(-\Delta)^{\frac{\nu}{2}}\varphi_T(x) = T^{-\nu}((-\Delta)^{\frac{\nu}{2}}\varphi)(x/T), \quad \text{for all } x \in \mathbb{R}^N. \tag{2.11}$$

In addition, for $p > 1$ and $T > 0$, then the following estimate holds

$$\int_{\mathbb{R}^N} \varphi_T^{-\frac{1}{p-1}}(x)|(-\Delta)^{\frac{\nu}{2}}\varphi_T(x)|^{\frac{p}{p-1}} \mathrm{d}x \leq CT^{N-\frac{p\nu}{p-1}}, \quad x \in \mathbb{R}^N. \tag{2.12}$$

Now, we are in position to announce the main results of this article.

Theorem 2.1. *Let $p > 1$, $\alpha, \beta \in (0,2)$, $\delta \in (0,1)$. Assume that $u_0 \equiv 0$, $u_1 \in L^1$ such that*

$$\int_{\mathbb{R}^N} u_1(x)\mathrm{d}x > 0. \tag{2.13}$$

Then there exists no global weak solutions of problem (1.1) *for any*

$$p \leq 1 + \frac{\delta\theta+\alpha}{(N+\theta-\alpha)_+}. \tag{2.14}$$

Moreover, if $[0, T_\rho)$ is the life span of u, then, for the initial data $u_1(x) = \rho g(x)$, $\rho > 0$, $g \in L^1$ and verifies (2.13), *there exists a constant $C > 0$ such that*

$$T_\rho \leq C\rho^{-\frac{\theta}{(\delta\theta+\alpha)q-(1+\delta)\theta-N}}, \quad \theta = \alpha - \min\left\{\beta, \frac{\alpha}{2}\right\},\ q = p/(p-1).$$

Theorem 2.2. *Let $p > 1$, $\alpha, \beta \in (0, 2)$, $\delta \in (0, 1)$. Assume that $u_0 \equiv 0$, $u_1 \in L^1$ such that* (2.13). *Then, there exists no global weak solutions of problem* (1.2) *for any*

$$p \leq 1 + \frac{(1+\delta)\sigma}{N}. \tag{2.15}$$

In addition, let $[0, T_\rho)$ be the life span of solution u of problem (1.2). *Then, for the initial data $u_1(x) = \rho g(x)$, $\rho > 0$, $g \in L^1$ and verifies* (2.13), *there exists a constant $C > 0$ such that*

$$T_\rho \leq C\rho^{-\frac{\sigma}{(1+\delta)(q-1)\sigma - N}}, \quad \sigma = \min\left\{\beta, \frac{\alpha}{2}\right\}, \ q = p/(p-1).$$

Theorem 2.3. *Let $p > 1$, $\alpha, \beta \in (0, 2)$, $\delta \in (0, 1)$. Assume that $u_0 \equiv 0$, $u_1 \in L^1_{\mathrm{Loc}}$ such that*

$$u_1(x) \geq \rho(1 + |x|)^{-\mu}, \quad \rho > 0 \text{ and } N < \mu < (\delta\theta + \alpha)q - (1+\delta)\theta. \tag{2.16}$$

Then, there exists no global weak solutions of problem (1.1) *for any $p > 1$ and satisfying inequality* (2.14). *Moreover, if $[0, T_\rho)$ is the life span of u, then there exists a positive constant C independent of ρ such that*

$$T_\rho \leq C\rho^{-\frac{\theta}{(\delta\theta+\alpha)q - (1+\delta)\theta - \mu}}, \quad \theta = \alpha - \min\left\{\beta, \frac{\alpha}{2}\right\}, \ q = p/(p-1).$$

Theorem 2.4. *Let $p > 1$, $\alpha, \beta \in (0, 2)$, $\delta \in (0, 1)$. Assume that $u_0 \equiv 0$, $u_1 \in L^1_{\mathrm{Loc}}$ such that*

$$u_1(x) \geq \rho(1 + |x|)^{-\mu}, \quad \rho > 0 \text{ and } N < \mu < (1+\delta)(q-1)\sigma. \tag{2.17}$$

Then, there exists no global weak solutions of problem (1.2) *for any $p > 1$ verifying the inequality* (2.15). *Moreover, there exists $C > 0$ such that*

$$T_\rho \leq C\rho^{-\frac{\sigma}{(1+\delta)(q-1)\sigma - \mu}}, \quad \sigma = \min\left\{\beta, \frac{\alpha}{2}\right\},$$

where $q = p/(p-1)$.

3 Proof of main results

In this section, we present the proofs of the results announced here earlier. For simplicity, we use C to denote a positive constant, which may vary from line to line, but it is not essential to the analysis of the problems. Before starting, we introduce the fractional Sobolev space for any $s \in \mathbb{R}_+$

$$H^s(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\mathcal{F}(u)(\xi)|^2 \mathrm{d}\xi < \infty\right\}.$$

Then, we present the definition of weak solutions for problems (1.1) and (1.2).

Definition 3.1. Let $p > 1$, $T > 0$ and (u_0, u_1). We say that

$$u \in L^p_{\mathrm{Loc}}([0, T) \times \mathbb{R}^N)$$

is a local weak solution to problem (1.1), or that

$$u \in L^1_{\mathrm{Loc}}([0, T) \times \mathbb{R}^N), \quad u_t \in L^p_{\mathrm{Loc}}([0, T) \times \mathbb{R}^N),$$

is a local weak solution to problem (1.2), if for any test function $\phi \in C([0,T]; H^{\alpha}(\mathbb{R}^N)) \cap C^1([0,T]; H^{\beta}(\mathbb{R}^N)) \cap C^2([0,T]; L^2(\mathbb{R}^N))$, with $\operatorname{supp}_x \phi \subset \mathbb{R}^N$ such that $\phi(\cdot,T) = 0$ and $\phi_t(\cdot,T) = 0$ it holds:

$$\begin{aligned} \mathcal{U} &= \int_0^T \int_{\mathbb{R}^N} u\Big(\phi_{tt}(x,t) + (-\Delta)^{\frac{\alpha}{2}}\phi(x,t) - (-\Delta)^{\frac{\beta}{2}}\phi_t(x,t)\Big)\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^N} u_1(x)\phi(x,0)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} u_0(x)(\phi_t(x,0) - (-\Delta)^{\frac{\beta}{2}}\phi(x,0))\,\mathrm{d}x, \end{aligned} \tag{3.1}$$

where

$$\mathcal{U} = \Gamma(\delta)\int_0^T\int_{\mathbb{R}^N} \mathbb{I}^{\delta}_{0|t}\, |u|^p \phi(x,t)\mathrm{d}x\mathrm{d}t,$$

for problem (1.1), and

$$\mathcal{U} = \Gamma(\delta)\int_0^T\int_{\mathbb{R}^N} \mathbb{I}^{\delta}_{0|t}\, |u_t|^p \phi(x,t)\mathrm{d}x\mathrm{d}t.$$

for problem (1.2). If $T = \infty$, we say that u is a global weak solution.

Actually, we are ready to prove the main results of the article.

Proof of Theorem 2.1. The proof is by contradiction. Let u be a global weak solution of problem (1.1). Using Definition (3.1), and recalling that $u_0 \equiv 0$, we obtain

$$\begin{aligned} &\Gamma(\delta)\int_{\Omega} \mathbb{I}^{\delta}_{0|t}\, |u|^p\phi(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_1(x)\phi(x,0)\,\mathrm{d}x \\ &= \int_{\Omega} u\phi_{tt}(x,t)\mathrm{d}x\mathrm{d}t + \int_{\Omega} u(-\Delta)^{\frac{\alpha}{2}}\phi(x,t)\mathrm{d}x\mathrm{d}t - \int_{\Omega} u(-\Delta)^{\frac{\beta}{2}}\phi_t(x,t)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.2}$$

for $\phi \in C([0,T]; H^{\alpha}(\mathbb{R}^N)) \cap C^1([0,T]; H^{\beta}(\mathbb{R}^N)) \cap C^2([0,T]; L^2(\mathbb{R}^N))$, with $\operatorname{supp}_x\phi \subset \mathbb{R}^N$, and $\phi(\cdot, T^{\theta}) = \phi_t(\cdot, T^{\theta}) = 0$, where $\Omega \coloneqq \mathbb{R}^N \times [0, T^{\theta})$.

Now, for all $x \in \mathbb{R}^N$ and $0 \le t \le T^{\theta}$, let

$$\phi(x,t) = \mathbb{D}^{\delta}_{t|T^{\theta}}\zeta(x,t) = \varphi_T(x)\, \mathbb{D}^{\delta}_{t|T^{\theta}}\psi(t), \quad T > 0, \delta \in (0,1),$$

where

$$\varphi_T(x) = \varphi\left(\frac{|x|}{T}\right), \quad \psi(t) = \left(1 - \frac{t}{T^{\theta}}\right)_+^m, \quad m \gg 1, \tag{3.3}$$

φ is given in (2.9), and θ is a positive parameter, which will be fixed later. Set

$$U \coloneqq \Gamma(\delta)\int_{\Omega} |u|^p\zeta(x,t)\mathrm{d}x\mathrm{d}t.$$

By using formulas of integration by parts (2.1)–(2.2), and from (2.4), we can write expression (3.2) as follows:

$$\begin{aligned} &U + \int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\mathbb{D}^{\delta}_{t|T^{\theta}}\psi(0)\,\mathrm{d}x \\ &= \int_{\Omega} u\varphi_T(x)\mathbb{D}^{\delta+2}_{t|T^{\theta}}\psi(t)\mathrm{d}x\mathrm{d}t + \int_{\Omega} u\, (-\Delta)^{\frac{\alpha}{2}}\varphi_T(x)\mathbb{D}^{\delta}_{t|T^{\theta}}\psi(t)\mathrm{d}x\mathrm{d}t - \int_{\Omega} u\, (-\Delta)^{\frac{\beta}{2}}\varphi_T(x)\mathbb{D}^{\delta+1}_{t|T^{\theta}}\psi(t)\mathrm{d}x\mathrm{d}t \\ &\coloneqq M_1 + M_2 + M_3. \end{aligned} \tag{3.4}$$

Next, we shall estimate each term of the right-hand side of equality (3.4). First, by applying ε-Young's inequality

$$ab \le \varepsilon a^p + C_{\varepsilon}b^q, \quad \text{with } 0 < \varepsilon < \frac{1}{3},\ a, b > 0, \text{ and } \frac{1}{p} + \frac{1}{q} = 1,$$

to M_1, we obtain

$$|M_1| \leq \int_\Omega |u| \varphi_T(x) \mathbb{D}^{\delta+2}_{t|T^\theta} \psi(t)\,\mathrm{d}x\mathrm{d}t$$
$$\leq \varepsilon U + C_\varepsilon \int_\Omega |\zeta(x,t)|^{-\frac{q}{p}} |\varphi_T(x)|^q |\mathbb{D}^{\delta+2}_{t|T^\theta}\psi(t)|^q\,\mathrm{d}x\mathrm{d}t.$$

Further, by using properties (2.6) and (2.7) with $n = 2$, and using the change of variables $y = x/T$, $r = t/T^\theta$, we obtain the estimate

$$|M_1| \leq \varepsilon U + CT^{-(2+\delta)\theta q+\theta+N}. \tag{3.5}$$

Similarly, we apply ε-Young's inequality to M_2 to obtain

$$|M_2| \leq \int_\Omega |u||(-\Delta)^{\frac{\alpha}{2}} \varphi_T(x)| \mathbb{D}^{\delta}_{t|T^\theta} \psi(t)\,\mathrm{d}x\mathrm{d}t$$
$$\leq \varepsilon U + C_\varepsilon \int_\Omega |\zeta(x,t)|^{-\frac{p}{q}} |(-\Delta)^{\frac{\alpha}{2}} \varphi_T(x)|^q |\mathbb{D}^{\delta}_{t|T^\theta}\psi(t)|^q\,\mathrm{d}x\mathrm{d}t.$$

Moreover, from formulas (2.6)–(2.7) with $n = 0$, and by passing to the new variables $y = x/T$, $r = t/T^\theta$, then using the scaling properties (2.11) and estimate (2.12), we obtain that

$$|M_2| \leq \varepsilon U + CT^{-(\delta\theta+\alpha)q+\theta+N}. \tag{3.6}$$

Repeating the same argument as earlier, we obtain for M_3 the following estimate:

$$|M_3| \leq \int_\Omega |u||(-\Delta)^{\frac{\beta}{2}} \varphi_T(x)| \mathbb{D}^{\delta+1}_{t|T^\theta} \psi(t)\,\mathrm{d}x\mathrm{d}t$$
$$\leq \varepsilon U + C_\varepsilon \int_\Omega |\zeta(x,t)|^{-\frac{q}{p}} ||(-\Delta)^{\frac{\beta}{2}} \varphi_T(x)|^q |\mathbb{D}^{\delta+1}_{t|T^\theta}\psi(t)|^q\,\mathrm{d}x\mathrm{d}t.$$

That is,

$$|M_3| \leq \varepsilon U + CT^{-((\delta+1)\theta+\beta)q+\theta+N}. \tag{3.7}$$

Now, at this stage, let us set

$$\theta = \max\left\{\alpha - \beta, \frac{\alpha}{2}\right\} = \alpha - \min\left\{\beta, \frac{\alpha}{2}\right\}. \tag{3.8}$$

Then, from (3.5)–(3.7), expression (3.4) can be written as follows:

$$(1 - 3\varepsilon)U + CT^{-\theta\delta} \int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \leq CT^{-(\delta\theta+\alpha)q+\theta+N}, \tag{3.9}$$

where we have used the fact that $\mathbb{D}^{\delta}_{t|T^\theta}\psi(0) = CT^{-\theta\delta}$. Consequently,

$$\int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \leq CT^{-(\delta\theta+\alpha)q+(1+\delta)\theta+N} = CT^{-\kappa}, \tag{3.10}$$

where

$$\kappa = (\delta\theta + \alpha)q - (1 + \delta)\theta - N. \tag{3.11}$$

Taking into account that inequality (2.14) is equivalent to $\kappa \geq 0$, we have to consider two cases:

- The case $\kappa > 0$: Passing to the limit in (3.10), as T goes to ∞, it follows that

$$\int_{\mathbb{R}^n} u_1(x)\,\mathrm{d}x \leq 0,$$

which contradicts assumption (2.13).

- The case $\kappa = 0$: We treat this case in a standard way as above by taking this time

$$\varphi_T(x) = \phi(|x|L/T),$$

where $1 \le L \le T$ is large enough such that when $T \to \infty$ we do not have $L \to \infty$ at the same time. Note that there exists a constant $C > 0$ independent of T and L such that

$$(1 - 3\varepsilon)U + CT^{-\theta\delta} \int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \le (CL^{-N} + CL^{\alpha q - N} + CL^{\beta q - N})T^{-(\delta\theta+\alpha)q+\theta+N}. \tag{3.12}$$

Whereupon,

$$\int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \le (CL^{-N} + CL^{\alpha q - N} + CL^{\beta q - N})T^{-\kappa}. \tag{3.13}$$

Thus, using $\kappa = 0$, $N > q\max\{\alpha, \beta\}$ and taking the limit when T tends to ∞ in inequality (3.13), then letting $L \to \infty$, we obtain

$$\int_{\mathbb{R}^N} u_1(x)\,\mathrm{d}x \le 0;$$

this contradicts again the assumption. Therefore, the solution u of problem (1.1) cannot be global.

- **Upper estimate of the life span of solution of problem** (1.1).

In the case $p < 1 + (\delta\theta + \alpha)/(N + \theta - \alpha) \iff \kappa > 0$. Also, noting that for the initial condition $u_1(x) = \varepsilon g(x) \in L^1$, with $g(x)$ verifying (2.13), there exists $\tilde{T} > 0$ such that

$$\int_{\mathbb{R}^N} g(x)\varphi_T(x)\,\mathrm{d}x \ge c > 0, \quad \text{for all } T \ge \tilde{T}.$$

Assume that u_ρ is a local solution of problem (1.1) in $[0, T_\rho]$, with $T_\rho \ge \tilde{T}^\theta$, then we have

$$(1 - 3\varepsilon)U + CT^{-\theta\delta} \int_{\mathbb{R}^N} \varepsilon g(x)\varphi_T(x)\,\mathrm{d}x \le CT^{-(\delta\theta+\alpha)q+\theta+N}.$$

Furthermore, setting $T = T_\rho^{\frac{1}{\theta}}$, we obtain

$$T_\rho^{-\delta} \int_{\mathbb{R}^N} \rho g(x)\varphi_T(x)\,\mathrm{d}x \le CT_\rho^{-\frac{(\delta\theta+\alpha)q-\theta-N}{\theta}}. \tag{3.14}$$

Hence, we have

$$\rho \int_{\mathbb{R}^N} g(x)\varphi_T(x)\,\mathrm{d}x \le CT_\rho^{-\frac{\kappa}{\theta}}.$$

Finally, for some positive constant C, independent of ρ, we obtain

$$T_\rho \le C\rho^{-\frac{\theta}{\kappa}},$$

where θ and κ are, respectively, given in (3.8) and (3.11). This completes the proof. □

Proof of Theorem 2.2. We proceed by contradiction. Supposing that u is global weak solution of problem (1.2), using Definition (3.1), recalling that $u_0 \equiv 0$, we have

$$\begin{aligned} &\Gamma(\delta)\int_\Omega \mathbb{I}^{\delta}_{0|t}\, |u_t|^p \phi(x,t)\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_1(x)\phi(x,0)\,\mathrm{d}x \\ &= \int_\Omega u\phi_{tt}(x,t)\,\mathrm{d}x\mathrm{d}t + \int_\Omega u(-\Delta)^{\frac{\alpha}{2}}\phi(x,t)\,\mathrm{d}x\mathrm{d}t - \int_\Omega u(-\Delta)^{\frac{\beta}{2}}\phi_t(x,t)\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.15}$$

for all test function $\phi \in C([0,T]; H^\alpha(\mathbb{R}^N)) \cap C^1([0,T]; H^\beta(\mathbb{R}^N)) \cap C^2([0,T]; L^2(\mathbb{R}^N))$, with $\operatorname{supp}_x \phi \subset \mathbb{R}^N$, and $\phi(\cdot, T^\sigma) = \phi_t(\cdot, T^\sigma) = 0$, where $\Omega := \mathbb{R}^N \times [0, T^\sigma)$.

Then, to achieve the aim we choose the test function $\phi(x,t)$, for all $x \in \mathbb{R}^N$ and $0 \le t \le T^\sigma$, as follows:

$$\phi(x,t) = \mathbb{D}^{\delta}_{t|T^\sigma}\zeta(x,t) = \varphi_T(x)\,\mathbb{D}^{\delta}_{t|T^\sigma}\psi(t), \quad T>0,\ \delta \in (0,1),$$

where $\phi_T(x)$ is given in (3.3), the function ψ is defined by

$$\psi(t) = \left(1 - \frac{t}{T^\sigma}\right)^m, \quad T>0,\ m \gg 1,$$

and σ is a positive parameter, which we will fixed later. Let us set

$$V \coloneqq \Gamma(\delta)\int_\Omega |u_t|^p \zeta(x,t)\,\mathrm{d}x\mathrm{d}t.$$

Thus, by formulas of integration by parts (2.1) and (2.2), and identinty (2.4), we can write expression (3.15) as follows:

$$\begin{aligned} V + \int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\mathbb{D}^{\delta}_{t|T^\sigma}\psi(0)\,\mathrm{d}x &= \int_\Omega u_t\ \varphi_T(x)\mathbb{D}^{\delta+1}_{t|T^\sigma}\psi(t)\,\mathrm{d}x\mathrm{d}t + \int_\Omega u_t(-\Delta)^{\frac{\alpha}{2}}\varphi_T(x)\Psi(t)\,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_\Omega u_t\ (-\Delta)^{\frac{\beta}{2}}\varphi_T(x)\mathbb{D}^{\delta}_{t|T^\sigma}\psi(t)\,\mathrm{d}x\mathrm{d}t \coloneqq K_1 + K_2 + K_3, \end{aligned} \tag{3.16}$$

where $\Psi(t) \in C^\infty([0,T])$ is test function defined by

$$\Psi(t) = \int_t^{T^\sigma} \mathbb{D}^{\delta}_{s|T^\sigma}\psi(s)\,\mathrm{d}s, \quad \text{and} \quad \Psi'(t) = -\mathbb{D}^{\delta}_{t|T^\sigma}\psi(t).$$

The difference, with respect to the proof of Theorem 2.1, is related to the estimate of the term containing $\Psi(t)$. Now applying ε-Young's inequality

$$ab \le \varepsilon a^p + C_\varepsilon b^q, \quad \text{with } 0 < \varepsilon < \frac{1}{3},\ a,b>0,\ \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1,$$

in each term of the right-hand side to (3.16), we obtain for the first term K_1 the following estimate:

$$\begin{aligned} |K_1| &\le \int_\Omega |u_t|\varphi_T(x)\mathbb{D}^{\delta+1}_{t|T^\sigma}\psi(t)\,\mathrm{d}x\mathrm{d}t \\ &\le \varepsilon V + C_\varepsilon \int_\Omega |\zeta(x,t)|^{-\frac{q}{p}}|\varphi_T(x)|^q|\mathbb{D}^{\delta+1}_{t|T^\sigma}\psi(t)|^q\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Then, from (2.6)–(2.7) with $n=1$, and by considering the change of variables $y = x/T$, $r = t/T^\sigma$, we obtain

$$|K_1| \le \varepsilon V + CT^{-(\delta+1)\sigma q + \sigma + N}. \tag{3.17}$$

Next, for the second term of the right-hand side of (3.16), we have

$$\begin{aligned} |K_2| &\le \int_\Omega |u_t||(-\Delta)^{\frac{\alpha}{2}}\varphi_T(x)||\Psi(t)|\,\mathrm{d}x\mathrm{d}t \\ &\le \varepsilon V + C_\varepsilon \int_\Omega |\zeta(x,t)|^{-\frac{q}{p}}|(-\Delta)^{\frac{\alpha}{2}}\varphi_T(x)|^q|\Psi(t)|^q\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Hence, by using the fact that $\Psi(t) \le \Psi(0)$, from (2.6) and (2.7) with $n=0$, and by introducing the following change of variables $y = x/T$, $r = t/T^\sigma$, then by using scaling properties (2.11) and estimate (2.12), it yields

$$|K_2| \le \varepsilon V + CT^{(1-\delta)\sigma q - \alpha q + \sigma + N}. \tag{3.18}$$

By applying similar arguments like the first and second term of (3.16) to the third term, we have

$$|K_3| \le \varepsilon V + CT^{-(\delta\sigma+\beta)q + \sigma + N}. \tag{3.19}$$

Whereupon, from (3.17) to (3.19), and using the fact that $\mathbb{D}^{\delta}_{t|T^{\sigma}}\psi(0) = CT^{-\delta\sigma}$, then expression (3.16) can be written as follows:

$$(1-3\varepsilon)V + CT^{-\delta\sigma}\int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \le CT^{-(1+\delta)q\sigma+\sigma+N}. \tag{3.20}$$

Consequently,

$$\int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \le CT^{-(1+\delta)(q-1)\sigma+N} = CT^{-\lambda}, \tag{3.21}$$

where

$$\lambda = (1+\delta)(q-1)\sigma - N, \tag{3.22}$$

with

$$\sigma = \min\left\{\frac{\alpha}{2}, \beta\right\}. \tag{3.23}$$

Note that inequality (2.15) is equivalent to $\lambda \ge 0$. So, we have to distinguish two cases.

- The case $\lambda > 0$: Passing to the limit in inequality (3.21) when T goes to ∞, we obtain

$$\int_{\mathbb{R}^N} u_1(x)\,\mathrm{d}x \le 0,$$

this contradicts assumption (2.13).

- The case $\lambda = 0$ is treated as in Theorem 2.1. Then, the solution of problem (1.2) cannot be global.

Proceeding as in the proof of Theorem 2.1, we give an upper bound estimate of the life span of solution of (1.2). By using inequality (3.20), and fixing $T = T_\rho^{\frac{1}{\sigma}}$, we can write

$$(1-3\varepsilon)V + CT_\rho^{-\delta}\int_{\mathbb{R}^N} \rho g(x)\varphi_T(x)\mathrm{d}x \le CT_\rho^{-\frac{(1+\delta)q\sigma-\sigma-N}{\sigma}}.$$

Whereupon,

$$\rho\int_{\mathbb{R}^N} g(x)\varphi_T(x)\mathrm{d}x \le CT_\rho^{-\frac{\lambda}{\sigma}},$$

where λ and σ are, respectively, given in (3.22) and (3.23). Note that λ and σ are two positive constants. Then, we can obtain the following upper bound estimate of life span of solutions of problem (1.2)

$$T_\rho \le C\rho^{-\frac{\sigma}{\lambda}}, \quad \sigma = \min\left\{\frac{\alpha}{2}, \beta\right\},$$

where we have used the assumption (2.13). This completes the proof. □

Proof of Theorem 2.3. We repeat the same calculation as in the proof of Theorem 2.1; we arrive at

$$(1-3\varepsilon)U + CT^{-\theta\delta}\int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \le CT^{-(\delta\theta+\alpha)q+\theta+N}. \tag{3.24}$$

First, the nonexistence of global weak solutions of problem (1.1) is obtained when inequality (2.14) holds. Next, to estimate the life span of solutions of problem (1.1), we assume that initial condition u_1 satisfy the assumption (2.17). Then we have

$$\int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \ge \rho\int_{\mathbb{R}^N} (1+|x|)^{-\mu}\varphi_T(x)\,\mathrm{d}x.$$

Further, by using the change of variable $y = x/T$, we obtain

$$\int_{\mathbb{R}^N} u_1(x)\varphi_T(x)\,\mathrm{d}x \geq CT^{-\mu+N}\rho. \tag{3.25}$$

Moreover, by combining (3.24) and (3.25), we obtain

$$\rho \leq CT^{-\kappa'},$$

where $\kappa' = (\delta\theta + \alpha)q - (1+\delta)\theta - \mu$. Note that assumption (2.17) guarantees the positivity of κ'. Finally, fixing $T = T_\rho^{\frac{1}{\theta}}$, where T_ρ is the maximal existence time of solution. Then we have

$$T_\rho \leq C\rho^{-\frac{\theta}{\kappa'}},$$

where C is a positive constant independent of ρ and θ is given in (3.8). This concludes the proof. □

Proof of Theorem 2.4. The proof is completely analogous to the proof of Theorem 2.3, but here we obtain for some positive constant C independent of ρ

$$T_\rho \leq C\rho^{-\frac{\sigma}{\lambda'}},$$

where $\lambda' = (1+\delta)(q-1)\sigma - \mu > 0$, and σ is given in (3.23). The conclusion follows. □

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