Research Article

Pablo M. Berná*, Andrea García, and David González

A note on 1-semi-greedy bases in p-Banach spaces with $0 < p \leq 1$

https://doi.org/10.1515/dema-2024-0047
received October 6, 2023; accepted July 9, 2024

Abstract: The purpose of this article is to discuss about the so-called semi-greedy bases in p-Banach spaces. Specifically, we will review existing results that characterize these bases in terms of almost-greedy bases, and, also, we analyze quantitatively the behavior of certain constants. As new results, by avoiding the use of certain classical results in p-convexity, we aim to quantitatively improve specific bounds for bi-monotone 1-semi-greedy bases.

Keywords: non-linear approximation, greedy algorithm, semi-greedy bases

MSC 2020: 41A65, 46B15

1 Introduction and background

In [1], Konyagin and Temlyakov introduced the thresholding Greedy algorithm (TGA), where, given a basis in a particular space $\mathbb{X}$ (a p-Banach or a quasi-Banach space) and one element $f \in \mathbb{X}$, the algorithm selects the largest coefficients in modulus of f with respect to the given basis. To formalize the algorithm, we introduce the notion of p-Banach spaces and bases (see, e.g., [1–3]).

1.1 p-Banach spaces and bases

Let $\mathbb{X}$ be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with a quasi-norm $\|\cdot\| : \mathbb{X} \to [0, +\infty)$, where the quasi-norm is an application verifying the following three conditions:

(a) $\|f\| \geq 0$ for all $f \in \mathbb{X}$ and $\|f\| = 0$ if and only if $f = 0$.

(b) $\|\lambda f\| = |\lambda|\|f\|$ for all $f \in \mathbb{X}$ and for all $\lambda \in \mathbb{F}$.

(c) There is $\kappa \geq 1$ such that for every $f, g \in \mathbb{X}$,

$$\|f + g\| \leq \kappa(\|f\| + \|g\|).$$

Given $0 < p \leq 1$, a *p-norm* is a map $\|\cdot\| : \mathbb{X} \to [0, +\infty)$ satisfying conditions (a), (b), and

(d) for every $f, g \in \mathbb{X}$,

$$\|f + g\|^p \leq \|f\|^p + \|g\|^p.$$

*** Corresponding author: Pablo M. Berná,** Departamento de Métodos Cuantitativos, CUNEF Universidad, Madrid, 28040, Spain, e-mail: pablo.berna@cunef.edu
Andrea García: Departamento de Métodos Cuantitativos, CUNEF Universidad, Madrid, 28040, Spain; Universidad San Pablo-CEU, CEU Universities, Madrid, 28003, Spain, e-mail: andrea.garciapons@usp.ceu.es
David González: Universidad San Pablo-CEU, CEU Universities, Madrid, 28003, Spain, e-mail: david.gonzalezmoro@usp.ceu.es

Of course, (d) implies (c) with $\kappa = 2^{1/p-1}$. If $\|\cdot\|$ is a quasi-norm (resp. a p-norm) on $\mathbb{X}$ such that defines a complete metrizable topology, then $\mathbb{X}$ is called a *quasi-Banach space* (resp. a *p-Banach space*) and thanks to the Aoki-Rolewicz's theory (see [4,5]), we know that a quasi-Banach space is p-convex for some $p \in (0, 1]$, that is, there is a constant $C > 0$ such that

$$\left\| \sum_{i=1}^{m} f_i \right\|^p \leq C \sum_{i=1}^{m} \|f_i\|^p, \quad \forall m \in \mathbb{N}, \forall f_i \in \mathbb{X}.$$

Then, a quasi-Banach space becomes p-Banach under a suitable renorming.

Now, given a quasi-Banach (or a p-Banach) space, a Schauder basis $\mathcal{B} = (\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is any collection of vectors in the space such that for every $f \in \mathbb{X}$, there is a unique sequence $(a_j(f))_{j\in\mathbb{N}}$ such that

$$\lim_{m\to+\infty} \left\| f - \sum_{j=1}^{m} a_j(f)\boldsymbol{x}_j \right\| = 0.$$

In other words, for every $f \in \mathbb{X}$, we have

$$f = \sum_{j=1}^{\infty} a_j(f)\boldsymbol{x}_j.$$

Moreover, given a Schauder basis, we have the collection of biorthogonal functionals (called dual basis) $\mathcal{B}^* = (\boldsymbol{x}_n^*)_{n\in\mathbb{N}} \subseteq \mathbb{X}^*$ such that $\boldsymbol{x}_n^*(\boldsymbol{x}_m) = \delta_{n,m}$. Using these functionals,

$$\boldsymbol{x}_n^*\left(\sum_{j=1}^{\infty} a_j(f)\boldsymbol{x}_j \right) = a_n(f),$$

so every element could be represented by

$$f = \sum_{j=1}^{\infty} \boldsymbol{x}_j^*(f)\boldsymbol{x}_j.$$

Equivalently, it is well known that $\mathcal{B} = (\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is a Schauder basis if and only if the partial sums are bounded, that is,

$$K_b \coloneqq \sup_m \|S_m\| < \infty,$$

where, given $m \in \mathbb{N}$ and $f \in \mathbb{X}$, $S_m(f)$ is the *mth partial sum of f*:

$$S_m[\mathcal{B}, \mathbb{X}](f) = S_m(f) \coloneqq \sum_{j=1}^{m} \boldsymbol{x}_j^*(f)\boldsymbol{x}_j,$$

and K_b is called the *basis constant*. To know more about Schauder bases in quasi-Banach spaces, we cite the book [6].

Definition 1.1. We say that a Schauder basis $\mathcal{B}$ in a p-Banach space (or quasi-Banach space) $\mathbb{X}$ is *monotone* if

$$\|S_m(f)\| \leq \|f\|, \quad \forall m \in \mathbb{N}, \forall f \in \mathbb{X}. \tag{1.1}$$

Moreover, we say that $\mathcal{B}$ is *bi-monotone* if (1.1) holds and

$$\|f - S_m(f)\| \leq \|f\|, \quad \forall m \in \mathbb{N}, \forall f \in \mathbb{X}. \tag{1.2}$$

A stronger condition than Schauder is unconditionality: Given a finite set of indices $A \subset \mathbb{N}$ and $f \in \mathbb{X}$, the projection operator P_A is defined as

$$P_A[\mathcal{B}, \mathbb{X}](f) = P_A(f) \coloneqq \sum_{n\in A} \boldsymbol{x}_n^*(f)\boldsymbol{x}_n.$$

Using that operator, we say that $\mathcal{B}$ is *K-unconditional* if

$$K \coloneqq \sup_{|A|<\infty} \|P_A\| < \infty.$$

Remark 1.2. There are several examples of bi-monotone bases. For instance, any 1-unconditional basis in a Banach space is bi-monotone. A particular basis that is bi-monotone but not unconditional could be the basis that appears in [7, Proposition 6.10].

From now on, we will consider that $\mathbb{X}$ is a p-Banach space for $0 < p \leq 1$ and $\mathcal{B}$ is a semi-normalized Schauder basis in $\mathbb{X}$.

1.2 TGA

In 1999, in [1], the authors introduced the *TGA* $(G_m)_{m=1}^{\infty}$. To define the algorithm, we take an element $f \in \mathbb{X}$ and $m \in \mathbb{N}$. A *greedy sum* of order m is a sum of the form

$$G_m[\mathcal{B}, \mathbb{X}](f) = G_m(f) := \sum_{n \in G} \boldsymbol{x}_n^*(f)\boldsymbol{x}_n,$$

where G is a finite set called a *greedy set* of f of order m and verifies that $|G| = m$ and

$$\min_{n \in G} |\boldsymbol{x}_n^*(f)| \geq \max_{n \notin G} |\boldsymbol{x}_n^*(f)|.$$

The collection $(G_m)_{m \in \mathbb{N}}$ is the TGA. Of course, any greedy sum is a particular projection since given a greedy sum of order m of $f \in \mathbb{X}$, using the corresponding greedy set G,

$$G_m(f) = P_G(f).$$

To talk about the convergence of the algorithm, we need the following definition.

Definition 1.3. We say that a basis $\mathcal{B}$ is *quasi-greedy* if for every $f \in \mathbb{X}$, there is $C > 0$ such that

$$\|f - P_G(f)\| \leq C\|f\|, \tag{1.3}$$

whenever G is a finite greedy set of f.

Thanks to a famous result of Wojtaszczyk (see [2,8]), we know that a basis is quasi-greedy if and only if

$$\lim_{m \to +\infty} \|f - G_m(f)\| = 0, \quad \forall f \in \mathbb{X}.$$

Remark 1.4. Since every greedy sum is a particular projection, if the basis is unconditional, automatically, the basis is quasi-greedy.

Although for quasi-greedy bases we have the convergence of the algorithm, we can ask when the algorithm produces the best possible approximation, that is, when $\|f - G_m(f)\|$ is comparable with the *best mth possible error in the approximation of f*

$$\sigma_m[\mathcal{B}, \mathbb{X}](f) = \sigma_m(f) := \inf_{y \in \mathbb{X} : |\mathrm{supp}(y)| \leq m} \|f - y\|,$$

where, for a general $g \in \mathbb{X}$, $\mathrm{supp}(g) = \{n \in \mathbb{N} : \boldsymbol{x}_n^*(g) \neq 0\}$.

In [1], we find *greedy bases*, where a basis is greedy if there is $C > 0$ such that for every element $f \in \mathbb{X}$,

$$\|f - P_G(f)\| \leq C\sigma_{|G|}(f),$$

whenever G is a finite greedy set of f. Of course, every greedy basis is quasi-greedy since we can take $y = 0$ in the definition of the error $\sigma_m(f)$.

Here, we study an intermediate notion between quasi-greediness and greediness, the so-called almost-greedy bases introduced in [9].

Definition 1.5. We say that $\mathcal{B}$ is *almost-greedy* if for every $f \in \mathbb{X}$, there is $C > 0$ such that

$$\|f - P_G(f)\| \leq C \inf_{B \subset \mathbb{N} : |B| \leq |G|} \|f - P_B(f)\|, \tag{1.4}$$

whenever G is a finite greedy set of f. The smallest constant verifying (1.4) is denoted by $C_{al} = C_{al}[\mathcal{B}, \mathbb{X}]$ and we say that $\mathcal{B}$ is C_{al}-almost-greedy.

The existence of almost-greedy and non-greedy bases is well known (see [10,11]). In fact, if $\mathcal{B}$ is almost-greedy in a Banach space, for every f and every finite greedy set G,

$$\|f - P_G(f)\| \lesssim \ln(|G|)\sigma_{|G|}(f),$$

and the logarithm is an optimal bound [12].

1.3 Thresholding Chebyshev Greedy algorithm (TCGA)

In 2003, Dilworth et al. introduced in [13] the notion of semi-greedy bases as an enhancement of the TGA to improve the rate of convergence. To define these bases, we need to introduce the TCGA: let $f \in \mathbb{X}$ and consider now G a finite greedy set of f. Then, a *Chebyshev greedy sum* of order $m := |G|$ of f is any element $\mathcal{CG}_m[\mathcal{B}, \mathbb{X}](f) = \mathcal{CG}_m(f)$ of the form $\sum_{n \in G} a_n \boldsymbol{x}_n \in \operatorname{span}\{\boldsymbol{x}_n : n \in G\}$ such that

$$\|f - \mathcal{CG}_m(f)\| = \min\left\{\left\| f - \sum_{n \in G} a_n \boldsymbol{x}_n \right\| : a_n \in \mathbb{F} \ \forall n \in G\right\}.$$

Definition 1.6. We say that $\mathcal{B}$ is *semi-greedy* if for every $f \in \mathbb{X}$, there is $C > 0$ such that

$$\min\left\{\left\| f - \sum_{n \in G} a_n \boldsymbol{x}_n \right\| : a_n \in \mathbb{F} \ \forall n \in G\right\} \leq C\sigma_{|G|}(f), \tag{1.5}$$

whenever G is a finite greedy set of f. The smallest constant verifying (1.5) is denoted by $C_{sg} = C_{sg}[\mathcal{B}, \mathbb{X}]$ and we say that $\mathcal{B}$ is C_{sg}-semi-greedy.

In the last few years, some authors have studied that these bases are equivalent to almost-greedy bases. Concretely, we can find the following results:

- In [13], the authors proved that if $\mathcal{B}$ is a Schauder basis in a Banach space with finite cotype, then $\mathcal{B}$ is almost-greedy if and only if the basis is semi-greedy.
- In [14], the author improved the last result by removing the condition of finite cotype.
- In [15], the authors improved the result from [14] by extending the condition of Schauder bases to semi-normalized Markushevich bases in Banach spaces.
- In [3], the authors showed the equivalence between almost-greedy and semi-greedy bases in the context of Schauder bases in p-Banach spaces using intermediate properties as quasi-greediness and democracy.

To introduce the notion of democracy, we need the indicator sums: let $A \subset \mathbb{N}$ be a finite set of indices and $\varepsilon \in \mathcal{E}_A$ a *sign*, where $\mathcal{E}_A$ is defined as follows:

$$\mathcal{E}_A = \{\varepsilon = (\varepsilon_n)_{n \in A} : |\varepsilon_n| = 1 \ \forall n \in A\}.$$

An *indicator sum* is any finite combination of the form

$$\mathbf{1}_{\varepsilon,A}[\mathcal{B}, \mathbb{X}] = \mathbf{1}_{\varepsilon,A} := \sum_{n \in A} \varepsilon_n \boldsymbol{x}_n,$$

where $\varepsilon \in \mathcal{E}_A$ and A is a finite set of indices. When $\varepsilon \equiv 1$, we use the notation $\mathbf{1}_A$.

Definition 1.7. We say that $\mathcal{B}$ is *super-democratic* if for every pair of finite sets A, B with $|A| \leq |B|$ and every choice of signs $\varepsilon \in \mathcal{E}_A$ and $\eta \in \mathcal{E}_B$, there is $C > 0$ such that

$$\|\mathbf{1}_{\varepsilon,A}\| \leq C\|\mathbf{1}_{\eta,B}\|. \tag{1.6}$$

The smallest constant verifying (1.6) is denoted by $\Delta_s = \Delta_s[\mathcal{B}, \mathbb{X}]$ and we say that $\mathcal{B}$ is Δ_s-super-democratic. When (1.6) is satisfied for $\varepsilon \equiv \eta \equiv 1$, we say that $\mathcal{B}$ is *democratic*.

Thanks to the combination of the results proved in [2,3], we have the following result in the context of p-Banach spaces for Schauder bases.

Theorem 1.8. *Let $\mathcal{B}$ be a Schauder basis with basis constant K_b in a p-Banach space with $0 < p \leq 1$. If $\mathcal{B}$ is C_{sg}-semi-greedy, then the basis is C_{al}-almost-greedy with*

$$C_{al} \leq C_1(p)(1 + [A_p C_2(p) C_1(p) \eta_p(C_1(p))]^p)^{1/p}, \tag{1.7}$$

where $C_1(p) = K_b C_{sg}(1 + (1 + K_b)^p C_{sg}^p)^{1/p}$, $C_2(p) = K_b(1 + K_b)C_{sg}^2$,

$$A_p = (2^p - 1)^{-1/p}, \quad 0 < p \leq 1$$

and

$$\eta_p(u) = \min_{0<t<1}(1 - t^p)^{-1/p}(1 - (1 + A_p^{-1}u^{-1}t)^{-p})^{-1/p}.$$

We want to focus our attention on the case $C_{sg} = 1$. The study of 1-greedy-like bases started in 2006 [16] where the authors characterized 1-greedy bases introducing the so-called Property (A) (or symmetry for largest coefficients). After this year, we can find some articles discussing the 1-almost-greedy and 1-quasi-greedy bases, but in no case is there mention of the 1-semi-greedy bases. In this article, we aim to combine known techniques about the characterization of semi-greediness to produce a new quantitative version for bounding certain constants in the general case of p-Banach spaces using bi-monotone and 1-semi-greedy bases. This new quantitative version implies a more refined bound for the constant of almost-greedy bases using semi-greediness in the general context of p-Banach spaces. Furthermore, the previous bound (1.7) is established by leveraging the concept of p-convexity, which introduces the constant A_p in p-Banach spaces, as we can see in [2]. In this work, we will avoid the use of this property, and we give an improvement of the constant C_{al} for bi-monotone 1-semi-greedy bases. Concretely, the main result that we show is the following one.

Theorem 1.9. *Let $\mathcal{B}$ be a bi-monotone basis in a p-Banach space with $0 < p \leq 1$. If $\mathcal{B}$ is C_{sg}-semi-greedy, then the basis is $3^{2/p}C_{sg}^5$-almost-greedy. In particular, for 1-semi-greedy bases, $C_{al} \leq 3^{2/p}$.*

The structure of this article is as follows: In Section 2, we prove some technical and necessary results to prove the main theorem. In Section 3, we show the proof of Theorem 1.9, and in Section 4, we discuss the bound proved in Theorem 1.9 in connection with some known results in the literature.

2 Some technical results about the TCGA

Proposition 2.1. *Let $\mathcal{B}$ be a C_{sg}-semi-greedy Schauder basis in a p-Banach space $\mathbb{X}$ with $0 < p \leq 1$. Let $f, g \in \mathbb{X}$ with $A = \operatorname{supp}(f)$, $B = \operatorname{supp}(g)$, $|\operatorname{supp}(f)| \leq |\operatorname{supp}(g)| < \infty$, and $\min_{n\in B}|\mathbf{x}_n^*(g)| \geq \max_{j\in A}|\mathbf{x}_j^*(f)|$.*

(*a*) *If the basis is monotone and $\operatorname{supp}(f) < \operatorname{supp}(g)$, then*

$$\|f\| \leq C_{sg}\|g\|.$$

(*b*) *If the basis is bi-monotone and* $A \cap B = \emptyset$, *then*

$$\|f\| \le C_{sg}^2 \|g\|.$$

Proof. We start with (a). We construct the element

$$h := f + g.$$

Using the hypothesis

$$\min_{n\in B} |x_n^*(g)| \ge \max_{j\in A} |x_j^*(f)|,$$

B is a greedy set of h. Hence, applying the TCGA, we obtain some coefficients $(a_n)_{n\in B}$ such that

$$h - C\mathcal{G}_m(h) = f + \sum_{n\in B} a_n x_n,$$

where $m := |B|$. Then,

$$\|f\| \underset{A<B \text{ and } (1.1)}{\le} \left\| f + \sum_{n\in B} a_n x_n \right\| \underset{\text{semi-greediness}}{\le} C_{sg}\sigma_m(h) \le C_{sg}\|h - f\| = C_{sg}\|g\|.$$

Hence, (a) is proved. Now, we show (b). For that, since we are playing with elements with finite support, we can take a set C such that $C > A \cup B$, $|C| = |B|$, and hence, taking $\alpha := \min_{n\in B}|x_n^*(g)| \ge \max_{j\in A}|x_j^*(f)|$ and applying (a),

$$\|f\| \le C_{sg}\|\alpha \mathbf{1}_C\|. \tag{2.1}$$

Now, we define the element

$$\delta := g + \alpha \mathbf{1}_C.$$

It is clear now that B is a greedy set for δ, and then, applying the TCGA, there is a sequence $(b_n)_{n\in B}$ such that

$$\delta - C\mathcal{G}_k(\delta) = \sum_{n\in B} b_n x_n + \alpha \mathbf{1}_C,$$

where $k := |B|$. Thus,

$$\begin{aligned} \|\alpha \mathbf{1}_C\| &\underset{(1.2)}{\le} \|\delta - C\mathcal{G}_k(\delta)\| \underset{\text{semi-greediness}}{\le} C_{sg}\sigma_k(\delta) \\ &\le C_{sg}\left\| \delta - \sum_{n\in C} \alpha x_n \right\| \\ &= C_{sg}\|g\|. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we conclude the result. □

Corollary 2.2. *Any* 1*-semi-greedy and bi-monotone basis in a p-Banach space with* $0 < p \le 1$ *is* 1*-super-democratic.*

Proof. Consider A, B two finite sets and $\varepsilon \in \mathcal{E}_A$ and $\eta \in \mathcal{E}_B$. Consider now a set $C > A \cup B$ with $|C| = |B|$. Applying (a) of Proposition 2.1,

$$\|\mathbf{1}_{\varepsilon,A}\| \le \|\mathbf{1}_C\|.$$

Using now item (b) of Proposition 2.1,

$$\|\mathbf{1}_C\| \le \|\mathbf{1}_{\eta,B}\|.$$

Thus, by both inequalities,

$$\|\mathbf{1}_{\varepsilon,A}\| \le \|\mathbf{1}_{\eta,B}\|,$$

so the basis is 1-super-democratic. □

Proposition 2.3. *Let $\mathbb{X}$ be a p-Banach space with $0 < p \le 1$ and $\mathcal{B}$ a bi-monotone C_{sg}-semi-greedy basis in $\mathbb{X}$. Then, for every greedy set G of f with $|\operatorname{supp}(f)| < \infty$ and $\varepsilon \in \mathcal{E}_G$,*

$$\|P_{G^c}(f) + \alpha\mathbf{1}_{\varepsilon,G}\| \le 3^{1/p}C_{sg}^3\|f\|,$$

where $\alpha := \min_{n\in G}|\mathbf{x}_n^*(f)|$.

Proof. Let $f \in \mathbb{X}$ with finite support and take G a greedy set of f, $\alpha := \min_{n\in G}|\mathbf{x}_n^*(f)|$ and $\varepsilon \in \mathcal{E}_G$. Define now the element

$$h := P_{G^c}(f) + \alpha\mathbf{1}_{\varepsilon,G} + \alpha\mathbf{1}_C,$$

where $C > \operatorname{supp}(f)$ and $|C| = |G|$. Hence, it is clear that C is a greedy set with cardinality $m := |C|$ of h. Applying the TCGA, there exists a sequence of coefficients $(a_n)_{n\in C}$ where

$$h - \mathcal{CG}_m(h) = P_{G^c}(f) + \alpha\mathbf{1}_{\varepsilon,G} + \sum_{n\in C} a_n\mathbf{x}_n.$$

Then,

$$\begin{aligned}
\|P_{G^c}(f) + \alpha\mathbf{1}_{\varepsilon,G}\|^p &\underset{(1.1)}{\le} \left\|P_{G^c}(f) + \alpha\mathbf{1}_{\varepsilon,G} + \sum_{n\in C} a_n\mathbf{x}_n\right\|^p \\
&= \|h - \mathcal{CG}_m(h)\|^p \\
&\underset{\text{semi-greediness}}{\le} C_{sg}^p(\sigma_m(h))^p \\
&\le C_{sg}^p\|h + P_G(f)\|^p \\
&= C_{sg}^p\|f + \alpha\mathbf{1}_{\varepsilon,G} + \alpha\mathbf{1}_C\|^p \\
&\le C_{sg}^p\|f\|^p + C_{sg}^p(\|\alpha\mathbf{1}_{\varepsilon,G}\|^p + \|\alpha\mathbf{1}_C\|^p) \\
&\underset{\text{Proposition 2.1}}{\le} C_{sg}^p\|f\|^p + 2C_{sg}^{2p}\|\alpha\mathbf{1}_C\|^p.
\end{aligned} \tag{2.3}$$

Now, take the element $g := f + \alpha\mathbf{1}_C$. Hence, G is a greedy set of g and then, applying the TCGA over G,

$$g - \mathcal{CG}_m(g) = \sum_{n\in G} d_n\mathbf{x}_n + P_{G^c}(f) + \alpha\mathbf{1}_C,$$

where the sequence $(d_n)_{n\in G}$ is given by the TCGA. Thus,

$$\begin{aligned}
\|\alpha\mathbf{1}_C\|^p &\underset{(1.2)}{\le} \left\|\sum_{n\in G} d_n\mathbf{x}_n + P_{G^c}(f) + \alpha\mathbf{1}_C\right\|^p = \|g - \mathcal{CG}_m(g)\|^p \\
&\underset{\text{semi-grediness}}{\le} C_{sg}^p(\sigma_m(g))^p \le C_{sg}^p\|g - \alpha\mathbf{1}_C\|^p = C_{sg}^p\|f\|^p.
\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we obtain that

$$\|P_G^c(f) + \alpha\mathbf{1}_{\varepsilon,G}\| \le 3^{1/p}C_{sg}^3\|f\|.$$ □

Proposition 2.4. *Let $\mathbb{X}$ be a p-Banach space with $0 < p \le 1$ and $\mathcal{B}$ a bi-monotone 1-semi-greedy basis in $\mathbb{X}$. For every $f \in \mathbb{X}$ with finite support,*

$$\|f\| \le 3^{1/p}C_{sg}^2\|f - P_A(f) + t\mathbf{1}_{\varepsilon,B}\|,$$

whenever $|A| \le |B| < \infty$, $B \cap \operatorname{supp}(f) = \emptyset$, $A \subset \operatorname{supp}(f)$, $t \ge \max_{n\in\operatorname{supp}(f)}|\mathbf{x}_n^*(f)|$, and $\varepsilon \in \mathcal{E}_B$.

Proof. Consider $f \in \mathbb{X}$ with finite support and t, A, B, ε as in the statement.

$$\|f\|^p \leq \|f - P_A(f) + t\mathbf{1}_{\varepsilon,B}\|^p + \|P_A(f)\|^p + \|t\mathbf{1}_{\varepsilon,B}\|^p. \tag{2.5}$$

Now, take an element $g \coloneqq t\mathbf{1}_C \in \mathbb{X}$ such that $\operatorname{supp}(g) > \operatorname{supp}(f) \cup B$ and $|B| = |C|$. Applying Proposition 2.1,

$$\|P_A(f)\| \leq C_{sg}\|g\|. \tag{2.6}$$

Also, applying the same result,

$$\|t\mathbf{1}_{\varepsilon,B}\| \leq C_{sg}\|g\|. \tag{2.7}$$

Now, since $\operatorname{supp}(g) > \operatorname{supp}(f) \cup B$, we can define the element

$$y \coloneqq f - P_A(f) + t\mathbf{1}_{\varepsilon,B} + t\mathbf{1}_C,$$

where one of the greedy sets of cardinality $n \coloneqq |B|$ is B so, applying the TGCA,

$$y - \mathcal{CG}_n(y) = f - P_A(f) + \sum_{n\in B} c_n\mathbf{x}_n + t\mathbf{1}_C.$$

Hence,

$$\begin{aligned} \|t\mathbf{1}_C\| &\underset{(1.2)}{\leq} \left\| f - P_A(f) + \sum_{n\in B} c_n\mathbf{x}_n + t\mathbf{1}_C \right\| \\ &\underset{\text{semi-greediness}}{\leq} C_{sg}\sigma_n(y) \leq \|y - t\mathbf{1}_C\| = C_{sg}\|f - P_A(f) + t\mathbf{1}_{\varepsilon,B}\|. \end{aligned} \tag{2.8}$$

Hence, combining (2.6), (2.7), and (2.8) in (2.5), we obtain the result. □

3 Proof of Theorem 1.9

Proof of Theorem 1.9. Consider $f \in \mathbb{X}$, G a finite greedy set of f and $B \subset \mathbb{N}$ such that $|B| \leq |G|$. Applying Proposition 2.4 over the element $f - P_G(f)$, we can obtain

$$\begin{aligned} \|f - P_G(f)\| &\leq 3^{1/p}C_{sg}^2\|f - P_G(f) - P_{B\setminus G}(f) + t\mathbf{1}_{\varepsilon,G\setminus B}\| \\ &= 3^{1/p}C_{sg}^2\|P_{(G\cup B)^c}(f - P_B(f)) + t\mathbf{1}_{\varepsilon,G\setminus B}\|, \end{aligned} \tag{3.1}$$

where $\varepsilon \equiv \operatorname{sign}\{(\mathbf{x}_n^*(f))\}$ and $t = \min_{n\in(G\setminus B)}|\mathbf{x}_n^*(f)|$. Now, taking into account that if G is a greedy set for f, then $G\setminus B$ is a greedy set for $f - P_B(f)$, applying Proposition 2.3, we have

$$\|P_{(G\cup B)^c}(f - P_B(f)) + t\mathbf{1}_{\varepsilon,G\setminus B}\| \leq 3^{1/p}C_{sg}^3\|f - P_B(f)\|. \tag{3.2}$$

Combining (3.1) and (3.2), we conclude the proof for elements with finite support. To conclude the result for general elements $f \in \mathbb{X}$, we can apply the density argument proved in [17, Lemma 3.7]. □

4 A comparison with previously known results

As we have commented in the introduction, in [3], the authors proved that for Schauder bases with basis constant K_b in general p-Banach spaces, if $\mathcal{B}$ is C_{sg}-semi-greedy, then

$$C_{al} \leq C_1(p)(1 + [A_pC_2(p)C_1(p)\eta_p(C_1(p))]^p)^{1/p},$$

where $C_1(p) = K_bC_{sg}(1 + (1 + K_b)^pC_{sg}^p)^{1/p}$, $C_2(p) = K_b(1 + K_b)C_{sg}^2$,

$$A_p = (2^p - 1)^{-1/p}, \quad 0 < p \leq 1$$

and

$$\eta_p(u) = \min_{0<t<1}(1 - t^p)^{-1/p}(1 - (1 + A_p^{-1}u^{-1}t)^{-p})^{-1/p}.$$

Since $\eta_p(u) \approx u^{1/p}$ (see [2, Remark 4.9]), in the case when the basis is bi-monotone and 1-semi-greedy, it is possible to derivate from the last bound that

$$C_{al} \lesssim 2^{1/p}(1 + (A_p)^p 2^{1+1/p})^{1/p} =: \mathbf{C}. \tag{4.1}$$

Using our Theorem 1.9, we have that $C_{al} \le 3^{2/p}$, that it is a better constant for a general $0 < p \le 1$ than (4.1) as Figure 1 shows.

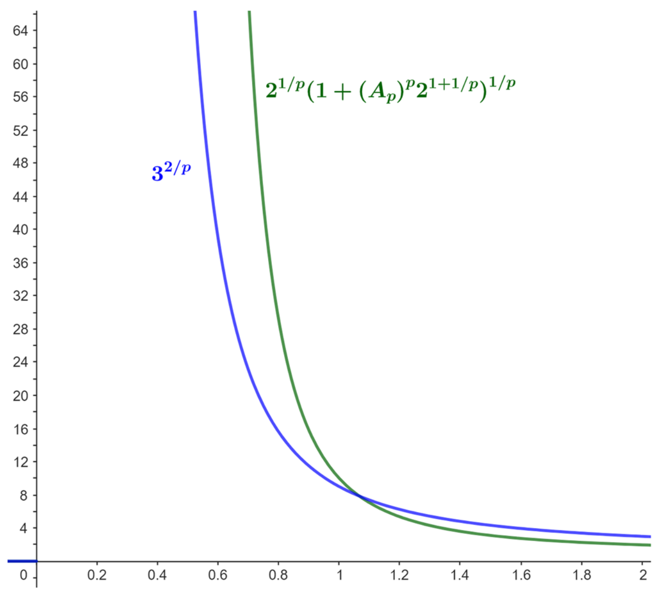

Figure 1: Comparation between $3^{2/p}$ and $\mathbf{C}$ of (4.1).

In the case of Banach spaces (i.e., a p-Banach space with $p = 1$), we can found in [14] some different estimates. Concretely, it is possible to deduce from this article that if $\mathcal{B}$ is bi-monotone and C_{sg}-semi-greedy, then the basis is almost-greedy with

$$C_{al} \le 3C_{sg}^2(1 + 4C_{sg}^2).$$

In particular, for $C_{sg} = 1$, we have that $C_{al} \le 15$, where, in our case, using the bound of Theorem 1.9 for $p = 1$, $C_{al} \le 9$.

Hence, to conclude, in any possible case, our estimates for the case of bi-monotone and 1-semi-greedy bases are the best so far.

Acknowledgements: The researcher would like to thank the reviewers for the valuable comments and suggestions.

Funding information: P. M. Berná and D. González are partially supported by Grant PID2022-142202NB-I00 (Agencia Estatal de Investigación, Spain) and D. González was partially supported by ESI International Chair@ CEU-UCH.

Author contributions: All authors contributed in the same proportion to the elaboration of this article.

Conflict of interest: Dr. Pablo M. Berná is a member of the Editorial Board of the Demonstratio Mathematica but was not involved in the review process of this article.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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