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Research Article

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(α, F)-Geraghty-type generalized F-contractions on non-Archimedean fuzzy metric-unlike spaces

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Abstract: In this study, we generalize fuzzy metric-like, non-Archimedean fuzzy metric-like, and all the variants of fuzzy metric spaces. We propose the idea of fuzzy metric-unlike and non-Archimedean fuzzy metric-unlike, respectively. We also propose the idea of (α, F)-Geraghty-type generalized F-contraction mappings utilizing fuzzy metric-unlike and non-Archimedean fuzzy metric-unlike spaces. We investigate the presence of unique fixed points using the recently introduced contraction mappings. In order to complement our study, we consider an application to dynamic market equilibrium.

Keywords: fuzzy metric-unlike space, α-admissibility, (α, F)-Geraghty-type generalized F-contraction mapping

MSC 2020: 47H10, 54H25

1 Introduction and preliminaries

In the domain of fixed point theory, the conclusions of Banach's [1] fixed point theorem have recently demonstrated significance across metric spaces and have been extensively studied in various newly introduced metric spaces with varied generalizations among scholars (for example, see [2–13]). Kramosil and Michalek's [14] description of the fuzzy metric was published in 1975, followed by George and Veeramani's [15] definition of the Hausdorff topology via fuzzy metric. Several well-known metric space results have been enhanced to fuzzy metric space. Gregory and Sapena [16] have established the fixed point theorem in the sense of both George and Veeramani, as well as Kramosil and Michalek in complete spaces. Later, for various contractions in fuzzy metric spaces, there are numerous fixed point outcomes in literature (see for instance, [17–20] and [21]). In 2017, Zhao et al. [22] extended the concept of fuzzy metric space to fuzzy metric-like space to establish some common fixed point conclusions. In this study, we generalize fuzzy metric-like, non-Archimedean fuzzy metric-like, and all the fuzzy metric variants and introduce fuzzy metric-unlike and non-Archimedean fuzzy metric-unlike. Furthermore, we propose (α, F)-Geraghty-type generalized F-contraction mappings in both spaces to investigate the presence of unique fixed points. To demonstrate the usefulness of our new space, we apply our conclusions to the problem of the dynamic market equilibrium.

The notations in the current document include the following: a collection of natural numbers is designated as $\mathbb{N}$. A collection of real numbers is designated as $\mathbb{R}$. A collection of real numbers that are positive is designated as $\mathbb{R}_+$.

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Now, we will recall some definitions, results, and examples.

The following is how Edelstein [23] explained Banach's fixed point principle [1] in 1962:

Theorem 1.1. [23] *In a complete metric space* (V, d), *if a self-mapping* $H : V \longrightarrow V$ *is such that*

$$d(Hv_1, Hv_2) < d(v_1, v_2),$$

for all $v_1, v_2 \in V$, *with* $v_1 \neq v_2$. *Then,* H *has a unique fixed point in* V.

A novel contraction known as the F-contraction was introduced in 2012 by Wardowski [7] to prove Theorem 1.2.

Definition 1.1. [7] If for some $\tau > 0$ and for every $v_1, v_2 \in V$, $d(Hv_1, Hv_2) > 0$ implies

$$\tau + F(d(Hv_1, Hv_2)) \leq F(d(v_1, v_2)),$$

then the self-mapping $H : V \longrightarrow V$ in the metric space (V, d) is referred to as a F-contraction, if $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is strictly increasing so that:

(F1) $\lim_{n\to\infty}\alpha_n = 0$ iff $\lim_{n\to\infty}F(\alpha_n) = -\infty$, where $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$,

(F2) $\lim_{\alpha\to 0^+}\alpha^\delta F(\alpha) = 0$, for some $\delta \in (0, 1)$.

Theorem 1.2. [7] *An* F*-contraction* $H : V \longrightarrow V$ *in a complete metric space* (V, d) *has a unique fixed point* $v^* \in V$ *and the sequence* $\{H^n v_0\}_{n\in\mathbb{N}}$ *converges to* v^*, *for each* $v_0 \in V$.

The requirement (F2) in the definition of "F-contraction" is recently replaced with the continuity of the F-type functions, by Piri and Kumam [13], to prove theorems in different types of generalized complete metric spaces.

In 2012, Samet et al. [24] defined α-admissibility as follows:

Definition 1.2. [24] Let there be two mappings: $\alpha : V \times V \longrightarrow [0, +\infty]$ and $H : V \longrightarrow V$. The self-mapping H will therefore be referred to as being α-admissible if for all $v_1, v_2 \in V$,

$$\alpha(Hv_1, Hv_2) \geq 1, \quad \text{whenever } \alpha(v_1, v_2) \geq 1.$$

Now, we state the definition of continuous t-norm before we begin with the terminology of fuzzy metric space.

Definition 1.3. [25] A continuous t-norm is a commutative and associative continuous binary mapping $\star : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ satisfying

(i) $\alpha_1 \star 1 = \alpha_1$, for every $\alpha_1 \in [0, 1]$,

(ii) $\alpha_1 \star \alpha_3 \leq \alpha_2 \star \alpha_4$, for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$ with $\alpha_1 \leq \alpha_2, \alpha_3 \leq \alpha_4$.

Definition 1.4. [14,15] An ordered triple $(V, d, \star)$, where V is a non-empty set, d is a fuzzy set on $V \times V \times (0, +\infty)$, and $\star$ is a continuous triangular norm, is a fuzzy metric space if it satisfies the subsequent hypotheses:

Fd1: $d(v_1, v_2, s_1) > 0$ (non-negativity),

Fd2: $d(v_1, v_2, s_1) = 1$ iff $v_1 = v_2$ (antisymmetric),

Fd3: $d(v_1, v_2, s_1) = d(v_2, v_1, s_1)$ (symmetric),

Fd4: $d(v_1, v_3, s_1 + s_2) \geq d(v_1, v_2, s_1) \star d(v_2, v_3, s_2)$ (triangular inequality),

Fd5: with respect to the third variable, $d(v_1, v_2, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous,

for all $v_1, v_2, v_3 \in V$ and $s_1, s_2 > 0$.

Definition 1.5. [15] By substituting any of the following for the criterion (Fd4) into the aforementioned definition, we have a definition of non-Archimedean fuzzy metric space $(V, d, \star)$:

Fd4′: $d(v_1, v_3, \max\{s_1, s_2\}) \geq d(v_1, v_2, s_1) \star d(v_2, v_3, s_2) \quad \forall v_1, v_2, v_3 \in V;\ s_1, s_2 > 0$,

or,
Fd4″: $d(v_1, v_3, s_1) \geq d(v_1, v_2, s_1) \star d(v_2, v_3, s_1) \quad \forall v_1, v_2, v_3 \in V;\ s_1 > 0$.

Remark 1.1. The aforementioned definitions make it apparent that non-Archimedean fuzzy metric spaces are likewise fuzzy metric spaces, and hence, inequality (Fd4′) entails inequality (Fd4).

Definition 1.6. [26] If we make the requirement (Fd2) one-sided in the definition of fuzzy metric space above by
Fd2′: $d(v_1, v_2, s_1) = 1$ implies $v_1 = v_2$,

then $(V, d, \star)$ will become a fuzzy metric-like space.

Remark 1.2. It is noticeable from the aforementioned definitions that each fuzzy metric is fuzzy metric-like. Because of this, each non-Archimedean fuzzy metric is a fuzzy metric.

Definition 1.7. [22] An ordered triple $(V, d, \star)$, where V is a non-empty set, $d : V^2 \times (0, +\infty) \longrightarrow [0, 1]$ and $\star$ is a continuous triangular norm, which satisfies
Fd1: $d(v_1, v_2, s_1) > 0$ (non-negativity),
Fd2: $d(v_1, v_2, s_1) = 1$ implies $v_1 = v_2$ (antisymmetric),
Fd3: $d(v_1, v_2, s_1) = d(v_2, v_1, s_1)$ (symmetric),
Fd4: $d(v_1, v_3, s_1 + s_2) \geq d(v_1, v_2, s_1) \star d(v_2, v_3, s_2)$ (triangular inequality),
Fd5: with respect to the third variable, $d(v_1, v_2, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous,

for all $v_1, v_2, v_3 \in V$ and $s_1 > 0$, is referred to as a fuzzy metric-like space.

Definition 1.8. [22] An ordered triple $(V, d, \star)$, where V is a non-empty set, $d : V^2 \times (0, +\infty) \longrightarrow [0, 1]$ and $\star$ is a continuous triangular norm, which satisfies
Fd1: $d(v_1, v_2, s_1) > 0$ (non-negativity),
Fd2: $d(v_1, v_2, s_1) = 1$ implies $v_1 = v_2$ (antisymmetric),
Fd3: $d(v_1, v_2, s_1) = d(v_2, v_1, s_1)$ (symmetric),
Fd4: $d(v_1, v_3, s_1) \geq d(v_1, v_2, s_1) \star d(v_2, v_3, s_1)$ (triangular inequality),
Fd5: with respect to the third variable, $d(v_1, v_2, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous,

for all $v_1, v_2, v_3 \in V$ and $s_1 > 0$, is referred to as a non-Archimedean fuzzy metric-like space.

2 Main results

Before presenting our main theorem, we introduce fuzzy metric-unlike space, which generalizes all the earlier fuzzy-type generalized metric spaces. Also, we will define (α, F)-Geraghty-type generalized F-contraction mapping in terms of both fuzzy metric-unlike and non-Archimedean fuzzy metric-unlike spaces.

Definition 2.1. An ordered triple $(V, d, \star)$, where V is a non-empty set, $d : V^2 \times (0, +\infty) \longrightarrow [0, 1]$ and $\star$ is a continuous triangular norm, which satisfies
Fd1: $d(v_1, v_2, s_1) > 0$ (non-negativity),
Fd2: $d(v_1, v_2, s_1) = d(v_2, v_1, s_1)$ (symmetric),
Fd3: $d(v_1, v_3, s_1 + s_2) \geq d(v_1, v_2, s_1) \star d(v_2, v_3, s_2)$ (triangular inequality),
Fd4: with respect to the third variable, $d(v_1, v_2, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous,

for all $v_1, v_2, v_3 \in V$ and $s_1 > 0$, is referred to as a fuzzy metric-unlike space.

Remark 2.1. The aforementioned definitions make it apparent that fuzzy metric-unlike spaces are not any of the fuzzy-type generalized metric spaces because it does not include antisymmetric property.

The following is an example of a fuzzy metric-unlike space. However, it is none of the fuzzy-type generalized metric spaces.

Example 2.1. Let $V = [0, +\infty)$. Define $d : V \times V \times (0, +\infty) \longrightarrow [0, 1]$ as

$$d(v_1, v_2, t) = \frac{t}{t + \min\{v_1, v_2\}}$$

and $\star : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ by $\delta_1 \star \delta_2 = \min\{\delta_1, \delta_2\}$. Then, $(V, d, \star)$ is a fuzzy metric-unlike space. Clearly, d does not satisfy the antisymmetric property.

Definition 2.2. An ordered triple $(V, d, \star)$, where V is a non-empty set, $d : V^2 \times (0, +\infty) \longrightarrow [0, 1]$ and $\star$ is a continuous triangular norm, which satisfies

nFd1: $d(v_1, v_2, s_1) > 0$ (non-negativity),
nFd2: $d(v_1, v_2, s_1) = d(v_2, v_1, s_1)$ (symmetric),
nFd3: $d(v_1, v_3, s_1) \geq d(v_1, v_2, s_1) \star d(v_2, v_3, s_1)$ (triangular inequality),
nFd4: with respect to the third variable, $d(v_1, v_2, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous,

for all $v_1, v_2, v_3 \in V$ and $s_1 > 0$, is referred to as a non-Archimedean fuzzy metric-unlike space.

Remark 2.2. The aforementioned definitions make it apparent that non-Archimedean fuzzy metric-unlike spaces are likewise fuzzy metric-unlike spaces, and hence, inequality (nFd3) entails inequality (Fd3).

Remark 2.3. Example 2.1 makes it apparent that fuzzy metric-unlike spaces may not be a non-Archimedean fuzzy metric-unlike space. Indeed, Example 2.1 does not satisfy the condition (nFd3). Also, it does not satisfy the antisymmetric property.

The following is an example of a non-Archimedean fuzzy metric-unlike space.

Example 2.2. Let $V = [0, +\infty)$. Define $d : V \times V \times (0, +\infty) \longrightarrow [0, 1]$ as

$$d(v_1, v_2, t) = \frac{t}{t + \max\{v_1, v_2\}}$$

and $\star : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ by $\delta_1 \star \delta_2 = \min\{\delta_1, \delta_2\}$. Then, $(V, d, \star)$ is a non-Archimedean fuzzy metric-unlike space.

Definition 2.3. In $(V, d, \star)$, a fuzzy metric-unlike space (respectively, non-Archimedean fuzzy metric-unlike space), consider any sequence $\{v_n\}_{n \in \mathbb{N}}$ and $\varepsilon > 0$ is arbitrary.

(i) Then, $\{v_n\}_{n \in \mathbb{N}}$ is convergent to $v \in V$ if there is $m \in \mathbb{N}$ so that $d(v_n, v, s_1) < \varepsilon$ for each $n \geq m, s_1 > 0$, in this case, we write

$$\lim_{n \to +\infty} v_n = v.$$

(ii) Then, $\{v_n\}_{n \in \mathbb{N}}$ is a Cauchy if $d(v_{n+p}, v_n, s_1) < \varepsilon$, for all $s_1 > 0$ and $p > 0$.
(iii) If each Cauchy sequence in $(V, d, \star)$ is convergent, then $(V, d, \star)$ is complete.

Definition 2.4. In a non-Archimedean fuzzy metric-unlike space $(V, d, \star)$, let $H : V \longrightarrow V, \alpha : V \times V \longrightarrow [0, +\infty)$ and $\beta : [0, +\infty) \longrightarrow [0, 1)$. If for some $\tau > 0$,

$$\alpha(v_1, v_2)(\tau + F(d(Hv_1, Hv_2, s_1))) \leq \beta(M_{v_1,v_2,s_1})F(M_{v_1,v_2,s_1}), \tag{1}$$

for all $v_1, v_2 \in V$ and $s_1 > 0$ with $d(Hv_1, Hv_2, s_1) > 0$ and $\alpha(v_1, v_2) \geq 1$, where F is the same as Definition 1.1 and $\lim_{n\to+\infty}\beta(r_n) = 1 \Rightarrow \lim_{n\to+\infty} r_n = 0$.

$$M_{v_1,v_2,s_1} = \max\left\{d(v_1, v_2, s_1), d(Hv_1, v_1, s_1), d(v_2, Hv_2, s_1), \frac{d(Hv_1, v_2, s_1) + d(v_1, Hv_2, s_1)}{4}, \frac{[1 + d(v_1, Hv_1, s_1)]d(v_2, Hv_2, s_1)}{d(v_1, v_2, s_1) + 1}\right\}.$$

Then, H is (α, F)-Geraghty-type generalized F-contraction.

Theorem 2.1. *An (α, F)-Geraghty-type generalized F-contraction self-mapping $H : V \longrightarrow V$, in a complete non-Archimedean fuzzy metric-unlike space $(V, d, \star)$ has a unique fixed point $u^* \in V$, and also, the sequence $\{H^n u\}_{n\in\mathbb{N}}$, for any $u \in V$, must converge to u^*.*

Proof. For an arbitrary point $u_0 \in V$, let us establish a sequence $\{u_n\}$ by

$$\begin{aligned} u_1 &= Hu_0,\\ u_2 &= Hu_1\\ &= H^2u_0,\\ &\vdots\\ u_n &= H^n u_0, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Now, if for some $n \in \mathbb{N}$, $u_{n+1} = u_n$, then, $Hu_n = u_n$ and u_n is a fixed point of H.

Hence, we can assume that the sequence consists of distinct points. Now, we prove sequence $\{u_n\}$ is Cauchy as follows:

Suppose $\{u_n\}$ is not Cauchy, then we can find subsequences $\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ for every $\varepsilon > 0$, $k \in \mathbb{N}$ with $p_n > q_n > k$ such that

$$d(u_{p_n}, u_{q_n}, r_0) > \varepsilon \quad \text{and} \quad d(u_{p_n-1}, u_{q_n}, r_0) \leq \varepsilon, \quad \text{for all } n > k \text{ and } r_0 > 0.$$

Again,

$$\begin{aligned} \varepsilon &\geq d(u_{p_n-1}, u_{q_n}, r_0)\\ &\geq d(u_{p_n}, u_{p_n-1}, r_0)\star d(u_{p_n}, u_{q_n}, r_0)\\ &> d(u_{p_n-1}, u_{p_n}, r_0)\star\varepsilon. \end{aligned}$$

When n is large enough, we have

$$\begin{aligned} \lim_{n\to+\infty} d(u_{p_n}, u_{q_n}, r_0) &= \varepsilon,\\ \lim_{n\to+\infty} d(u_{p_n-1}, u_{q_n}, r_0) &= \varepsilon,\\ \lim_{n\to+\infty} d(u_{p_n}, u_{q_n-1}, r_0) &= \varepsilon,\\ \lim_{n\to+\infty} d(u_{p_n-1}, u_{q_n-1}, r_0) &= \varepsilon. \end{aligned}$$

Again,

$$\begin{aligned} M_{u_{p_n-1},u_{q_n-1},r_0} &= \max\Bigg\{d(u_{p_n-1}, u_{q_n-1}, r_0), d(Hu_{p_n-1}, u_{p_n-1}, r_0), d(u_{q_n-1}, Hu_{q_n-1}, r_0),\\ &\quad \frac{d(Hu_{p_n-1}, u_{q_n-1}, r_0) + d(u_{p_n-1}, Hu_{q_n-1}, r_0)}{4}, \frac{[1 + d(u_{p_n-1}, Hu_{p_n-1}, r_0)]d(u_{q_n-1}, Hu_{q_n-1}, r_0)}{d(u_{p_n-1}, u_{q_n-1}, r_0) + 1}\Bigg\}\\ &= \max\Bigg\{d(u_{p_n-1}, u_{q_n-1}, r_0), d(u_{p_n}, u_{p_n-1}, r_0), d(u_{q_n-1}, u_{q_n}, r_0), \frac{d(u_{p_n}, u_{q_n-1}, r_0) + d(u_{p_n-1}, u_{q_n}, r_0)}{4},\\ &\quad \frac{[1 + d(u_{p_n-1}, u_{p_n}, r_0)]d(u_{q_n-1}, u_{q_n}, r_0)}{d(u_{p_n-1}, u_{q_n-1}, r_0) + 1}\Bigg\}. \end{aligned}$$

This implies

$$\lim_{n\to+\infty} M_{u_{p_n-1},u_{q_n-1},r_0} = \varepsilon.$$

Since $d(u_{p_n}, u_{q_n}, r_0) > \varepsilon > 0$, we have

$$\begin{aligned}\tau + F(d(u_{p_n}, u_{q_n}, r_0)) &\leq \alpha(u_{p_n-1}, u_{q_n-1})(\tau + F(d(u_{p_n}, u_{q_n}, r_0)))\\ &\leq \beta(M_{u_{p_n-1},v_{q_n-1},r_0})F(M_{u_{p_n-1},v_{q_n-1},r_0}), \quad \text{by (1)}.\end{aligned}$$

Since $\beta(r) < 1$, limiting $n \to +\infty$, we have

$$\tau + F(\varepsilon) < F(\varepsilon), \quad \text{for all } r \in [0, +\infty), \text{ a contradiction to the hypothesis } \tau > 0.$$

Hence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ becomes Cauchy and $(V, d, \star)$ is complete space. So the sequence $\{u_n\}$ must be convergent in $(V, d, \star)$, i.e., for some $u^* \in V$, we have

$$\lim_{n\to+\infty} u_n = u^*.$$

Next, we demonstrate that u^* is our point such that $Hu^* = u^*$. Let $\{u_{q_n}\}$ be any subsequence of $\{u_n\}$ such that $\lim_{n\to\infty} u_{q_n} = Hu^*$, so that $Hu^* = u^*$ and we are done. Suppose $Hu_{q_n} \neq Hu_*$ for any subsequence $\{u_{q_n}\}$. Then, $Hu^* \neq u^*$, i.e., $d(Hu_{q_n}, Hu^*, r_0) > 0$ and $d(Hu^*, u^*, r_0) > 0$ for any $r_0 > 0$.

Utilizing (1), we obtain

$$\begin{aligned}\tau + F(d(Hu_{q_n}, Hu^*, r_0)) &\leq \alpha(u_{q_n}, Hu^*, r_0)(\tau + F(d(u_{q_n+1}, Hu^*, r_0)))\\ &\leq \beta(M_{u_{q_n},u^*,r_0})F(M_{u_{q_n},u^*,r_0}),\end{aligned}$$

where

$$\begin{aligned}M_{u_{q_n},u^*,r_0} = \max\Big\{&d(u_{q_n}, u^*, r_0), d(u_{q_n+1}, u_{q_n}, r_0), d(u^*, Hu^*, r_0),\\ &\frac{d(u_{q_n+1}, u^*, r_0) + d(u_{q_n}, Hu^*, r_0)}{4}, \frac{[1 + d(u_{q_n}, u_{q_n+1}, r_0)]d(u^*, Hu^*, r_0)}{d(u_{q_n}, u^*, r_0) + 1}\Big\}.\end{aligned}$$

As $d(Hu^*, u^*, r_0) > 0$, we have $\lim_{n\to+\infty} M_{u_{q_n},u^*,r_0} = d(Hu^*, u^*, r_0)$ and

$$\begin{aligned}\tau + F(d(u^*, Hu^*, r_0)) &\leq \beta(d(Hu^*, u^*, r_0))F(d(Hu^*, u^*, r_0))\\ &< F(d(Hu^*, u^*, r_0)), \quad \text{a contradiction to the hypothesis } \tau > 0.\end{aligned}$$

Hence, $Hu^* = u^*$.

We shall now demonstrate the uniqueness of point u^*. Instead, imagine that we have two distinct fixed points u^* and u^{**}. In this case, sequence u_n converges to u^* as well as u^{**}. Also, $Hu^* = u^{**}$ and $Hu^{**} = u^{**}$.

Since $u_* \neq u_{**}$, we have $d(Hu^*, Hu^{**}, r_0) = d(u^*, u^{**}, r_0) > 0$ for $r_0 > 0$, and this implies

$$\begin{aligned}\tau + F(d(Hu^*, Hu^{**}, r_0)) &\leq \alpha(u^*, u^{**})(\tau + F(d(Hu^*, Hu^{**}, r_0)))\\ &\leq \beta(M_{u^*,u^{**},r_0})F(M_{u^*,u^{**},r_0}), \quad \text{using (1)},\end{aligned}$$

where

$$\begin{aligned}M_{u^*,u^{**},r_0} &= \max\Big\{d(u^*, u^{**}, r_0), d(Hu^*, u^*, r_0), d(u^{**}, Hu^{**}, r_0), \frac{d(Hu^*, u^{**}, r_0) + d(u^*, Hu^{**}, r_0)}{4},\\ &\qquad \frac{[1 + d(u^*, Hu^*, r_0)]d(u^{**}, Hu^{**}, r_0)}{d(u^*, u^{**}, r_0) + 1}\Big\}\\ &= \max\Big\{d(u^*, u^{**}, r_0), \frac{d(u^*, u^{**}, r_0)}{2}\Big\}\\ &= d(u^*, u^{**}, r_0).\end{aligned}$$

Thus,

$$\begin{aligned}\tau + F(d(u^*, u^{**}, r_0)) &\le \beta(d(u^*, u^{**}, r_0))F(d(u^*, u^{**}, r_0))\\ &< F(d(u^*, u^{**}, r_0)), \quad \text{since } \beta(d(u^*, u^{**}, r_0)) < 1,\end{aligned}$$

which implies $\tau \le 0$, a contradiction to the hypothesis $\tau > 0$. Hence, $u^* = u^{**}$ implies that H has a unique fixed point u^*. □

Example 2.3. Let $V = \left\{0, \frac{1}{3}, \frac{1}{2}\right\}$. Then, V, with the usual metric in $\mathbb{R}$, is a complete metric space. Define $d : V^2 \times (0, +\infty) \longrightarrow [0, 1]$ by

$$d(v_1, v_2, r) = e^{-\frac{|v_1 - v_2|}{r}}, \quad \forall v_1, v_2 \in V, r > 0,$$

and $\star : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ by

$$\delta_1 \star \delta_2 = \min\{\delta_1, \delta_2\}, \quad \forall \delta_1, \delta_2 \in [0, 1].$$

Then, V with respect to defined d and $\star$ forms a complete non-Archimedean fuzzy metric-unlike space.

Suppose a self-mapping H on V is defined by

$$H(0) = H\left(\frac{1}{3}\right) = 0, H\left(\frac{1}{2}\right) = \frac{1}{3}.$$

Now, considering the mappings $\alpha : V \times V \longrightarrow [0, +\infty)$ so that

$$\alpha(v_1, v_2) = \begin{cases} 1, & \text{if } v_1, v_2 \in V, \\ 0, & \text{otherwise,} \end{cases}$$

$$F : \mathbb{R}_+ \longrightarrow \mathbb{R} \quad \text{so that} \quad F(u) = -\frac{1}{\sqrt{u}},$$

and

$$\beta : (0, +\infty) \longrightarrow [0, 1), \quad \text{given by} \quad \beta(r) = \frac{1}{1 + r},$$

we found H as an (α, F)-Geraghty-type generalized F-contraction in the complete non-Archimedean fuzzy metric-unlike space $(V, d, \star)$ for $0 < \tau \le \frac{1}{2}$. And from the definition of H, we see that the self-mapping H possesses 0 as the only fixed point.

Remark 2.4. Since fuzzy metric-like or non-Archimedean fuzzy metric-like spaces are neither fuzzy metric-unlike nor non-Archimedean fuzzy metric-unlike spaces, our results cannot hold in standard variants of fuzzy spaces. This fact is also supported by Example 2.3 since d does not satisfy the antisymmetric property.

Remark 2.5. In a fuzzy metric-unlike space, we can show a different version of Theorem 2.1, using the same notion of (α, F)-Geraghty-type generalized F-contraction.

Theorem 2.2. *An (α, F)-Geraghty-type generalized F-contraction $H : V \longrightarrow V$, in a complete fuzzy metric-unlike space $(V, d, \star)$ has a unique fixed point $u^* \in V$; also, the sequence $\{H^n u\}_{n \in \mathbb{N}}$, for any $u \in V$, converges to u^*.*

Proof. Since non-Archimedean fuzzy metric-unlike spaces are fuzzy metric-unlike spaces, proceeding analogous to the proof of Theorem 2.1, we can demonstrate this theorem. □

Remark 2.6. In a non-Archimedean fuzzy metric space (respectively, non-Archimedean fuzzy metric-like space), we can show a different version of Theorem 2.1, using the same notion of (α, F)-Geraghty-type generalized F-contraction.

Theorem 2.3. *An (α, F)-Geraghty-type generalized F-contraction $H : V \longrightarrow V$, in a complete non-Archimedean fuzzy metric space (respectively, non-Archimedean fuzzy metric-like space) $(V, d, \star)$ has a unique fixed point $u^* \in V$; also, the sequence $\{H^n u\}_{n\in\mathbb{N}}$, for any $u \in V$, converges to u^*.*

Remark 2.7. In a fuzzy metric space (respectively, fuzzy metric-like space), we can show a different version of Theorem 2.2, using the same notion of (α, F)-Geraghty-type generalized F-contraction.

Theorem 2.4. *An (α, F)-Geraghty-type generalized F-contraction $H : V \longrightarrow V$, in a complete fuzzy metric space (respectively, fuzzy metric-like space) $(V, d, \star)$ has a unique fixed point $u^* \in V$; also, the sequence $\{H^n u\}_{n\in\mathbb{N}}$, for any $u \in V$, converges to u^*.*

Corollary 2.1. *In a complete non-Archimedean fuzzy metric-unlike space $(V, d, \star)$, let $H : V \longrightarrow V$, $\alpha : V \times V \longrightarrow [0, +\infty)$, and $\beta : [0, +\infty) \longrightarrow [0, 1]$. If for some $\tau > 0$*

$$\alpha(v_1, v_2)(\tau + F(d(Hv_1, Hv_2, s_1))) \le \beta(M'_{v_1,v_2,s_1})F(M'_{v_1,v_2,s_1}), \tag{2}$$

for all $v_1, v_2 \in V$, and $s_1 > 0$ with $d(Hv_1, Hv_2, s_1) > 0$ and $\alpha(v_1, v_2) \ge 1$, where F is the same as Definition 1.1, $\lim_{n\to+\infty} \beta(r_n) = 1 \Rightarrow \lim_{n\to+\infty} r_n = 0$ *and*

$$M'_{v_1,v_2,r} = \max\{d(v_1, v_2, r), d(Hv_1, v_1, r), d(v_2, Hv_2, r)\}.$$

Then, there exists unique $u^* \in V$ so that $Hu^* = u^*$. Moreover, the sequence $\{H^n u\}$ converges to u^*, for every $u \in V$.

Proof. Continuing analogously to the proof of Theorem 2.1 here, we have

$$\begin{aligned} M'_{u_{p_n-1},u_{q_n-1},r_0} &= \max\{d(u_{p_n-1}, u_{q_n-1}, r_0), d(Hu_{p_n-1}, u_{p_n-1}, r_0), d(u_{q_n-1}, Hu_{q_n-1}, r_0)\} \\ &= \max\{d(u_{p_n-1}, u_{q_n-1}, r_0), d(u_{p_n}, u_{p_n-1}, r_0), d(u_{q_n-1}, u_{q_n}, r_0)\}. \end{aligned}$$

This implies

$$\lim_{n\to+\infty} M'_{u_{p_n-1},u_{q_n-1},r_0} = \varepsilon.$$

Again,

$$M'_{u_{q_n},u^*,r_0} = \max\{d(u_{q_n}, u^*, r_0), d(u_{q_n+1}, u_{q_n}, r_0), d(u^*, Hu^*, r_0)\}$$

implies $\lim_{n\to+\infty} M'_{u_{q_n},u^*,r_0} = d(Hu^*, u^*, r_0)$ and

$$\begin{aligned} M'_{u^*,u^{**},r_0} &= \max\{d(u^*, u^{**}, r_0), d(Hu^*, u^*, r_0), d(u^{**}, Hu^{**}, r_0)\} \\ &= \max\{d(u^*, u^{**}, r_0)\} \\ &= d(u^*, u^{**}, r_0). \end{aligned}$$

This eventually proves that u^* is the only fixed point of H. □

Corollary 2.2. *In a complete fuzzy metric-unlike space $(V, d, \star)$, let $H : V \longrightarrow V$, $\alpha : V \times V \longrightarrow [0, +\infty)$, and $\beta : [0, +\infty) \longrightarrow [0, 1]$. If for some $\tau > 0$*

$$\alpha(v_1, v_2)(\tau + F(d(Hv_1, Hv_2, s_1))) \le \beta(M'_{v_1,v_2,s_1})F(M'_{v_1,v_2,s_1}), \tag{3}$$

for all $v_1, v_2 \in V$, and $s_1 > 0$ with $d(Hv_1, Hv_2, s_1) > 0$ and $\alpha(v_1, v_2) \ge 1$, where F is the same as Definition 1.1, $\lim_{n\to+\infty} \beta(r_n) = 1 \Rightarrow \lim_{n\to+\infty} r_n = 0$ *and*

$$M'_{v_1,v_2,r} = \max\{d(v_1, v_2, r), d(Hv_1, v_1, r), d(v_2, Hv_2, r)\}.$$

Then, there exists unique $u^* \in V$ so that $Hu^* = u^*$. Moreover, the sequence $\{H^n u\}$ converges to u^*, for every $u \in V$.

3 Application

In order to complement our study, this section provides an application to the dynamic market equilibrium [27]. Current pricing and price patterns (i.e., whether prices are increasing or decreasing at an increasing or decreasing rate) have an impact on supply and demand in many marketplaces. Then, for the given supply and demand functions:

$$\mathbb{S} = \zeta_1 + \zeta_2 u(t) + \zeta_3 u'(t) + \zeta_4 u''(t)$$

and

$$\mathbb{D} = \eta_1 + \eta_2 u(t) + \eta_3 u'(t) + \eta_4 u''(t),$$

the economist is curious about $u(t)$, $u'(t)$ and $u''(t)$, where $u(t)$ is current price at time t and $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \eta_1, \eta_2, \eta_3, \eta_4$ are the constants. In equilibrium, $\mathbb{S} = \mathbb{D}$, putting $\zeta_1 - \eta_1 = -d$, $\zeta_2 - \eta_2 = c$, $\zeta_3 - \eta_3 = b$, $\zeta_4 - \eta_4 = a$ we have the initial value problem

$$\begin{cases} u''(t) + \dfrac{b}{a}u'(t) + \dfrac{c}{a}u(t) = -\dfrac{d}{a}, \\ u(0) = 0, \\ u'(0) = 0, \end{cases} \tag{4}$$

and the corresponding equivalent integral equation

$$u(t) = \int_0^1 G(t,s) f(t,s,u(t))\mathrm{d}s,$$

where Green's function

$$G(t,s) = \begin{cases} s e^{\frac{c}{2b}}(t-s), & \text{if } 0 \le s \le t \le 1 \\ t e^{\frac{c}{2b}}(s-t), & \text{if } 0 \le t \le s \le 1 \end{cases}.$$

Let us consider the complete non-Archimedean fuzzy metric-unlike space $(V, d, \star)$, where $V = C[0,1]$, the set of all real-valued continuous functions defined on $[0,1]$,

$$d(u_1, u_2, r) = \max_{t\in[0,1]} \frac{\inf\{u_1, u_2\} + r}{\sup\{u_1, u_2\} + r}, \quad \forall r > 0, u_1, u_2 \in C[0,1],$$

and $\star$ is continuous t-norm given by $u_1 \star u_2 = u_1 u_2$.

Consider $H : C[0,1] \longrightarrow C[0,1]$ as

$$Hu(t) = \int_0^1 G(t,s) f(t,s,u(t))\mathrm{d}s.$$

Theorem 3.1. *The initial value problem defined in* (4) *has a unique solution if for* $\tau > 0$, *the following conditions are fulfilled*:

(*a*) $G : [0,1] \times [0,1] \longrightarrow \mathbb{R}^+$ *is continuous and* $\int_0^1 G(t,s)\mathrm{d}s \le 1$.

(*b*) $f : [0,1] \times [0,1] \times C[0,1] \longrightarrow \mathbb{R}^+$ *is continuous.*

(*c*) $\dfrac{\inf\{f(t,s,u_1(t)), f(t,s,u_2(t))\} + r}{\sup\{f(t,s,u_1(t)), f(t,s,u_2(t))\} + r} \le e^{-\tau}\left(\dfrac{\inf\{u_1(t), u_2(t)\} + r}{\sup\{u_1(t), u_2(t)\} + r}\right)^{\frac{1}{1+M'_{u_1,u_2,r}}}$,

where $M'_{u_1,u_2,r}$ *is defined in Corollary* 2.1.

Proof. We have

$$
\begin{aligned}
\frac{\inf\{Hu_1, Hu_2\} + r}{\sup\{Hu_1, Hu_2\} + r} &= \frac{\inf\left\{\int_0^1 G(t,s)f(t,s,u_1(t))\mathrm{d}s, \int_0^1 G(t,s)f(t,s,u_2(t))\mathrm{d}s\right\} + r}{\sup\left\{\int_0^1 G(t,s)f(t,s,u_1(t))\mathrm{d}s, \int_0^1 G(t,s)f(t,s,u_2(t))\mathrm{d}s\right\} + r}\\
&= \frac{\int_0^1 \inf\{G(t,s)f(t,s,u_1(t)), G(t,s)f(t,s,u_2(t))\}\mathrm{d}s + r}{\int_0^1 \sup\{G(t,s)f(t,s,u_1(t)), G(t,s)f(t,s,u_2(t))\}\mathrm{d}s + r}\\
&\le \frac{\inf\{f(t,s,u_1(t)), f(t,s,u_2(t))\}\int_0^1 G(t,s)\mathrm{d}s + r}{\sup\{f(t,s,u_1(t)), f(t,s,u_2(t))\}\int_0^1 G(t,s)\mathrm{d}s + r}\\
&\le \frac{\inf\{f(t,s,u_1(t)), f(t,s,u_2(t))\} + r}{\sup\{f(t,s,u_1(t)), f(t,s,u_2(t))\} + r}\\
&\le e^{-\tau}\left(\frac{\inf\{u_1(t), u_2(t)\} + r}{\sup\{u_1(t), u_2(t)\} + r}\right)^{\frac{1}{1+M'_{u_1,u_2,r}}}
\end{aligned}
$$

$$
\Rightarrow \max_{t\in[0,1]} \frac{\inf\{Hu_1, Hu_2\} + r}{\sup\{Hu_1, Hu_2\} + r} \le e^{-\tau}\left(\max_{t\in[0,1]} \frac{\inf\{u_1(t), u_2(t)\} + r}{\sup\{u_1(t), u_2(t)\} + r}\right)^{\frac{1}{1+M'_{u_1,u_2,r}}}. \tag{5}
$$

Now, if $F(r) = \ln(r)$, $\beta(r) = \frac{1}{1+r}$, and $\alpha : V \times V \longrightarrow \mathbb{R}$ defined by

$$
\alpha(u_1, u_2) = \begin{cases} 1, & \text{if } u_1 \in V,\\ 0, & \text{otherwise}, \end{cases}
$$

then from inequality (5), we have

$$
\begin{aligned}
\alpha(u_1, u_2)(\tau + F(d(Hu_1, Hu_2, r))) &= \tau + F(d(Hu_1, Hu_2, r))\\
&= \ln\left(e^{\tau} \max_{t\in[0,1]} \frac{\inf\{Hu_1, Hu_2\} + r}{\sup\{Hu_1, Hu_2\} + r}\right)\\
&\le \frac{1}{1 + M'_{u_1,u_2,r}} \ln\left(\max_{t\in[0,1]} \frac{\inf\{u_1(t), u_2(t)\} + r}{\sup\{u_1(t), u_2(t)\} + r}\right)\\
&= \beta(M'_{u_1,u_2,r})F(d(u_1, u_2, r))\\
&\le \beta(M'_{u_1,u_2,r})F(M'_{u_1,u_2,r}).
\end{aligned}
$$

Consequently, we have our conclusion from Corollary 2.1. □

Remark 3.1. We can utilize the non-Archimedean fuzzy metric-unlike and non-Archimedean fuzzy metric-unlike spaces in a variety of related domains, such as nonlinear analysis, differential equations, and fractional calculus models, in the future. Our outcomes can be utilized for further expansions.

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