Research Article

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Existence of solutions for nonlinear problems involving mixed fractional derivatives with $p(x)$-Laplacian operator

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Abstract: In this article, a functional boundary value problem involving mixed fractional derivatives with $p(x)$-Laplacian operator is investigated. Based on the fixed point theorems and Mawhin's coincidence theory's extension theory, some existence theorems are obtained in the case of non-resonance and the case of resonance. Some examples are supplied to verify our main results.

Keywords: functional boundary condition, $p(x)$-Laplacian, mixed fractional derivatives, fixed point theorem, coincidence degree theory

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1 Introduction

A typical model of elliptic equations with $p(x)$-growth conditions is

$$-\text{div}((\alpha + |\nabla u|^2)^{(p(x)-2)/2}\nabla u) = f(x, u).$$

Especially, when $\alpha = 0$, the operator $-\Delta_{p(x)}u \coloneqq -\text{div}(|\nabla u|^{(p(x)-2)}\nabla u)$ is called the one-dimensional $p(x)$-Laplacian, which is a natural generalization of the p-Laplacian, if $p(x) \equiv p$(a constant). Because of the non-homogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated nonlinearity than those of p-Laplacian.

For fractional differential equations and variational problems under non-standard $p(x)$-growth conditions, they have been applied in many research fields in recent years [1–5]. There are many results on the related issues raised by the aforementioned discussions, e.g., [6–11].

Shen et al. [9] dealt with the following fractional boundary value problem (BVP) with $p(t)$-Laplacian operator and obtained the uniqueness of its solution by the method in cone

$$\begin{cases} D_{0^+}^{\beta}\varphi_{p(t)}(D_{0^+}^{\alpha}x(t)) + f(t, x(t)) = 0, & t \in [0, 1], \\ x'(0) = x'(1) = x''(0) = 0, D_{0^+}^{\alpha}x(0) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $2 < \alpha < 3$, $0 < \beta < 1$, and $\varphi_{p(t)}(\cdot)$ is the $p(t)$-Laplacian operator with $p(t) \in C^1[0, 1]$ such that $p(t) > 1$. Moreover, f does not need to satisfy the Lipschitz condition.

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Tang et al. [10], using the continuation theorem of coincidence degree theory, studied the following mixed fractional resonant BVP with $p(t)$-Laplacian operator:

$$\begin{cases} {}^{C}D_{0^+}^{\beta}\varphi_{p(t)}(D_{0^+}^{\alpha}u(t)) = f(t, u(t), D_{0^+}^{\alpha}u(t)), \quad t \in [0, T], \\ t^{1-\alpha}u(t)|_{t=0} = 0, D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(T), \end{cases}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, ${}^{C}D_{0^+}^{\beta}$ is the Caputo fractional derivative and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $\varphi_{p(t)}(\cdot)$ is a $p(t)$-Laplacian operator, $p(t) > 1, p(t) \in C^1[0, 1]$ with $p(0) = p(T), f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$.

Recently, it has been found that the mixed operator fractional problem [12–15] can describe many mathematical models. For example, Blaszczyk [12] numerically studied linear oscillatory equations with mixed fractional derivatives. Leszczynski and Blaszczyk [13] studied the following fractional mathematical model:

$${}^{C}D_{T^-}^{\alpha}D_{a^+}^{\alpha}h^*(t) + \beta h^*(t) = 0, \quad t \in [0, T],$$

where ${}^{C}D_{T^-}^{\alpha}$ and $D_{a^+}^{\alpha}$ are, respectively, the right Caputo and left Riemann-Liouville fractional derivatives of order $\alpha \in (0, 1)$, which can be used to describe the height of granular material decreasing over time in a silo. Moreover, a certain amount of research has also been obtained on the BVPs of mixed operators [16–23].

Guezane Lakoud and Kilicman [19] considered the existence of resonant solutions for the following type of equation based on Mawhin's coincidence degree theory:

$$\begin{cases} D_{1^-}^{\theta}D_{0^+}^{\upsilon}x(t) = f(t, x(t)), \quad t \in (0, 1), \\ x(0) = 0, D_{0^+}^{\upsilon}x(1) = D_{0^+}^{\upsilon}x(0), \end{cases}$$

where $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $0 < \theta, \upsilon < 1$ such that $\theta + \upsilon > 1$, while the notations $D_{1^-}^{\theta}$ and $D_{0^+}^{\upsilon}$ refer to the right and left fractional derivatives in the Caputo sense, respectively.

In [23], considering some excellent results [24–29] on fields such as functional problems, $p(x)$-operator, fractional-order operators, we studied the following functional BVP at resonance:

$$\begin{cases} -{}^{C}D_{1^-}^{\alpha}D_{0^+}^{\beta}u(t) + f(t, u(t), D_{0^+}^{\beta}u(t), D_{0^+}^{\beta+1}u(t)) = 0, \quad t \in (0, 1), \\ D_{0^+}^{\beta-1}u(0) = 0, I_{0^+}^{2-\beta}u(0) = 0, \ T_1(u) = 0, \ T_2(u) = 0, \end{cases}$$

where $f \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$, ${}^{C}D_{1^-}^{\alpha}$, and $D_{0^+}^{\beta}$ denote the right Caputo fractional derivative of order $\alpha \in (1, 2]$ and the left Riemann-Liouville fractional derivative of order $\beta \in (1, 2]$ such that $\alpha + \beta > 3$. T_1 and T_2 are the continuous linear functionals.

Based on the aforementioned discussions, and further considering the great extension of $p(x)$-Laplacian operator and functional, as well as the applicability of the mixed operator, we realized that the existence of solutions for the functional BVP (1.1) involving the mixed fractional derivatives with $p(x)$-Laplacian operator has not been studied and contributes to the theoretical and applied nature. So, we investigate the following functional BVP:

$$\begin{cases} {}^{C}D_{1^-}^{\upsilon}\varphi_{p(x)}(D_{0^+}^{\theta}\psi(x)) = f(x, \psi(x), D_{0^+}^{\theta-1}\psi(x), D_{0^+}^{\theta}\psi(x)), \quad x \in (0, 1), \\ D_{0^+}^{\theta}\psi(1) = 0, \ \psi(0) = 0, \ T(\psi) = 0, \end{cases} \tag{1.1}$$

where $0 < \upsilon \leq 1$, $1 < \theta \leq 2$, T is a continuous linear functional, and $\varphi_{p(x)}(\cdot)$ is a $p(x)$-Laplacian operator, $p(x) > 1, p(x) \in C^1[0, 1]$.

Let the classical Banach space $Y = C[0, 1]$ with norm $\|g\|_\infty = \max_{x\in[0,1]}|g(x)|$ and the Banach space $X = \{\psi | \psi, D_{0^+}^{\theta}\psi \in C[0, 1]\}$ with norm $\|\psi\|_X = \max\{\|\psi\|_\infty, \|D_{0^+}^{\theta-1}\psi\|_\infty, \|D_{0^+}^{\theta}\psi\|_\infty\}$ [37].

Definition 1.1. We say $\psi \in X$ is a solution to functional BVP (1.1), which means that ψ satisfies the equation and boundary conditions in (1.1).

The structure of this article is organized as follows. In Section 2 of this work, we introduce some basic definitions and preliminaries later used. Section 3 discusses two types of problems. Subsection 3.1, by means of the Banach fixed point theorem, discusses the non-resonant case and yields the solvability results of the problem. Subsection 3.2 discusses the existence of solution in the resonance sense by Mawhin's coincidence theory's extension theory. Meanwhile, some examples are given separately to illustrate our main results.

2 Preliminaries

We recall the following definitions and auxiliary lemmas related to fractional calculus theory (for details, see [30–33]).

Definition 2.1. Let g be a real function defined on $[0, 1]$ and $\gamma > 0$. Then, the left and right Riemann-Liouville fractional integrals of order $\gamma > 0$ of a function g are defined, respectively, by

$$I_{0^+}^{\gamma} g(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} g(s) \mathrm{d}s,$$

$$I_{1^-}^{\gamma} g(t) = \frac{1}{\Gamma(\gamma)} \int_t^1 (s-t)^{\gamma-1} g(s) \mathrm{d}s.$$

The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order $\gamma > 0$ of a function $g \in AC^n([0, 1], \mathbb{R})$ are defined, respectively, by

$$D_{0_+}^{\gamma} g(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} (I_{0^+}^{n-\gamma} g)(t),$$

$${}^C D_{1^-}^{\gamma} g(t) = (-1)^n I_{1^-}^{n-\gamma} g^{(n)}(t),$$

where $n = [\gamma] + 1$, $[\gamma]$ denotes the integer part of number γ.

Lemma 2.2. *Assume $f \in L[0, 1]$, $p > 0$, and $q > 0$; then, the following relations hold almost everywhere on $[0, 1]$:*

$$I_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{p+q} f(t), \quad \text{and} \quad I_{1^-}^{p} I_{1^-}^{q} f(t) = I_{1^-}^{p+q} f(t).$$

For the properties of Riemann-Liouville and Caputo fractional derivatives, we mention the followings.

Lemma 2.3. *Let $n - 1 < \gamma < n$, $f \in L[0, 1]$, and $I_{0^+}^{n-\alpha} f(t) \in AC^n[a, b]$, then*

(1) $I_{0^+}^{\gamma} D_{0_+}^{\gamma} f(t) = f(t) - \sum_{k=1}^{n} \frac{(I_{0^+}^{n-\gamma} f(t))^{(n-k)}(0)}{\Gamma(\gamma-k+1)} t^{\gamma-k}$;

If $f(t) \in AC^n[a, b]$ or $f(t) \in C^n[a, b]$, then for all $t \in [a, b]$,

(2) $I_{1^-}^{\gamma} {}^C D_{1^-}^{\gamma} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(1)}{k!} (1-t)^k$.

In addition, the following properties are correct:

$$D_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{q-p} f(t), \; q \geq p \geq 0,$$

$$D_{0_+}^{\theta} (t-a)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\theta)} (t-a)^{\gamma-\theta-1},$$

and

$${}^C D_{b\_}^{\theta} (b-t)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\theta)} (b-t)^{\gamma-\theta-1}, \quad \gamma > [\theta] + 1.$$

Lemma 2.4. *If the fractional derivatives $D_{0^+}^{\gamma} g(t)$ and $D_{0^+}^{\gamma+m} g(t)$ exist, then*
$(D_{0^+}^{\gamma} g(t))^{(m)} = D_{0^+}^{\gamma+m} g(t)$, where $\gamma > 0$, m is a positive integer.

Lemma 2.5. [6] *For any* $(x, u) \in [0, 1] \times \mathbb{R}$, $\varphi_{p(x)}(u) = |u|^{p(x)-2}u$, *is a homeomorphism from* $\mathbb{R}$ *to* $\mathbb{R}$ *and strictly monotone increasing for any fixed t. Moreover, its inverse operator* $\varphi^{-1}_{p(x)}(\cdot)$ *is defined by*

$$\begin{cases} \varphi^{-1}_{p(x)}(u) = |u|^{\frac{2-p(x)}{p(x)-1}}u, & u \in \mathbb{R}\backslash\{0\}, \\ \varphi^{-1}_{p(x)}(0) = 0, & u = 0, \end{cases}$$

which is continuous and sends bounded sets to bounded sets.

Next, we need the following definitions and a theorem for the development of our results.

Definition 2.6. (see [34–37]) Let X and Y be two Banach spaces. A continuous operator $M\colon X \cap \mathrm{dom}M \to Y$ is said to be quasi-linear if

(i) $\mathrm{Im}M := M(X \cap \mathrm{dom}M)$ is a closed subset of Y;
(ii) $\mathrm{Ker}M := \{\psi \in X \cap \mathrm{dom}M : M\psi = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,

where $\mathrm{dom}M$ denotes the domain of the operator M.

Let $X_1 = \mathrm{Ker}M$ and X_2 be the complement space of X_1 in X. Then, $X = X_1 \oplus X_2$. Let $P : X \to X_1$ be the projector and $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$.

Definition 2.7. (See [37]) Suppose that $N_\lambda : \overline{\Omega} \to Y, \lambda \in [0, 1]$ is a continuous and bounded operator. Denote N_1 by N. Let $\Sigma_\lambda = \{\psi \in \overline{\Omega} : M\psi = N_\lambda\psi\}$. N_λ is said to be M-quasi-compact in $\overline{\Omega}$ if there exists a vector subspace Y_1 of Y satisfying $\dim Y_1 = \dim X_1$ and two operators Q and R such that for $\lambda \in [0, 1]$,

(a) $\mathrm{Ker}Q = \mathrm{Im}M$;
(b) $QN_\lambda\psi = \theta, \lambda \in (0, 1) \Leftrightarrow QN\psi = \theta$;
(c) $(R\cdot,0)$ is the zero operator and $(R\cdot,\lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$;
(d) $M[P + R(\cdot,\lambda)] = (I - Q)N_\lambda$,

where $Q : Y \to Y_1$, $QY = Y_1$ is continuous, bounded and satisfies $Q(I - Q) = 0$ and $R : \overline{\Omega} \times [0, 1] \to X_2$ is continuous and compact with $P\psi + R(\psi, \lambda) \in \mathrm{dom}M$, $\psi \in \overline{\Omega}, \lambda \in [0, 1]$.

Theorem 2.8. (See [34– 37]) *Let X and Y be two Banach spaces and* $\Omega \subset X$ *be an open and bounded nonempty set. Suppose*

$$M : X \cap \mathrm{dom}M \to Y$$

is a quasi-linear operator and that $N_\lambda : \overline{\Omega} \to Y, \lambda \in [0, 1]$ *is M-quasi-compact. In addition, if the following conditions hold*:

(C′) $M\psi \neq N_\lambda\psi, \forall \psi \in \mathrm{dom}M \cap \partial\Omega, \lambda \in (0, 1)$;
(C″) $\deg(JQN, \Omega \cap \mathrm{Ker}M, 0) \neq 0$,

then the abstract equation $M\psi = N\psi$ *has at least one solution in* $\mathrm{dom}M \cap \overline{\Omega}$, *where* $N = N_1, J : \mathrm{Im}Q \to \mathrm{Ker}M$ *is a homeomorphism with* $J(\theta) = \theta$ *and deg is the Brouwer degree.* □

3 Main results

First, consider the following two conditions:
(B_1): $T(x^{\theta-1}) \neq 0$.
(B_2): $T(x^{\theta-1}) = 0$.

We shall prove that: If B_1 holds, then $\mathrm{Ker}M = \{0\}$. It is so-called non-resonance case. If B_2 holds, then $\mathrm{Ker}M = \{cx^{\theta-1} | c \in \mathbb{R}\}$.

3.1 Non-resonance case

As to this case, we always assume that the following conditions hold:
(H): Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfy the following conditions:
(1) $f(\cdot, u)$ is measurable for each fixed $u \in \mathbb{R}^3$ and $f(x, \cdot)$ is continuous for a.e., $x \in [0,1]$.
(2) $\forall r > 0$, there must exist $g(x) \in C[0,1]$ such that $|f(x,u,v,w)| \le g(x)$, $|u| \le r$, $|v| \le r$, $|w| \le r$.

Then, BVP (1.1) can be transformed into an operator equation.

Lemma 3.1. *If B_1 holds, then BVP* (1.1) *has a unique solution if and only if the following operator $A : X \to X$ has a unique fixed point, where*

$$A(\psi)(x) = \frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_1^{\upsilon} f(y,\psi(y),D_{0^+}^{\theta-1}\psi(y),D_{0^+}^{\theta}\psi(y)))\mathrm{d}y - \frac{T\left(\frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_1^{\upsilon} f(y,\psi(y),D_{0^+}^{\theta-1}\psi(y),D_{0^+}^{\theta}\psi(y)))\mathrm{d}y\right)}{T(x^{\theta-1})}x^{\theta-1}. \tag{3.1}$$

Proof. If ψ is a solution to $A\psi = \psi$, we obtain

$$^{C}D_{1^-}^{\upsilon}\varphi_{p(x)}(D_{0^+}^{\theta}\psi(x)) = f(x,\psi(x),D_{0^+}^{\theta-1}\psi(x),D_{0^+}^{\theta}\psi(x)).$$

Considering $\psi \in X$, i.e., $\psi \in C[0,1], D_{0^+}^{\theta}\psi \in C[0,1]$, we have $\psi(0) = 0$ and $D_{0^+}^{\theta}\psi(1) = \varphi_{p(x)}^{-1}(I_1^{\upsilon} f(x,\psi(x), D_{0^+}^{\theta-1}\psi(x),D_{0^+}^{\theta}\psi(x)))|_{x=1} = 0$.

Based on the linearly of T and (3.1), we have

$$T(\psi) = T\left(\frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_1^{\upsilon} f(y,\psi(y),D_{0^+}^{\theta-1}\psi(y),D_{0^+}^{\theta}\psi(y)))\mathrm{d}y\right) - \frac{T\left(\frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_1^{\upsilon} f(y,\psi(y),D_{0^+}^{\theta-1}\psi(y),D_{0^+}^{\theta}\psi(y)))\mathrm{d}y\right)}{T(x^{\theta-1})}T(x^{\theta-1}) = 0.$$

So, we have ψ, which is a solution to BVP (1.1). If ψ is a solution to BVP (1.1), then

$$\begin{aligned} A(\psi)(x) &= \frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_1^{\upsilon} f(y,\psi(y),D_{0^+}^{\theta-1}\psi(y),D_{0^+}^{\theta}\psi(y)))\mathrm{d}y \\ &\quad - \frac{T\left(\frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_1^{\upsilon} f(y,\psi(y),D_{0^+}^{\theta-1}\psi(y),D_{0^+}^{\theta}\psi(y)))\mathrm{d}y\right)}{T(x^{\theta-1})}x^{\theta-1} \\ &= \frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}\,{}^{C}D_{1^-}^{\upsilon}\varphi_{p(y)}(D_{0^+}^{\theta}\psi(y)))\mathrm{d}y \\ &\quad - \frac{T\left(\frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}\,{}^{C}D_{1^-}^{\upsilon}\varphi_{p(y)}(D_{0^+}^{\theta}\psi(y)))\mathrm{d}y\right)}{T(x^{\theta-1})}x^{\theta-1} \\ &= \frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}D_{0^+}^{\theta}\psi(y)\mathrm{d}y - \frac{T\left(\frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}D_{0^+}^{\theta}\psi(y)\mathrm{d}y\right)}{T(x^{\theta-1})}x^{\theta-1} \\ &= \psi(x) - \frac{D_{0^+}^{\theta-1}\psi(0)}{\Gamma(\theta)}x^{\theta-1} - \frac{T\left(\psi(x) - \frac{D_{0^+}^{\theta-1}\psi(0)}{\Gamma(\theta)}x^{\theta-1}\right)}{T(x^{\theta-1})}x^{\theta-1} \\ &= \psi(x). \end{aligned}$$

From the aforementioned two arguments, we obtain that BVP (1.1) has a unique solution in X if and only if the operator equation $A\psi = \psi$ has a unique solution in X. □

By making use of Lemma 3.1, we can obtain the following existence theorem for BVP (1.1) at non-resonance.

Theorem 3.2. *Assume B_1, (H), and the following conditions hold:*
(C_1): *For each fixed $u_1, u_2 \in \mathbb{R}$, there exists a constant κ_1 such that*

$$|\varphi_{p(x)}^{-1}(u_1) - \varphi_{p(x)}^{-1}(u_2)| \leq \kappa_1 |u_1 - u_2|,$$

$$\kappa_1 < \min\left\{\frac{T(x^{\theta-1})\Gamma(\theta)\Gamma(\upsilon+1)(\theta+\upsilon)}{T(1)+T(x^{\theta-1})}, \frac{T(x^{\theta-1})\Gamma(\theta)\Gamma(\upsilon+2)(\theta+\upsilon)}{(\upsilon+1)T(1)+(\theta+\upsilon)\Gamma(\theta)T(x^{\theta-1})}, \Gamma(\upsilon+1)\right\}.$$

(C_2): *For almost every $x \in [0,1]$, then, $\forall\ (u_1, v_1, w_1), (u_2, v_2, w_2) \in \mathbb{R}^3$,*

$$|f(x,u_1,v_1,w_1) - f(x,u_2,v_2,w_2)| \leq \max\{|u_1-u_2|, |v_1-v_2|, |w_1-w_2|\}.$$

(C_3): *If each $\psi_1, \psi_2 \in X$ satisfies $|\psi_1(x)| \leq |\psi_2(x)|$, $\forall x \in [0,1]$, then $|T(\psi_1)| \leq |T(\psi_2)|$.* □

Then, BVP (1.1) has a unique solution in X.

Proof. We shall prove that $A\psi = \psi$ has a unique solution in X.

For each $\psi_1, \psi_2 \in X$, by making use of (C_1)–(C_3) and the linearly of T, we have

$$\begin{aligned}|(A\psi_1)(x) - (A\psi_2)(x)| &= \Bigg| I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_1(x), D_{0^+}^{\theta-1}\psi_1(x), D_{0^+}^{\theta}\psi_1(x))) \\ &\quad - I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_2(x), D_{0^+}^{\theta-1}\psi_2(x), D_{0^+}^{\theta}\psi_2(x))) \\ &\quad + \frac{T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_2(x), D_{0^+}^{\theta-1}\psi_2(x), D_{0^+}^{\theta}\psi_2(x))))}{T(x^{\theta-1})}x^{\theta-1} \\ &\quad - \frac{T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_1(x), D_{0^+}^{\theta-1}\psi_1(x), D_{0^+}^{\theta}\psi_1(x))))}{T(x^{\theta-1})}x^{\theta-1}\Bigg| \\ &\leq \kappa_1\|\psi_1-\psi_2\|_X |I_{0^+}^{\theta}I_{1^-}^{\upsilon}1| + \left|\frac{T(\kappa_1\|\psi_1-\psi_2\|_X I_{0^+}^{\theta}I_{1^-}^{\upsilon}1)}{T(x^{\theta-1})}x^{\theta-1}\right| \\ &\leq \frac{\kappa_1\|\psi_1-\psi_2\|_X |I_{0^+}^{\theta}(1-x)^{\upsilon}|}{\Gamma(\upsilon+1)} + \frac{\kappa_1\|\psi_1-\psi_2\|_X |T(I_{0^+}^{\theta}(1-x)^{\upsilon})|}{|T(x^{\theta-1})|\Gamma(\upsilon+1)}x^{\theta-1} \\ &\leq \frac{\kappa_1\|\psi_1-\psi_2\|_X}{\Gamma(\theta)\Gamma(\upsilon+1)(\theta+\upsilon)} + \frac{|T(1)|}{|T(x^{\theta-1})|}\frac{\kappa_1\|\psi_1-\psi_2\|_X}{\Gamma(\theta)\Gamma(\upsilon+1)(\theta+\upsilon)},\end{aligned}$$

$$\begin{aligned}&|(D_{0^+}^{\theta-1}A\psi_1)(x) - (D_{0^+}^{\theta-1}A\psi_2)(x)| \\ &= \Bigg| I_{0^+}^{1}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_1(x), D_{0^+}^{\theta-1}\psi_1(x), D_{0^+}^{\theta}\psi_1(x))) - I_{0^+}^{1}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_2(x) D_{0^+}^{\theta-1}\psi_2(x), D_{0^+}^{\theta}\psi_2(x))), \\ &\quad + \frac{\Gamma(\theta)T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_2(x), D_{0^+}^{\theta-1}\psi_2(x), D_{0^+}^{\theta}\psi_2(x))))}{T(x^{\theta-1})} \\ &\quad - \frac{\Gamma(\theta)T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1}^{\upsilon}f(x,\psi_1(x), D_{0^+}^{\theta-1}\psi_1(x), D_{0^+}^{\theta}\psi_1(x))))}{T(x^{\theta-1})}\Bigg| \\ &\leq \kappa_1\|\psi_1-\psi_2\|_X |I_{0^+}^{1}I_{1^-}^{\upsilon}1| + \left|\frac{\Gamma(\theta)|T(\kappa_1\|\psi_1-\psi_2\|_X I_{0^+}^{\theta}I_{1^-}^{\upsilon}1)|}{T(x^{\theta-1})}\right| \\ &\leq \frac{\kappa_1\|\psi_1-\psi_2\|_X |I_{0^+}^{1}(1-x)^{\upsilon}|}{\Gamma(\upsilon+1)} + \frac{\kappa_1\|\psi_1-\psi_2\|_X |T(I_{0^+}^{\theta}(1-x)^{\upsilon})|}{|T(x^{\theta-1})|\Gamma(\upsilon+1)} \\ &\leq \frac{\kappa_1\|\psi_1-\psi_2\|_X}{\Gamma(\upsilon+2)} + \frac{|T(1)|}{|T(x^{\theta-1})|}\frac{\kappa_1\|\psi_1-\psi_2\|_X}{\Gamma(\theta)\Gamma(\upsilon+1)(\upsilon+\theta)},\end{aligned}$$

and

$$\begin{aligned}
&|(D_{0^+}^{\theta}A\psi_1)(x) - (D_{0^+}^{\theta}A\psi_2)(x)| \\
&= |\varphi_{p(x)}^{-1}(I_1^{\upsilon}f(x,\psi_1(x),D_{0^+}^{\theta-1}\psi_1(x),D_{0^+}^{\theta}\psi_1(x))) - \varphi_{p(x)}^{-1}(I_1^{\upsilon}f(x,\psi_2(x),D_{0^+}^{\theta-1}\psi_2(x),D_{0^+}^{\theta}\psi_2(x)))\\
&\le \kappa_1|(I_1^{\upsilon}f(x,\psi_1(x),D_{0^+}^{\theta-1}\psi_1(x),D_{0^+}^{\theta}\psi_1(x))) - I_1^{\upsilon}f(x,\psi_2(x),D_{0^+}^{\theta-1}\psi_2(x),D_{0^+}^{\theta}\psi_2(x))|\\
&\le \frac{\kappa_1\|\psi_1-\psi_2\|_X}{\Gamma(\upsilon+1)}.
\end{aligned}$$

Considering (C_1), the aforementioned inequality implies that T is a contaction. Using *Banach's contaction principle*, $A\psi = \psi$ has a unique solution in X. From Lemma 3.1, BVP (1.1) has a unique solution in X. □

In the following, we give a more specific result.

Theorem 3.3. *Assume B_1, (H), (C_3), and the following conditions hold*:
(C_4): *If each fixed $u_1, u_2 > 1 \in \mathbb{R}$, there exists a constant κ_2 such that*

$$|\varphi_{p(x)}^{-1}(u_1) - \varphi_{p(x)}^{-1}(u_2)| \le \kappa_2|u_1-u_2|,$$

$$\kappa_2 < \min\left\{\frac{T(x^{\theta-1})\Gamma(\theta)\Gamma(\upsilon+1)(\theta+\upsilon)}{T(1)+T(x^{\theta-1})},\ \frac{T(x^{\theta-1})\Gamma(\theta)\Gamma(\upsilon+2)(\theta+\upsilon)}{(\upsilon+1)T(1)+(\theta+\upsilon)\Gamma(\theta)T(x^{\theta-1})},\Gamma(\upsilon+1)\right\}.$$

(C_5): *For almost every $x \in [0,1]$, $\forall(u_1,v_1,w_1),(u_2,v_2,w_2) \in \mathbb{R}^3$,*

$$|f(x,u_1,v_1,w_1) - f(x,u_2,v_2,w_2)| \le \max\{|u_1-u_2|,|v_1-v_2|,|w_1-w_2|\}.$$

In addition, $|f(x,u_i,v_i,w_i)| > \frac{\Gamma(\upsilon+1)}{(1-x)^{\upsilon}}$, for a.e. $x \in [0,1]$.

Then, BVP (1.1) has a unique solution in X.

Proof. Since the conditions of the theorem are further conditions of Theorem 3.2, the proof process is similar. We will not go into too much detail here. □

Example 3.4. Considering the following FBVP:

$$\begin{cases}
{}^{C}D_{1^-}^{\frac12}\varphi_{p(x)}(D_{0^+}^{\frac32}\psi(x)) = f(x,\psi(x),D_{0^+}^{\frac12}\psi(x),D_{0^+}^{\frac32}\psi(x)), \quad x \in (0,1),\\
D_{0^+}^{\frac32}\psi(1) = 0,\ \ \psi(0) = 0,\ \ T(\psi) = \int_0^1 \psi(y)\mathrm{d}y = 0,
\end{cases}$$

where $f(x,\psi(x),D_{0^+}^{\frac12}\psi(x),D_{0^+}^{\frac32}\psi(x)) = \frac13\mathrm{sgn}\{\psi(x)\}\psi(x) + \frac13\mathrm{sgn}\{D_{0^+}^{\frac12}\psi(x)\}D_{0^+}^{\frac12}\psi(x) + \frac13|D_{0^+}^{\frac32}\psi(x)| + \frac{\sqrt{\pi}}{2}(1-x)^{-\frac12}$, $p(x) = x^2+2$.

It is easy to see that $T(x^{\frac12}) = \frac23 \neq 0$. The problem is at non-resonance.

The assumption (C_3) is fulfilled with $T(\psi) = \int_0^1\psi(y)\mathrm{d}y$.

Since $\varphi_{p(x)}^{-1}(u) = 1$, $u > 1$,

$$|\varphi_{p(x)}^{-1}(u_1) - \varphi_{p(x)}^{-1}(u_2)| = 0 < \kappa_2|u_1-u_2| = \frac{3\pi}{4\sqrt{\pi}+9}|u_1-u_2|,$$

the assumption (C_4) holds. Moreover, with a little effort, we can compute,

$$\begin{aligned}&|f(x,\psi_1(x),D_{0^+}^{\frac{1}{2}}\psi_1(x),D_{0^+}^{\frac{3}{2}}\psi_1(x))-f(x,\psi_2(x),D_{0^+}^{\frac{1}{2}}\psi_2(x),D_{0^+}^{\frac{3}{2}}\psi_2(x))|\\&=|\frac{1}{3}(|\psi_1(x)|-|\psi_2(x)|)+\frac{1}{3}(|D_{0^+}^{\frac{1}{2}}\psi_1(x)|-|D_{0^+}^{\frac{1}{2}}\psi_2(x)|)+\frac{1}{3}(|D_{0^+}^{\frac{3}{2}}\psi_1(x)|-|D_{0^+}^{\frac{3}{2}}\psi_2(x)|)\\&\leq\max\left\{|\psi_1(x)|-|\psi_2(x)|,|D_{0^+}^{\frac{1}{2}}\psi_1(x)|-|D_{0^+}^{\frac{1}{2}}\psi_2(x)|,|D_{0^+}^{\frac{3}{2}}\psi_1(x)|-|D_{0^+}^{\frac{3}{2}}\psi_2(x)|\right\}.\end{aligned}$$

It is obvious that $|f(x,\psi(x),D_{0^+}^{\frac{1}{2}}\psi(x),D_{0^+}^{\frac{3}{2}}\psi(x))|>\frac{\sqrt{\pi}}{2}(1-x)^{-\frac{1}{2}}$. At this point, all the conditions of Theorem 3.3 have been verified, which means that the non-resonance problem has a unique solution.

3.2 Resonance case

In this part, noting that if B_2 holds, BVP (1.1) turns into resonance case.

We will always suppose that $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is continuous. $P_m=\min_{x\in[0,1]}p(x)$, $P_M=\max_{x\in[0,1]}p(x)$.

Define operators $M:X\cap\mathrm{dom}M\to Y$, $N_\lambda:X\to Y$ and $F:Y\to\mathbb{R}$ by

$$M\psi={}^CD_{1^-}^{\upsilon}\varphi_{p(x)}(D_{0^+}^{\theta}\psi),$$

$$N_\lambda\psi(x)=\lambda f(x,\psi(x),D_{0^+}^{\theta-1}\psi(x),D_{0^+}^{\theta}\psi(x)),\ x,\lambda\in[0,1],$$

and

$$F(g)=T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)),\tag{3.2}$$

where $\mathrm{dom}M=\{{}^CD_{1^-}^{\upsilon}\varphi_{p(x)}(D_{0^+}^{\theta}\psi)\in Y,D_{0^+}^{\theta}\psi(1)=0,\psi(0)=0,T(\psi)=0\}$.

Then, BVP (1.1) is equivalent to the operator equation $M\psi=N\psi$, $\psi\in\mathrm{dom}M$.

In order to make it easier to read, we will introduce the following assumptions:

(H_1): The functional $T:X\to\mathbb{R}$ is linear continuous with norm $|T(\psi)|\leq\|T\|\|\psi\|_X$.

(H_2): For $g_1,g_2\in Y$, there exists a constant c such that $F(g-c)=0$, if $g_1(x)\leq g_2(x)$, $x\in[0,1]$, $g_1\not\equiv g_2$, then either $F(g_1)<F(g_2)$, or $F(g_1)>F(g_2)$ (i.e., F is strictly monotonous).

Lemma 3.5. (See [31]) *Define two functions $A:C[0,1]\to\mathbb{R}$, $A(g)=c$, where g and c satisfy $F(g-c)=0$. For $g\in C[0,1]$, there is only one constant $c\in\mathbb{R}$ corresponding to it such that $A(g)=c$ with $|c|\leq\|g\|_\infty$ and $A:C[0,1]\to\mathbb{R}$ is continuous and bounded.*

Remark. (See [31]) $A(c)=c$, $A(g+c)=A(g)+c$, $A(cg)=cA(g)$, $c\in\mathbb{R}$, $g\in C[0,1]$.

Lemma 3.6. *If B_2 holds, the operator M is quasi-linear. Moreover,*

$$\mathrm{Ker}M=\{cx^{\theta-1}:c\in\mathbb{R}\},\tag{3.3}$$

and

$$\mathrm{Im}M=\{g|g\in Y,A(g)=0\}.\tag{3.4}$$

Proof. For $\psi\in\mathrm{dom}M$, if $M\psi=g$ and $D_{0^+}^{\theta}\psi(1)=0$, then we have

$$\psi(x)=\frac{1}{\Gamma(\theta)}\int_0^x(x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}g(y))\mathrm{d}y+c_1x^{\theta-1}+c_2x^{\theta-2},\tag{3.5}$$

where c_1 and c_2 are two arbitrary constants. By substituting the boundary conditions $\psi(0)=0$, $T(\psi)=0$ and B_2 into (3.5), one has

$$T(\psi(x)) = T\left(\frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}g(y))\mathrm{d}y\right) = 0,$$

i.e.,

$$\mathrm{Im}M \subseteq \{g \in Y : T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)) = 0\}. \tag{3.6}$$

Conversely, if $g \in Y$ satisfies $\{g \in Y : T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)) = 0\}$, then take

$$\psi(x) = \frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}g(y))\mathrm{d}y.$$

Then, we conclude that $M\psi = g(x)$, $D_{0^+}^{\theta}\psi(1) = \varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)|_{x=1} = 0$, $\psi(0) = 0$, and $T(\psi) = T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)) = 0$, i.e., $\psi \in \mathrm{dom}M$, and hence, $g \in \mathrm{Im}M$.

In conclusion,

$$\mathrm{Im}M = \{g \in Y : T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)) = 0, \quad \text{i.e.}, A(g) = 0\}. \tag{3.7}$$

Also, setting g = 0, recalling equation (3.5) and the boundary conditions, we clearly obtain that

$$\mathrm{Ker}M = \{cx^{\theta-1} : c \in \mathbb{R}\} \coloneqq X_1.$$

Obviously, $\mathrm{Ker}M$ is a linearly homeomorphic to $\mathbb{R}$.

Let $g_n \in \mathrm{Im}M \subset Y$ and $g_n \to g \in Y$. By the continuity of $\varphi_{p(x)}^{-1}(\cdot)$, we have

$$|\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g_n) - \varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)| \to 0, \quad \text{as } n \to \infty.$$

Furthermore, $\|I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g_n) - I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)\|_{\infty} \to 0$ and $\|I_{0^+}^{1}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g_n) - I_{0^+}^{1}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)\|_{\infty} \to 0$, as $n \to \infty$. Therefore, $|T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g_n)) - T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g))| \le \|T\|\|I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g_n) - I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}g)\|_X \to 0$, as $n \to \infty$. This, together with $g_n \in \mathrm{Im}M$, shows that $g \in \mathrm{Im}M$. Hence, $\mathrm{Im}M$ is a closed subset of Y. Thus, M is *quasi-linear*.

Define $Q : Y \to Y_1 \coloneqq \mathbb{R}$ as follows: $Qg = c$, where c satisfies $c = A(g)$.

By Lemma 3.5, $Q : Y \to Y_1$ is continuous and bounded with $|Qg| \le \|g\|_{\infty}$ and $\dim Y_1 = \dim X_1$. Obviously, $Q(I - Q) = 0$, $QY = Y_1$, and $\mathrm{Ker}Q = \mathrm{Im}M$.

Take $P : X \to X$ as follows:

$$P\psi(x) = \frac{D_{0^+}^{\theta-1}\psi(0)}{\Gamma(\theta)}x^{\theta-1}.$$

For $\psi \in X$, set $\psi = \psi - P\psi + P\psi$. It is easy to check that $P^2\psi = P\psi$, $\psi \in X$, and it is also elementary to confirm the identity $\mathrm{Im}P = \mathrm{Ker}M$ and $\mathrm{Im}P \cap \mathrm{Ker}P = \{0\}$. So, $X = \mathrm{Ker}M \oplus \mathrm{Ker}P$. □

Lemma 3.7. *Define an operator* $R : X \times [0,1] \to X_2$ *as*

$$R(\psi,\lambda)(x) = \frac{1}{\Gamma(\theta)}\int_0^x (x-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_{\lambda}\psi(y))\mathrm{d}y,$$

where $\mathrm{Ker}M \oplus X_2 = X$. *Then,* $R : \overline{\Omega} \times [0,1] \to X_2$ *is continuous and compact with* $Pu + R(u,\lambda) \in \mathrm{dom}M$, $u \in \overline{\Omega}$, $\lambda \in [0,1]$, *where* $\Omega \subset X$ *is an open bounded set.*

Proof. Since $R(\psi,\lambda)(x) = I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_{\lambda}\psi(x))$, we easily deduce that $D_{0^+}^{\theta-1}R(\psi,\lambda)(x) = I_{0^+}^{1}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_{\lambda}\psi(x))|_{x=0} = 0$, so, $PR(\psi,\lambda) = 0$, i.e., $R(\psi,\lambda) \in X_2$.

Considering the continuity of Q and f, we can simply indicate that $R(\psi,\lambda)$ is continuous.

Clearly, for $\psi \in X$, $\lambda \in [0,1]$, we show the fact that

$$
\begin{aligned}
{}^{C}D_{1^-}^{\upsilon}\varphi_{p(x)}(D_{0^+}^{\theta}(P\psi + R(\psi,\lambda))) &= (I-Q)N_\lambda\psi \in C[0,1],\\
D_{0^+}^{\theta}(P\psi + R(\psi,\lambda))(1) &= \varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x))|_{x=1} = 0,\ (P\psi + R(\psi,\lambda))(0) = 0,\\
T(P\psi + R(\psi,\lambda)) &= T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)) + \frac{D_{0^+}^{\theta-1}\psi(0)T(x^{\theta-1})}{\Gamma(\theta)}\\
&= T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)).
\end{aligned}
$$

Note that $(I-Q)N_\lambda\psi \in \operatorname{Ker}Q = \operatorname{Im}M$, we obtain

$$T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)) = 0.$$

Therefore, $P\psi + R(\psi,\lambda) \in \operatorname{dom}M$. Next, we will prove that R is compact.

By $|A\psi| \le \|\psi\|_\infty$ and the continuity of f and $\varphi_{p(x)}^{-1}(\cdot)$, we obtain that there exists a constant $M > 0$ such that $\|\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)\|_\infty \le M$ in $\overline{\Omega}$ for all $\lambda \in [0,1]$, which implies that

$$
\begin{aligned}
\|R(\psi,\lambda)\|_X &= \|I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)\|_X\\
&= \max\{\|I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)\|_\infty, \|I_{0^+}^{1}\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)\|_\infty, \|\varphi_{p(x)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi)\|_\infty\}\\
&\le \max\left\{\frac{M}{\Gamma(\theta+1)}, M\right\} = M,
\end{aligned}
$$

i.e., $R(u,\lambda)$ is uniformly bounded in $\overline{\Omega}$. For $(\psi,\lambda) \in \overline{\Omega} \times [0,1]$, $0 \le x_1 < x_2 \le 1$, we have

$$
\begin{aligned}
&|R(\psi,\lambda)(x_2) - R(\psi,\lambda)(x_1)|\\
&= \left|\frac{1}{\Gamma(\theta)}\int_0^{x_2}(x_2-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(y))dy - \frac{1}{\Gamma(\theta)}\int_0^{x_1}(x_1-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(y))dy\right|\\
&= \left|\frac{1}{\Gamma(\theta)}\int_0^{x_1}((x_2-y)^{\theta-1} - (x_1-y)^{\theta-1})\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(y))dy + \frac{1}{\Gamma(\theta)}\int_{x_1}^{x_2}(x_2-y)^{\theta-1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(y))dy\right|\\
&\le \left|\frac{M}{\Gamma(\theta)}\int_0^{x_1}((x_2-y)^{\theta-1} - (x_1-y)^{\theta-1})dy\right| + \left|\frac{M}{\Gamma(\theta)}\int_{x_1}^{x_2}(x_2-y)^{\theta-1}dy\right|\\
&= \frac{M}{\Gamma(\theta+1)}(|x_2^\theta - x_1^\theta - (x_2-x_1)^\theta| + |(x_2-x_1)^\theta|) \to 0,\ x_1 \to x_2,
\end{aligned}
$$

$$
\begin{aligned}
&|D_{0^+}^{\theta-1}R(\psi,\lambda)(x_2) - D_{0^+}^{\theta-1}R(\psi,\lambda)(x_1)|\\
&= \left|\int_0^{x_2}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(y))dy \int_0^{x_1}\varphi_{p(y)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(y))dy\right|\\
&\le M(x_2-x_1) \to 0,\ x_1 \to x_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_{0^+}^{\theta}R(\psi,\lambda)(x_2) - D_{0^+}^{\theta}R(\psi,\lambda)(x_1)|\\
&= |\varphi_{p(x_2)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2)) - \varphi_{p(x_1)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_1))|\\
&= |\varphi_{p(x_2)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2)) - \varphi_{p(x_1)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2)) + \varphi_{p(x_1)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2)) - \varphi_{p(x_1)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_1))|\\
&\le |I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2)|\left||(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2))^{\frac{2-p(x_2)}{p(x_2)-1}}| - |(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2))^{\frac{2-p(x_1)}{p(x_1)-1}}|\right|\\
&\quad + |\varphi_{p(x_1)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_2)) - \varphi_{p(x_1)}^{-1}(I_{1^-}^{\upsilon}(I-Q)N_\lambda\psi(x_1))| \to 0,\ x_1 \to x_2,
\end{aligned}
$$

since $p(x) \in C^1[0,1]$ and the continuity of $\varphi^{-1}_{p(x)}(\cdot)$. So, we obtain that $\{R(\psi,\lambda) : \psi \in \overline{\Omega}, \lambda \in [0,1]\}$, $\{D^{\theta-1}_{0^+}R(\psi,\lambda) : \psi \in \overline{\Omega}, \lambda \in [0,1]\}$, and $\{D^{\theta}_{0^+}R(\psi,\lambda) : \psi \in \overline{\Omega}, \lambda \in [0,1]\}$ are equicontinuous. The compactness of the operator R follows from the Arzela-Ascoli theorem. □

Now, we will show that N_λ is M-quasi-compact in $\overline{\Omega}$, where $\Omega \subset X$ is an open and bounded set with $\theta \in \Omega$. Obviously, N_λ is continuous, bounded and $\dim X_1 = \dim Y_1 = 1$.

Lemma 3.8. *Assume that $\Omega \subset X$ is an open and bounded set. Then, N_λ is M-quasi-compact in $\overline{\Omega}$.*

Proof. It is clear that $\mathrm{Im}P = \mathrm{Ker}M$, $\dim\mathrm{Ker}M = \dim\mathrm{Im}Q = 1$, $Q(I-Q) = 0$, $\mathrm{Ker}Q = \mathrm{Im}M$, $M(Pu + R(\psi,\lambda)) = (I-Q)N_\lambda u$, and $R(\cdot,0) = 0$ is the zero operator.

$QN_\lambda\psi = 0$, $\lambda \in (0,1) \Leftrightarrow A(N_\lambda\psi) = 0 \Leftrightarrow \lambda A(N\psi) = 0 \Leftrightarrow Q(N\psi) = 0$.

Next, we need only to prove that $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}$.

For $\psi \in \Sigma_\lambda = \{\psi \in \overline{\Omega}, M\psi = N_\lambda\psi\}$, we have $QN_\lambda\psi = 0$ and $N_\lambda\psi = {}^C D^{\upsilon}_{1^-}(\varphi_{p(x)}(D^{\theta}_{0^+}\psi))$. It follows from $D^{\theta}_{0^+}\psi(1) = 0$ and $\psi(0) = 0$ that

$$\begin{aligned} R(\psi,\lambda)(x) &= I^{\theta}_{0^+}\varphi^{-1}_{p(x)}(I^{\upsilon}_{1^-}(I-Q)N_\lambda\psi) = I^{\theta}_{0^+}\varphi^{-1}_{p(x)}(I^{\upsilon}_{1^-}N_\lambda\psi) \\ &= I^{\theta}_{0^+}\varphi^{-1}_{p(x)}(I^{\upsilon}_{1^-}{}^C D^{\upsilon}_{1^-}\varphi_{p(x)}(D^{\theta}_{0^+}\psi)) \\ &= I^{\theta}_{0^+}\varphi^{-1}_{p(x)}(\varphi_{p(x)}D^{\theta}_{0^+}\psi) = \psi(x) - \frac{D^{\theta-1}_{0^+}\psi(0)}{\Gamma(\theta)}x^{\theta-1} = \psi(x) - P\psi(x). \end{aligned}$$

So, Definition 2.7(a)–(d) hold. Therefore, N_λ is M-quasi-compact in $\overline{\Omega}$. □

Lemma 3.9. *Assume that $a \neq 0$ and the following conditions hold*:

(H_3): *There exists a constant $M_1 > 0$ such that if $|D^{\theta-1}_{0^+}\psi(x)| > M_1$, for all $x \in [0,1]$, then*

$$A(N\psi) \neq 0.$$

(H_4): *There exist constants nonnegative a, b, c, and $d \geq 0$ such that*

$$|f(t,x,y,z)| \leq a + b\,|x|^{r-1} + c\,|y|^{r-1} + d\,|z|^{r-1}, r \in (1, P_m], x, y, z \in \mathbb{R},$$

with

$$2^{\frac{1}{p_m-1}}\max\{A_2^{\frac{1}{p_m-1}}, A_2^{\frac{1}{p_M-1}}\} < 1,$$

where $A_2 := \frac{1}{\Gamma(\upsilon+1)}[b(\frac{2(\theta+2)}{\Gamma(\theta+1)})^{r-1} + c2^{r-1} + d]$.

Then, the set $\Omega_1 = \{\psi \in \mathrm{dom}M : M\psi = N_\lambda\psi, \lambda \in (0,1)\}$ *is bounded.*

Proof. Since $\psi \in \Omega_1$, we have $N\psi \in \mathrm{Im}M = \mathrm{Ker}Q$. Thus, $Q(N\psi) = 0$, i.e., $A(N\psi) = 0$.

By (H_3), there exists $x_1 \in [0,1]$ such that $|D^{\theta-1}_{0^+}\psi(x_1)| \leq M_1$.

Considering $D^{\theta-1}_{0^+}\psi(x) = D^{\theta-1}_{0^+}\psi(x_1) + \int_{x_1}^{x} D^{\theta}_{0^+}\psi(y)\mathrm{d}y$, then

$$|D^{\theta-1}_{0^+}\psi(x)| \leq M_1 + \|D^{\theta}_{0^+}\psi\|_\infty, x \in [0,1]. \tag{3.8}$$

By $\psi(0) = 0$, we have $\psi(x) = I^{\theta}_{0^+}D^{\theta}_{0^+}\psi(x) + C_1x^{\theta-1}$ and $D^{\theta-1}_{0^+}\psi(x) = \int_0^x D^{\theta}_{0^+}\psi(y)\mathrm{d}y + C_1\Gamma(\theta)$.

Hence, we can obtain $|C_1| \leq \frac{1}{\Gamma(\theta)}(\|D^{\theta-1}_{0^+}\psi\|_\infty + \|D^{\theta}_{0^+}\psi\|_\infty)$ and $|\psi(x)| \leq \frac{(\theta+1)}{\Gamma(\theta+1)}\|D^{\theta}_{0^+}\psi\|_\infty + \frac{1}{\Gamma(\theta)}\|D^{\theta-1}_{0^+}\psi\|_\infty$. Based on $M\psi = N_\lambda\psi$ and $D^{\theta}_{0^+}\psi(1) = 0$, we obtain that

$$\varphi_{p(x)}(D^{\theta}_{0^+}\psi(x)) = \lambda I^{\upsilon}_{1^-}N\psi.$$

From (H_4) and $\lambda \in (0,1)$, we have

$$\begin{aligned}|D_{0^+}^{\theta}\psi(x)|^{p(x)-1} &\le \lambda\frac{1}{\Gamma(\upsilon)}\int_x^1 (y-x)^{\upsilon-1}|N\psi(y)|dy\\ &\le \lambda\frac{1}{\Gamma(\upsilon+1)}\|N\psi\|_\infty\\ &\le \frac{1}{\Gamma(\upsilon+1)}[a+b\|\psi\|_\infty^{r-1}+c\|D_{0^+}^{\theta-1}\psi\|_\infty^{r-1}+d\|D_{0^+}^{\theta}\psi\|_\infty^{r-1}]\\ &\le \frac{1}{\Gamma(\upsilon+1)}\left[a+b\left(\frac{\theta+2}{\Gamma(\theta+1)}\|D_{0^+}^{\theta}\psi\|_\infty+\frac{M_1}{\Gamma(\theta)}\right)^{r-1}+c(M_1+\|D_{0^+}^{\theta}\psi\|_\infty)^{r-1}+d\|D_{0^+}^{\theta}\psi\|_\infty^{r-1}\right].\end{aligned}$$

Furthermore, by the basic inequality $(x+y)^p \le 2^p(x^p+y^p)$, $x, y, p > 0$, we have

$$\begin{aligned}|D_{0^+}^{\theta}\psi(x)|^{p(x)-1} &\le \frac{1}{\Gamma(\upsilon+1)}\left[a+b2^{r-1}\left(\frac{\theta+2}{\Gamma(\theta+1)}\right)^{r-1}\|D_{0^+}^{\theta}\psi\|_\infty^{r-1}+\left(\frac{M_1}{\Gamma(\theta)}\right)^{r-1}\right.\\ &\quad\left.+c2^{r-1}(M_1^{r-1}+\|D_{0^+}^{\theta}\psi\|_\infty^{r-1})+d\|D_{0^+}^{\theta}\psi\|_\infty^{r-1}\right]\\ &= A_1+A_2\|D_{0^+}^{\theta}\psi\|_\infty^{r-1},\end{aligned}$$

where $A_1 := \frac{a+(\frac{b}{\Gamma(\theta)^{r-1}}+c)(2M_1)^{r-1}}{\Gamma(\upsilon+1)}$, $A_2 := \frac{1}{\Gamma(\upsilon+1)}[b(\frac{2(\theta+2)}{\Gamma(\theta+1)})^{r-1}+c2^{r-1}+d]$. Hence, we can obtain

$$\begin{aligned}\|D_{0^+}^{\theta}\psi(x)\|_\infty &\le [A_1+A_2\|D_{0^+}^{\theta}\psi\|_\infty^{r-1}]^{\frac{1}{p(x)-1}}\\ &\le 2^{\frac{1}{p(x)-1}}\left(A_1^{\frac{1}{p(x)-1}}+A_2^{\frac{1}{p(x)-1}}\|D_{0^+}^{\theta}\psi\|_\infty^{\frac{r-1}{p(x)-1}}\right).\end{aligned}$$

Since $\frac{r-1}{p(x)-1} \in (0,1]$ and $x^\iota \le x+1$, for $x > 0$, $\iota \in [0,1]$, we have

$$\|D_{0^+}^{\theta}\psi(x)\|_\infty \le 2^{\frac{1}{p(x)-1}}\left(A_1^{\frac{1}{p(x)-1}}+A_2^{\frac{1}{p(x)-1}}(\|D_{0^+}^{\theta}\psi\|_\infty+1)\right).$$

In view of $2^{\frac{1}{p_m-1}}\max\{A_2^{\frac{1}{p_m-1}}, A_2^{\frac{1}{p_M-1}}\} < 1$, we can obtain that there exists a constant $M_2 > 0$ such that $\|D_{0^+}^{\theta}\psi\|_\infty \le M_2$, $\|D_{0^+}^{\theta-1}\psi\|_\infty \le M_1+M_2 := M_3$, and $\|\psi\|_\infty \le \frac{\theta+1}{\Gamma(\theta+1)}M_2+\frac{1}{\Gamma(\theta)}M_3 := M_4$. So, $\|\psi\|_X = \max\{\|\psi\|_\infty, \|D_{0^+}^{\theta-1}\psi\|_\infty, \|D_{0^+}^{\theta}\psi\|_\infty\} \le \max\{M_2, M_3, M_4\} := \mathbf{M}$, i.e., Ω_1 is bounded. So, the proof is complete. □

Lemma 3.10. *Assume that the following condition holds*:

(H_5) *There exists a constant* $a_0 > 0$ *such that for* $|a_0| > 0$ *such that if* $|c| > a_0$, *then either*

$$cQ(N(cx^{\theta-1})) > 0, \tag{3.9}$$

$$cQ(N(cx^{\theta-1})) < 0. \tag{3.10}$$

Then, the set $\Omega_2 = \{\psi \in \mathrm{Ker}M : QN\psi = 0\}$ *and* $\Omega_3 = \{\psi \in \mathrm{Ker}M : \rho\delta I\psi + (1-\delta)JQN\psi = 0, \delta \in [0,1]\}$ *are bounded, where* $J : \mathrm{Im}Q \to \mathrm{Ker}M$ *is a homeomorphism with* $J(c) = cx^{\theta-1}$,

$$\rho = \begin{cases}1, & \text{if (3.9) holds,}\\ -1, & \text{if (3.10) holds.}\end{cases} \tag{3.11}$$

Proof. If $\psi_1 \in \Omega_2$, then $\psi_1(x) = cx^{\theta-1}$ and $A(N\psi_1) = 0$. By (H_5), we obtain $|\psi_1(x)| \le a_0$, $|D_{0^+}^{\theta-1}\psi_1(x)| \le a_0\Gamma(\theta)$ and $|D_{0^+}^{\theta}\psi_1(x)| = 0$. These mean that Ω_2 is bounded. □

For $\psi_2 \in \Omega_3$, $\psi_2 = cx^{\theta-1}$ and $\rho\delta I\psi + (1-\delta)JQN\psi = 0$.

If $\delta = 1$, then $\psi_2 \equiv 0$. If $\delta = 0$, then $QN\psi_2 = 0$, i.e., $A(N\psi_2) = 0$, which follows from the proof of boundness of Ω_2 that $||\psi_2||_X \le a_0 < \infty$.

If $\delta \in (0,1)$, we can have $\rho\delta c x^{\theta-1} + (1-\delta)JQN(cx^{\theta-1}) = 0$, then taking $|c| > a_0$, we have $\rho\delta c^2 x^{\theta-1} + (1-\delta)cQN(cx^{\theta-1})x^{\theta-1} = 0$, i.e., $\rho\delta c^2 = -(1-\delta)cQN(cx^{\theta-1})$, which contradicts with same sign for left and right of equation. So, $|c| \le a_0$, i.e., Ω_3 is also bounded.

Theorem 3.11. *Assume B_2, (H), and $(H_1 - H_5)$ hold. Then, the functional BVP* (1.1) *has at least one solution in X.*

Proof. Choose R_0 large enough such that $\Omega = \{\psi \in X : ||u||_X < R_0\} \supset \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \{\varnothing\}$ and $R_0 \ge \max\{\mathbf{M}, a_0\} + 1$, where $\mathbf{M}$ is introduced in Lemma 3.9. From Lemmas 3.9 and 3.10, $M\psi \ne N_\lambda\psi$ for $\psi \in \partial\Omega \cap \mathrm{dom}M$, $\lambda \in (0,1)$ and $QN\psi \ne 0$, $\psi \in \mathrm{Ker}M \cap \partial\Omega$. So, (C') of Theorem 2.3 holds.

Let $H(\psi,\delta) = \rho\delta I\psi + (1-\delta)JQN\psi$, $\delta \in [0,1]$, $\psi \in \mathrm{Ker}M \cap \partial\Omega$, noting $\Omega_3 \subset \Omega$, we know $H(\psi,\delta) \ne 0$, $\psi \in \mathrm{Ker}M \cap \partial\Omega$, $\delta \in [0,1]$.

For $u \in \mathrm{Ker}M \cap \partial\Omega$, we have $u(t) = c(at^{\alpha-1} - bt^{\alpha-2}) \ne 0$, $H(u,1) = \rho c(at^{\alpha-1} - bt^{\alpha-2}) \ne 0$. By Lemma 3.10, we know that $H(u,0) = QN(c(at^{\alpha-1} - bt^{\alpha-2}))(at^{\alpha-1} - bt^{\alpha-2}) \ne 0$.

Thus, by invariance of degree under a homotopy, we obtain that

$$\begin{aligned}\deg(JQN, \Omega \cap \mathrm{Ker}M, 0) &= \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}M, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}M, 0)\\ &= \deg(\rho I, \Omega \cap \mathrm{Ker}M, 0) = \pm 1 \ne 0.\end{aligned}$$

Therefore, the condition (C'') of Theorem 2.8 holds. By Theorem 2.8, Problem (1.1) has at least one solution in $\overline{\Omega}$. □

Example 3.12. Now, we illustrate Theorem 3.11 by the following example. Considering the non-local BVP:

$$\begin{cases} {}^C D_{1^-}^{\frac{1}{3}}\varphi_{x^2+2}\left(D_{0^+}^{\frac{5}{3}}\psi(x)\right) = f\left(x, \psi(x), D_{0^+}^{\frac{2}{3}}\psi(x), D_{0^+}^{\frac{5}{3}}\psi(x)\right), x \in (0,1),\\ D_{0^+}^{\frac{5}{3}}\psi(1) = 0, \psi(0) = 0, T(\psi) = D_{0^+}^{\frac{5}{3}}\psi\left(\frac{1}{2}\right) = 0, \end{cases}$$

where $f(x, \psi(x), D_{0^+}^{\frac{2}{3}}\psi(x), D_{0^+}^{\frac{5}{3}}\psi(x)) = \frac{1}{5} + \frac{1}{20}\sin\psi(x) + \frac{1}{20}D_{0^+}^{\frac{2}{3}}\psi(x) + \frac{1}{20}\sin D_{0^+}^{\frac{5}{3}}\psi(x)$.

It is easy to see that $a(t) = \frac{1}{5}$, $b = c = d = \frac{1}{20}$, $P_m = 2$, $P_M = 3$, $||T|| = 1$, and $T(x^{\frac{2}{3}}) = 0$. The problem is at resonance and $\mathrm{Ker}M = \{cx^{\frac{2}{3}} : c \in \mathbb{R}\}$. The assumption (H_2) is fulfilled with $r = 2$ and

$$2^{\frac{1}{p_m-1}}\max\left\{A_2^{\frac{1}{p_m-1}}, A_2^{\frac{1}{p_M-1}}\right\} = 2\max\left\{A_2, A_2^{\frac{1}{2}}\right\} = 2\frac{1}{\Gamma\left(\frac{4}{3}\right)}\left[\frac{1}{20}\left(7 + \frac{2}{\Gamma\left(\frac{8}{3}\right)}\right)\right] \approx 0.22044 < 1.$$

If $D_{0^+}^{\frac{2}{3}}\psi(x) > -2$, then $f(x, \psi(x), D_{0^+}^{\frac{2}{3}}\psi(x), D_{0^+}^{\frac{5}{3}}\psi(x)) > 0$, and if $D_{0^+}^{\frac{2}{3}}\psi(x) < -6$, then $f(x, \psi(x), D_{0^+}^{\frac{2}{3}}\psi(x), D_{0^+}^{\frac{5}{3}}\psi(x)) < 0$.

Hence, if $|D_{0^+}^{\frac{2}{3}}\psi(x)| > M_1 = 6$,

$$T(I_{0^+}^{\theta}\varphi_{p(x)}^{-1}(I_{1_-}^{\upsilon}N\psi)) = D_{0^+}^{\frac{5}{3}}I_{0^+}^{\frac{5}{3}}\varphi_{x^2+1}^{-1}\left(I_{1^-}^{\frac{1}{3}}N\psi\right)\left(\frac{1}{2}\right) = \left|I_{1^-}^{\frac{1}{3}}N\psi\right|^{\frac{-x^2}{x^2+2}}\Bigg|_{x=\frac{1}{2}} \cdot I_{1^-}^{\frac{1}{3}}N\psi\Bigg|_{x=\frac{1}{2}} \ne 0,$$

i.e., (H_1) is satisfied.

Finally, for $\psi_c(x) = cx^{\frac{2}{3}}$, one can choose $a_0 = 6 > 0$ such that $cQN\psi_c = c\varphi^{-1}_{x^2+2}(I_1^{\frac{1}{3}}N\psi_c) > 0$, which shows that (H_3) is confirmed, since $\varphi^{-1}_{x^2+2}(I_1^{\frac{1}{3}}N\psi_c) = |I_1^{\frac{1}{3}}N\psi_c|^{\frac{-x^2}{x^2+2}}|_{x=\frac{1}{2}} \cdot cI_1^{\frac{1}{3}}N\psi_c\,|_{x=\frac{1}{2}}$, and

$$
\begin{aligned}
cI_1^{\frac{1}{3}}N\psi_c\,|_{x=\frac{1}{2}} &= \frac{1}{\Gamma\left(\frac{1}{3}\right)}\int_x^1 (y-x)^{-\frac{2}{3}} cf\left(y, cy^{\frac{2}{3}}, c\Gamma\left(\frac{5}{3}\right), 0\right)\mathrm{d}y\,\bigg|_{x=\frac{1}{2}} \\
&= \frac{1}{\Gamma\left(\frac{1}{3}\right)}\int_x^1 (y-x)^{-\frac{2}{3}}\left(\frac{c}{5} + \frac{c}{20}\sin(cy^{\frac{2}{3}}) + \frac{c^2}{20}\Gamma\left(\frac{5}{3}\right)\right)\mathrm{d}y\,\bigg|_{x=\frac{1}{2}} \\
&> \frac{1}{\Gamma\left(\frac{1}{3}\right)}\int_x^1 (y-x)^{-\frac{2}{3}}\left(-\frac{|c|}{5} - \frac{|c|}{20} + \frac{c^2}{20}\Gamma\left(\frac{5}{3}\right)\right)\mathrm{d}y\,\bigg|_{x=\frac{1}{2}} > 0, \quad |c| > 6.
\end{aligned}
$$

It follows from Theorem 3.11 that there must be at least one solution in X. □

4 Conclusion

Resonance problems are one of the more interesting and popular topics in differential BVPs. And nonlinear mixed fractional-order BVPs with generalized functional boundary conditions certainly add to the difficulty, where we consider the complexity of the $p(x)$-Laplacian operator, which is a generalization of the p-Laplacian operator. The mixed operator fractional problem can describe some mathematical models, for example, the mathematical model for the height of granular material decreasing over time in a silo can be well described, so there is a good application background for studying the mixed fractional order with linear functional conditions, which also generalizes recent work on multi-point and integral BVPs. As far as we know, the solvability of the functional BVPs for mixed fractional differential equation with $p(x)$-Laplacian has not been well studied till now, so, we consider both resonant and non-resonant cases of problems in these scenarios and find some novel results.

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