Research Article

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Superposition operator problems of Hölder-Lipschitz spaces

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Abstract: Let f be a function defined on the real line, and T_f be the corresponding superposition operator which maps h to $T_f(h)$, i.e., $T_f(h) = f \circ h$. In this article, the sufficient and necessary conditions such that T_f maps periodic Hölder-Lipschitz spaces H_p^α into itself with $0 < \alpha < \frac{1}{p}$ and $\frac{1}{p} < \alpha < 1$, where α is the smoothness index, are shown. Our result in the case $0 < \alpha < \frac{1}{p}$ may be the first result about the superposition operator problems of smooth function space containing unbounded functions.

Keywords: Hölder-Lipschitz spaces, superposition operator problem, locally Lipschitz function, Lipschitz function

MSC 2020: 26A15, 26A16, 47H30

1 Introduction

Given a real-valued function f defined on the real line $\mathbb{R}$, the (autonomous) superposition operator T_f is defined by

$$T_f h(x) = (f \circ h)(x) = f(h(x)), \quad x \in [a, b], \tag{1.1}$$

where h is a real-valued function on $[a, b]$.

Let $H \equiv H[a, b]$ be a Banach space of real-valued functions $h : [a, b] \mapsto \mathbb{R}$ equipped with some norm $\|\cdot\|_H$. The superposition operator problem (SOP) refers to determining conditions on a function f on $\mathbb{R}$, possibly both necessary and sufficient, under which the superposition operator T_f maps H into itself. This problem is well known in many fields of nonlinear analysis and has its applications to operator theory, ordinary or partial differential equations, integral or integro-differential equations, considerations conducted in nonlinear functional analysis, and even physics and engineering [1,2]. The solution to SOP for a given H is sometimes very easy, sometimes highly nontrivial.

For many common Banach spaces H, the SOPs have been solved ([3,4] and references therein). For example, denote by $\mathrm{WBV}_p([a, b])$ $(1 \le p < \infty)$ the space of all functions of bounded p-variation in Wiener's sense with finite norm

$$\|h\|_{\mathrm{WBV}_p} = |h(a)| + \mathrm{Var}_p^W(h; [a, b]),$$

where

$$\mathrm{Var}_p^W(h; [a, b]) = \sup_{\{I_n\}} \left(\sum_{n=1}^{\infty} |h(I_n)|^p \right)^{1/p},$$

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and the supremum is taken over all sequences $\{I_n\} = \{[a_n, b_n]\}$ of nonoverlapping intervals in $[a, b]$, $h(I_n) = h(b_n) - h(a_n)$. Denote by $\mathrm{RBV}_p([a, b])(1 \le p < \infty)$ the space of all functions of bounded p-variation in Riesz's sense with finite norm

$$\|h\|_{\mathrm{RBV}_p} = |h(a)| + \mathrm{Var}_p^R(h;\ [a, b]),$$

where

$$\mathrm{Var}_p^R(h;\ [a, b]) = \sup_{\{I_n\}} \left(\sum_{n=1}^{\infty} \frac{|h(I_n)|^p}{|b_n - a_n|^{p-1}} \right)^{1/p}.$$

Particularly, when $p = 1$, the spaces $\mathrm{WBV}_p([a, b])$ and $\mathrm{RBV}_p([a, b])$ recede to the well-known space $BV([a, b])$ of functions of bounded variation.

Denote by $\mathrm{Lip}(\alpha, C[a, b])$ $(0 < \alpha \le 1)$ the space of all Lipschitz continuous functions of α-order with finite norm

$$\|h\|_{\mathrm{Lip}\alpha} = |h(a)| + \sup_{x \ne y,\ x, y \in [a,b]} \frac{|h(x) - h(y)|}{|x - y|^\alpha}.$$

We remark that when $H = \mathrm{WBV}_p([a, b])$ [5,6], $\mathrm{RBV}_p([a, b]), 1 \le p < \infty$ [7], $\mathrm{Lip}(\alpha, C[a, b])$ [8], and $AC[a, b]$ consisting of absolutely continuous functions on $[a, b]$ [9], the superposition operator T_f maps H into itself if and only if f is locally Lipschitz, i.e., there exists a positive constant $K(r)$ depending only on $r > 0$ such that

$$|f(x) - f(y)| \le K(r)|x - y|, \quad |x|, |y| \le r. \tag{1.2}$$

We denote such f by $f \in \mathrm{Lip}_{\mathrm{loc}}$. If $K(r)$ in (1.2) is replaced by a constant K independent of r, then f is said to be Lipschitz and is denoted by $f \in \mathrm{Lip}$.

Note that all the above spaces can be continuously embedded in the space $B[a, b]$ consisting of bounded functions. For SOPs of smooth function spaces $H[a, b]$ containing unbounded functions, we have few results.

Denote by $\mathrm{Lip}(\alpha, L_p(A))$ the Hölder-Lipschitz spaces consisting of all functions $h \in L_p(A)$ for which

$$\|\triangle_t h\|_p = \left(\int_{A_t} |h(x + t) - h(x)|^p \mathrm{d}x \right)^{1/p} \le M t^\alpha, \quad t > 0,$$

where $A_t \coloneqq [a, b - t]$, if $A = [a, b]$, $0 < t \le b - a$, and $A_t = A$, if A is the torus $\mathbb{T} = \{e^{2\pi i x}, x \in \mathbb{R}\}$. We remark that the spaces $\mathrm{Lip}(\alpha, L_p(A))$ contain unbounded functions if $0 < \alpha < 1/p$.

When $\alpha = 1$, $\mathrm{Lip}(\alpha, L_p[a, b]) = W_p^1[a, b] = \mathrm{RBV}_p([a, b])$ if $1 < p < \infty$ ([10, Theorem 9.3] and [11, Chapter IX §4, Theorem 7]), $\mathrm{Lip}(\alpha, L_p[a, b]) = BV[a, b]$ if $p = 1$ ([10, Theorem 9.3]), where $W_p^1[a, b] = \{f \mid f \in AC[a, b]$ and $f' \in L_p[a, b]\}$. It follows that for $\alpha = 1$, the superposition operator T_f maps $\mathrm{Lip}(\alpha, L_p[a, b])$ $(1 \le p < \infty)$ into itself if and only if f is locally Lipschitz.

This work is devoted to studying the SOPs of the Hölder-Lipschitz spaces $H_p^\alpha \equiv \mathrm{Lip}(\alpha, L_p(\mathbb{T}))$, $1 \le p < \infty$, $0 < \alpha < 1$. For $1/p < \alpha < 1$, $1 < p < \infty$, the space H_p^α can be embedded into the continuous function space $C(\mathbb{T})$ and any function in H_p^α is bounded. However, for $0 < \alpha \le 1/p$, $1 \le p < \infty$, $\alpha < 1$, the space H_p^α contains unbounded functions [12]. The investigation in these two cases is completely different.

Our main results can be formulated as follows.

Theorem 1.1. *Let* $\frac{1}{p} < \alpha < 1$, $1 < p < \infty$, *and let* T_f *be the superposition operator defined by* (1.1). *Then,* T_f *maps* H_p^α *into itself if and only if f is locally Lipschitz.*

Theorem 1.2. *Let* $0 < \alpha < \frac{1}{p}$, $1 \le p < \infty$. *Then,* T_f *maps* H_p^α *into itself if and only if f is Lipschitz.*

Theorem 1.3. *Let* $\alpha = \frac{1}{p}$, $1 < p < \infty$. *Then, the sufficient condition for which* T_f *maps* H_p^α *into itself is that f is Lipschitz, and the necessary but not sufficient condition is that f is locally Lipschitz.*

The proofs of the sufficiency parts of Theorems 1.1–1.3 are standard and easy. For the proof of necessity of Theorem 1.1, we first show the continuity of f, and then the locally Lipschitz continuity of f. Our proof of Theorem 1.1 makes essential use of Terekhin's estimates of L_p-moduli of continuity in terms of p-moduli of continuity of a function in H_p^α. Such idea is proposed and used in [12,13].

For the proof of the necessity part of Theorem 1.2, apart from the one of locally Lipschitz continuity of f, we also show Lipschitz continuity of f in the end. We give and use estimates of L_p-moduli of continuity of a step function to prove Theorem 1.2.

Remark 1.4. Theorem 1.2 may be the first result about the SOPs of smooth function space containing unbounded functions.

Remark 1.5. We do not know what is the sufficient and necessary condition for which T_f maps $H_p^{1/p}$ $(1 < p < \infty)$ into itself. It is open. We conjecture that the sufficient and necessary condition for which T_f maps $H_p^{1/p}$ $(1 < p < \infty)$ into itself is that f is Lipschitz.

2 Proof of Theorem 1.1

Let g be a 1-periodic function on the real line and $1 < p < \infty$. Following Terekhin [14], we define the modulus of p-continuity of g by

$$\omega_{1-1/p}(g;\, \delta) = \sup_{\|\mathcal{I}\| \le \delta} \left(\sum_{n=1}^{\infty} |g(I_n)|^p \right)^{1/p} \quad (0 < \delta \le 1), \tag{2.1}$$

where the supremum takes over all sequences $\mathcal{I} = \{I_n\} = \{[a_n, b_n]\}$ of nonoverlapping intervals contained in a period with $\|\mathcal{I}\| = \sup_n |I_n| = \sup_n |b_n - a_n| \le \delta$. It follows from Hölder's inequality that if f is absolutely continuous and $f' \in L^p$, then

$$\omega_{1-1/p}(f;\delta) \le \delta^{1/p'} \|f'\|_p \quad (0 < \delta \le 1), \tag{2.2}$$

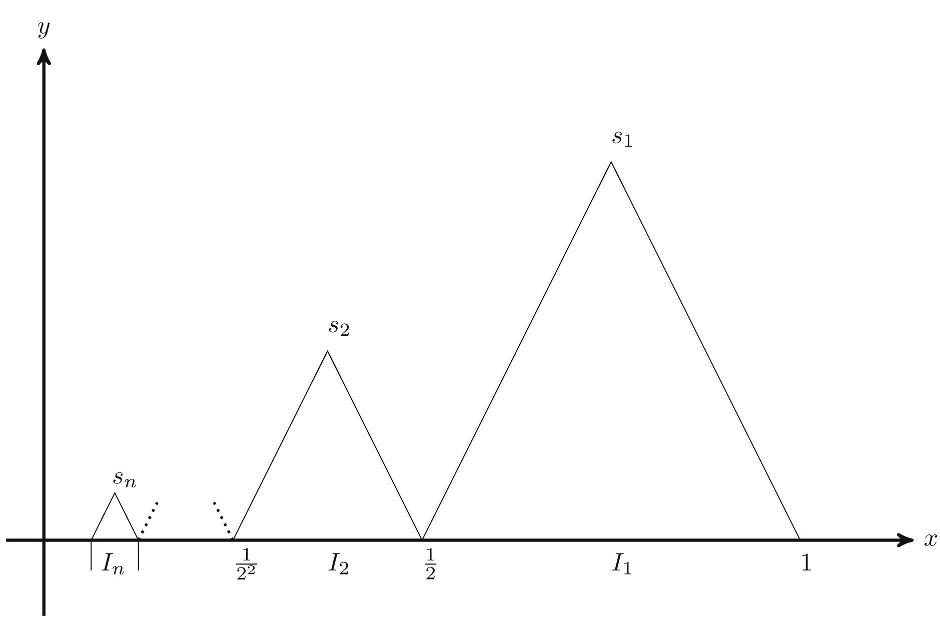

Figure 1: The graph of g.

where $\frac{1}{p} + \frac{1}{p'} = 1$. It is proved in [14, Corollary 1] that $f \in H_p^\alpha, 1/p < \alpha < 1$ if and only if f can be modified on a set of zero measures in such a way that f is continuous and

$$\omega_{1-1/p}(f;\delta) \ll \delta^{\alpha-1/p}, \tag{2.3}$$

where $A \asymp B$ means $A \ll B$ and $A \gg B$, $A \ll B$ means that there exists a nonessential constant $c > 0$ such that $A \leq cB$, and $A \gg B$ means $B \ll A$. Hence, we assume that every function in H_p^α, $\alpha \in (1/p, 1)$ is continuous.

Proof of Theorem 1.1. *Sufficiency.*

Assume that f is locally Lipschitz. Let g be an arbitrary function in $H_p^\alpha, 1/p < \alpha < 1, 1 < p < \infty$. Then, g is continuous and bounded. So there exists a constant $r > 0$ such that $\|g\|_\infty = \sup_{x\in\mathbb{T}} |g(x)| \leq r$. Since the function f is locally Lipschitz, there exists a positive constant $k(r)$ such that for any $x, y \in [-r, r]$,

$$|f(x) - f(y)| \leq k(r)|x - y|.$$

It follows that

$$\|\triangle_t(f \circ g)(\cdot)\|_p = \|f(g(\cdot + t)) - f(g(\cdot))\|_p \leq k(r)\|g(\cdot + t) - g(\cdot)\|_p \ll |t|^\alpha,$$

which means that $f \circ g \in H_p^\alpha$. Hence, T_f maps H_p^α into itself if f is locally Lipschitz.

Necessity.

Let $\alpha \in (1/p, 1), 1 < p < \infty$. Suppose that $T_f(h) = f \circ h \in H_p^\alpha$ for any $h \in H_p^\alpha$. We want to show $f \in \mathrm{Lip}_{\mathrm{loc}}$. We recall that any function h in H_p^α $(\alpha > 1/p)$ is continuous on $\mathbb{T}$. First, we prove that f is continuous on $\mathbb{R}$. Otherwise, f is not continuous at some point $x \in \mathbb{R}$. Without loss of generality we may assume that $x = 0$, $f(0) = 0$, and there exists a decreasing positive sequence $\{s_n\}$ satisfying $s_n < 2^{-n}$ and $f(s_n) > 1$.

Denote $I_n = [2^{-n}, 2^{-n+1}]$, $n = 1, 2, \ldots$. Set $g(0) = g(2^{-n}) = 0$. On each interval I_n, we define $g(x)$ as a tent function such that $g\left(\frac{2^{-n} + 2^{-n+1}}{2}\right) = g(2^{-n} \cdot \frac{3}{2}) = s_n$. The graph of g is shown in Figure 1.

For any $x, y \in [0, 1]$, we assume that $x \in I_n$, $y \in I_m$. Then, we have $\frac{|g(x)-g(y)|}{|x-y|} \leq \max\left\{\frac{s_n}{2^{-n} \cdot \frac{1}{2}}, \frac{s_m}{2^{-m} \cdot \frac{1}{2}}\right\} \leq 2.$

Hence, $g \in \mathrm{Lip}(1, C(\mathbb{T})) \subseteq H_p^\alpha$. However, we also have $f \circ g\left(2^{-n} \cdot \frac{3}{2}\right) = f(s_n) > 1$ and $f \circ g(0) = 0$, which imply that $f \circ g$ is not continuous at $x = 0$. This leads to a contradiction. Hence, f is continuous.

We now prove that f is locally Lipschitz. Assume that f is not locally Lipschitz. Then, there exists a real number $r > 0$ and a sequence of intervals $\{[a_n, b_n]\}$ which are subsets of $[-r, r]$ and satisfies

$$|f(b_n) - f(a_n)| \geq k_n|b_n - a_n| = 2^{n+2}|b_n - a_n|. \tag{2.4}$$

(We take $k_n = 2^{n+2}$.) From the continuity of f and compactness of $[-r, r]$, we may assume that

$$\max_{x,y\in[-r,r]} |f(x) - f(y)| \leq 1.$$

Hence, we have

$$|b_n - a_n| \leq \frac{1}{k_n}.$$

Furthermore, there are a subsequence of $\{[a_n, b_n]\}$ (without loss of generality, we still denote it as $\{[a_n, b_n]\}$) and a constant ξ satisfying $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \xi$. If $a_n < \xi < b_n$,

$$k_n \leq \frac{|f(b_n) - f(a_n)|}{|b_n - a_n|} \leq \max\left\{\frac{|f(\xi) - f(a_n)|}{|\xi - a_n|}, \frac{|f(\xi) - f(b_n)|}{|\xi - b_n|}\right\}.$$

Thus, we can assume that $\xi < a_n < b_n$. Considering subsequence if necessary, we may assume that

$$\xi < a_n < b_n < a_{n-1} < b_{n-1},$$

$$|a_n - \xi| \le \frac{1}{k_n} = 2^{-n-2}, \tag{2.5}$$

and

$$b_n - a_n < b_{n-1} - a_{n-1} < \frac{1}{k_{n-1}}.$$

Set

$$S_n = \left[(b_n - a_n)^{1-\frac{1}{\alpha}}\right], \quad l_n = \left[\frac{1}{k_n(b_n - a_n)}\right] S_n \asymp \frac{1}{k_n(b_n - a_n)^{1/\alpha}}, \tag{2.6}$$

and

$$\delta_n = A\frac{b_n - a_n}{2S_n} \asymp (b_n - a_n)^{\frac{1}{\alpha}}, \quad l_0 = 0, \quad \delta_0 = 1, \quad a_0 = a_1, \tag{2.7}$$

where $[x]$ denotes the largest integer not exceeding x, A is a positive constant satisfying

$$\sum_{n=1}^{\infty} 2l_n\delta_n = \sum_{n=1}^{\infty} A(b_n - a_n)\left[\frac{1}{k_n(b_n - a_n)}\right] = 1/4.$$

(The above equality is required just to ensure that the function g_1, which is defined below, belongs to Lip.) The existence of A is guaranteed by the following inequality:

$$\sum_{n=1}^{\infty} (b_n - a_n)\left[\frac{1}{k_n(b_n - a_n)}\right] \le \sum_{n=1}^{\infty} \frac{1}{k_n} = 1/4.$$

We denote

$$J_n^1 = \left[\sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}),\ \sum_{j=0}^{n}(2l_j\delta_j + a_j - a_{j+1})\right],$$

$$J_{n,1}^1 = \left[\sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}),\ \sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}) + 2l_n\delta_n\right],$$

and

$$J_{n,2}^1 = \left[\sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}) + 2l_n\delta_n,\ \sum_{j=0}^{n}(2l_j\delta_j + a_j - a_{j+1})\right].$$

We define $\beta(x)$ as follows:

$$\beta(x) = \begin{cases} a_n, & x = \sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}) + 2k\delta_n, \quad 0 \le k \le l_n, \\ b_n, & x = \sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}) + (2k+1)\delta_n, \quad 0 \le k \le l_n - 1, \\ \xi, & x = \sum_{j=0}^{\infty}(2l_j\delta_j + a_j - a_{j+1}) = 1/4 + a_1 - \xi, \\ a_1, & x = 1 \\ \text{linear}, & \text{otherwise}. \end{cases}$$

The graph of β is shown in Figure 2.

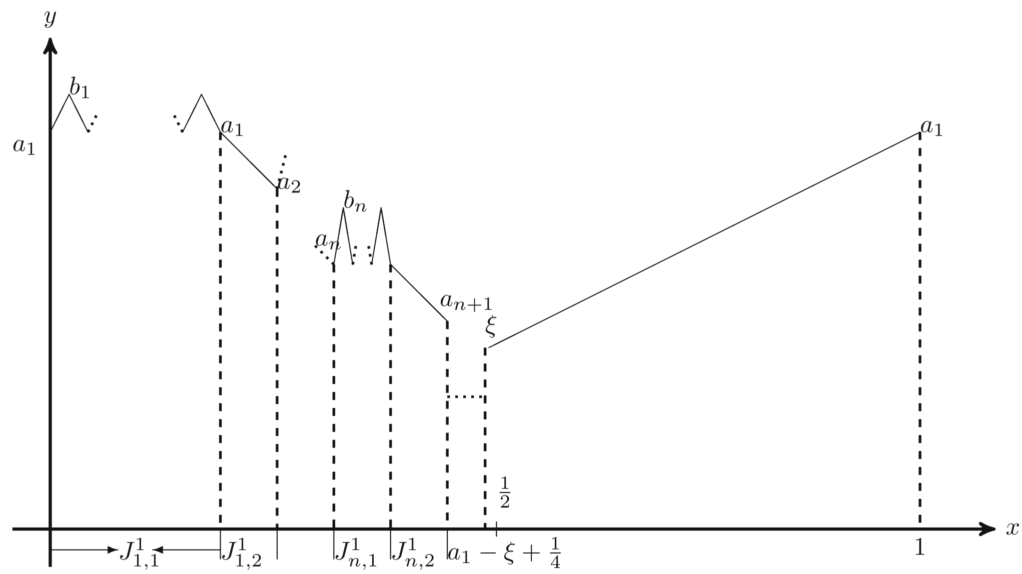

Figure 2: The graph of β.

Let

$$\beta_2(x) = \sum_{n=1}^{\infty} (\beta(x) - a_n)\chi_{J^1_{n,1}}(x), \quad \beta_1(x) = \beta(x) - \beta_2(x),$$

where χ_A denotes the characteristic function of a set A. Then,

$$\beta_1(x) = \begin{cases} a_n, & x \in J^1_{n,1}, \\ \xi, & x = 1/4 + a_1 - \xi, \\ a_1, & x = 1 \\ \text{linear}, & \text{otherwise}. \end{cases}$$

The graph of β_1 is shown in Figure 3.

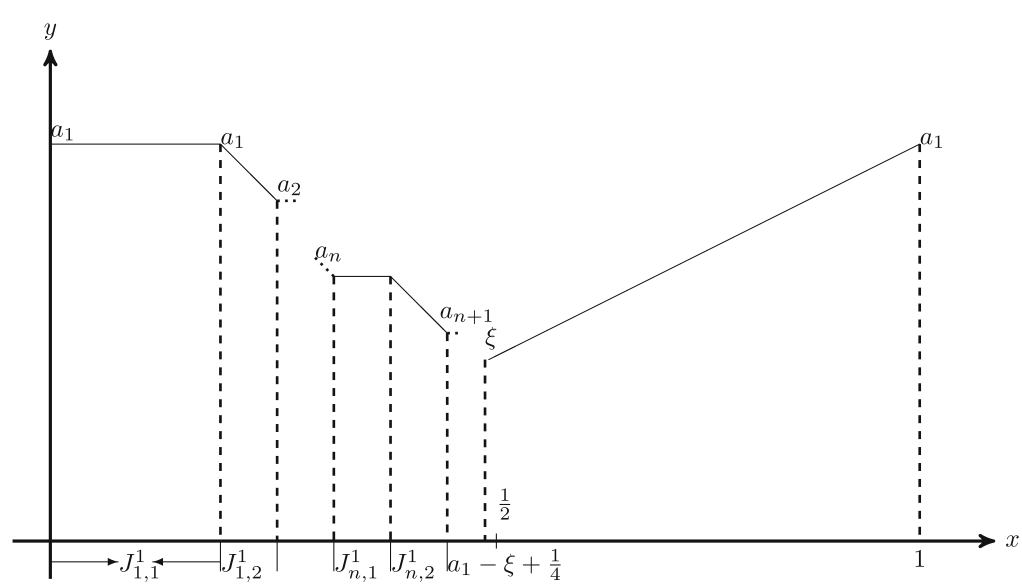

Figure 3: The graph of β_1.

Since $\frac{a_1-\xi}{1-(a_1-\xi+1/4)} \le 1$ (by (2.5)) and the length of $J^1_{n,2}$ is $a_n - a_{n+1}$, we obtain that for any $x, y \in [0,1]$,

$$\frac{|\beta_1(x)-\beta_1(y)|}{|x-y|} \le 1.$$

It follows that

$$\beta_1 \in \mathrm{Lip}(1, C(\mathbb{T})) \subseteq H^\alpha_p.$$

Now we show that $\beta_2 \in H^\alpha_p$. We note that

$$\beta_2(x) = \begin{cases} 0, & x = \sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}) + 2k\delta_n, \quad 0 \le k \le l_n, \\ b_n - a_n, & x = \sum_{j=0}^{n-1}(2l_j\delta_j + a_j - a_{j+1}) + (2k+1)\delta_n, \quad 0 \le k \le l_n - 1, \\ 0, & x \in J^1_{n,2} \cup [1/4 + a_1 - \xi, 1], \\ \text{linear}, & \text{otherwise.} \end{cases}$$

The graph of β_2 is shown in Figure 4.

For any $0 < \delta < 1$, take n_0 such that $\delta_{n_0+1} < \delta \le \delta_{n_0}$. We set

$$\beta_{2_1}(x) = \beta_2(x)\chi_{[0,\sum_{j=0}^{n_0}(2l_j\delta_j + a_j - a_{j+1})]}, \quad \beta_{2_2}(x) = \beta_2(x) - \beta_{2_1}(x).$$

We recall from (2.6) and (2.7) that

$$\delta_n \asymp (b_n - a_n)^{1/\alpha}, \quad l_n \asymp \frac{1}{k_n(b_n-a_n)^{1/\alpha}}, \quad \text{and} \quad l_n\delta_n \asymp 1/k_n \asymp 2^{-n}.$$

Noting that $(b_n - a_n)^{-1} \asymp \delta_n^{-\alpha} \le \delta^{-\alpha}$ for $n \le n_0$ and $1 - \frac{1}{\alpha} < 0$, by (2.2) we obtain

$$\begin{aligned} \omega_{1-\frac{1}{p}}(\beta_{2_1}; \delta) &\le \delta^{1-\frac{1}{p}}\|\beta'_{2_1}\|_p = \delta^{1-\frac{1}{p}}\left(\sum_{n=1}^{n_0}\left(\frac{b_n-a_n}{\delta_n}\right)^p \delta_n \cdot 2l_n\right)^{1/p} \\ &\asymp \delta^{1-\frac{1}{p}}\left(\sum_{n=1}^{n_0}(b_n-a_n)^{\left(1-\frac{1}{\alpha}\right)p}\frac{1}{k_n}\right)^{\frac{1}{p}} \\ &\le \delta^{1-\frac{1}{p}}\delta^{\alpha-1}\left(\sum_{n=1}^{n_0}\frac{1}{k_n}\right)^{\frac{1}{p}} \ll \delta^{\alpha-\frac{1}{p}}. \end{aligned}$$

For any $n > n_0$ and $p > \frac{1}{\alpha}$, by (2.7) and (2.1) we have $(b_n - a_n) \asymp \delta_n^\alpha \le \delta^\alpha$ and

$$\omega_{1-\frac{1}{p}}(\beta_{2_2}; \delta) = \sup_{\|\mathcal{I}\|\le\delta}\left(\sum_{n=1}^{\infty}|\beta_{2_2}(I_n)|^p\right)^{\frac{1}{p}} = \left(\sum_{n=n_0+1}^{\infty}(b_n-a_n)^p \cdot 2l_n\right)^{\frac{1}{p}} \ll \left(\sum_{n=n_0}^{\infty}(b_n-a_n)^{p-\frac{1}{\alpha}}\frac{1}{k_n}\right)^{\frac{1}{p}} \ll \delta^{\alpha-\frac{1}{p}}.$$

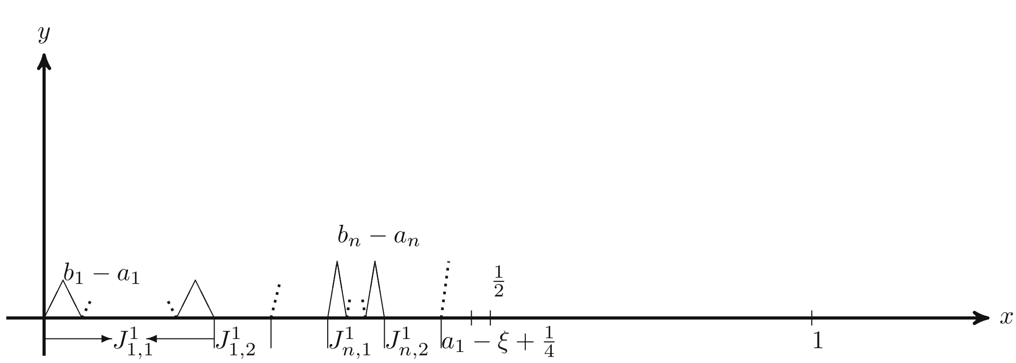

Figure 4: The graph of β_2.

Hence,

$$\omega_{1-\frac{1}{p}}(\beta_2;\ \delta) \le \omega_{1-\frac{1}{p}}(\beta_{2_1};\ \delta) + \omega_{1-\frac{1}{p}}(\beta_{2_2};\ \delta) \ll \delta^{\alpha-\frac{1}{p}},$$

which combining with (2.3) implies that $\beta_2 \in H_p^\alpha$. Hence, $\beta = \beta_1 + \beta_2 \in H_p^\alpha$. However, by (2.4) we have

$$\omega_{1-\frac{1}{p}}(f \circ \beta;\ \delta_n) \ge |f(b_n) - f(a_n)| \cdot (2l_n)^{\frac{1}{p}} \ge \delta_n^{\alpha-\frac{1}{p}} k_n^{1-\frac{1}{p}}.$$

Since $k_n^{1-\frac{1}{p}} \to \infty$ as $n \to \infty$, by (2.3) we obtain $T_f(\beta) = f \circ \beta \notin H_p^\alpha$, which contradicts our assumption.

Theorem 1.1 is proved. □

3 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. *Sufficiency.*

Assume that f is Lipschitz. Then, there exists a constant $k > 0$ such that for any $x, y \in \mathbb{R}$, $|f(x) - f(y)| \le k|x - y|$. Thus, for any function $g \in H_p^\alpha$, $0 < \alpha \le 1/p$, $1 \le p < \infty$, and $t > 0$, we have

$$\|\triangle_t(f \circ g)(\cdot)\|_p = \|f \circ g(\cdot + t) - f \circ g(\cdot)\|_p \le k\|g(\cdot + t) - g(\cdot)\|_p \ll t^\alpha,$$

which means that $f \circ g \in H_p^\alpha$. Hence, T_f maps H_p^α, $0 < \alpha \le 1/p$, $1 \le p < \infty$ into itself if f is Lipschitz.

Necessity.

Let $\alpha \in (0, 1/p]$, $1 \le p < \infty$, $\alpha < 1$. Suppose that $T_f(h) = f \circ h \in H_p^\alpha$ for any $h \in H_p^\alpha$. We want to show that $f \in \mathrm{Lip}_{\mathrm{loc}}$ if $0 < \alpha \le 1/p$ and $f \in \mathrm{Lip}$ if $0 < \alpha < 1/p$.

First, we show that f is continuous. Otherwise, f is not continuous at some point x. Without loss of generality, we may assume that $x = 0$, $f(0) = 0$, and there exists a nonincreasing sequence $\{s_n\}$ satisfying $0 < s_n < 2^{-n}$ and $f(s_n) > 1$. We want to construct a function ϕ such that $\phi \in H_p^\alpha$ but $f \circ \phi \notin H_p^\alpha$.

Choose two positive numbers r and η such that

$$0 < r < p, \quad 0 < \eta < \alpha r p < p.$$

Set

$$l_n = [s_n^{-(1-\alpha p)r-\eta}] + 1, \quad t_n = s_n^r A, \tag{3.1}$$

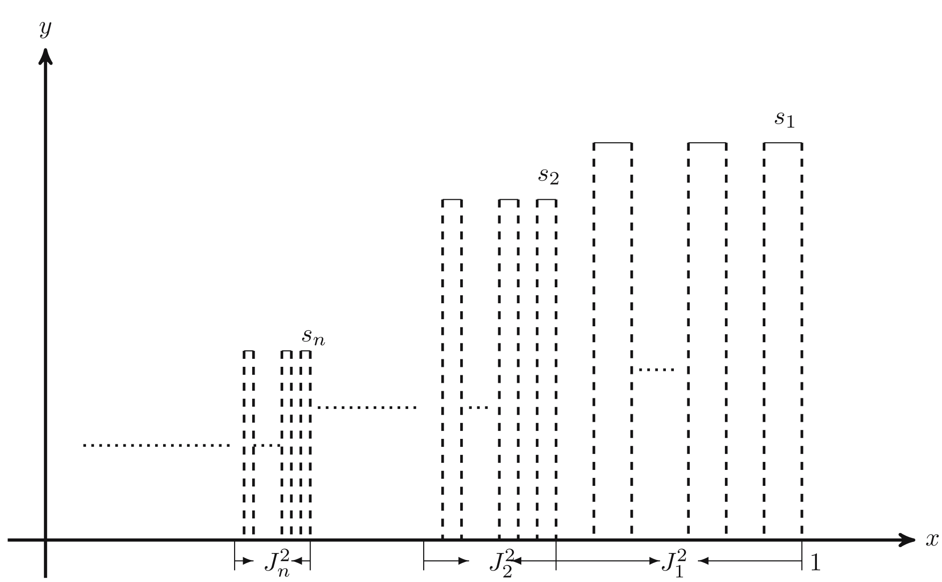

Figure 5: The graph of ϕ.

where A is a positive constant such that

$$\sum_{n=1}^{\infty} 2l_n t_n = 1.$$

The existence of A is guaranteed by the following inequality:

$$\sum_{n=1}^{\infty} 2([s_n^{-(1-\alpha p)r-\eta}]+1)s_n^r \ll \sum_{n=1}^{\infty} s_n^{\alpha rp-\eta} \le \sum_{n=1}^{\infty} 2^{-n(\alpha rp-\eta)} < \infty.$$

Set

$$J_n^2 = \left[1-\sum_{j=1}^{n} 2l_j t_j,\, 1-\sum_{j=1}^{n-1} 2l_j t_j\right].$$

We divide J_n equally into $2l_n$ subintervals and denote every part by

$$J_{n,m}^2 = \left[1-\sum_{j=1}^{n} 2l_j t_j + (m-1)t_n,\, 1-\sum_{j=1}^{n} 2l_j t_j + m t_n\right], \quad m = 1,\dots,2l_n.$$

Now, we define $\phi(x)$ as

$$\phi(x) = \begin{cases} 0, & x \in J_{n,2j-1}^2, \\ s_n, & x \in J_{n,2j}^2, \end{cases} \quad n = 1,2,\dots,\ j = 1,\dots,l_n.$$

Denote $\phi_n(x) = \phi(x)\chi_{J_n}(x)$. The graph of ϕ is shown in Figure 5.

For fixed $n \ge 1$, we estimate $\|\triangle_t\phi_n\|_p^p$, $0 < t < 1$. Noting that $t_n \asymp s_n^r$, by (3.1) we have for $0 < t < t_n$,

$$\begin{aligned} \|\triangle_t\phi_n\|_p^p &= \int_{[0,1]} |\phi_n(x+t)-\phi_n(x)|^p \mathrm{d}x \\ &= s_n^p \cdot |\{x : |\phi_n(x+t)-\phi_n(x)| = s_n\}| \\ &= s_n^p \cdot (2tl_n) \asymp t s_n^{(p-r)+(\alpha pr-\eta)} \le t^{\alpha p} s_n^{(p-r)+(\alpha pr-\eta)}, \end{aligned} \tag{3.2}$$

where $|A|$ denotes the Lebesgue measure of a set A. Similarly for $t \ge t_n$ we have

$$\|\triangle_t\phi_n\|_p^p = s_n^p \cdot |\{x : |\phi_n(x+t)-\phi_n(x)| = s_n\}| \le s_n^p \cdot (2l_n t_n) \asymp s_n^{\alpha rp-\eta+p} \ll t^{\alpha p} s_n^{p-\eta}. \tag{3.3}$$

For any $t \in (0,1)$, it follows from (3.2) and (3.3) that

$$\begin{aligned} \|\triangle_t\phi\|_p &\le \sum_{n=1}^{\infty} \|\Delta_t\phi_n\|_p = \sum_{t_n>t} \|\triangle_t\phi_n\|_p + \sum_{t_n\le t} \|\triangle_t\phi_n\|_p \\ &\ll \sum_{t_n>t} t^\alpha s_n^{(1-\frac{r}{p})+\frac{\alpha pr-\eta}{p}} + \sum_{t_n\le t} t^\alpha s_n^{-\frac{\eta}{p}+1} \\ &\ll t^\alpha\left(\sum_{t_n>t} 2^{-n\left((1-\frac{r}{p})+\frac{\alpha pr-\eta}{p}\right)} + \sum_{t_n\le t} 2^{-n(-\frac{\eta}{p}+1)}\right) \ll t^\alpha, \end{aligned}$$

which means that $\phi \in H_p^\alpha$.

However, we have

$$\begin{aligned} \|\triangle_{t_n}(f\circ\phi)\|_p &\ge \left(\int_{\cup_{j=1}^{l_n} J_n^{2j-1}} |f\circ\phi(x+t_n) - f\circ\phi(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &= f(s_n)(l_n t_n)^{\frac{1}{p}} \ge (l_n t_n)^{\frac{1}{p}} \gg s_n^{\alpha r-\frac{\eta}{p}} \asymp t_n^\alpha s_n^{\frac{-\eta}{p}}. \end{aligned}$$

Since $s_n^{\frac{-\eta}{p}} > 2^{\frac{n\eta}{p}} \to \infty$ as $n \to \infty$, we obtain $f\circ\phi \notin H_p^\alpha$. This contradicts our assumption. So f is continuous on $\mathbb{R}$.

Second, we show that $f \in \mathrm{Lip}_{\mathrm{loc}}$ if $0 < \alpha \le 1/p,\ 1 \le p < \infty,\ \alpha < 1$. Assume that f is not locally Lipschitz. As in the proof of Theorem 1.1, we can assume that there exist sequences $\{a_n\}$, $\{b_n\}$, and ξ satisfying

$$\max\{|a_n - \xi|, |b_n - a_n|\} \le \frac{1}{k_n} = 2^{-n}, \tag{3.4}$$

$$\xi < a_n < b_n < a_{n-1} < b_{n-1},$$

and

$$1 \ge |f(b_n) - f(a_n)| \ge k_n(b_n - a_n) = 2^n(b_n - a_n). \tag{3.5}$$

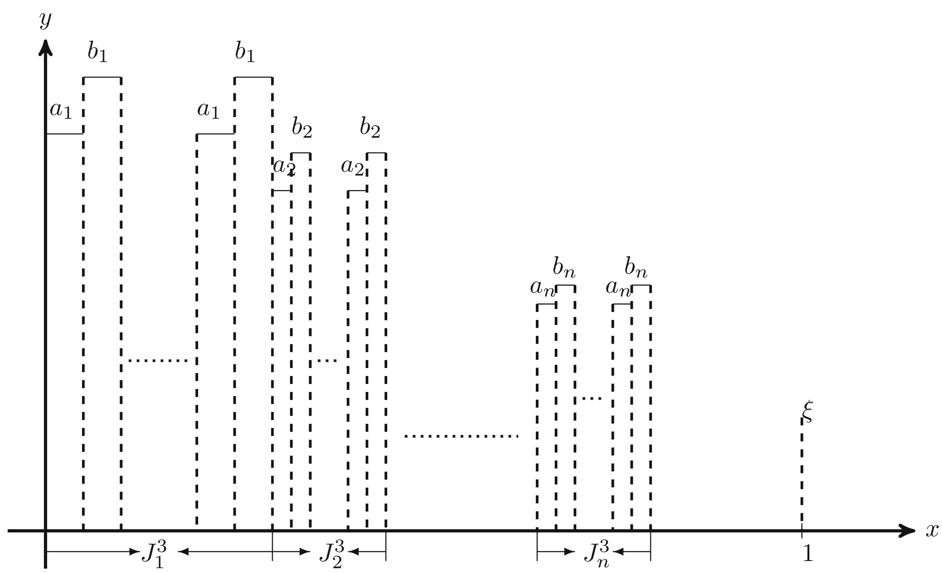

Figure 6: The graph of ψ.

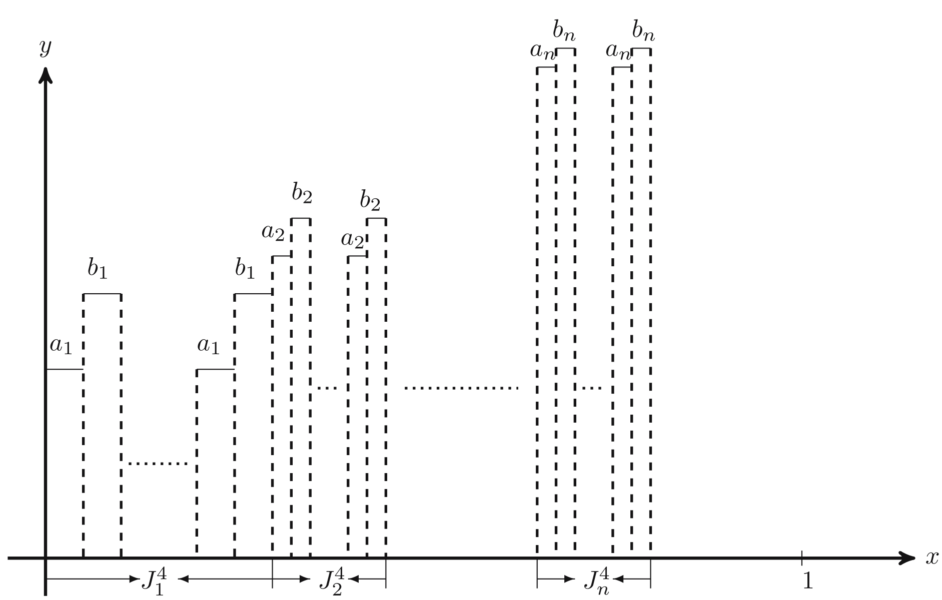

Figure 7: The graph of γ.

(We take $k_n = 2^n$.) Choose two positive numbers r and η such that

$$r\alpha > 1, \quad (r\alpha - 1)p < \eta < p.$$

For example, we may take $r = \frac{3}{2\alpha}$, $\eta = \frac{3p}{4}$. Set

$$t_n = (b_n - a_n)^r A, \quad l_n = \left[\frac{(b_n - a_n)^{r\alpha p - r - p}}{k_n^\eta}\right] + 1, \quad t_0 = l_0 = 0, \tag{3.6}$$

where A is a positive constant satisfying

$$\sum_{n=1}^{\infty} 2 l_n t_n = 1.$$

We remark by (3.4) that for $n \geq 1$,

$$l_n \geq \frac{(b_n - a_n)^{r(\alpha p - 1) - p}}{k_n^\eta} \geq k_n^{r(-\alpha p + 1) + (p - \eta)} > 1.$$

The existence of A is derived from

$$\sum_{n=1}^{\infty} 2\left(\left[\frac{(b_n - a_n)^{r\alpha p - r - p}}{k_n^\eta}\right] + 1\right)(b_n - a_n)^r \asymp \sum_{n=1}^{\infty} \frac{(b_n - a_n)^{r\alpha p - p}}{k_n^\eta} \leq \sum_{n=1}^{\infty} k_n^{p(-r\alpha + 1) - \eta} < \infty.$$

Denote $J_n^3 = \left[\sum_{j=0}^{n-1} 2 l_j t_j, \sum_{j=0}^{n} 2 l_j t_j\right]$ and

$$J_{n,k}^3 = \left[\sum_{j=0}^{n-1} 2 l_j t_j + (k-1) t_n, \sum_{j=0}^{n-1} 2 l_j t_j + k t_n\right], \quad (k = 1, \ldots, 2l_n,\ n \geq 1).$$

We define $\psi(x)$ as follows:

$$\psi(x) = \begin{cases} a_n, & x \in J_{n,2j-1}^3, \\ b_n, & x \in J_{n,2j}^3, \end{cases} \quad n = 1, 2, \ldots,\ j = 1, \ldots, l_n.$$

The graph of ψ is shown in Figure 6.

Set

$$\psi_1(x) = \xi + \sum_{n=1}^{\infty} (a_n - \xi) \chi_{J_n^3}(x), \quad \psi_2(x) = \psi(x) - \psi_1(x)$$

and

$$\psi_{1,n}(x) = (a_n - \xi) \chi_{J_n^3}(x), \quad \psi_{2,n}(x) = \psi_2(x) \chi_{J_n^3}(x).$$

We note that for $t \in (0, 1)$,

$$\|\triangle_t \chi_{J_n^3}\|_p = |\{x : |\chi_{J_n^3}(x + t) - \chi_{J_n^3}(x)| = 1\}|^{\frac{1}{p}} \leq \min\{(2t)^{1/p}, (4 l_n t_n)^{1/p}\}.$$

It follows that for $t \in (0, 1)$,

$$\begin{aligned} \|\triangle_t \psi_1\|_p &\leq \sum_{n=1}^{\infty} \|\triangle_t \psi_{1,n}\|_p \\ &\ll \sum_{2 l_n t_n > t} (a_n - \xi) t^{\frac{1}{p}} + \sum_{2 l_n t_n \leq t} (a_n - \xi)(l_n t_n)^{\frac{1}{p}} \\ &\ll \sum_{n=1}^{\infty} \frac{1}{k_n} t^{\frac{1}{p}} = t^{1/p} \ll t^\alpha, \end{aligned}$$

which gives $\psi_1 \in H_p^\alpha$.

We now prove that $\psi_2 \in H_p^\alpha$. Similar to the proofs of (3.2) and (3.3), we can show that

$$\begin{aligned}\|\triangle_t\psi_{2,n}\|_p &= ((b_n - a_n)^p|\{x : |\psi_{2,n}(x+t) - \psi_{2,n}(x)| = b_n - a_n\}|)^{\frac{1}{p}}\\ &\ll ((b_n - a_n)^p \min\{l_n t, l_n t_n\})^{\frac{1}{p}}.\end{aligned}$$

Noting that $t_n \asymp (b_n - a_n)^r$ and $0 < \alpha \le 1/p$, by (3.6) we have for $0 < t < t_n$,

$$(b_n - a_n)(l_n t)^{1/p} \ll t^\alpha\ t_n^{1/p-\alpha}\frac{(b_n - a_n)^{r(\alpha-\frac{1}{p})}}{k_n^{\frac{\eta}{p}}} \ll \frac{t^\alpha}{k_n^{\frac{\eta}{p}}},$$

and for $1 > t \ge t_n$,

$$(b_n - a_n)(l_n t_n)^{1/p} \le t^\alpha, \quad (b_n - a_n)(l_n)^{1/p}(t_n)^{1/p-\alpha} \ll \frac{t^\alpha}{k_n^{\frac{\eta}{p}}}.$$

Hence, we have

$$\|\triangle_t\psi_2\|_p \le \sum_{n=1}^{\infty}\|\triangle_t\psi_{2,n}\|_p \le \sum_{t_n>t}\|\triangle_t\psi_{2,n}\|_p + \sum_{t_n\le t}\|\triangle_t\psi_{2,n}\|_p \ll \sum_{t_n>t}\frac{t^\alpha}{k_n^{\frac{\eta}{p}}} + \sum_{t_n\le t}\frac{t^\alpha}{k_n^{\frac{\eta}{p}}} \ll t^\alpha,$$

which implies $\psi_2 \in H_p^\alpha$. Finally, we obtain $\psi = \psi_1 + \psi_2 \in H_p^\alpha$.

However, we have

$$\begin{aligned}\|\triangle_{t_n}(f\circ\psi)\|_p &\ge \left(\int_{\cup_{j=1}^{l_n} J_{n,2j-1}^3} |f\circ\psi(x+t_n) - f\circ\psi(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &\gg k_n(b_n - a_n)(l_n t_n)^{\frac{1}{p}} \asymp k_n^{1-\frac{\eta}{p}} t_n^\alpha.\end{aligned}$$

Since $k_n^{1-\frac{\eta}{p}} \to \infty$ as $n \to \infty$, we obtain $f\circ\psi \notin H_p^\alpha$, which contradicts our assumption. So f is locally Lipschitz.

It remains to show that f is Lipschitz if $0 < \alpha < 1/p$, $1 \le p < \infty$. Assume that f is locally Lipschitz and is not Lipschitz. Then, $f \notin \mathrm{Lip}_{(-\infty,-2)}$ or $f \notin \mathrm{Lip}_{(2,\infty)}$, where $\mathrm{Lip}_I = \left\{f \mid \sup_{x,y\in I}\frac{|f(x)-f(y)|}{|x-y|} < +\infty\right\}$ for a given interval I. There is no loss of generality in assuming that $f \notin \mathrm{Lip}_{(2,\infty)}$. Set $k_n = 2^n$. There exist $2 < a_1 < b_1$ such that

$$|f(b_1) - f(a_1)| \ge k_1|b_1 - a_1|.$$

Since f is not Lipschitz, we have $f \notin \mathrm{Lip}_{(\max\{b_1,k_2\},\infty)}$. So there exist $\max\{b_1, k_2\} < a_2 < b_2$ such that

$$|f(b_2) - f(a_2)| \ge k_2|b_2 - a_2|.$$

In the same manner, we can obtain that there exists a sequence $\{[a_n, b_n]\}$ of intervals satisfying

$$\begin{aligned}&|f(b_n) - f(a_n)| \ge k_n|b_n - a_n|,\\ &2 < a_1 < b_1 < \ldots < a_n < b_n < \ldots, \quad k_n < a_n < b_n.\end{aligned}$$

From the continuity of f and the compactness of $[a_n, b_n]$, we know that f is uniformly continuous on $[a_n, b_n]$. Hence, there is $\delta_n > 0$ such that for any $x, y \in [a_n, b_n]$, $|x - y| < \delta_n$, we have

$$|f(x) - f(y)| < 1.$$

We equally divide $[a_n, b_n]$ into $v_n = \left[\frac{b_n - a_n}{\delta_n}\right] + 1$ parts and denote as $d_{n,j} = a_n + j\frac{b_n - a_n}{v_n}$ $(0 \le j \le v_n)$, thus $a_n = d_{n,0} < d_{n,1} < \ldots < d_{n,v_n-1} < d_{n,v_n} = b_n$ and $|f(d_{n,j}) - f(d_{n,j+1})| < 1$, $j = 0, 1, \ldots, v_n - 1$. We obtain

$$k_n \le \frac{|f(b_n) - f(a_n)|}{|b_n - a_n|} \le \max_{0\le j\le v_n-1}\left\{\frac{|f(d_{n,j}) - f(d_{n,j+1})|}{|d_{n,j} - d_{n,j+1}|}\right\} = \frac{|f(d_{n,j_n}) - f(d_{n,j_n+1})|}{|d_{n,j_n} - d_{n,j_n+1}|},$$

for some j_n, $0 \le j_n \le v_n - 1$. We replace $[a_n, b_n]$ by $[d_{n,j_n}, d_{n,j_n+1}]$ which is also denoted by $[a_n, b_n]$. Thus, we have

$$k_n|b_n - a_n| \le |f(b_n) - f(a_n)| < 1.$$

Take $l_0 = t_0 = 0$ and

$$l_n = \left[\left(\frac{b_n}{b_n - a_n}\right)^{\frac{1}{\alpha}}\right] + 1, \quad t_n = A\left(k_n^{-\eta} b_n^{-\frac{1}{\alpha p}} (b_n - a_n)^{\frac{1-\alpha p}{\alpha p}}\right)^{\frac{p}{1-\alpha p}}, \quad n \ge 1, \tag{3.7}$$

where $0 < \eta < 1$, and A is a positive constant satisfying

$$\sum_{n=1}^{\infty} 2 l_n t_n = 1.$$

The existence of A follows from the following inequality:

$$\sum_{n=1}^{\infty} 2\left(\left[\left(\frac{b_n}{b_n - a_n}\right)^{\frac{1}{\alpha}}\right] + 1\right)\left(k_n^{-\eta} b_n^{-\frac{1}{\alpha p}} (b_n - a_n)^{\frac{1-\alpha p}{\alpha p}}\right)^{\frac{p}{1-\alpha p}} \ll \sum_{n=1}^{\infty} b_n^{\frac{-p}{1-\alpha p}} k_n^{\frac{-\eta p}{1-\alpha p}} \le \sum_{n=1}^{\infty} k_n^{\frac{-\eta p - p}{1-\alpha p}} < \infty.$$

Set $J_n^4 = \left[\sum_{j=0}^{n-1} 2 l_j t_j, \sum_{j=0}^{n} 2 l_j t_j\right]$, $n = 1, 2, \dots$ and

$$J_{n,m}^4 = \left[\sum_{j=0}^{n-1} 2 l_j t_j + (m-1) t_n, \sum_{j=0}^{n-1} 2 l_j t_j + m t_n\right], \quad m = 1, \dots, 2 l_n, \ n = 1, 2, \dots.$$

Similar to the construction as we did in the proof of locally Lipschitz continuity of f, we define $\gamma(x)$ as follows:

$$\gamma(x) = \begin{cases} a_n, & x \in J_{n,2k-1}^4, \quad 1 \le k \le l_n, \\ b_n, & x \in J_{n,2k}^4, \quad 1 \le k \le l_n, \end{cases} \quad n = 1, 2, \dots.$$

We assert that $\gamma \in L_p$, since

$$\|\gamma\|_p^p \le \sum_{n=1}^{\infty} b_n^p 2 l_n t_n \ll \sum_{n=1}^{\infty} k_n^{\frac{-\eta p}{1-\alpha p}} b_n^{\frac{-\alpha p^2}{1-\alpha p}} \ll \sum_{n=1}^{\infty} k_n^{\frac{-\eta p - \alpha p^2}{1-\alpha p}} < \infty.$$

The graph of γ is shown in Figure 7.

Write

$$\gamma_1(x) = \sum_{n=1}^{\infty} a_n \, \chi_{J_n^4}(x), \quad \gamma_2(x) = \gamma(x) - \gamma_1(x) = \sum_{n=1}^{\infty}\left((b_n - a_n)\sum_{j=1}^{l_n} \chi_{J_{n,2j}^4}(x)\right)$$

and

$$\gamma_{n,1}(x) = a_n \chi_{J_n^4}(x), \quad \gamma_{n,2}(x) = (b_n - a_n)\sum_{j=1}^{l_n} \chi_{J_{n,2j}^4}(x).$$

We now prove that $\gamma \in H_p^\alpha$. The proof is similar to the one as in the proof of $f \in \mathrm{Lip}_{\mathrm{loc}}$. For $0 < t < 1$ we have

$$\begin{aligned} \|\triangle_t \gamma_1\|_p &\le \sum_{n=1}^{\infty} \|\triangle_t \gamma_{n,1}\|_p \\ &= \sum_{n=1}^{\infty} (a_n^p |\{x : \gamma_{n,1}(x+t) - \gamma_{n,1}(x) = a_n\}|)^{\frac{1}{p}} \\ &\le \sum_{2 l_n t_n > t} a_n (2t)^{\frac{1}{p}} + \sum_{2 l_n t_n \le t} a_n (4 l_n t_n)^{\frac{1}{p}} \\ &\ll t^\alpha \sum_{2 l_n t_n > t} b_n (l_n t_n)^{\frac{1}{p} - \alpha} + t^\alpha \sum_{2 l_n t_n \le t} b_n (l_n t_n)^{\frac{1}{p} - \alpha} \\ &\asymp t^\alpha \sum_{n=1}^{\infty} k_n^{-\eta} \ll t^\alpha \end{aligned}$$

and

$$\begin{aligned}\|\triangle_t \gamma_2\|_p &\leq \sum_{n=1}^{\infty}\|\triangle_t \gamma_{n,2}\|_p = \sum_{n=1}^{\infty}((b_n-a_n)^p|\{x:\gamma_{n,2}(x+t)-\gamma_{n,2}(x)=b_n-a_n\}|)^{\frac{1}{p}}\\ &\ll \sum_{t_n>t}(b_n-a_n)(l_n t)^{\frac{1}{p}} + \sum_{t_n\leq t}(b_n-a_n)(l_n t_n)^{\frac{1}{p}}\\ &\ll t^{\alpha}\sum_{t_n>t}(b_n-a_n)l_n^{\frac{1}{p}}t_n^{\frac{1}{p}-\alpha} + t^{\alpha}\sum_{t_n\leq t}(b_n-a_n)l_n^{\frac{1}{p}}t_n^{\frac{1}{p}-\alpha}\\ &\asymp t^{\alpha}\sum_{n=1}^{\infty}k_n^{-\eta} \ll t^{\alpha},\end{aligned}$$

from which we deduce that $\gamma=\gamma_1+\gamma_2\in H_p^{\alpha}$. However, we obtain

$$\|\triangle_{t_n}(f\circ\gamma)\|_p \geq k_n|b_n-a_n|(l_n t_n)^{\frac{1}{p}} \gg k_n^{1-\eta}t_n^{\alpha}.$$

Since $k_n^{1-\eta}\to\infty$ as $n\to\infty$, we have $(f\circ\gamma)\notin H_p^{\alpha}$, which contradicts our assumption. So f is Lipschitz.

The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. The proof of the sufficient condition and the necessary condition for which T_f maps $H_p^{1/p}$ $(1<p<\infty)$ into itself was given in the proof of Theorem 1.2. So it suffices to find out a locally Lipschitz function f for which T_f does not map $H_p^{1/p}$ $(1<p<\infty)$ into itself.

Let $1<p<\infty$. We set

$$\zeta(x)=\begin{cases}\ln\dfrac{1}{|x|}, & x\in[-1/2,1/2]\backslash\{0\}\\ 0, & x=0\end{cases} \tag{3.8}$$

and $f(x)=|x|^a$, $a>1$, $x\in\mathbb{R}$. Then, f is locally Lipschitz. We want to prove that $\zeta\in H_p^{1/p}$ and $f\circ\zeta\notin H_p^{1/p}$. First, we prove that $\left|\ln\left(1+\frac{1}{u}\right)\right|^p$ and $\left|\ln\left(\frac{1}{u}-1\right)\right|^p$ are Lebesgue integrable on $[0,1]$.

Let $A=\int_0^1\left|\ln\left(1+\frac{1}{u}\right)\right|^p du=\int_0^1\left(\ln\left(1+\frac{1}{u}\right)\right)^p du$ and

$$B=\int_0^1\left|\ln\left(\frac{1}{u}-1\right)\right|^p du=\int_0^{\frac{1}{2}}\left(\ln\left(\frac{1}{u}-1\right)\right)^p du+\int_{\frac{1}{2}}^1\left|\ln\left(\frac{1}{u}-1\right)\right|^p du=D+C.$$

For C, making the change in variable $x=\frac{u}{1-u}$ yields:

$$C=\int_{\frac{1}{2}}^1\left|\ln\left(\frac{1}{u}-1\right)\right|^p du=\int_{\frac{1}{2}}^1\left(\ln\left(\frac{u}{1-u}\right)\right)^p du=\int_1^{+\infty}\frac{(\ln x)^p}{(x+1)^2}dx.$$

By the comparison theorem, and the convergence of integrals $\int_0^1 u^{-\frac{1}{2}}du$ and $\int_1^{+\infty}u^{-\frac{3}{2}}du$, as well as:

$$\lim_{x\to+\infty}\frac{\frac{(\ln x)^p}{(x+1)^2}}{x^{-\frac{3}{2}}}=0,\quad \lim_{u\to0^+}\frac{\left(\ln\left(\frac{1}{u}-1\right)\right)^p}{u^{-\frac{1}{2}}}=0,\quad \lim_{u\to0^+}\frac{\left(\ln\left(1+\frac{1}{u}\right)\right)^p}{u^{-\frac{1}{2}}}=0.$$

we obtain that $\left|\ln\left(1+\frac{1}{u}\right)\right|^p$ and $\left|\ln\left(\frac{1}{u}-1\right)\right|^p$ are Rieman integrable on $[0,1]$. Furthermore, by [15, Theorem 2.28] and the monotone convergence theorem [15, Theorem 2.14], we deduce that $\left|\ln\left(1+\frac{1}{u}\right)\right|^p$ and $\left|\ln\left(\frac{1}{u}-1\right)\right|^p$ are Lebesgue integrable on $[0,1]$.

Next we prove $\zeta \in H_p^{1/p}$. Without loss of generality we can assume that $0 < t < 1/8$. We have

$$\|\triangle_t \zeta\|_p^p = \left(\int_{-\frac{1}{2}}^{-2t} + \int_{-2t}^{-t} + \int_{-t}^{0} + \int_{0}^{t} + \int_{t}^{\frac{1}{2}-t} + \int_{\frac{1}{2}-t}^{\frac{1}{2}} \right) |\triangle_t \zeta|^p \mathrm{d}x = I + II + III + IV + V + VI.$$

Using the transformation $x = -u$ in I and the inequality $|\ln(1-s)| \le \frac{s}{1-s}$, $0 < s < 1$, we obtain

$$I = \int_{-\frac{1}{2}}^{-2t} \left| \ln \left| \frac{x+t}{x} \right| \right|^p \mathrm{d}x = \int_{2t}^{\frac{1}{2}} \left| \ln\left(1 - \frac{t}{u}\right) \right|^p \mathrm{d}u \le \int_{2t}^{\frac{1}{2}} \left(\frac{t}{u-t} \right)^p \mathrm{d}u \ll t.$$

Applying the transformation $x = -(u+1)t$ in II, $x = -ut$ in III, $x = ut$ in IV, and noting that the functions $\left|\ln\left(1 + \frac{1}{u}\right)\right|^p$ and $\left|\ln\left(\frac{1}{u} - 1\right)\right|^p$ are integrable on $[0, 1]$, we have

$$II = \int_{-2t}^{-t} \left| \ln \left| \frac{x+t}{x} \right| \right|^p \mathrm{d}x = t \int_0^1 \left(\ln\left(1 + \frac{1}{u}\right) \right)^p \mathrm{d}u \ll t,$$

$$III = \int_{-t}^{0} \left| \ln \left| \frac{x+t}{x} \right| \right|^p \mathrm{d}x = t \int_0^1 \left| \ln\left(-1 + \frac{1}{u}\right) \right|^p \mathrm{d}u \ll t,$$

and

$$IV = \int_0^t \left(\ln \left| \frac{x+t}{x} \right| \right)^p \mathrm{d}x = t \int_0^1 \left(\ln\left(1 + \frac{1}{u}\right) \right)^p \mathrm{d}u \ll t.$$

Using the inequality $\ln(1 + x) \le x$, $x > 0$ in V, we obtain

$$V = \int_t^{\frac{1}{2}-t} \left(\ln \left| \frac{x+t}{x} \right| \right)^p \mathrm{d}x = \int_t^{\frac{1}{2}-t} \left(\ln\left(1 + \frac{t}{x}\right) \right)^p \mathrm{d}x \le \int_t^{\frac{1}{2}-t} \left(\frac{t}{x} \right)^p \mathrm{d}x \ll t.$$

Finally, we note that $x, |x + t - 1| \in [1/4, 1/2]$ if $x \in [1/2 - t, 1/2]$. This gives that

$$VI = \int_{\frac{1}{2}-t}^{\frac{1}{2}} \left(\ln \left| \frac{x+t-1}{x} \right| \right)^p \mathrm{d}x \le (\ln 2)^p\, t \ll t.$$

Hence, we have

$$\|\triangle_t \zeta\|_p = (I + II + III + IV + V + VI)^{\frac{1}{p}} \ll t^{\frac{1}{p}},$$

which implies that $\zeta \in H_p^{1/p}$.

Next we show that $f \circ \zeta \notin H_p^{1/p}$. We have for $t \in (0, 1/(3e))$,

$$\begin{aligned}
\|\triangle_t (f \circ \zeta)\|_p^p &\ge \int_t^{2t} \left| \int_0^t (f \circ \zeta)'(u + x) \mathrm{d}u \right|^p \mathrm{d}x \\
&= \int_t^{2t} \left| (-a) \int_0^t \left(\ln \frac{1}{u+x} \right)^{a-1} \frac{1}{x+u} \mathrm{d}u \right|^p \mathrm{d}x \\
&\ge \int_t^{2t} \left| a \int_0^t \left(\ln \frac{1}{3t} \right)^{a-1} \frac{1}{3t} \mathrm{d}u \right|^p \mathrm{d}x = \left(\frac{a}{3} \right)^p t \left(\ln \frac{1}{3t} \right)^{p(a-1)}.
\end{aligned}$$

Since $\left(\ln \frac{1}{3t} \right)^{a-1} \to \infty$ as $t \to 0+$, we obtain $f \circ \zeta \notin H_p^{1/p}$.

The proof of Theorem 1.3 is complete. □

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