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Research Article

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Asymptotic analysis of Leray solution for the incompressible NSE with damping

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Abstract: In 2008, Cai and Jiu showed that the Cauchy problem of the Navier-Stokes equations, with damping $\alpha\,|u|^{\beta-1}u$ for $\alpha > 0$ and $\beta \geq 1$ has global weak solutions in $L^2(\mathbb{R}^3)$. In this article, we study the uniqueness and the continuity in $L^2(\mathbb{R}^3)$ of this global weak solution. We also prove the large time decay for this global solution for $\beta \geq \frac{10}{3}$.

Keywords: Navier-Stokes equations, Friedrich method, global weak solution

MSC 2020: Primary 35-XX, 35Q30, 76D05, 76N10

1 Introduction

Consider the three-dimensional Navier-Stokes equations (NSEs) with nonlinear damping

$$\begin{cases} \partial_t u - \nu\Delta u + u.\nabla u + \alpha|u|^{\beta-1}u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \text{div } u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,x) = u^0(x) & \text{in } \mathbb{R}^3 \\ \alpha > 0, \;\; \beta > 1, & \end{cases} \tag{NSD}$$

where $u = u(t,x) = (u_1, u_2, u_3)$ and $p = p(t,x)$ are, respectively, the unknown velocity and the pressure of the fluid at $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, ν represents the viscosity of the fluid, and $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ represents the initial given velocity.

The NSEs govern the motion of incompressible fluids. The damping is from the resistance to the motion of the flow and describes various physical situations (see [1] and references therein).

The global existence of a weak solution to the initial value problem of the classical incompressible Navier-Stokes was proved by Leray and Hopf (see [2,3]) long before. The uniqueness remains an open problem for the dimensions $d \geq 3$.

The case of polynomial damping $\alpha|u|^{\beta-1}u$ was studied in 2005 by Zhou [4] for $\alpha = 1$. He proved the existence of global strong solution for $\beta \geq 3$.

In 2008, Cai and Jiu [5] proved the existence of global weak solution for $\beta \geq 1$, the existence of global strong solution for $\beta \geq \frac{7}{2}$, and the uniqueness of the strong solution for $\frac{7}{2} \leq \beta \leq 5$. They employ the Galerkin approximations to construct the global solution of the system (NSD).

In 2011, Jia et al. [6] studied the L^2 decay of the weak solutions with $\beta \geq \frac{10}{3}$ and $\alpha = 1$.

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For $\beta > 3$ and u_0 in a suitable space, Zhang et al. [7] proved the existence of global strong solution and they prove the uniqueness if $3 < \beta \le 5$. They also stated the existence and uniqueness of the strong solution to the system (NSD) for $\beta > 3$, $\alpha > 0$, and $\alpha \ge \frac{1}{2}$ if $\beta = 3$.

In 2017, Liu and Gao [8] established the L^2 decay of the weak solutions for $\beta > 2$ and $\alpha > 0$. They also stated the asymptotic stability of the solution if $\beta > 3$ and $\alpha > 0$ and if $\beta = 3$ for $\alpha \ge \frac{1}{2}$.

In this article, we study the uniqueness, continuity, and the large time decay of the global solution of the system (NSD). We use the Friedrich method to prove the continuity and the uniqueness of such solution for $\beta > 3$. We also study the large time decay for $\beta \ge \frac{10}{3}$. Precisely, our main result is the following:

Theorem 1.1. *Let $\beta > 3$ and $u^0 \in L^2(\mathbb{R}^3)$ a divergence-free vector field, then there is a unique global solution of (NSD): $u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3))$. Moreover, for all $t \ge 0$*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2\alpha \int_0^t \|u(s)\|_{L^{\beta+1}}^{\beta+1} ds \le \|u^0\|_{L^2}^2. \tag{1.1}$$

Also, if $\beta \ge \frac{10}{3}$, we have

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{1.2}$$

Remark 1.2.

(1) In [6], the authors study the L^2 decay of weak solutions by developing the Fourier splitting method [9, 10] for $\alpha = 1$ and $\beta \ge \frac{10}{3}$. In this article, we use a different method called the Friedrich method to study the L^2 decay of the weak solution. We prove the uniqueness, the continuity of the global solution in $L^2(\mathbb{R}^3)$, and the asymptotic behavior (1.2). This inequality is proved in [5].

(2) The case of fractional order NSEs with damping can be studied based on the previous work [11,12].

2 Preliminary results

In the following, we recall some preliminary results. The reader is referred to [12–14] for more details of the proofs.

Proposition 2.1.

(1) *In a Hilbert space H, the unit ball is weakly compact, i.e., if $(x_n)_n$ is a sequence in the unit ball of H, then there is a subsequence $(x_{\varphi(n)})_n$ such that*

$$\lim_{n\to+\infty} (x_{\varphi(n)}|y) = (x|y), \quad \forall y \in H.$$

(2) *If $x \in H$ is a weak limit of a bounded sequence $(x_n)_n$ in H, then*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

(3) *If $x \in H$ is a weak limit of a bounded sequence $(x_n)_n$ in H, and $\limsup_{n\to\infty}\|x_n\| \le \|x\|$, then $\lim_{n\to\infty}\|x_n - x\| = 0$.*

Lemma 2.2. *Let s_1 and s_2 be two real numbers and $N \in \mathbb{N}$.*

- *If $s_1 < \frac{N}{2}$ and $s_1 + s_2 > 0$, there exists a constant $C_1 = C_1(N, s_1, s_2)$, such that if $f, g \in \dot{H}^{s_1}(\mathbb{R}^N) \cap \dot{H}^{s_2}(\mathbb{R}^N)$, then $f.\,g \in \dot{H}^{s_1+s_2-\frac{N}{2}}(\mathbb{R}^N)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{N}{2}}} \le C_1(\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}}\|g\|_{\dot{H}^{s_1}}).$$

- *If $s_1, s_2 < \frac{N}{2}$ and $s_1 + s_2 > 0$, there exists a constant $C_2 = C_2(N, s_1, s_2)$ such that if $f \in \dot{H}^{s_1}(\mathbb{R}^N)$ and $g \in \dot{H}^{s_2}(\mathbb{R}^N)$, then $f.\, g \in \dot{H}^{s_1+s_2-\frac{N}{2}}(\mathbb{R}^N)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{N}{2}}} \leq C_2\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}}.$$

Lemma 2.3. *Let $r > 0$, then, for all $x, y \in \mathbb{R}^N$, we have*

$$\langle |x|^r x - |y|^r y, x - y\rangle \geq \frac{1}{2}(|x|^r + |y|^r)|x - y|^2. \tag{2.1}$$

The Leray projector $\mathbb{P} : (L^2(\mathbb{R}^3))^3 \to (L^2_\sigma(\mathbb{R}^3))^3$ is defined by

$$\mathcal{F}(\mathbb{P}f) = \widehat{f}(\xi) - \left(\widehat{f}(\xi).\, \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} = M(\xi)\widehat{f}(\xi),$$

where $M(\xi)$ is the matrix $\left(\delta_{k,\ell} - \frac{\xi_k\xi_\ell}{|\xi|^2}\right)_{1\leq k,\ell\leq 3}$ and $L^2_\sigma(\mathbb{R}^3)$ represents the space of divergence-free vector fields in $L^2(\mathbb{R}^3)$. In particular, if u is in the Schwartz space $(\mathcal{S}(\mathbb{R}^3))^3$, the Leray projector $\mathbb{P}$ is given by

$$\mathbb{P}(u)_k(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\left(\delta_{kj} - \frac{\xi_k\xi_j}{|\xi|^2}\right)\hat{u}_j(\xi)\, e^{i\xi\cdot x}\mathrm{d}\xi.$$

For $R > 0$, the Friedrich operator J_R is defined on $L^2(\mathbb{R}^3)$ by $J_R(D)f = \mathcal{F}^{-1}(\chi_{B_R}\widehat{f})$ and the operator $A_R(D)$ is defined by

$$A_R(D)u = \mathbb{P}J_R(D)u = \mathcal{F}^{-1}(M(\xi)\chi_{B_R}(\xi)\hat{u}),$$

with B_R the ball of center 0 and radius $R > 0$.

The following result is a generalization of Proposition 3.1 in [15]. The proof is similar to that of Proposition 2.4 in [12].

Proposition 2.4. *Let $\nu_1, \nu_2, \nu_3 \in [0, \infty)$, $r_1, r_2, r_3 \in (0, \infty)$, and $f^0 \in L^2_\sigma(\mathbb{R}^3)$.*

For $n \in \mathbb{N}$, let $F_n : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function in $C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ such that $A_n(D)F_n = F_n$, $F_n(0, x) = A_n(D)f^0(x)$, and

$$\partial_t F_n + \sum_{k=1}^{3}\nu_k\, |D_k|^{2r_k}F_n + A_n(D)\,\mathrm{div}(F_n \otimes F_n) + A_n(D)h(|F_n|)F_n = 0.$$

$$\|F_n(t)\|^2_{L^2} + 2\sum_{k=1}^{3}\nu_k\int_0^t\||D_k|^{r_k}F_n(s)\|^2_{L^2}\mathrm{d}s + 2\int_0^t\|h(|F_n(s)|)|F_n(s)|^2\|_{L^1}\mathrm{d}s \leq \|f^0\|^2_{L^2},$$

where $h(z) = \alpha z^{\beta-1}$, with $\alpha > 0$ and $\beta > 3$. Then: for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, \alpha, \beta, \nu_1, \nu_2, \nu_3, r_1, r_2, r_3, \|f^0\|_{L^2}) > 0$ such that for all $t_1, t_2 \in \mathbb{R}^+$, we have

$$(|t_2 - t_1| < \delta \Rightarrow \|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} < \varepsilon),\ \ \forall n \in \mathbb{N},$$

with $s_0 \geq \max(3, 2r_1, 2r_2, 2r_3)$.

3 Proof of Theorem 1.1

3.1 Existence of solution

Consider the approximate system:

$$(\mathrm{NSD}_n)\begin{cases}\partial_t u - \Delta J_n u + J_n(J_n u.\nabla J_n u) + \alpha J_n[|J_n u|^{\beta-1}J_n u] = -\nabla p_n \ \text{ in } \mathbb{R}^+ \times \mathbb{R}^3\\ p_n = (-\Delta)^{-1}(\mathrm{div}\, J_n(J_n u.\nabla J_n u) + \alpha\mathrm{div}\, J_n[|J_n u|^{\beta-1}J_n u])\\ \mathrm{div}\ u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3\\ u(0, x) = J_n u^0(x) \quad \text{in } \mathbb{R}^3.\end{cases}$$

- By the Cauchy-Lipschitz theorem, there exists a unique solution $u_n \in C^1(\mathbb{R}^+, L^2_\sigma(\mathbb{R}^3))$ of the system (NSD$_n$) such that $J_n u_n = u_n$ and

$$\|u_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u_n(s)\|_{L^2}^2 ds + 2\alpha\int_0^t \|u_n(s)\|_{L^{\beta+1}}^{\beta+1} ds \le \|u^0\|_{L^2}^2. \tag{3.1}$$

- The sequence $(u_n)_n$ is bounded on $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$. Hence, by Proposition 2.4 and the interpolation method, the sequence $(u_n)_n$ is equicontinuous on $H^{-1}(\mathbb{R}^3)$.
- For a strictly increasing sequence $(T_k)_k$ such that $\lim_{k\to+\infty} T_k = \infty$, consider a sequence of functions $(\delta_k)_k$ in $C_0^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} \delta_k(x) = 1, & \text{for } |x| \le k + \frac{5}{4} \\ \delta_k(x) = 0, & \text{for } |x| \ge k + 2 \\ 0 \le \delta_k \le 1. \end{cases}$$

  Using the energy estimate (3.1), the equicontinuity of the sequence $(u_n)_n$ on $H^{-1}(\mathbb{R}^3)$ and classical argument by combining Ascoli's theorem and the Cantor diagonal process, there exists a subsequence $(u_{\varphi(n)})_n$ and $u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3))$ such that for all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \|\delta_k(u_{\varphi(n)} - u)\|_{L^\infty([0,T_k],H^{-4})} = 0. \tag{3.2}$$

  In particular, the sequence $(u_{\varphi(n)}(t))_n$ is weakly convergent in $L^2(\mathbb{R}^3)$ to $u(t)$ for all $t \ge 0$.
- Using the same method as in [15], we obtain

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2\alpha\int_0^t \|u(s)\|_{L^{\beta+1}}^{\beta+1} ds \le \|u^0\|_{L^2}^2, \tag{3.3}$$

  for all $t \ge 0$, and u a solution of the system (NSD).

3.2 Continuity of the solution in L^2

Using inequality (3.3), we obtain $\limsup_{t\to 0}\|u(t)\|_{L^2} \le \|u^0\|_{L^2}$ and by Proposition 2.1-(3), we have $\limsup_{t\to 0}\|u(t) - u^0\|_{L^2} = 0$. This ensures the continuity of the solution u at 0. To prove the continuity on $\mathbb{R}^+$, consider the functions $v_{n,\varepsilon}(t) = u_{\varphi(n)}(t+\varepsilon)$, $p_{n,\varepsilon}(t) = p_{\varphi(n)}(t+\varepsilon)$, for $n \in \mathbb{N}$ and $\varepsilon > 0$. We have

$$\begin{aligned} \partial_t u_{\varphi(n)} - \Delta u_{\varphi(n)} + J_{\varphi(n)}(u_{\varphi(n)}.\nabla u_{\varphi(n)}) + aJ_{\varphi(n)}(|u_{\varphi(n)}|^{\beta-1}u_{\varphi(n)}) &= -\nabla p_{\varphi(n)} \\ \partial_t v_{n,\varepsilon} - \Delta v_{n,\varepsilon} + J_{\varphi(n)}(v_{n,\varepsilon}.\nabla v_{n,\varepsilon}) + aJ_{\varphi(n)}(|v_{n,\varepsilon}|^{\beta-1}v_{n,\varepsilon}) &= -\nabla p_{n,\varepsilon}. \end{aligned}$$

The function $w_{n,\varepsilon} = u_{\varphi(n)} - v_{n,\varepsilon}$ verifies the following:

$$\begin{aligned} &\partial_t w_{n,\varepsilon} - \Delta w_{n,\varepsilon} + aJ_{\varphi(n)}(|u_{\varphi(n)}|^{\beta-1}u_{\varphi(n)} - |v_{\varphi(n)}|^{\beta-1}v_{n,\varepsilon}) \\ &\quad = -\nabla(p_{\varphi(n)} - p_{n,\varepsilon}) + J_{\varphi(n)}(w_{n,\varepsilon}.\nabla w_{n,\varepsilon}) - J_{\varphi(n)}(w_{n,\varepsilon}.\nabla u_{\varphi(n)}) - J_{\varphi(n)}(u_{\varphi(n)}.\nabla w_{n,\varepsilon}). \end{aligned}$$

Taking the scalar product with $w_{n,\varepsilon}$ in $L^2(\mathbb{R}^3)$ and using that $\text{div} w_{n,\varepsilon} = 0$ and $\langle w_{n,\varepsilon}.\nabla w_{n,\varepsilon}, w_{n,\varepsilon}\rangle = 0$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_{n,\varepsilon}(t)\|_{L^2}^2 + \|\nabla w_{n,\varepsilon}(t)\|_{L^2}^2 + \alpha\langle J_{\varphi(n)}(|u_{\varphi(n)}|^{\beta-1}u_{\varphi(n)} - |v_{n,\varepsilon}|^{\beta-1}v_{n,\varepsilon}); w_{n,\varepsilon}\rangle_{L^2} \\ &\quad = -\langle J_{\varphi(n)}(w_{n,\varepsilon}.\nabla u_{\varphi(n)}); w_{n,\varepsilon}\rangle_{L^2}. \end{aligned} \tag{3.4}$$

By inequality (2.1), we have

$$\begin{aligned}
&\langle J_{\varphi(n)}(|u_{\varphi(n)}|^{\beta-1}u_{\varphi(n)} - |v_{n,\varepsilon}|^{\beta-1}v_{n,\varepsilon});\ w_{n,\varepsilon}\rangle_{L^2}\\
&\quad = \langle |u_{\varphi(n)}|^{\beta-1}u_{\varphi(n)} - |v_{n,\varepsilon}|^{\beta-1}v_{n,\varepsilon}; J_{\varphi(n)}w_{n,\varepsilon}\rangle_{L^2}\\
&\quad = \langle |u_{\varphi(n)}|^{\beta-1}u_{\varphi(n)} - |v_{n,\varepsilon}|^{\beta-1}v_{n,\varepsilon};\ w_{n,\varepsilon}\rangle_{L^2}\\
&\quad \geq \frac{1}{2}\int_{\mathbb{R}^3}(|u_{\varphi(n)}|^{\beta-1} + |v_{n,\varepsilon}|^{\beta-1})|w_{n,\varepsilon}|^2\\
&\quad \geq \frac{1}{2}\int_{\mathbb{R}^3}|u_{\varphi(n)}|^{\beta-1}|w_{n,\varepsilon}|^2,
\end{aligned}$$

which implies

$$\alpha\langle J_{\varphi(n)}(|u_{\varphi(n)}|^{\beta-1}u_{\varphi(n)} - (|v_{n,\varepsilon}|^{\beta-1}v_{n,\varepsilon}));w_{n,\varepsilon}\rangle_{L^2} \geq \frac{\alpha}{2}\int_{\mathbb{R}^3}|u_{\varphi(n)}|^{\beta-1}|w_{n,\varepsilon}|^2. \tag{3.5}$$

Also, we have

$$|\langle J_{\varphi(n)}(w_{n,\varepsilon}.\ \nabla u_{\varphi(n)});\ w_{n,\varepsilon}\rangle_{L^2}| \leq \int_{\mathbb{R}^3}|w_{n,\varepsilon}|.\ |u_{\varphi(n)}|.\ |\nabla w_{n,\varepsilon}| \leq \frac{1}{2}\int_{\mathbb{R}^3}|w_{n,\varepsilon}|^2|u_{\varphi(n)}|^2 + \frac{1}{2}\|\nabla w_{n,\varepsilon}\|_{L^2}^2.$$

The convex inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq a^p + b^q$, with $p = \frac{\beta-1}{2}$, $q = \frac{\beta-1}{\beta-3}$, $a = |w_{n,\varepsilon}|^2(\frac{\alpha}{2})^{\frac{2}{\beta-1}}$, $b = (\frac{2}{\alpha})^{\frac{2}{\beta-1}}$, yields

$$|\langle J_{\varphi(n)}(w_{n,\varepsilon}.\ \nabla u_{\varphi(n)});\ w_{n,\varepsilon}\rangle_{L^2}| \leq \frac{\alpha}{4}\int_{\mathbb{R}^3}|w_{n,\varepsilon}|^{\beta-1}|u_{\varphi(n)}|^2 + C_{\alpha,\beta}\|w_{n,\varepsilon}\|_{L^2}^2 + \frac{1}{2}\|\nabla w_{n,\varepsilon}\|_{L^2}^2,$$

where $C_{\alpha,\beta} = \frac{1}{2}(\frac{2}{\alpha})^{\frac{2}{\beta-3}}$. Combining this inequality and inequalities (3.4), (3.5), and (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_{n,\varepsilon}\|_{L^2}^2 + \|\nabla w_{n,\varepsilon}\|_{L^2}^2 \leq 2C_{\alpha,\beta}\|w_{n,\varepsilon}\|_{L^2}^2.$$

By the Gronwall lemma, we deduce

$$\|w_{n,\varepsilon}(t)\|_{L^2} \leq \|w_{n,\varepsilon}(0)\|_{L^2}e^{C_{\alpha,\beta}t}$$

and

$$\|u_{\varphi(n)}(t+\varepsilon) - u_{\varphi(n)}(t)\|_{L^2} \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}e^{C_{\alpha,\beta}t}.$$

For $t_0 > 0$ and $\varepsilon \in (0, t_0)$, we have

$$\|u_{\varphi(n)}(t_0+\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2} \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}e^{C_{\alpha,\beta}t_0}.$$

$$\|u_{\varphi(n)}(t_0-\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2} \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}e^{C_{\alpha,\beta}t_0}.$$

So

$$\begin{aligned}
\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 &= \|J_{\varphi(n)}u_{\varphi(n)}(\varepsilon) - J_{\varphi(n)}u_{\varphi(n)}(0)\|_{L^2}^2\\
&= \|J_{\varphi(n)}(u_{\varphi(n)}(\varepsilon) - u^0)\|_{L^2}^2\\
&\leq \|u_{\varphi(n)}(\varepsilon) - u^0\|_{L^2}^2\\
&\leq \|u_{\varphi(n)}(\varepsilon)\|_{L^2}^2 + \|u^0\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u^0\rangle\\
&\leq 2\|u^0\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u^0\rangle.
\end{aligned}$$

But $\lim_{n\to+\infty}\langle u_{\varphi(n)}(\varepsilon), u^0\rangle = \langle u(\varepsilon), u^0\rangle$, hence

$$\liminf_{n\to\infty}\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \leq 2\|u^0\|_{L^2}^2 - 2\mathrm{Re}\langle u(\varepsilon);\ u^0\rangle_{L^2}.$$

Moreover, for all $k, N \in \mathbb{N}$

$$\|J_N(\delta_k.\,(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)))\|_{L^2}^2 \leq \|\delta_k.\,(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2}^2 \\ \leq \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.$$

Using (3.2) we obtain, for k big enough,

$$\|J_N(\delta_k.\,(u(t_0 \pm \varepsilon) - u(t_0)))\|_{L^2} \leq \liminf_{n\to\infty} \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}.$$

Then

$$\|J_N(\delta_k.\,(u(t_0 \pm \varepsilon) - u(t_0)))\|_{L^2}^2 \leq 2(\|u^0\|_{L^2}^2 - \mathrm{Re}\langle u(\varepsilon);\, u^0\rangle_{L^2})e^{2C_{\alpha,\beta}t_0}.$$

By applying the monotone convergence theorem in the order N and q, we obtain

$$\|u(t_0 \pm \varepsilon) - u(t_0)\|_{L^2}^2 \leq 2(\|u^0\|_{L^2}^2 - Re\langle u(\varepsilon);\, u^0\rangle_{L^2})e^{2C_{\alpha,\beta}t_0}.$$

Using the continuity at 0, we deduce the continuity at t_0 by making $\varepsilon \to 0$.

3.3 Uniqueness

Let u and v be two solutions of (NSD) in the space

$$C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)).$$

The function $w = u - v$ satisfies the following:

$$\partial_t w - \Delta w + \alpha(|u|^{\beta-1}u - |v|^{\beta-1}v) = -\nabla(p - \tilde{p}) + w.\nabla w - w.\nabla u - u.\nabla w.$$

Taking the scalar product in L^2 with w, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \alpha\langle(|u|^{\beta-1}u - |v|^{\beta-1}v);\, w\rangle_{L^2} = -\langle w.\nabla u;\, w\rangle_{L^2}.$$

By adapting the same method for the proof of the continuity of such solution in $L^2(\mathbb{R}^3)$, with u, v, w instead of $u_{\varphi(n)}, v_{n,\varepsilon}, w_{n,\varepsilon}$ in order, we find

$$\alpha\langle(|u|^{\beta-1}u - |v|^{\beta-1}v);\, w\rangle_{L^2} \geq \frac{\alpha}{2}\int_{\mathbb{R}^3} |u|^{\beta-1}|w|^2$$

and

$$|\langle w.\nabla u;\, w\rangle_{L^2}| \leq \frac{\alpha}{4}\int_{\mathbb{R}^3} |w|^2|u|^2 + C_{\alpha,\beta}\|w\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2.$$

Combining the above inequalities, we obtain the following energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \leq 2C_{\alpha,\beta}\|w(t)\|_{L^2}^2.$$

By Gronwall lemma inequality, we obtain

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 \leq \|w^0\|_{L^2}^2 e^{2C_{\alpha,\beta}t}.$$

As $w^0 = 0$, then $w = 0$ and $u = v$, which implies the uniqueness.

3.4 Asymptotic study of the global solution

To prove the asymptotic behavior (1.2), we need some preliminary lemmas:

Lemma 3.1. *If u is a global solution of (NSD) with $\beta \geq \frac{10}{3}$, then $u \in L^\beta(\mathbb{R}^+ \times \mathbb{R}^3)$.*

Proof. Let $E_1 = \{(t,x) : |u(t,x)| \leq 1\}$, $E_2 = \{(t,x) : |u(t,x)| > 1\}$, $L_1 = \int_{E_1} |u(s,x)|^\beta dx ds$, and $L_2 = \int_{E_2} |u(s,x)|^\beta dx ds$. We have

$$L_1 = \int_{E_1} |u(s,x)|^\beta dx ds = \int_{E_1} |u(s,x)|^{\beta - \frac{10}{3}} |u(s,x)|^{\frac{10}{3}} dx ds \leq \int_0^\infty \|u(s)\|_{L^{\frac{10}{3}}}^{\frac{10}{3}} ds.$$

Using the Sobolev injection $\dot{H}^{\frac{3}{5}}(\mathbb{R}^3) \hookrightarrow L^{\frac{10}{3}}(\mathbb{R}^3)$, we obtain

$$L_1 \leq C \int_0^\infty \|u(s)\|_{\dot{H}^{\frac{3}{5}}}^{\frac{10}{3}} ds.$$

By interpolation inequality $\|u(s)\|_{\dot{H}^{\frac{3}{5}}} \leq \|u(s)\|_{\dot{H}^0}^{\frac{2}{5}} \|u(s)\|_{\dot{H}^1}^{\frac{3}{5}}$, we obtain

$$L_1 \leq C \int_0^\infty \|u(s)\|_{L^2}^{\frac{4}{3}} \|\nabla u(s)\|_{L^2}^2 \leq C \|u^0\|_{L^2}^{\frac{4}{3}} \int_0^\infty \|\nabla u(s)\|_{L^2}^2.$$

Moreover, for the term L_2, we have

$$L_2 = \int_{E_2} |u(s,x)|^\beta dx ds \leq \int_0^\infty \int_{\mathbb{R}^3} |u(s,x)|^{\beta+1} dx ds.$$

Hence

$$\|u\|_{L^\beta(\mathbb{R}^+ \times \mathbb{R}^3)} \leq C \|u^0\|_{L^2}^{\frac{4}{3}} \int_0^\infty \|\nabla u(s)\|_{L^2}^2 ds + \int_0^\infty \int_{\mathbb{R}^3} |u(s,x)|^{\beta+1} dx ds.$$

Therefore, $u \in L^\beta(\mathbb{R}^+ \times \mathbb{R}^3)$. □

Lemma 3.2. *If u is a global solution of (NSD), with $\beta \geq \frac{10}{3}$, then $\lim_{t\to\infty} \|u(t)\|_{H^{-2}} = 0$.*

Proof. For $\varepsilon > 0$, using the energy inequality (1.1) and Lemma 3.1, there exists $t_0 \geq 0$ such that

$$\|\nabla u\|_{L^2([t_0,\infty) \times \mathbb{R}^3)} < \frac{\varepsilon}{4} \tag{3.6}$$

and

$$\|u\|_{L^\beta([t_0,\infty) \times \mathbb{R}^3)} < \frac{\varepsilon}{4}. \tag{3.7}$$

Now, consider the following system:

$$\begin{cases} \partial_t v - \nu \Delta v + v.\nabla v + \alpha\, |v|^{\beta-1} v = -\nabla q & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ v(0,x) = u(t_0,x) & \text{in } \mathbb{R}^3. \end{cases} \tag{NSD'}$$

By the existence and uniqueness part, the system (NSD′) has a unique global solution $v \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3))$ such that $v(t_0) = u(t_0,x)$ and $q(t) = p(t_0 + t)$. The energy estimate for this system is as follows:

$$\|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s)\|_{L^2}^2 ds + 2a \int_0^t \|v(s)\|_{L^{\beta+1}}^{\beta+1} \leq \|u(t_0)\|_{L^2}^2 \leq \|u^0\|_{L^2}^2.$$

By the Duhamel formula, we obtain

$$v(t,x) = e^{t\Delta}v^0(x) + f(t,x) + g(t,x),$$

where

$$f(t,x) = -\int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}(v\otimes v)(s,x)\mathrm{d}s$$

and

$$g(t,x) = -\alpha\int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,|v(s,x)|^{\beta-1}v(s,x)\mathrm{d}s.$$

By dominated convergence theorem, $\lim_{t\to\infty}\|e^{t\Delta}v^0\|_{L^2} = 0$ and hence $\lim_{t\to\infty}\|e^{t\Delta}v^0\|_{H^{-2}} = 0$. Moreover,

$$\begin{aligned}
\|f(t)\|^2_{H^{-2}} \le \|f(t)\|^2_{H^{-\frac{1}{2}}} &\le \|f(t)\|^2_{\dot{H}^{-\frac{1}{2}}}\\
&\le \int_{\mathbb{R}^3}|\xi|^{-1}\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}\mathrm{div}(v\otimes\ v)(s,\xi)|\mathrm{d}s\right)^2\mathrm{d}\xi\\
&\le \int_{\mathbb{R}^3}|\xi|\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}(v\otimes\ v)(s,\xi)|\mathrm{d}s\right)^2\mathrm{d}\xi.
\end{aligned}$$

Since

$$\begin{aligned}
\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}(v\otimes\ v)(s,\xi)|\mathrm{d}s\right)^2 &\le \left(\int_0^t e^{-2(t-s)|\xi|^2}\mathrm{d}s\right)\int_0^t|\mathcal{F}(v\otimes\ v)(s,\xi)|^2\mathrm{d}s\\
&\le |\xi|^{-2}\int_0^t|\mathcal{F}(v\otimes\ v)(s,\xi)|^2\mathrm{d}s,
\end{aligned}$$

then

$$\begin{aligned}
\|f(t)\|^2_{H^{-2}}\mathrm{d}t &\le \int_{\mathbb{R}^3}|\xi|^{-1}\int_0^t|\mathcal{F}(v\otimes\ v)(s,\xi)|^2\mathrm{d}s\mathrm{d}\xi\\
&\le \int_0^t\left(\int_{\mathbb{R}^3}|\xi|^{-1}|(v\otimes\ v)(s,\xi)|^2\mathrm{d}\xi\right)\mathrm{d}s = \int_0^t\|(v\otimes\ v)(s)\|^2_{\dot{H}^{-\frac{1}{2}}}\mathrm{d}s.
\end{aligned}$$

Using the product law in homogeneous Sobolev spaces, with $s_1 = 0$, $s_2 = 1$, we obtain

$$\|f(t)\|^2_{H^{-2}}\mathrm{d}t \le C\int_0^t\|v(s)\|^2_{L^2}\|\nabla v(s)\|^2_{L^2}\mathrm{d}s.$$

Using inequalities (3.6) and (3.7), we obtain

$$\begin{aligned}
\|f(t)\|^2_{H^{-2}}\mathrm{d}t &\le C\|u^0\|^2_{L^2}\int_0^t\|\nabla u(t_0+s)\|^2_{L^2}\mathrm{d}s\\
&\le C\|u^0\|^2_{L^2}\int_0^\infty\|\nabla u(t_0+s)\|^2_{L^2}\mathrm{d}s\\
&\le C\|u^0\|^2_{L^2}\int_{t_0}^\infty\|\nabla u(s)\|^2_{L^2}\mathrm{d}s\\
&\le C\|u^0\|^2_{L^2}\frac{\varepsilon^2}{9(C\|u^0\|^2_{L^2}+1)},
\end{aligned}$$

which implies that $\|f(t)\|_{H^{-2}} < \frac{\varepsilon}{3},\ \forall t \ge 0$.

For an estimation of $\|g(t)\|_{H^{-2}}$, using the injection $L^1(\mathbb{R}^3) \hookrightarrow H^{-s}(\mathbb{R}^3),\ \forall s > 3/2$, with $s = 2$, we obtain

$$\begin{aligned}\|g(t)\|^2_{H^{-2}}\mathrm{d}t &\le \int_{\mathbb{R}^3}(1+|\xi|^2)^{-2}\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}(|v|^{\beta-1}v)(s,\xi)|\mathrm{d}s\right)^2\mathrm{d}\xi\\ &\le C\left(\int_0^t \|(|v|^{\beta-1}v)(s,.)\|_{L^1(\mathbb{R}^3)}\mathrm{d}s\right)^2\\ &\le C\left(\int_0^t \||v(s,.)|^\beta\|_{L^1(\mathbb{R}^3)}\mathrm{d}s\right)^2\\ &\le C\|v\|^2_{L^\beta(\mathbb{R}^+\times\mathbb{R}^3)},\end{aligned}$$

where $C = \int_{\mathbb{R}^3}(1+|\xi|^2)^{-2}\mathrm{d}\xi$.

Also, using inequality (3.7), we obtain

$$\|g(t)\|^2_{H^{-2}}\mathrm{d}t \le C\||u(t_0+.)\|^2_{L^\beta(\mathbb{R}^+\times\mathbb{R}^3)} \le C\|u\|^2_{L^\beta([t_0,\infty)\times\mathbb{R}^3)} \le C\frac{\varepsilon^2}{9C},$$

which implies that $\|g(t)\|_{H^{-2}} < \frac{\varepsilon}{3},\ \forall t \ge 0$. Combining the above inequalities, we obtain

$$\lim_{t\to\infty}\|u(t)\|_{H^{-2}} = 0.$$

□

Lemma 3.3. *If u is a global solution of (NSD) and $\beta \ge \frac{10}{3}$, then* $\lim_{t\to\infty}\|u(t)\|_{L^2} = 0$.

Proof. Let $w_1 = \mathbf{1}_{|D|<1}u = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|<1}\hat{u})$ and $w_2 = \mathbf{1}_{|D|\ge1}u = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\ge1}\hat{u})$.

Using the second step, we obtain

$$\|w_1(t)\|_{L^2} = c_0\|w_1(t)\|_{H^0} \le 2c_0\||w_1(t)\|_{H^{-2}} \le 2\||u(t)\|_{H^{-2}},$$

which implies $\lim_{t\to\infty}\|w_1(t)\|_{L^2} = 0$.

For $\varepsilon > 0$, there is $t_1 > 0$ such that $\|w_1(t)\|_{L^2} < \frac{\varepsilon}{2},\ \forall t \ge t_1$, we have

$$\int_{t_1}^\infty \|w_2(t)\|^2_{L^2}\mathrm{d}t \le \int_{t_1}^\infty \|\nabla w_2(t)\|^2_{L^2}\mathrm{d}t \le \int_{t_1}^\infty \|\nabla u(t)\|^2_{L^2}\mathrm{d}t < \infty.$$

Since the map $t \longmapsto \|w_2(t)\|_{L^2}$ is continuous, there exists $t_2 \ge t_1$ such that $\|w_2(t_2)\|_{L^2} < \frac{\varepsilon}{2}$. Hence

$$\|u(t_2)\|^2_{L^2} = \|w_1(t_2)\|^2_{L^2} + \|w_2(t_2)\|^2_{L^2} < \frac{\varepsilon^2}{2}.$$

Using the following energy estimate

$$\|u(t)\|^2_{L^2} + 2\int_{t_2}^t \|\nabla u(s)\|^2_{L^2}\mathrm{d}s + 2\alpha\int_{t_2}^t \|u(s)\|_{L^{\beta+1}}\mathrm{d}s \le \|u(t_2)\|^2_{L^2},\ \forall t \ge t_2,$$

we obtain $\|u(t)\|_{L^2} < \varepsilon,\ \forall t \ge t_2$, and the proof is completed.

□

4 Conclusion

Cai and Jiu [5] consider the incompressible NSEs with damping (NSD) and they proved the existence of global weak solution for $\beta \ge 1$. The problem of uniqueness and asymptotic behavior of the solution is not addressed. In this article, we prove by a new and different method, the uniqueness and the continuity of the global solution in $L^2(\mathbb{R}^3)$ for $\beta > 3$. We also study the asymptotic behavior for this global solution for $\beta \ge \frac{10}{3}$.

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