Research Article

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Decreasing and complete monotonicity of functions defined by derivatives of completely monotonic function involving trigamma function[#]

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Abstract: In this study, using convolution theorem of the Laplace transforms, a monotonicity rule for the ratio of two Laplace transforms, Bernstein's theorem for completely monotonic functions, and other analytic techniques, the authors verify decreasing property of a ratio between three derivatives of a function involving trigamma function and find the necessary and sufficient conditions for a function defined by three derivatives of a function involving trigamma function to be completely monotonic. These results confirm previous guesses posed by Qi and generalize the corresponding known conclusions.

Keywords: decreasing monotonicity, complete monotonicity, completely monotonic function, trigamma function, derivative, ratio, convolution theorem, inequality, monotonicity rule, Laplace transform, Bernstein's theorem, exponential function guess

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1 Introduction

In the literature [1, Section 6.4], the function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t, \quad \Re(z) > 0$$

and its logarithmic derivative $\psi(z) = [\ln\Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ are called Euler's gamma function and digamma function, respectively. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as the trigamma, tetragamma, pentagamma, and hexagamma functions, respectively. All the derivatives $\psi^{(k)}(z)$ for $k \ge 0$ are known as polygamma functions.

Dedicated to Professor Dr. Mourad E. H. Ismail at University of Central Florida.

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Recall from Chapter XIII in [2], Chapter 1 in [3], and Chapter IV in [4] that if a function $f(t)$ on an interval I has derivatives of all orders on I and satisfies inequalities $(-1)^n f^{(n)}(t) \ge 0$ for $t \in I$ and $n \in \{0\} \cup \mathbb{N}$, then we call $f(t)$ a completely monotonic function on I. There have been plenty of literature dedicated to the study and applications of completely monotonic functions, logarithmically completely monotonic functions, and completely monotonic degrees.

Let

$$\Phi(x) = x\psi'(x) - 1 = x\left[\psi'(x) - \frac{1}{x}\right], \quad x \in (0, \infty).$$

Lemma 2 in [5] states that the function $(-1)^k \Phi^{(k)}(x)$ for $k \ge 0$ is completely monotonic on $(0, \infty)$. The completely monotonic function $\Phi(x)$ and its derivatives have been initially investigated by Qi and his coauthors in a series of papers such as [5–7]. This topic was reviewed and surveyed in the article [8].

In [6, Theorem 4.1] and [7, Theorem 4], Qi turned out the following necessary and sufficient conditions and double inequality:

(1) if and only if $\alpha \ge 2$, the function $\mathfrak{H}_\alpha(x) = \Phi'(x) + \alpha\Phi^2(x)$ is completely monotonic on $(0, \infty)$;
(2) if and only if $\alpha \le 1$, the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0, \infty)$;
(3) the double inequality $-2 < \frac{\Phi'(x)}{\Phi^2(x)} < -1$ is valid and sharp in the sense that the lower and upper bounds -2 and -1 cannot be replaced by any bigger and smaller ones, respectively.

In [6, Theorem 1.1], Qi found the following necessary and sufficient conditions and limits:

(1) if and only if $\beta \ge 2$, the function $H_\beta(x) = \frac{\Phi'(x)}{\Phi^\beta(x)}$ is decreasing on $(0, \infty)$, with the limits

$$\lim_{x\to 0^+} H_\beta(x) = \begin{cases} -1, & \beta = 2; \\ 0, & \beta > 2 \end{cases} \quad \text{and} \quad \lim_{x\to\infty} H_\beta(x) = \begin{cases} -2, & \beta = 2; \\ -\infty, & \beta > 2; \end{cases}$$

(2) if $\beta \le 1$, the function $H_\beta(x)$ is increasing on $(0, \infty)$, with the limits

$$H_\beta(x) \to \begin{cases} -\infty, & x \to 0^+; \\ 0, & x \to \infty. \end{cases}$$

For $k \in \{0\} \cup \mathbb{N}$ and $\lambda_k, \mu_k \in \mathbb{R}$, let

$$\mathfrak{J}_{k,\lambda_k}(x) = \Phi^{(2k+1)}(x) + \lambda_k[\Phi^{(k)}(x)]^2 \quad \text{and} \quad J_{k,\mu_k}(x) = \frac{\Phi^{(2k+1)}(x)}{[(-1)^k\Phi^{(k)}(x)]^{\mu_k}}$$

on $(0, \infty)$. In [5, Theorems 1 and 2], Qi presented the following necessary and sufficient conditions, limits, and double inequality:

(1) if and only if $\lambda_k \ge \frac{(2k+2)!}{k!(k+1)!}$, the function $\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0, \infty)$;
(2) if and only if $\lambda_k \le \frac{1}{2}\frac{(2k+2)!}{k!(k+1)!}$, the function $-\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0, \infty)$;
(3) if and only if $\mu_k \ge 2$, the function $J_{k,\mu_k}(x)$ is decreasing on $(0, \infty)$, with the limits

$$\lim_{x\to 0^+} J_{k,\mu_k}(x) = \begin{cases} -\dfrac{1}{2}\dfrac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2; \\ 0, & \mu_k > 2 \end{cases}$$

and

$$\lim_{x\to\infty} J_{k,\mu_k}(x) = \begin{cases} -\dfrac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2; \\ -\infty, & \mu_k > 2; \end{cases}$$

(4) if $\mu_k \le 1$, the function $J_{k,\mu_k}(x)$ is increasing on $(0,\infty)$, with the limits

$$J_{k,\mu_k}(x) \to \begin{cases} -\infty, & x \to 0^+; \\ 0, & x \to \infty; \end{cases}$$

(5) the double inequality

$$-\frac{(2k+2)!}{k!(k+1)!} < \frac{\Phi^{(2k+1)}(x)}{[\Phi^{(k)}(x)]^2} < -\frac{1}{2}\frac{(2k+2)!}{k!(k+1)!}$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.

For $k \ge m \ge 0$, let

$$\mathcal{J}_{k,m}(x) = \frac{\Phi^{(2k+2)}(x)}{\Phi^{(k-m)}(x)\Phi^{(k+m+1)}(x)}$$

on $(0,\infty)$. In [5, Remark 3], Qi guessed that the function $\mathcal{J}_{k,m}(x)$ for $k \ge m \ge 0$ should be decreasing on $(0,\infty)$ and that the double inequality

$$-\frac{2(2k+2)!}{k!(k+1)!} < \mathcal{J}_{k,0}(x) < -\frac{(2k+2)!}{k!(k+1)!} \tag{1}$$

for $k \ge 0$ should be valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.

For $m,n \in \{0\} \cup \mathbb{N}$ and $\omega_{m,n} \in \mathbb{R}$, let

$$Y_{m,n}(x) = \frac{\Phi^{(m+n+1)}(x)}{\Phi^{(m)}(x)\Phi^{(n)}(x)} \tag{2}$$

and

$$\mathcal{Y}_{m,n;\omega_{m,n}}(x) = \Phi^{(m+n+1)}(x) + \omega_{m,n}\Phi^{(m)}(x)\Phi^{(n)}(x). \tag{3}$$

It is clear that

$$Y_{m,n}(x) = Y_{n,m}(x), \quad \mathcal{Y}_{m,n;\omega_{m,n}}(x) = \mathcal{Y}_{n,m;\omega_{n,m}}(x),$$
$$Y_{k-m,k+m+1}(x) = \mathcal{J}_{k,m}(x), \quad \mathcal{Y}_{k,k;\omega_{k,k}}(x) = \mathfrak{J}_{k,\omega_{k,k}}(x), \quad \mathcal{Y}_{0,0;\omega_{0,0}}(x) = \mathfrak{H}_{\omega_{0,0}}(x).$$

In this study, we will prove decreasing property of the function $Y_{m,n}(x)$ and find necessary and sufficient conditions on $\omega_{m,n}$ for $\pm(-1)^{m+n+1}\mathcal{Y}_{m,n;\omega_{m,n}}(x)$ to be completely monotonic on $(0,\infty)$. These results confirm the above guesses and generalize corresponding ones in [5–7] mentioned above.

2 Lemmas

The following lemmas are necessary in this study.

Lemma 1. (Convolution theorem for the Laplace transforms [4, pp. 91–92]) *Let $f_k(t)$ for $k = 1,2$ be piecewise continuous in arbitrary finite intervals included in $(0,\infty)$. If there exist some constants $M_k > 0$ and $c_k \ge 0$ such that $|f_k(t)| \le M_k e^{c_k t}$ for $k = 1,2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t-u)\,\mathrm{d}u\right] e^{-st}\,\mathrm{d}t = \int_0^\infty f_1(u)e^{-su}\,\mathrm{d}u \int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}v.$$

Lemma 2. [9, Lemma 4] *Let the functions $A(t)$ and $B(t) \ne 0$ be defined on $(0,\infty)$ such that their Laplace transforms $\int_0^\infty A(t)e^{-xt}\mathrm{d}t$ and $\int_0^\infty B(t)e^{-xt}\mathrm{d}t$ exist. If the ratio $\frac{A(t)}{B(t)}$ is increasing, then the ratio $\frac{\int_0^\infty A(t)e^{-xt}\mathrm{d}t}{\int_0^\infty B(t)e^{-xt}\mathrm{d}t}$ is decreasing on $(0,\infty)$.*

Lemma 3. *Let $x, y \in \mathbb{R}$ such that $0 < 2x < y$.*

(1) *When $y > 2x > 2\left(2 + \frac{1}{\ln 2}\right) = 6.885390\ldots$, the function*

$$F(x,y) = 2\left(\frac{1}{x} - \frac{1}{y-x}\right) + \frac{1}{2}\left(\frac{2^{y-x}}{y-x} - \frac{2^x}{x}\right) - \frac{2^{y-x} - 2^x}{(y-x)x}$$

is positive.

(2) *For $k, m \in \mathbb{N}$ such that $6 \le 2m < k$, the sequence $F(m,k)$ is positive.*

Proof. The function $F(x,y)$ can be rearranged as

$$F(x,y) = \frac{2(y-2x) + 2^{x-1}[2 - y + x + (x-2)2^{y-2x}]}{x(y-x)}.$$

Therefore, it suffices to prove $2 - y + x + (x-2)2^{y-2x} > 0$, that is,

$$2^{y-2x} > \frac{y-x-2}{x-2}. \tag{4}$$

Replacing $y - 2x$ by t in (4) leads to

$$2^t > \frac{t+x-2}{x-2} = 1 + \frac{t}{x-2} \tag{5}$$

for $t > 0$ and $x > 2$. Inequality (5) can be reformulated as $x > 2 + \frac{t}{2^t - 1}$. Since the function $\frac{t}{2^t-1}$ is decreasing from $(0, \infty)$ onto $\left(0, \frac{1}{\ln 2}\right)$, it is sufficient for $x > 2 + \frac{1}{\ln 2} = 3.442695\ldots$.

Repeating those arguments before Inequality (4) hints us that, for proving $F(m,k) > 0$, it is sufficient to show

$$2^{k-2m} > \frac{k-m-2}{m-2} = 1 + \frac{k-2m}{m-2},$$

which can be rewritten as

$$\frac{k-2m}{2^{k-2m} - 1} < m - 2. \tag{6}$$

Since $\frac{t}{2^t-1}$ is decreasing in $t \in (0, \infty)$ and $k - 2m \ge 1$, the largest value of the left-hand side in inequality (6) is $\frac{1}{2^1-1} = 1$ which means that the strict inequality (6) is valid for all $m \ge 4$. As a result, the sequence $F(m,k)$ is positive for all $m \ge 4$.

When $m = 3$, sequence $F(3,k)$ is

$$F(3,k) = \frac{2^k - 32k + 128}{48(k-3)} = \frac{2^5[2^{k-5} - (k-4)]}{48(k-3)},$$

which is positive for all $k > 2 \cdot 3 = 6$. The proof of Lemma 3 is complete. □

Lemma 4. *Let*

$$h(t) = \begin{cases} \dfrac{e^t(e^t - 1 - t)}{(e^t-1)^2}, & t \ne 0; \\ \dfrac{1}{2}, & t = 0 \end{cases}$$

on $(-\infty, \infty)$. Then, for any fixed $s \in (0,1)$, the ratio $\frac{h(st)}{h^s(t)}$ is increasing in t from $(0, \infty)$ onto $\left(\frac{1}{2^{1-s}}, 1\right)$.

Proof. It is easy to see that

$$\lim_{t\to 0}\frac{h(st)}{h^s(t)}=\frac{\lim_{t\to 0}h(st)}{\lim_{t\to 0}h^s(t)}=\frac{\frac{1}{2}}{\frac{1}{2^s}}=\frac{1}{2^{1-s}}$$

and

$$\lim_{t\to\infty}H_s(t)=\frac{\lim_{t\to\infty}h(st)}{\lim_{t\to 0}h^s(t)}=\frac{1}{1^s}=1.$$

Direct differentiating and expanding to power series give

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{h(st)}{h^s(t)}\right]&=-\frac{se^{(1+s)t}\left[\begin{gathered}(t-2)e^{(1+2s)t}+(t+2)e^{2st}+(2-st)e^{(2+s)t}+4(s-1)te^{(1+s)t}-(2st^2+3st+2)e^{st}\\ -(st+2)e^{2t}+(2st^2+3t+2)e^t+(s-1)t\end{gathered}\right]}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}\\
&=\frac{se^{(1+s)t}\sum_{k=7}^{\infty}\left(\begin{gathered}(3k+2)s^k+2(2s+1)^k+ks(s+2)^{k-1}+2k^2s^{k-1}+4k(s+1)^{k-1}+2k(1+2^{k-2})s+2^{k+1}\\ -[2^{k+1}s^k+2(s+2)^k+4ks(s+1)^{k-1}+2k(1+2^{k-2})s^{k-1}+k(2s+1)^{k-1}+2k^2s+3k+2]\end{gathered}\right)\frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}\\
&=\frac{se^{(1+s)t}\sum_{k=7}^{\infty}\left(\begin{gathered}\sum_{m=1}^{k-1}\left[k2^{k-m}\binom{k-1}{m-1}+4k\binom{k-1}{m}+2^{m+1}\binom{k}{m}\right]s^m+2k(1-k+2^{k-2})(s-s^{k-1})\\ -\sum_{m=1}^{k-1}\left[k2^m\binom{k-1}{m}+4k\binom{k-1}{m-1}+2^{k-m+1}\binom{k}{m}\right]s^m\end{gathered}\right)\frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}\\
&=\frac{se^{(1+s)t}\sum_{k=7}^{\infty}\left(\sum_{m=1}^{k-1}\left[k2^{k-m}\binom{k-1}{m-1}+4k\binom{k-1}{m}+2^{m+1}\binom{k}{m}\right](s^m-s^{k-m})+2k(1-k+2^{k-2})(s-s^{k-1})\right)\frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}\\
&=\frac{se^{(1+s)t}\sum_{k=7}^{\infty}\left(\begin{gathered}\sum_{3\le m<\frac{k}{2}}\left[k2^{k-m}\binom{k-1}{m-1}-k2^m\binom{k-1}{k-m-1}+4k\binom{k-1}{m}-4k\binom{k-1}{k-m}\right.\\ \left.+2^{m+1}\binom{k}{m}-2^{k-m+1}\binom{k}{k-m}\right](s^m-s^{k-m})\end{gathered}\right)\frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}\\
&=\frac{se^{(1+s)t}\sum_{k=7}^{\infty}\left(\begin{gathered}2\sum_{3\le m<\frac{k}{2}}\frac{k!}{(m-1)!(k-m-1)!}\left[2\left(\frac{1}{m}-\frac{1}{k-m}\right)\right.\\ \left.+\frac{1}{2}\left(\frac{2^{k-m}}{k-m}-\frac{2^m}{m}\right)-\frac{2^{k-m}-2^m}{(k-m)m}\right](s^m-s^{k-m})\end{gathered}\right)\frac{t^k}{k!}}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}\\
&=\frac{s^4e^{(1+s)t}\left(\begin{gathered}\frac{1-s}{36}t^7+\frac{1-s^2}{45}t^8+\frac{22(1-s^3)+15(s-s^2)}{2160}t^9+\frac{52(1-s^4)+63(s-s^3)}{15120}t^{10}\\ +\frac{285(1-s^5)+470(s-s^4)+238(s^2-s^3)}{302400}t^{11}+\dotsb\end{gathered}\right)}{(e^t-1)^3(e^{st}-1)^3h^{s+1}(t)}.
\end{aligned}
$$

Utilizing Lemma 3 reveals that the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{h(st)}{h^s(t)}\right]$ is positive for $s\in(0,1)$ and $t>0$. Consequently, for $s\in(0,1)$, the ratio $\frac{h(st)}{h^s(t)}$ is increasing in $t>0$. The proof of Lemma 4 is complete. □

Lemma 5. [5, Lemma 2] *For $k \ge 0$, the function $(-1)^k\Phi^{(k)}(x)$ is completely monotonic on $(0,\infty)$, with the limits*

$$(-1)^k x^{k+1}\Phi^{(k)}(x) \to \begin{cases} k!, & x \to 0^+; \\ \dfrac{k!}{2}, & x \to \infty. \end{cases} \tag{7}$$

Lemma 6. (Bernstein's theorem [4, p. 161, Theorem 12b]) *A function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt}\mathrm{d}\sigma(t), \quad x \in (0,\infty), \tag{8}$$

where $\sigma(s)$ is non-decreasing and the integral in (8) converges for $x \in (0,\infty)$.

3 Decreasing property

In this section, we prove that the function $Y_{m,n}(x)$ defined in (2) is decreasing.

Theorem 1. *For $m,n \in \{0\}\cup\mathbb{N}$, the function $Y_{m,n}(x)$ defined in (2) is decreasing in x from $(0,\infty)$ onto the interval $\left(-\frac{2(m+n+1)!}{m!n!}, -\frac{(m+n+1)!}{m!n!}\right)$. Consequently, for $m,n \in \{0\}\cup\mathbb{N}$, the double inequality*

$$-\frac{2(m+n+1)!}{m!n!} < Y_{m,n}(x) < -\frac{(m+n+1)!}{m!n!} \tag{9}$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.

Proof. In the proof of [7, Theorem 4], Qi established that

$$\Phi(x) = \int_0^\infty h(t)e^{-xt}\mathrm{d}t. \tag{10}$$

Then, the ratio $Y_{m,n}(x)$ can be rewritten as

$$Y_{m,n}(x) = -\frac{\int_0^\infty t^{m+n+1}h(t)e^{-xt}\mathrm{d}t}{\int_0^\infty t^m h(t)e^{-xt}\mathrm{d}t\int_0^\infty t^n h(t)e^{-xt}\mathrm{d}t} = -\frac{\int_0^\infty t^{m+n+1}h(t)e^{-xt}\mathrm{d}t}{\int_0^\infty\left[\int_0^t u^m(t-u)^n h(u)h(t-u)\mathrm{d}u\right]e^{-xt}\mathrm{d}t},$$

where we used Lemma 1. Based on Lemma 2, in order to prove decreasing property of $Y_{m,n}(x)$, it suffices to show that the ratio

$$\mathfrak{Y}_{m,n}(t) = \frac{t^{m+n+1}h(t)}{\int_0^t u^m(t-u)^n h(u)h(t-u)\mathrm{d}u} \tag{11}$$

is decreasing in $t \in (0,\infty)$. By changing the variable $u = \frac{1+v}{2}t$, the denominator of $\mathfrak{Y}_{m,n}(t)$ becomes

$$\left(\frac{t}{2}\right)^{m+n+1}\int_{-1}^{1}(1+v)^m(1-v)^n h\left(\frac{1+v}{2}t\right)h\left(\frac{1-v}{2}t\right)\mathrm{d}v.$$

Accordingly, we obtain

$$\frac{1}{\mathfrak{Y}_{m,n}(t)} = \frac{\int_{-1}^{1}(1+v)^m(1-v)^n h\left(\frac{1+v}{2}t\right)h\left(\frac{1-v}{2}t\right)\mathrm{d}v}{2^{m+n+1}h(t)}$$

$$
\begin{aligned}
&= \frac{1}{2^{m+n+1}} \int_{-1}^{1} (1+v)^m (1-v)^n \frac{h\left(\frac{1+v}{2}t\right) h\left(\frac{1-v}{2}t\right)}{h(t)} \mathrm{d}v \\
&= \frac{1}{2^{m+n+1}} \int_{-1}^{1} (1+v)^m (1-v)^n \frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)} \mathrm{d}v,
\end{aligned} \tag{12}
$$

where $s = \frac{1+v}{2} \in (0,1)$. From Lemma 4, we find that the function $\frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)}$ is increasing in $t \in (0,\infty)$ for any fixed $s \in (0,1)$. Hence, the function $\mathfrak{Y}_{m,n}(t)$ is decreasing on $(0,\infty)$. Therefore, the function $Y_{m,n}(x)$ for $m,n \in \{0\} \cup \mathbb{N}$ is decreasing on $(0,\infty)$.

Making use of the limits in (7) in Lemma 5 yields

$$
Y_{m,n}(x) = -\frac{(-1)^{m+n+1} x^{m+n+2} \Phi^{(m+n+1)}(x)}{[(-1)^m x^{m+1} \Phi^{(m)}(x)][(-1)^n x^{k+m+2} \Phi^{(n)}(x)]} \to \begin{cases} -\dfrac{(m+n+1)!}{m!n!}, & x \to 0^+; \\ -\dfrac{2(m+n+1)!}{m!n!}, & x \to \infty. \end{cases}
$$

The proof of Theorem 1 is complete. □

4 Necessary and sufficient conditions of complete monotonicity

In this section, we discover necessary and sufficient conditions on $\omega_{m,n}$ for the function $\pm(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ defined in (3) to be completely monotonic.

Theorem 2. *For $m,n \in \{0\} \cup \mathbb{N}$ and $\omega_{m,n} \in \mathbb{R}$,*

(1) *if and only if $\omega_{m,n} \le \frac{(m+n+1)!}{m!n!}$, the function $(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0,\infty)$;*
(2) *if and only if $\omega_{m,n} \ge \frac{2(m+n+1)!}{m!n!}$, the function $(-1)^{m+n} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0,\infty)$;*
(3) *the double inequality (9) is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers, respectively.*

Proof. As done in the proof of Theorem 1, by virtue of the integral representation (10) and Lemma 1, we acquire

$$
\begin{aligned}
(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x) &= \left[\int_0^\infty t^{m+n+1} h(t) e^{-xt} \mathrm{d}t - \omega_{m,n} \int_0^\infty t^m h(t) e^{-xt} \mathrm{d}t \int_0^\infty t^n h(t) e^{-xt} \mathrm{d}t \right] \\
&= \int_0^\infty \left[t^{m+n+1} h(t) - \omega_{m,n} \int_0^t u^m (t-u)^n h(u) h(t-u) \mathrm{d}u \right] e^{-xt} \mathrm{d}t \\
&= \int_0^\infty \left[1 - \frac{\omega_{m,n}}{\mathfrak{Y}_{m,n}(t)} \right] t^{m+n+1} h(t) e^{-xt} \mathrm{d}t,
\end{aligned}
$$

where $\mathfrak{Y}_{m,n}(t)$ is defined by (11) and it has been proved in the proof of Theorem 1 to be decreasing on $(0,\infty)$. From Lemma 4, we conclude that the function $\frac{h(st)}{h^s(t)} \frac{h((1-s)t)}{h^{1-s}(t)}$ is increasing in t from $(0,\infty)$ onto $\left(\frac{1}{2}, 1\right)$. Accordingly, by virtue of (12), we arrive at the sharp inequalities

$$
\frac{1}{2^{m+n+2}} \int_{-1}^{1} (1+v)^m (1-v)^n \mathrm{d}v < \frac{1}{\mathfrak{Y}_{m,n}(t)} < \frac{1}{2^{m+n+1}} \int_{-1}^{1} (1+v)^m (1-v)^n \mathrm{d}v.
$$

Since

$$
\begin{aligned}
\int_{-1}^{1} (1+v)^m (1-v)^n \mathrm{d}v &= \int_0^1 [(1+v)^m (1-v)^n + (1-v)^m (1+v)^n] \mathrm{d}v \\
&= 2^{m+n+1} B(m+1, n+1) = 2^{m+n+1} \frac{m!n!}{(m+n+1)!},
\end{aligned}
$$

where we used the formula

$$\int_0^1 [(1+x)^{\mu-1}(1-x)^{\nu-1} + (1+x)^{\nu-1}(1-x)^{\mu-1}]\mathrm{d}x = 2^{\mu+\nu-1}B(\mu,\nu) = 2^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

for $\Re(\mu), \Re(\nu) > 0$ in [10, p. 321, 3.214], the double inequality

$$\frac{1}{2}\frac{m!n!}{(m+n+1)!} < \frac{1}{\mathfrak{Y}_{m,n}(t)} < \frac{m!n!}{(m+n+1)!}$$

is valid and sharp on $(0,\infty)$. Consequently, by virtue of Lemma 6, if and only if $\omega_{m,n} \le \frac{(m+n+1)!}{m!n!}$, the function $(-1)^{m+n+1}\mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0,\infty)$; if and only if $\omega_{m,n} \ge \frac{2(m+n+1)!}{m!n!}$, the function $(-1)^{m+n}\mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0,\infty)$.

The double inequality (9) follows from complete monotonicity of the functions $\pm(-1)^{m+n+1}\mathcal{Y}_{m,n;\omega_{m,n}}(x)$. The proof of the sharpness of the double inequality (9) is the same as done in the proof of Theorem 1. The proof of Theorem 2 is complete. □

5 Remarks

In this section, we list several remarks related to our main results and their proofs in this study.

Remark 1. Lemma 4 in this study generalizes a conclusion in [6, Lemma 2.3], which reads that the function $\frac{h(2t)}{h^2(t)}$ is decreasing from $(0,\infty)$ onto $(1,2)$.

Remark 2. The function $F(x,y)$ discussed in Lemma 3 can be reformulated as

$$F(x,y) = \left(\frac{1}{x} - \frac{1}{y-x}\right)\left[2 - \frac{1}{2}\frac{\frac{2^{y-x}}{y-x} - \frac{2^x}{x}}{\frac{1}{y-x} - \frac{1}{x}} - \frac{2^{y-x} - 2^x}{(y-x) - x}\right],$$

in which the functions

$$\frac{\frac{2^{y-x}}{y-x} - \frac{2^x}{x}}{\frac{1}{y-x} - \frac{1}{x}} \quad \text{and} \quad \frac{2^{y-x} - 2^x}{(y-x) - x}$$

can be regarded as special means [11,12].

Let $x, y \in \mathbb{R}$ such that $0 < x < \frac{y}{2}$. Motivated by Lemma 3, we guess that

(1) when $2 < x < \frac{y}{2}$, the function $F(x,y)$ is positive;
(2) when $y > 4$ and $0 < x < 2$, the function $F(x,y)$ is negative.

Furthermore, one can discuss positivity and negativity of the function $F(x,y)$ for all x, y satisfying $0 < x < \frac{y}{2}$.

Remark 3. When taking $m = k$ and $n = k+1$, the double inequality (9) in Theorem 1 becomes the double inequality (1) guessed by the corresponding and third author in [5, Remark 3].

Remark 4. For $m, n \in \{0\} \cup \mathbb{N}$, direct differentiation gives

$$Y'(x) = \frac{\Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)] - \Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]'}{[\Phi^{(m)}(x)\Phi^{(n)}(x)]^2}$$

on $(0,\infty)$. The decreasing monotonicity of $Y_{m,n}(x)$ in Theorem 1 implies that, for $m, n \in \{0\} \cup \mathbb{N}$, the inequality

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' > \Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)],$$

equivalently,

$$\frac{[\Phi^{(m)}(x)\Phi^{(n)}(x)]'}{\Phi^{(m)}(x)\Phi^{(n)}(x)} < \frac{\Phi^{(m+n+2)}(x)}{\Phi^{(m+n+1)}(x)},$$

is valid on $(0, \infty)$.

We guess that, for $m, n \in \{0\} \cup \mathbb{N}$, the function

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' - \Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]$$

is completely monotonic in $x \in (0, \infty)$.

One can also consider necessary and sufficient conditions on $\Lambda_{m,n} \in \mathbb{R}$ for $m, n \in \{0\} \cup \mathbb{N}$ such that the function

$$\Phi^{(m+n+1)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]' - \Lambda_{m,n}\Phi^{(m+n+2)}(x)[\Phi^{(m)}(x)\Phi^{(n)}(x)]$$

and its opposite are, respectively, completely monotonic on $(0, \infty)$.

Remark 5. This work has two electronic preprints at the sites https://hal.archives-ouvertes.fr/hal-02998203v1 and https://doi.org/10.48550/arxiv.2405.19361.

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