Research Article

Choonkil Park, Mohammad Amin Tareeghee, Abbas Najati*, and Batool Noori

Hyers-Ulam stability of Davison functional equation on restricted domains

https://doi.org/10.1515/dema-2024-0039
received August 5, 2022; accepted June 7, 2024

Abstract: In this article, we study the Hyers-Ulam stability of Davison functional equation

$$f(xy) + f(x + y) = f(xy + x) + f(y)$$

on some unbounded restricted domains. Using the obtained results, we study an interesting asymptotic behavior of Davison functions. We also investigate the Hyers-Ulam stability of Davison functional equation and its generalized form given by

$$f(xy) + g(x + y) = h(xy + x) + k(y),$$

for $x, y \in \mathbb{R}^{\geqslant 0} = \{t \in \mathbb{R} : t \geqslant 0\}$.

Keywords: Hyers-Ulam stability, Davison functional equation, Davison function, asymptotic behavior

MSC 2020: 39B82, 39B62

1 Introduction and preliminaries

During the 17th International Symposium on functional equations, the following functional equation

$$f(xy) + f(x + y) = f(xy + x) + f(y), \tag{1}$$

was introduced by Davison [1]. He specifically asked about the general solution of (1) when the domain and range of f are assumed to be (commutative) fields. Benz [2] proved that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of (1), then f is of the form $f(x) = ax + b$, where a and b are arbitrary constants. The general solution (without any regularity assumptions on $f : \mathbb{R} \to \mathbb{R}$) of the Davison functional equation (1) was obtained by Girgensohn and Lajkó [3]. They proved that if $f : \mathbb{R} \to \mathbb{R}$ is a solution (1), then f has the form $f(x) = A(x) + b$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $b \in \mathbb{R}$ is an arbitrary constant. Moreover, they obtained the general solution of the generalized version of (1) as follows.

Theorem 1.1. [3] *The functions* $f, g, h, k : \mathbb{R} \to \mathbb{R}$ *satisfy*

$$f(xy) + g(x + y) = h(xy + x) + k(y), \quad x, y \in \mathbb{R}, \tag{2}$$

*** Corresponding author: Abbas Najati**, Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran, e-mail: a.nejati@yahoo.com
Choonkil Park: Department of Mathematics, Research Institute for Convergence of Basic Science, Hanyang University, Seoul, Korea, e-mail: baak@hanyang.ac.kr
Mohammad Amin Tareeghee: Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran, e-mail: amintareeghee30@gmail.com
Batool Noori: Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran, e-mail: noori.batool@yahoo.com

if and only if they have the form

$$f(x) = A(x) + b_1, \quad g(x) = A(x) + b_2, \quad h(x) = A(x) + b_3, \quad k(x) = A(x) + b_4,$$

where $A : \mathbb{R} \to \mathbb{R}$ *is additive and* b_1, b_2, b_3, b_4 *are real constants with* $b_1 + b_2 = b_3 + b_4$.

It should be noted that the aforementioned results can be extended to functions $f : \mathcal{K} \to \mathcal{G}$, where $\mathcal{K}$ is a commutative field of characteristic different from 2 and 3, and $\mathcal{G}$ is an abelian group.

Girgensohn and Lajkó [3] also presented the general solution of (1) and (2) for $x, y \in (0, +\infty)$.

Davison [4] solved the functional equation (1) in two cases for functions with values in an abelian group $\mathcal{G}$. First $f : \mathbb{N} \to \mathcal{G}$, and second $f : \mathbb{Z} \to \mathcal{G}$, where $\mathbb{N}$ and $\mathbb{Z}$ are, respectively, the set of natural numbers and the set of integer numbers. Indeed, he proved that every solution of (1) is a linear combination (in the codomain) of four functions.

In 1940, Ulam [5] posed the following problem concerning the stability of group homomorphisms: Let $(\mathcal{G}_1, *)$ be a group, $(\mathcal{G}_2, \circ)$ a metric group with a metric $d(.,.)$ and $\varepsilon > 0$. Find a $\delta > 0$ such that if $f : \mathcal{G}_1 \to \mathcal{G}_2$ fulfills $d(f(x*y), f(x) \circ f(y)) \leqslant \delta$ for all $x, y \in \mathcal{G}_1$, then there exists a homomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ such that $d(f(x), h(x)) \leqslant \varepsilon$ for all $x \in \mathcal{G}_1$. This problem was first solved by Hyers [6] for the case where $\mathcal{G}_1$ and $\mathcal{G}_2$ are Banach spaces. The Hyers-Ulam stability problem for various functional equations have been investigated by numerous mathematicians. For more information on this area we refer the reader to [7–9].

The Hyers-Ulam stability of the Davison functional equation was first treated by Jung and Sahoo [10]. They investigated the Hyers-Ulam stability of Davison functional equation (1) by following an idea of Girgensohn and Lajkó [3] (see also [11]).

Theorem 1.2. [10] *If the function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the inequality*

$$|f(xy) + f(x + y) - f(xy + x) - f(y)| \leqslant \varepsilon, \quad x, y \in \mathbb{R},$$

for some $\varepsilon \geqslant 0$, *then there exists an additive function* $A : \mathbb{R} \to \mathbb{R}$ *such that*

$$|f(x) - A(x) - f(0)| \leqslant 12\varepsilon, \quad x \in \mathbb{R}.$$

It was shown in [12,13] that the estimate 12ε in Theorem 1.2 can be improved to 9ε.

Kim [14] studied the Hyers-Ulam stability of two generalized Davison functional equations, namely,

$$f(xy) + f(x + y) = g(xy + x) + g(y) \tag{3}$$

and

$$f(xy) + g(x + y) = f(xy + x) + g(y)$$

for all $x, y \in \mathcal{X}$ and $f, g : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ is a normed algebra with a unit element and $\mathcal{Y}$ is a Banach space. The Hyers-Ulam stability of (3) has been investigated in [12].

It will also be interesting to study the Hyers-Ulam stability of Davison functional equations (1) and (2) on restricted domains. In this article, we prove the Hyers-Ulam stability of the Davison functional equation on unbounded restricted domains by following ideas of [3] and [12]. We apply the obtained results to the study of an interesting asymptotic behavior of Davison functions.

2 Hyers-Ulam stability

Throughout this article, we denote by $\mathcal{A}$ a normed algebra with the unit element 1 and by $\mathcal{B}$ a Banach space.

Theorem 2.1. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be a function which for* $\varepsilon \geqslant 0$ *and* $d > 0$ *satisfies*

$$\|f(xy) + f(x + y) - f(xy + x) - f(y)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, y \in \mathcal{A}. \tag{4}$$

Then there is a unique additive function $\varphi : \mathcal{A} \to \mathcal{B}$, *such that*

$$\|\varphi(x) - f(x) + f(0)\| \leqslant 28\varepsilon, \quad x \neq -1;$$
$$\|\varphi(-1) - f(-1) + f(0)\| \leqslant 19\varepsilon.$$

Proof. Replace y by $y + 1$ in (4) to obtain

$$\|f(xy + x) + f(x + y + 1) - f(xy + 2x) - f(y + 1)\| \leqslant \varepsilon, \tag{5}$$

for all $x, y \in \mathcal{A}$ with $\|x\| \geqslant d$. Thus, it follows from (4) and (5) that

$$\begin{aligned}&\|f(xy) + f(x + y) + f(x + y + 1) - f(y) - f(xy + 2x) - f(y + 1)\| \\ &\quad \leqslant \|f(xy) + f(x + y) - f(xy + x) - f(y)\| + \|f(xy + x) + f(x + y + 1) - f(xy + 2x) - f(y + 1)\| \\ &\quad \leqslant 2\varepsilon,\end{aligned} \tag{6}$$

for all $x, y \in \mathcal{A}$ with $\|x\| \geqslant d$. Replacing y by $4y$ in (6), we obtain

$$\|f(4xy) + f(x + 4y) + f(x + 4y + 1) - f(4y) - f(4xy + 2x) - f(4y + 1)\| \leqslant 2\varepsilon,$$

for all $x, y \in \mathcal{A}$ with $\|x\| \geqslant d$. From (4) and the last inequality, we obtain

$$\begin{aligned}&\|f(x + 4y) + f(x + 4y + 1) - f(2x + 2y) - f(4y) - f(4y + 1) + f(2y)\| \\ &\quad \leqslant \|f(4xy) + f(x + 4y) + f(x + 4y + 1) - f(4y) - f(4xy + 2x) - f(4y + 1)\| + \|f(4xy) + f(2x + 2y) \\ &\qquad - f(4xy + 2x) - f(2y)\| \\ &\quad \leqslant 3\varepsilon,\end{aligned}$$

for all $x, y \in \mathcal{A}$ with $\|x\| \geqslant d$. If we replace x by $x - y$ in the above inequality and then substitute $3x$ for x in the resultant inequality, we obtain

$$\|f(3x + 3y) + f(3x + 3y + 1) - f(6x) - f(4y) - f(4y + 1) + f(2y)\| \leqslant 3\varepsilon, \tag{7}$$

for all $x, y \in \mathcal{A}$ with $\|3x - y\| \geqslant d$. Letting $y = 0$ and $y = -x$ separately in (7), we obtain

$$\|f(3x) + f(3x + 1) - f(6x) - f(1)\| \leqslant 3\varepsilon, \tag{8}$$

$$\|f(0) + f(1) - f(6x) - f(-4x) - f(-4x + 1) + f(-2x)\| \leqslant 3\varepsilon, \tag{9}$$

for all $x \in \mathcal{A}$ with $\|x\| \geqslant d$. If we replace x, y by $2x, -x$, respectively in (7), we have

$$\|-f(3x) - f(3x + 1) + f(12x) + f(-4x) + f(-4x + 1) - f(-2x)\| \leqslant 3\varepsilon, \tag{10}$$

for all $x \in \mathcal{A}$ with $\|x\| \geqslant d$. Adding (8), (9), and (10), we infer that

$$\|f(12x) - 2f(6x) + f(0)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant d.$$

Let $g(x) \coloneqq f(6x) - f(0)$. Then the last inequality means

$$\|g(2x) - 2g(x)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant d.$$

It is easy to see that

$$\left\| \frac{g(2^{n+1}x)}{2^{n+1}} - \frac{g(2^m x)}{2^m} \right\| \leqslant \sum_{k=m}^{n} \frac{9\varepsilon}{2^{k+1}}, \quad \|x\| \geqslant d. \tag{11}$$

This implies that $\left\{\frac{g(2^n x)}{2^n}\right\}_n$ is a Cauchy sequence for $\|x\| \geqslant d$. It is easy to show that $\left\{\frac{g(2^n x)}{2^n}\right\}_n$ is a Cauchy sequence for each $x \in \mathcal{A}$ and thus converges. Therefore, we can define

$$T : \mathcal{A} \to \mathcal{B}, \quad T(x) = \lim_n \frac{g(2^n x)}{2^n}.$$

Obviously,

$$T(0) = 0, \quad T(2x) = 2T(x), \quad T(x) = \lim_n \frac{f(2^n \cdot 6x)}{2^n}, \quad x \in \mathcal{A}. \tag{12}$$

By (8), we obtain

$$T(x) = \lim_n \frac{f(2^n \cdot 6x + 1)}{2^n}, \quad x \in \mathcal{A}. \tag{13}$$

Using (7), (12), and (13), we obtain

$$T(3x + 3y) = T(3x) + 3T(y), \quad x, y \in \mathcal{A}, y \neq 3x.$$

Letting $x = 0$ in the last equation and using $T(0) = 0$, we obtain $T(3y) = 3T(y)$ for all $y \in \mathcal{A}$. So, T is additive. Setting $m = 0$ and taking the limit on both sides of (11) as $n \to \infty$, we obtain

$$\|T(x) - f(6x) + f(0)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant d.$$

Since T is additive, this inequality implies

$$\|\varphi(x) - f(x) + f(0)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant 6d, \tag{14}$$

where $\varphi(x) = \frac{1}{6}T(x)$. Obviously, (14) holds for $x = 0$. Let $y \in \mathcal{A}$ and $y \neq -1$. We can choose $x \in \mathcal{A}$ such that $\min\{\|x\|, \|xy\|, \|x + y\|, \|xy + x\|\} \geqslant 6d$. By (14), we have

$$\begin{aligned} \|\varphi(xy) - f(xy) + f(0)\| &\leqslant 9\varepsilon, \\ \|\varphi(x + y) - f(x + y) + f(0)\| &\leqslant 9\varepsilon, \\ \|f(xy + x) - \varphi(xy + x) - f(0)\| &\leqslant 9\varepsilon. \end{aligned}$$

Adding these inequalities and (4), we obtain

$$\|\varphi(xy) + \varphi(x + y) - \varphi(xy + x) - f(y) + f(0)\| \leqslant 28\varepsilon.$$

Since φ is additive, the last inequality yields

$$\|\varphi(y) - f(y) + f(0)\| \leqslant 28\varepsilon.$$

In the case $y = -1$, we choose $x \in \mathcal{A}$ such that $\|x\| \geqslant 6d + \|1\|$. By (4) and (14), we have

$$\begin{aligned} \|f(-x) + f(x - 1) - f(0) - f(-1)\| &\leqslant \varepsilon, \\ \|\varphi(x - 1) - f(x - 1) + f(0)\| &\leqslant 9\varepsilon, \\ \|\varphi(-x) - f(-x) + f(0)\| &\leqslant 9\varepsilon. \end{aligned}$$

Adding these inequalities and using the additivity of φ, we obtain

$$\|\varphi(-1) - f(-1) + f(0)\| \leqslant 19\varepsilon.$$

The uniqueness of φ is obvious. Hence, the proof is complete. □

With a proof similar to the one presented in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let $\mathcal{Y}$ be a linear space and $f : \mathcal{A} \to \mathcal{Y}$ a function satisfying*

$$f(xy) + f(x + y) = f(xy + x) + f(y), \quad \|x\| \geqslant d.$$

Then f is affine, that is, $f - f(0)$ is additive on $\mathcal{A}$.

Remark 2.3. Since

$$\begin{aligned} \{(x, y) \in \mathcal{A} \times \mathcal{A} : \|x\| \geqslant 2d\} &\subseteq \{(x, y) \in \mathcal{A} \times \mathcal{A} : \|x\| + \|y\| \geqslant 2d\} \\ &\subseteq \{(x, y) \in \mathcal{A} \times \mathcal{A} : \max\{\|x\|, \|y\|\} \geqslant d\}, \end{aligned}$$

the above results are valid if the condition $\|x\| \geqslant d$ is replaced by $\|x\| + \|y\| \geqslant d$ or $\max\{\|x\|, \|y\|\} \geqslant d$.

Corollary 2.4. *Let $\mathcal{Y}$ be a normed linear space. A function $f:\mathcal{A}\to\mathcal{Y}$ is affine on $\mathcal{A}$ if and only if one of the following conditions hold*:
(*i*) $\lim_{\|x\|\to\infty}\sup_{y\in\mathcal{A}}\|f(xy)+f(x+y)-f(xy+x)-f(y)\|=0$;
(*ii*) $\lim_{\|x\|+\|y\|\to\infty}[f(xy)+f(x+y)-f(xy+x)-f(y)]=0$;
(*iii*) $\lim_{\max\{\|x\|,\|y\|\}\to\infty}[f(xy)+f(x+y)-f(xy+x)-f(y)]=0$.

Proof. It is clear that an affine function $f:\mathcal{A}\to\mathcal{Y}$ fulfills $(i),(ii),(iii)$. Obviously, $(ii)\Rightarrow(i)$ and $(iii)\Rightarrow(i)$. Let f satisfy (i) and $\varepsilon>0$ be arbitrary. Then there exists $d>0$ such that

$$\|f(xy)+f(x+y)-f(xy+x)-f(y)\|\leqslant\varepsilon,\quad \|x\|\geqslant d, y\in\mathcal{A}.$$

Let $\widetilde{\mathcal{Y}}$ be the completion of $\mathcal{Y}$. By Theorem 2.1, there exists an additive function $\varphi:\mathcal{A}\to\widetilde{\mathcal{Y}}$ satisfying

$$\|f(x)-f(0)-\varphi(x)\|\leqslant 28\varepsilon,\quad x\in\mathcal{A}.$$

Let $g(x)=f(x)-f(0)$. Then

$$\begin{aligned}\|g(x+y)-g(x)-g(y)\|&\leqslant\|g(x+y)-\varphi(x+y)\|+\|g(x)-\varphi(x)\|+\|g(y)-\varphi(y)\|\\&\leqslant 84\varepsilon,\quad x,y\in\mathcal{A}.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we obtain g is additive. This means f is affine. □

Corollary 2.5. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by*

$$f(x):=\frac{1}{1+x^2},\quad x\in\mathbb{R}.$$

It is clear that

$$\lim_{|y|\to\infty}[f(xy)+f(x+y)-f(xy+x)-f(y)]=0,$$

but f is not affine.

Note that, as a consequence of Corollary 2.4, we obtain the result that every function $f:\mathbb{R}\to\mathbb{R}$ is either affine or satisfies the condition

$$\sup_{x,y\in\mathbb{R}}|f(xy)+f(x+y)-f(xy+x)-f(y)||x|^p=\infty,$$

for all $p>0$.

3 Stability on $\mathbb{R}^{\geqslant 0}$

The aim of this section is to present the Hyers-Ulam stability of Davison functional equation and its generalized form (2) for $x,y\in\mathbb{R}^{\geqslant 0}=\{t\in\mathbb{R}:t\geqslant 0\}$.

Theorem 3.1. *Let $\varepsilon\geqslant 0$ and $f:[0,+\infty)\to\mathcal{B}$ be a function satisfying*

$$\|f(xy)+f(x+y)-f(xy+x)-f(y)\|\leqslant\varepsilon,\quad x,y\geqslant 0. \tag{15}$$

Then there is an affine function $\psi:\mathbb{R}\to\mathcal{B}$ such that

$$\|\psi(x)-f(x)\|\leqslant 55\varepsilon,\quad x>0.$$

Proof. Replacing y by $y+1$ in (15), we obtain

$$\|f(xy+x)+f(x+y+1)-f(xy+2x)-f(y+1)\|\leqslant\varepsilon,\quad x,y\geqslant 0.$$

Adding this inequality and (15), we obtain

$$\|f(xy) + f(x+y) + f(x+y+1) - f(y) - f(xy+2x) - f(y+1)\| \leqslant 2\varepsilon, \quad x, y \geqslant 0.$$

Now, replacing x by $\frac{x}{2}$ and y by $2y$, we obtain

$$\left\| f(xy) + f\left(\frac{x}{2} + 2y\right) + f\left(\frac{x}{2} + 2y + 1\right) - f(2y) - f(xy + x) - f(2y+1) \right\| \leqslant 2\varepsilon, \quad x, y \geqslant 0.$$

By this inequality and (15), one obtains

$$\left\| f\left(\frac{x}{2} + 2y\right) + f\left(\frac{x}{2} + 2y + 1\right) - f(x+y) - f(2y) - f(2y+1) + f(y) \right\| \leqslant 3\varepsilon, \tag{16}$$

for all $x, y \geqslant 0$. Replacing x by $x - \frac{y}{3}$ and y by $\frac{y}{3}$ in (16), we obtain

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x+y}{2} + 1\right) - f\left(\frac{2}{3}y\right) - f\left(\frac{2}{3}y + 1\right) + f\left(\frac{1}{3}y\right) - f(x) \right\| \leqslant 3\varepsilon,$$

for all $x \geqslant \frac{y}{3} \geqslant 0$. The last inequality can be written as follows:

$$\|G(x+y) - f(x) - H(y)\| \leqslant 3\varepsilon, \quad x \geqslant \frac{y}{3} \geqslant 0, \tag{17}$$

where

$$G(t) \coloneqq f\left(\frac{t}{2}\right) + f\left(\frac{t}{2} + 1\right),$$
$$H(t) \coloneqq f\left(\frac{2}{3}t\right) + f\left(\frac{2}{3}t + 1\right) - f\left(\frac{1}{3}t\right).$$

Letting $y = x$ and $y = 0$ separately in (17), we obtain

$$\|f(x+1) - H(x)\| \leqslant 3\varepsilon, \quad x \geqslant 0, \tag{18}$$

$$\|G(x) - f(x) - f(1)\| \leqslant 3\varepsilon, \quad x \geqslant 0. \tag{19}$$

By (19), we obtain

$$\|G(x+y) - f(x+y) - f(1)\| \leqslant 3\varepsilon, \quad x, y \geqslant 0. \tag{20}$$

It follows from (17), (18), and (20) that

$$\|f(x+y) - f(x) - f(y+1) + f(1)\| \leqslant 9\varepsilon, \quad x \geqslant \frac{y}{3} \geqslant 0. \tag{21}$$

Putting $y = 1$ in (21), we obtain

$$\|f(x+1) - f(x) - f(2) + f(1)\| \leqslant 9\varepsilon, \quad x \geqslant \frac{1}{3}. \tag{22}$$

It follows from (21) and (22) that

$$\|f(x+y) - f(x) - f(y) - f(2) + 2f(1)\| \leqslant 18\varepsilon, \quad x \geqslant \frac{y}{3}, y \geqslant \frac{1}{3}. \tag{23}$$

Letting $y = x$ in (23), we obtain

$$\|f(2x) - 2f(x) + c\| \leqslant 18\varepsilon, \quad x \geqslant \frac{1}{3},$$

where $c = 2f(1) - f(2)$. Then

$$\left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} + \sum_{k=m}^{n} \frac{c}{2^{k+1}} \right\| \leqslant \sum_{k=m}^{n} \frac{9\varepsilon}{2^k}, \quad x \geqslant \frac{1}{3}. \tag{24}$$

Then $\left\{\frac{f(2^n x)}{2^n}\right\}_n$ is a Cauchy sequence for all $x \geqslant 0$. Define

$$A : [0, +\infty) \to \mathcal{B}, \quad A(x) \coloneqq \lim_n \frac{f(2^n x)}{2^n}.$$

It follows from (19) and the definition of G that

$$A(x) \coloneqq \lim_n \frac{G(2^n x)}{2^n} = \lim_n \frac{f(2^n x + 1)}{2^n}, \quad x \geqslant 0.$$

Hence, (18) yields

$$A(x) \coloneqq \lim_n \frac{H(2^n x)}{2^n}, \quad x \geqslant 0.$$

Therefore, (17) implies

$$A(x + y) = A(x) + A(y), \quad (x, y) \in D \coloneqq \left\{(x, y) \in \mathbb{R}^2 : x > \frac{y}{3} > 0\right\}.$$

Since D is an open connected set in $\mathbb{R}^2$, by the extension theorem of Rimán [15] there is an additive function $\varphi : \mathbb{R} \to \mathcal{B}$ and a constant $a \in \mathbb{R}$ such that $A(x) = \varphi(x) + a$ for all $x > 0$. Putting $m = 0$ and letting $n \to \infty$ in (24), it follows

$$\|\varphi(x) - f(x) + a + 2f(1) - f(2)\| \leqslant 18\varepsilon, \quad x \geqslant \frac{1}{3}. \tag{25}$$

Let $y > 0$ and choose $x > 0$ such that $\min\{xy, x + y, xy + x\} > \frac{1}{3}$. By (25), we have

$$\|\varphi(xy) - f(xy) + a + 2f(1) - f(2)\| \leqslant 18\varepsilon,$$
$$\|\varphi(x + y) - f(x + y) + a + 2f(1) - f(2)\| \leqslant 18\varepsilon,$$
$$\|f(xy + x) - \varphi(xy + x) - a - 2f(1) + f(2)\| \leqslant 18\varepsilon.$$

Adding these inequalities and (15), we obtain

$$\|\varphi(xy) + \varphi(x + y) - \varphi(xy + x) - f(y) + a + 2f(1) - f(2)\| \leqslant 55\varepsilon.$$

Since φ is additive, we infer

$$\|\varphi(y) - f(y) + a + 2f(1) - f(2)\| \leqslant 55\varepsilon.$$

This completes the proof. □

Theorem 3.2. *Let $\varepsilon \geqslant 0$ and $f, g, h, k : [0, +\infty) \to \mathcal{B}$ be functions satisfying*

$$\|f(xy) + g(x + y) - h(xy + x) - k(y)\| \leqslant \varepsilon, \quad x, y \geqslant 0. \tag{26}$$

Then there is an affine function $\varphi : \mathbb{R} \to \mathcal{B}$ such that

$$\|\varphi(x) - g(x)\| \leqslant 330\varepsilon, \quad x > 0; \tag{27}$$

$$\|\varphi(x) - h(x) + f(0) - k(0)\| \leqslant 331\varepsilon, \quad x > 0; \tag{28}$$

$$\|\varphi(x) - k(x) + f(0) - h(0)\| \leqslant 331\varepsilon, \quad x > 0; \tag{29}$$

$$\|\varphi(x) - f(x) + 2f(0) - h(0) - k(0)\| \leqslant 333\varepsilon, \quad x > 0. \tag{30}$$

Proof. Letting $x = 0$ and $y = 0$ separately in (26), we obtain

$$\|f(0) + g(x) - h(x) - k(0)\| \leqslant \varepsilon, \quad x \geqslant 0, \tag{31}$$

$$\|f(0) + g(y) - h(0) - k(y)\| \leqslant \varepsilon, \quad y \geqslant 0. \tag{32}$$

It follows from (31) that

$$\|f(0) + g(xy + x) - h(xy + x) - k(0)\| \leqslant \varepsilon, \quad x, y \geqslant 0. \tag{33}$$

By (26), (32), and (33), one obtains

$$\|f(xy) + g(x + y) - g(xy + x) - g(y) - 2f(0) + h(0) + k(0)\| \leqslant 3\varepsilon, \quad x, y \geqslant 0. \tag{34}$$

Setting $x = 1$ in (34), we have

$$\|f(y) - g(y) - 2f(0) + h(0) + k(0)\| \leqslant 3\varepsilon, \quad y \geqslant 0. \tag{35}$$

Then

$$\|g(xy) - f(xy) + 2f(0) - h(0) - k(0)\| \leqslant 3\varepsilon, \quad x, y \geqslant 0. \tag{36}$$

Adding (34) and (36), we obtain

$$\|g(xy) + g(x + y) - g(xy + x) - g(y)\| \leqslant 6\varepsilon, \quad x, y \geqslant 0.$$

By Theorem 3.1, there is an affine function $\varphi : \mathbb{R} \to \mathcal{B}$ such that

$$\|\varphi(x) - g(x)\| \leqslant 330\varepsilon, \quad x > 0.$$

This inequality with (31), (32), and (35) yield (28), (29), and (30). □

Corollary 3.3. *Let $\varepsilon \geqslant 0$ and $f, g, h, k : [0, +\infty) \to \mathcal{B}$ be functions satisfying*

$$f(xy) + g(x + y) - h(xy + x) - k(y) = 0, \quad x, y \geqslant 0.$$

Then f, g, h, k are affine functions on $(0, +\infty)$. Moreover,

$$\begin{aligned} h(x) &= g(x) + f(0) - k(0),\\ k(x) &= g(x) + f(0) - h(0),\\ f(x) &= g(x) + f(0) - g(0), \quad x > 0. \end{aligned}$$

4 Conclusions

The functional equation $f(xy) + f(x + y) = f(xy + x) + f(y)$ is known as the Davison functional equation. We treated the Hyers-Ulam stability of the Davison functional equation in two cases. The first is for functions defined on a normed unitary algebra with values in a Banach space and the functional inequality associated with Davison equation is valid only on a restricted unbounded domain. The second one is for the case of functions defined on interval $[0, +\infty)$ with values in a Banach space. In the second case, we considered also a generalization form of Davison functional equation. Finally, as an application of the obtained results, the asymptotic behavior of Davison functions has been investigated. Investigation and solving Davison's functional equation and its generalization in other certain restricted domains can be future research prospects.

Acknowledgments: The authors would like to thank the reviewers for their valuable suggestions and comments.

Funding information: The authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors declare that they have no competing interests.

Data availability statement: Not applicable.

References

[1] T. M. K. Davison, *191. Probem*, Aequationes Math. **20** (1980), 306.

[2] W. Benz, *191R1. Remark*, Aequationes Math. **20** (1980), 307.

[3] R. Girgensohn and K. Lajkó, *A functional equation of Davison and its generalization*, Aequationes Math. **60** (2000), no. 3, 219–224, DOI: https://doi.org/10.1007/s000100050148.

[4] T. M. K. Davison, *A Hosszú-like functional equation*, Publ. Math. Debrecen **58** (2001), 505–513, DOI: https://doi.org/10.5486/PMD.2001.2326.

[5] S. M. Ulam, *Problems in Modern Mathematics (Science Editions)*, John Wiley & Sons, New York, 1964.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224, DOI: https://doi.org/10.1073/pnas.27.4.222.

[7] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[8] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, Dordrecht, Heidelberg, London, 2011.

[9] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, New York, 2009.

[10] S.-M. Jung and P. K. Sahoo, *Hyers-Ulam-Rassias stability of an equation of Davison*, J. Math. Anal. Appl. **238** (1999), no. 1, 297–304, DOI: https://doi.org/10.1006/jmaa.1999.6545.

[11] S.-M. Jung and P. K. Sahoo, *On the Hyers-Ulam stability of a functional equation of Davison*, Kyungpook Math. J. **40** (2000), no. 1, 87–92.

[12] K.-W. Jun, S.-M. Jung, and Y.-H. Lee, *A generalization of the Hyers-Ulam-Rassias stability of a functional equation of Davison*, J. Korean Math. Soc. **41** (2004), no. 3, 501–511, DOI: https://doi.org/10.4134/JKMS.2004.41.3.501.

[13] S.-M. Jung and P. K. Sahoo, *Hyers-Ulam-Rassias stability of a functional equation of Davison in rings*, Nonlinear Funct. Anal. Appl. **11** (2006), no. 5, 891–896.

[14] Y.-H. Kim, *On the Hyers-Ulam-Rassias stability of an equation of Davison*, Indian J. Pure Appl. Math. **33** (2002), 713–726.

[15] J. Rimán, *On an extension of Pexider's equation*, Zb. Rad. (Beogr.) **1** (1976), no. 9, 65–72.