Research Article

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Topological structure of the solution sets to non-autonomous evolution inclusions driven by measures on the half-line

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Abstract: In this article, we investigate a class of measure differential inclusions of evolution type involving non-autonomous operator with nonlocal condition defined on the half-line. By fixed point theorem, we first obtain some sufficient conditions to ensure the solution set is nonempty, compact, and R_δ-set on compact interval. Subsequently, by means of the inverse limit method, we generalize the results on compact interval to noncompact interval. Finally, an example is given to demonstrate the effectiveness of obtained results.

Keywords: measure evolution inclusions, R_δ-set, inverse limit, noncompact interval, nonlocal condition

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1 Introduction

When dealing problems in control theory, economics, or game theory, one has to consider set-valued functions, and consequently, the models may involve multivalued differential equations (i.e., differential inclusions). Differential inclusions play a crucial role in the theory of differential equations with discontinuities on the right-hand side. The investigation of such equations is of great importance since they model the performance of various mechanical and electrical devices as well as the behavior of automatic control systems. Differential inclusions are also used to describe some systems with hysteresis [1–3]. In recent decades, the theory of differential inclusions is highly developed and constitutes an important branch of nonlinear analysis.

Recently, the study of measure differential inclusions has become popular because it encompasses special cases such as differential and difference inclusions, as well as impulsive and hybrid problems [3–5]. Measure differential inclusions have become important in recent decades in mathematical models of real processes, and they arise in hybrid systems which with dry friction, processes of controlled heat transfer, obstacle problems, and others can be described with the help of various differential inclusions, both linear and nonlinear.

Since differential inclusions often have multiple solutions at a given starting point, topological properties of solution sets have also attracted more and more attention. In the study of the topological structure of solution sets for differential inclusions, an important aspect is the R_δ-property [6–11]. Today, the topological structure of the solution set continues to arouse research enthusiasm. It is important not only from the viewpoint of academic interest, but also has a wide range of applications in practice. It is known that an R_δ-set is acyclic and particular, nonempty, compact, and connected. The topological structure of solution sets

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of differential inclusions on compact intervals has been investigated intensively by many authors (please see Bothe [12], Deimling [13], Hu and Papageorgiou [14], De Blasi and Pianigiani [15], Zhou and Peng [16] and references therein). Moreover, one can find results on the topological structure of solution sets for differential inclusions defined on non-compact intervals (including infinite intervals) from Andres et al. [17], Gabor [18], Wang et al. [19], and references therein.

Zhou and Peng [16] studied the topological properties of solution sets for partial functional evolution inclusions on a compact interval:

$$\begin{cases} d[x(t) - h(t, x_t)] \in Ax(t)\mathrm{d}t + F(t, x_t)\mathrm{d}t, \quad t \in [0, b], \\ x(0) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $x(\cdot)$ is the take value in Banach space X with norm $|\cdot|$, A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ of contractions on a Banach space X, which is uniformly continuous on $(0, \infty)$ and $F : [0, b] \to P(X)$ is a multimap. For any continuous function x defined on $[-\tau, b]$ and any $t \in [0, b]$, we denote by x_t the element of $C([-\tau, 0], X)$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. Here, $x_t(\cdot)$ represents the history of the state from time $t - \tau$ up to the present time t. It is shown that the solution set is nonempty, compact, and an R_δ-set.

Chen and Wang [8] studied the topological properties of the following nonlinear delay differential evolution inclusion on noncompact interval using the inverse limit:

$$\begin{cases} \mathrm{d}u(t) = Au(t)\mathrm{d}t + f(t), \quad t \in R^+, \\ f(t) \in F(t, u(t), u_t), \quad t \in R^+, \\ u(0) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.2}$$

where $A : D(A) \subset X \to 2^X$ is an m-dissipative operator and $F : R^+ \times \overline{D(A)} \times C[-\tau, 0], \overline{D(A)} \to 2^X$ is multi-valued function with convex, closed values for which $F(t, \cdot, \cdot)$ is weakly upper semi-continuous (weakly u.s.c., see Definition 2.4) for a.e. $t \in R^+$ and and $F(\cdot, x, v)$ has a strongly measurable selection for each $(x, v) \in \overline{D(A)} \times C([-\tau, 0]; D(A))$, and every $\phi \in C([-\tau, 0], \overline{D(A)})$, $u_t \in C([-\tau, 0], \overline{D(A)})$ is defined by $u_t(s) = u(t + s)$, $s \in [-\tau, 0]$ for every $u \in \widetilde{C}([-\tau, \infty], \overline{D(A)})$ and $t \in R^+$, in which $\widetilde{C}([-\tau, \infty], \overline{D(A)})$ is a subset of Frechet space $\widetilde{C}([-\tau, \infty], X)$.

Studies on the theory for nonlinear evolution inclusion of different types have many research achievements [20,21]. However, to the best of our knowledge, there are few works investigating the topological properties of solution sets for measure evolution inclusion of autonomous type with nonlocal condition. In this article, we apply the inverse system method, which was introduced in [22] for analyzing the topology of fixed point sets in function spaces. This method was developed in [17] and then also in [6]. It is observed that differential problems on noncompact intervals can be reformulated as fixed point problems in Frechet (function) spaces, which are inverse limits of Banach spaces that appear when we consider these differential problems on compact intervals.

We point out that among the previous studies, all of the researchers focus on the case that the differential operators in the main parts are independent of time t, which means that the problems under considerations are autonomous. However, when dealing with some parabolic evolution equations, it is often assumed that the partial differential operators are time-dependent because these types of operators frequently arise in practical applications; for more information, see references [23–25]. Therefore, it is interesting and significant to investigate non-autonomous evolution equations, i.e., the differential operators in the main parts of the considered problems are dependent of time t.

Therefore, based on the aforementioned study, in this article, we will study the following inclusion on non-compact intervals:

$$\begin{cases} \mathrm{d}u(t) \in A(t)u(t)\mathrm{d}t + F(t, u(t))dg(t), \quad t \in [0, \infty), \\ u(0) = u_0 + H(\sigma(u), u), \end{cases} \tag{1.3}$$

where $A(t)$ is a family of (possibly unbounded) linear operators depending on time and having the domains $D(A(t))$ for every $t \in [0, \infty)$, $g : [0, \infty) \to R$ is a left continuous and non-decreasing function. $F : [0, \infty) \times$

$G([0,\infty),X)\to P(X)$, where $P(X)$ is the subset of X. The set $G([0,\infty),X)$ is a space of regulated functions on $[0,\infty)$, which will be defined the nonlocal function $H:[0,\infty)\times G([0,\infty),X)\to X$, and $\sigma: G([0,\infty),X)\to[0,\infty)$ are both specified functions.

Evolution equations with nonlocal conditions were motivated by physical problems. It is observed that nonlocal condition in (1.3), which was proposed and discussed in [26], is clearly more general than the ones [27]. Actually, as pointed out in [26,28], the nonlocal function H in (1.3) is state-dependent and generalizes several types of nonlocal conditions considered in the literature mentioned earlier [27]. Moreover, from the mathematical point of view, some initial and boundary conditions, such as integral initial condition and periodic boundary condition, are also special cases of the nonlocal condition in (1.3).

The motives and highlights in this article are as follows:

(1) The result is extended to study topological properties of non-autonomous evolution inclusion system with nonlocal conditions. As far as we know, there are few works that have studied the topological structure of the measure evolution inclusion systems.

(2) Using the fixed point theorem and the concept of R_δ-set, we have established the topological properties of (1.3) within the compact interval $I=[0,T]$. Our results indicate that the solution set is nonempty, compact, and an R_δ-set. As far as we know, there are few articles investigating this property.

(3) Another result of this article is proved the R_δ topological structure of solution sets for (1.3) on noncompact intervals $I_\infty=[0,\infty)$ by inverse limit.

The rest of this article is organized as follows. In Section 2, some notations and preparation are given. A suitable concept on mild solution for our problem is introduced. In Section 3, on the one hand, it is devoted to proving that the solution set for inclusion (1.3) is nonempty and compact in the case that the evolution system of non-autonomous operator is compact on compact interval I, then proceed to study the R_δ-structure of the solution set. On the other hand, it is provided that the solution set for inclusion (1.3) is nonempty compact and the R_δ-structure of the solution set of (1.3) on noncompact interval by inverse system. Finally, an example is given to illustrate the obtained theory.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this article.

Let $(X,\|\cdot\|)$ be a Banach space, denote by $L(X)$ the space of bounded linear operators from X to X, and refer the reader to [29] or [30] for necessary notions of operator theory. In particular, the operator norm $\|\cdot\|$ is defined on $L(X)$, and $L(dg,[0,T],X)$ stands for the Banach space consisting of all Bocher integrable functions w.r.t. g (with respect to g) from $[0,T]$ to X equipped with the norm

$$\|f\|=\int_0^T\|f(s)\|dg(s).$$

2.1 Regulated functions

Definition 2.1. [31] If a function $f:[0,T]\to X$ satisfies the limits

$$\lim_{s\to t^-}f(s)=f(t^-),\ t\in(0,T]\quad\text{and}\quad\lim_{s\to t^+}f(s)=f(t^+),\quad t\in[0,T),$$

then the function f is called regulated function on $[0,T]$. It was proved that the set of discontinuous points of a regulated function is at most countable, regulated functions are bounded, and the space $G([0,T],X)$ of regulated functions is a Banach space when endowed with the norm

$$\|f\|_\infty=\sup_{t\in[0,T]}\|f(t)\|.$$

In order to obtain compactness in $G([0,T],X)$, we need the concept of equiregulated sets.

Definition 2.2. [32] A set $\mathscr{A} \subset G([0,T],X)$ is said to be equi-regulated, if for every $\varepsilon > 0$ and every $t_0 \in [0,T]$, there exists $\delta > 0$ such that:
(1) for any $t_0 - \delta < t' < t_0 : \|x(t') - x(t_0 - 0)\| \leq \varepsilon$,
(2) for any $t_0 - \delta < t'' < t_0 + \delta : \|x(t'') - x(t_0 + 0)\| \leq \varepsilon$,
for all $x \in A$.

Definition 2.3. [33] If $f : [0,T] \to X$ is called Kurzweil-Stieltjes integrable with respect to $g : [0,T] \to R$ if there is a function denoted by $(\mathrm{KS})\int_0^{\cdot} f(s)dg(s) : [0,T] \to X$ such that, for every $\varepsilon > 0$, there exists a gauge δ_ε on $[0,T]$ with

$$\left\| \sum_{i=1}^{n} f(\xi_i)(g(t_i) - g(t_{i-1})) - (\mathrm{KS})\int_0^T f(s)dg(s) \right\| < \varepsilon,$$

for every δ_ε-fine partition $\{(\xi_i, [t_{i-1}, t_i)) : i = 1, 2, \dots, n\}$ of $[0,1]$. A partition $\{(\xi_i, [t_{i-1}, t_i)) : i = 1, 2, \dots, n\}$ is δ_ε-fine if for all $i = 1, 2, \dots, n$, $[t_{i-1}, t_i] \subset (\xi_i - \delta_\varepsilon(\xi_i), \xi_i + \delta_\varepsilon(\xi_i))$.

Lemma 2.1. [33] *Consider the functions $f : [0,T] \to X$, g is regulated and $g : [0,T] \to R$ such that Kurzweil-Stieltjes integral $\int_0^T f(s)dg(s)$ exists. Then, for every $t \in [0,T]$, the function $h(t) = \int_0^t f(s)dg(s)$, $t \in [0,T]$, is regulated and satisfies*

$$h(t^-) = h(t) - f(t)\triangle^- g(t), \quad t \in (0,T],$$
$$h(t^+) = h(t) + f(t)\triangle^+ g(t), \quad t \in [0,T),$$

where $\triangle^+ g(t) = g(t^+) - g(t)$, $\triangle^- g(t) = g(t) - g(t^-)$, $g(t^+)$, and $g(t^-)$ denote the right and left limits of function g at point t.

Now, we present a short overview on Kurzweil-Stieltjes integration on compact intervals, but the same definition works on unbound intervals as well, as it can be seen in [34].

The Kurzweil-Stieltjes integral is a generalized form of integration that offers certain advantages over other forms of integration. Specifically, the advantages of the Kurzweil-Stieltjes integral may be reflected in the following aspects: the Kurzweil-Stieltjes integral allows for discontinuities in both the integrand and the integrator at certain points. This is because it can handle functions with common points of discontinuity, which might not be possible with other forms of integration. Besides, the Kurzweil-Stieltjes integral has applications in functional analysis, distribution theory, generalized elementary functions, and various generalized differential equations including those on time scales. These characteristics make the Kurzweil-Stieltjes integral a useful tool in both theoretical research and practical applications due to its ability to handle discontinuous functions, applicability to bounded variation functions, and its wide range of applications [34].

Proposition 2.1. [35] *A set K of regulated functions is relatively compact in the space $G([0,T],X)$ if and only if it is equi-regulated, and for every $t \in [0,T]$, $\{f(t), t \in [0,T]\}$ is relatively compact in X.*

Proposition 2.2. [35] *A sequence $(u_n)_n \subset G([0,T],X)$ is weakly convergent to a function $u \in G([0,T],X)$ if and only if is (norm) bounded, and for any $t \in [0,T]$, the sequence $(u_n)_n$ is weakly convergent to $u(t)$ in X.*

Lemma 2.2. [35] *Let $(u_n)_n$ be a sequence of functions from $[0,T]$ to X. If (u_n) converges pointwisely to u_0 as $n \to \infty$ and the sequence $(u_n)_n$ is equiregulated, then $(u_n)_n$ converges uniformly to u_0.*

Lemma 2.3. [35] *Let X be a Banach space and μ be a finite measure. Then, a bounded, uniformly integrable subset $\mathcal{A}$ of $L^1(\mu, X)$ such that for each $t \in [0,T]$, $\mathcal{A}(t)$ is relatively weakly compact in X, is weakly relatively compact.*

Theorem 2.1. [36] *Let* $\mathbf{g} : [0, T] \to R$ *be a function of bounded variation. Assume that* $f, f_n : [0, T] \to R, n \in \mathbb{N}$, *satisfy the following conditions*:

(*i*) *The integral* $\int_0^T f_n(s)d\mathbf{g}(s)$ *exists for every* $n \in \mathbb{N}$;

(*ii*) *For each* $\tau \in [0, T]$, $\lim_{n\to\infty} f_n(\tau) = f(\tau)$;

(*iii*) *There exists a constant* $\mathcal{K} > 0$ *such that for every division* $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_l = T$ *of* $[0, T]$ *and every finite sequence* $m_1, \ldots, m_l \in \mathbb{N}$, *we have*

$$\left\| \sum_{i=1}^{l} \int_{\sigma_{i-1}}^{\sigma_i} f_{m_i}(s)d\mathbf{g}(s) \right\| \leq \mathcal{K}.$$

Then, the integral $\int_0^T f(s)d\mathbf{g}(s)$ *exists and*

$$\int_0^T f(s)d\mathbf{g}(s) = \lim_{n\to\infty} \int_0^T f_n(s)d\mathbf{g}(s).$$

Theorem 2.2. [37] *Let* $\mathbf{D}$ *be a nonempty, compact, and convex subset of a Banach space, and let the correspondence* $N : \mathbf{D} \to B(\mathbf{D})$ *an upper semi-continuous (u.s.c.) multi-valued mapping with contractible values. Then, the set of fixed points of* N *is nonempty.*

2.2 Multi-valued analysis and properties of non-autonomous operator

Let X and Y be two metric spaces. We denote by $P(Y)$ the family of all nonempty subsets of Y, and we set:

$P_{cl}(Y) = \{A \in P(Y),\ \text{closed}\}$,

$P_{cl,cv}(Y) = \{A \in P(Y),\ \text{closed and convex}\}$,

$P_{cp}(Y) = \{A \in P(Y),\ \text{compact}\}$,

$P_{cp,cv}(Y) = \{A \in P(Y),\ \text{compact and convex}\}$.

We say that a multivalued map $F : X \to P(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. A map F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty$). The mapping F is u.s.c on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subset N$. The mapping F is completely continuous if $F(B)$ is relatively compact for each $B \in P_b(X)$.

Definition 2.4. [38] A multi-valued mapping $F : X \to P(X)$ is called weakly u.s.c. if $F^{-1}(\bar{B})$ is closed for each weakly closed set $\bar{B} \subset X$.

Evidently, u.s.c. is stronger than weakly u.s.c., and simple examples show that a weakly u.s.c. function with compact convex values may fail to be u.s.c.

Let us recall the definition of evolution family below, sometimes also called an evolution process or evolution system. For a detailed account and bibliographic references, see, e.g., the survey by Acquistapace [29], Acquistapace and Terreni [30], Engel and Nagel [39], and Goldstein [40].

Let $[0, T]$ be a fixed interval of real line, $\triangle = \{(t, s) \in [0, T] \times [0, T] : 0 \leq s \leq t \leq T\}$, $\triangle_\infty = \{(t, s) \in [0, \infty) \times [0, \infty) : 0 \leq s \leq t\}$. Let us recall that two-parameter family $\{T(t, s)\}_{(t,s)\in\triangle}$, $\{T(t, s)\}_{(t,s)\in\triangle_\infty}$, $T(t, s) : X \to X$ bounded linear operator, $(t, s) \in \triangle$, $((t, s) \in \triangle_\infty)$ is called an evolution system if the following lemma of conditions are satisfied:

Lemma 2.4. [29] *The family of the linear operator* $\{T(t, s)\}_{(t,s)\in\triangle}$ *satisfies the following properties.*

(*i*) $T(t, r)T(r, s) = T(t, s)$, $T(t, t) = I$ *for* $0 \leq s \leq t \leq T$;

(*ii*) *The map* $(t, s) \to T(t, s)x$ *is strong continuous for all* $x \in X$ *and* $0 \leq s \leq t \leq T$;

(iii) $T(\cdot, s) \in C^1((s, \infty), L(E))$, $\frac{\partial T(t,s)}{\partial t} = A(t)T(t,s)$ *for* $t > s$, *and* $\|A^k(t)T(t,s)\| \le M(t-s)^{-k}$ *for* $0 < t - s \le 1$ *and* $k = 0, 1$;

(iv) $\frac{\partial T(t,s)}{\partial t} = -T(t,s)A(s)x$ *for* $t > s$ *and* $x \in D(A(s))$ *with* $A(s)x \in D(A(s))$.

Definition 2.5. [29] An evolution family $\{T(t,s)\}_{(t,s)\in\triangle}$ is said to be compact if for all $0 \le s \le t \le T$, $T(t,s)$ is continuous and maps bounded subsets of X into precompact subsets of X.

Lemma 2.5. [29] *Let* $\{T(t,s)\}_{(t,s)\in\triangle}$ *be a compact evolution family on X. Then, for each* $s \in [0, T]$, *the function* $t \to T(t,s)$ *is continuous by operator norm for* $t \in (s, T]$.

Lemma 2.6. [41] *For each* $t \in [0, T]$ *and some* $\lambda \in \rho(A(t))$, *if the resolvent* $R(\lambda, A(t))$ *is a compact operator, then* $T(t,s)$ *is a compact operator whenever* $0 \le s \le t \le T$.

From now on, what will be assumed throughout is that $\|T(t,s)\| \le 1$ for all $t > s \ge 0$.

Lemma 2.7. [38] *Let* $F : X \to P_{cl,cv}(X)$ *be a closed and quasi-compact multi-valued mapping. Then, F is u.s.c.*

Recall that a subset $\mathbb{A} \subset X$ is called a retract of X if there exists a continuous function $\gamma : X \to \mathbb{A}$, such that $\gamma(x) = x$, for every $x \in \mathbb{A}$. $\mathbb{A}$ is called a neighborhood retract of X if there exists an open subset $U \subset X$ such that $\mathbb{A} \subset U$ and $\mathbb{A}$ is retract of U. If we have two spaces X and Y, then every homeomorphism $\psi : X \to Y$ such that $\psi(X)$ is a closed subset of Y is called an embedding. We say that X is an absolute retract (is an absolute neighborhood retract) if and only if for any space Y and for any embedding $\psi : X \to Y$, the set $\psi(X)$ is a retract of Y ($\psi(X)$ is a neighborhood retract of Y) . We write $X \in AR$ (resp. $X \in ANR$). Obviously, if Y is an AR-space, then it is an ANR-space. Any absolute retract is contractible.

The following hierarchy holds for nonempty subsets of a metric space: compact + convex ⊂ compact AR-space ⊂ compact + contractible ⊂ R_d-set.

Definition 2.6. [42] A nonempty subset D of a metric space is said to be contractible if there exists a point $y_0 \in D$ and a continuous function $h : [0,1] \times D \to D$ such that $h(0, y) = y_0$ and $h(1, y) = y$ for every $y \in D$.

Definition 2.7. [42] A subset D of a metric space is called an R_δ-set if there exists a decreasing sequence D_n of compact and contractible sets such that

$$D = \bigcap_{n=1}^{\infty} D_n.$$

Let X be a Banach space and $B(X)$ the family of all bounded subsets of X. Then, the function $\beta : B(E) \to R^+$ defined by

$$\beta(A) = \inf\{r > 0 \mid A \text{ can be covered by finitely many balls of radius } r\}$$

is called the (Hausdorff) measure of noncompactness. It is monotone, nonsingular, real, and regular [43].

The following characterization of R_δ-sets, which develops the well-known Hyman theorem [42], was shown by Bothe.

Theorem 2.3. [12] *Let X be a complete metric space,* β *denote the Hausdorff measure of noncompactness in X, and let* $\emptyset \ne \tilde{A} \subset X$.

Then, the following statements are equivalent:

(*a*) $\tilde{A}$ *is an* R_δ-set,

(*b*) $\tilde{A}$ *is an intersection of a decreasing sequence* $\{\tilde{A}_n\}$ *of closed contractible spaces with* $\beta(\tilde{A}_n) \to 0$,

(*c*) $\tilde{A}$ *is compact and absolutely neighborhood contractible, i.e.,* $\tilde{A}$ *is contractible in each neighborhood in* $Y \in ANR$, *where* $\tilde{A}$ *is embedded.*

2.3 Inverse limit

Let us recall that an inverse system of topological spaces is a family $S = \{X_\alpha, \pi_\alpha^\beta, \sum\}$, where $\sum$ is a directed set ordered by the relation " $\leq$ ", X_α is a topological (Hausdorff) space, for every $\alpha \in \sum$, and $\pi_\alpha^\beta : X_\beta \to X_\alpha$ is a continuous mapping for each two elements $\alpha, \beta \in \sum$ such that $\alpha \leq \beta$. Moreover, for each $\alpha \leq \beta \leq \gamma$, the following conditions should hold: $\pi_\alpha^\alpha = idX_\alpha$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$.

A subspace of the product $\pi_\alpha \in \Pi_{\alpha\in\sum} X_\alpha$ is called a limit of the inverse system S, and it is denoted by $\lim_\leftarrow S$ whenever

$$\lim_{\leftarrow} S = \left\{ (x_\alpha) \in \Pi_{\alpha\in\sum} X_\alpha | \pi_\alpha^\beta(x_\beta) = x_\alpha, \ \text{ for all } \ \alpha \leq \beta \right\}.$$

Consider two inverse systems $S = \{X_\alpha, \pi_\alpha^\beta, \sum\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha,}^{\beta'}, \sum\}$. Let us recall (see [17]) that by a multi-valued map of the system S into the system S', we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \sum \to \sum$, i.e., $\sigma(\alpha') \leq \sigma(\beta')$ for $\alpha' \leq \beta'$, and of multivalued maps $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \to Y_{\alpha'}$ with nonempty values, defined for every $\alpha' \in \sum'$ and such that

$$\pi_\alpha^\beta \varphi_{\sigma(\beta')} = \varphi_{\alpha'} \pi_{\sigma(\alpha')}^{\sigma(\beta')},$$

for each $\alpha' \leq \beta'$.

A map of systems $\{\sigma, \pi_{\sigma(\alpha')}\}$ induces a limit map $\varphi : \lim_\leftarrow S \to \lim_\leftarrow S'$ defined as follows:

$$\varphi(x) = \prod_{\alpha'\in\sum'} \varphi_{\sigma(\alpha')}(x_{\sigma(\alpha')}) \cap \lim_{\leftarrow} S'.$$

In other words, a limit map is the one such that

$$\pi_{\alpha'}\varphi = \varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')},$$

for each $\alpha' \leq \beta'$.

Now, we summarize some useful properties of limits of inverse systems.

Proposition 2.3. [44] *Let $S = \{X_\alpha, \pi_\alpha^\beta, \sum\}$ be an inverse system. If for every $\alpha \in \sum$, X_α is compact and nonempty, then* $\lim_\leftarrow S$ *is compact and non-empty.*

Proposition 2.4. [22] *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system. If for every $n \in \mathbb{N}$, X_n is an R_δ-set, then* $\lim_\leftarrow S$ *is R_δ-set, as well.*

Theorem 2.4. [6] *Let $S = \{X_\alpha, \pi_\alpha^\beta, \sum\}$ be an inverse system and φ : $\lim_\leftarrow S \to \lim_\leftarrow S'$ be a limit map induced by a map $\{id, \varphi_\alpha\}$, where $\varphi_\alpha : X_\alpha \to X_\alpha$. Then, the fixed point set of φ is a limit of the inverse system generated by the sets* $\mathrm{Fix}(\varphi_\alpha)$. *In particular, if the sets* $\mathrm{Fix}(\varphi_\alpha)$ *are compact acyclic (resp. R_δ), then it is compact acyclic (resp. R_δ), as well.*

Below, we introduce the following two inverse systems and their limits. For more details about the inverse system and its limit, we refer the reader to [22,45].

Let $a \in R$ and $\mathbb{N}(a) = \{m \in \mathbb{N}\backslash\{0\}; m > a\}$. For each $p, m \in \mathbb{N}(a)$ with $p \geq m$, let us consider a projection $\pi_{a,m}^p : G([a, p], X) \to G([a, m], X)$, defined by

$$\pi_{a,m}^p(u) = u \mid_{[a,m]}, \quad u \in G([a, p], X).$$

Then, it is readily checked that $\{G([a, m], X), \pi_{a,m}^p, \mathbb{N}(a)\}$ is an inverse system and its limit is

$$G([a, \infty), X) = \lim_{\leftarrow}\{G([a, m], X), \pi_{a,m}^p, \mathbb{N}(a)\}.$$

Let $L_{\text{loc}}(dg, [0, \infty), X)$ be the separately locally convex space consisting of all locally Bocher integrable functions w.r.t. g from $[0, \infty)$ to X endow with a family of seminorms $\{\|\cdot\|_m^1; m \in \mathbb{N}\backslash\{0\}\}$, defined by

$$\|u\|_m^1 = \int_0^m \|u(t)\| dg(t), \quad m \in \mathbb{N}\backslash\{0\}.$$

In a similar manner as earlier, we also obtain that $\{L(dg, [0, m], X), \dot{\pi}_{a,m}^p, \mathbb{N}\backslash\{0\}\}$ is an inverse system, where $p \geq m$ and

$$\dot{\pi}_{a,m}^p(f) = f\,|_{[0,m]}, \quad f \in L(dg, [0, p], X).$$

Moreover,

$$L(dg, [0, \infty), X) = \varprojlim\{L(dg, [0, m], X), \dot{\pi}_{0,m}^p, \mathbb{N}\backslash\{0\}\}.$$

Remark 2.1. [46] Let us recall that by a Frechet space, we mean a locally convex space, which is metrizable and complete. Every Banach space is Frechet space, and every Frechet space is the limit of an inverse system of Banach spaces.

3 Topological structure of solution sets

In this section, we will study the existence result of the mild solution and the R_δ-set of solution set for system (1.3) on I. Further, we will study the R_δ-set for system (1.3), which is defined on I_∞. Before stating and proving the main results, we introduce some assumptions:

(H0) $\{A(t)\}_{t\in I}$ is a family of linear not necessarily bounded operators ($A(t) : D(A) \subset X \to X, t \in I,$ $D(A)$ a dense subset of X not depending on t) generating an evolution operator $T : \triangle \to L(X)$, where $L(X)$ is the space of all bounded linear operators in X.

(H1) The multivalued $F : I \times G(I, X) \to P_{cl,cv}(X)$ satisfies $F(t, \cdot)$ is weakly u.s.c. for a.e. $t \in I$, and $F(\cdot, u)$ has strongly measurable selection for every $u \in G(I, X)$.

(H2) There exists a constant $b > 0$, a function $a(t) \in L^1(dg, I, R^+)$, and a compact set $S \subset X$ such that $F(t, u) \subseteq a(t)(1 + b\|u\|)S$ for a.e. $t \in I$ and $u \in G(I, X)$.

(H3) The nonlocal function $H(\cdot,\cdot) : [0, \infty) \times G(I, X) \to X$ and $\sigma(\cdot) : G(I, X) \to [0, \infty)$ are both continuous. $H(\sigma(\cdot), \cdot) : G(I, X) \to X$ is a compact mapping, and there is a constant $K > 0$ such that, for $u \in G(I, X)$,

$$\|H(\sigma(u), u)\| \leq K\|u\|.$$

For each $u \in G(I, X)$, let us denote

$$\text{Sel}_F(u) = \{f \in L^1(dg, I, X) : f(t) \in F(t, u(t)), \text{ for a.e. } t \in I\}.$$

Definition 3.1. A regulated function $u : I \to X$ is called a mild solution on I of system (1.3) if it is a solution of the integral equation

$$u(t) = T(t, 0)[u_0 + H(\sigma(u), u)] + \int_0^t T(t, s)f(s)dg(s), \text{ for } t \in I,$$

where $f \in \text{Sel}_F(u)$.

When maintaining fundamental properties (including continuity, local boundedness, local maxima and minima, etc.); mapping compact sets to compact sets as well as mapping connected sets to connected sets; degree of smoothness; monotonicity preservation, etc., the contents that can be established for general continuous functions can also hold for regular functions. Therefore, Lemma 3.1 can be deduced from [6] (Lemma 3.3) as being obviously true.

Lemma 3.1. *Let Hypothesis (H1) and (H2) be satisfied. Then, there exists a sequence* $\{F_n\}$ *with* $F_n : I \times G(I,X) \to P_{cl,cv}(X)$ *such that*

(*i*) $F(t,u) \subset \ldots \subset F_{n+1}(t,u) \subset F_n(t,u) \ldots \subset \overline{co}(F(t, B_{3^{1-n}}(u)))$, $n \geq 1$, *for each* $u \in G(I,X)$;

(*ii*) $\|F_n(t,u)\| \leq \alpha(t)(3 + \|u\|)$, $n \geq 1$, *for each* $u \in G(I,X)$;

(*iii*) *There exists* $E \subset I$ *with* $mes(E) = 0$ *such that for each* $x^* \in X^*$, $\varepsilon > 0$, *and* $(t,u) \in I \backslash E \times G(I,X)$, *there exists* $\mathcal{N} > 0$ *such that for all* $n \geq \mathcal{N}$,

$$x^*(F_n(t,u)) \subset x^*(F(t,u)) + (-\varepsilon, \varepsilon);$$

(*iv*) $F_n(t,\cdot) : X \times G(I,X) \to P_{cl,cv}(X)$ *is continuous for a.e.* $t \in [0,T]$ *with respect to Hausdorff-Pompeiu metric for each* $n \geq 1$;

(*v*) *For each* $n \geq 1$, *there exists a selection* $g_n : [0,T] \times G(I,X) \to X$ *of* F_n *such that* $g_n(\cdot,u)$ *is strongly measurable for each* $u \in G(I,X)$ *and for any compact subset* $\mathcal{D} \subset G(I,X)$, *there exist constants* $C_V > 0$ *and* $\delta > 0$ *for which the estimate*

$$\|g_n(t,u_1) - g_n(t,u_2)\| \leq C_V \alpha(t)(\|u_1 - u_2\|)$$

holds for a.e. $t \in [0,T]$ *and each* $u_1, u_2 \in V$ *with* $V \coloneqq \mathcal{D} + B_d(0)$;

(*vi*) F_n *verifies the condition (H2) with* F_n *instead of* F *for each* $n \geq 1$, *provided that* X *is reflexive.*

3.1 Topological structure on compact intervals

Lemma 3.2. *Let conditions (H1) and (H2) be satisfied. Then,* $\mathrm{Sel}_F(u)$ *is nonempty.*

Proof. Let us show that $\mathrm{Sel}_F(u) \neq \emptyset$ for each $u \in G(I,X)$. For this purpose, we assume that $u \in G(I,X)$ and $\{u_n\}$ is the sequence of step functions from I to $G(I,X)$ such that

$$\sup_{t \in I} \|u_n(t) - u(t)\| \to 0, \quad n \to \infty. \tag{3.1}$$

By (H1), we can see that for each n, $F(\cdot, u_n)$ has a strongly measurable selection $f_n(\cdot)$. Furthermore, by (3.1), for each $n > 0$, $t \in I$, there exists $M > 0$, such that $\|u_n(t)\| \leq M$. It follows from (H2) that f_n is integrably bounded in $L^1(dg, I, X)$. Making use of Lemma 2.3, one can conclude that this family is weakly relatively compact, and so, we can find a subsequence f_n (which we do not relabel), which is weakly $\sigma(L^1(dg,I,X), L^\infty(dg,I,X))$ convergent to some dg-measurable function f, i.e., for every $x^* \in X^*$, we have

$$\langle x^*, f_n \rangle \to \langle x^*, f \rangle, \quad n \to \infty.$$

And using Lebesgue controlled convergence theorem, we obtain

$$\left\langle x^*, \int_0^t f_n(s) dg(s) \right\rangle \to \left\langle x^*, \int_0^t f(s) dg(s) \right\rangle, \quad n \to \infty.$$

Therefore, for $n \geq 1$, $\{\int_0^t f_n(s) dg(s)\}$ is relatively weakly compact.

By Mazur's theorem, there exists a sequence of convex combinations convergent in the norm of $L^1(dg, I, X)$, i.e., for every $n \geq 1$, $\tilde{f}_n \in co\{f_k ; k \geq n\}$, and $\tilde{f}_n(t) \to f(t)$ for a.e. $t \in I$.

The hypotheses of Theorem 2.3 are satisfied by the sequence $\{\tilde{f}_n\}$. It follows that f is KS-integrable w.r.t. g and

$$\int_0^t \tilde{f}_n(s) dg(s) \to \int_0^t f(s) dg(s), \quad n \to \infty, \forall t \in I.$$

It is sufficient to check $f(t) \in F(t, u(t))$ for a.e. $t \in I$. Let J be a set and $J \subseteq I$ such that $\tilde{f}_n(t) \to f(t)$ and $f_n(t) \in F(t, u_n(t))$ for all $t \in I$. For all $n \geq 1$, let $x^* \in X^*$, $\varepsilon > 0$, and $t \in I$ fixed, from (H1), which implies that $(x^* \circ F(t,\cdot)) : X \to 2^R$ is an u.s.c. and with convex valued. To see this, let us write that for every $s \in J$, and $\varepsilon > 0$, there exists $n_{\varepsilon,s} \in N$ such that

$$x^*(F(s,u_n(s))) \subset x^*(F(s,u(s))) + (-\varepsilon,\varepsilon), \quad \forall n \geq n_{\varepsilon,s}.$$

Since for every $n \in N$, $x^*(\tilde{f}_n(s)) \in co\{x^*(f_k(t));\ k \geq n\} \subset x^*(F(s,u_n(s)))$, it comes that

$$x^*(\tilde{f}_n(s)) \in x^*(F(s,u(s))) + (-\varepsilon,\varepsilon), \quad \forall n \geq n_{\varepsilon,s},$$

take the limit from both sides, the arbitrariness of x^*, and F has convex valued, so $f(s) \in F(s,u(s))$, for every $t \in I$. Hence, we obtain $f \in \mathrm{Sel}_F(u)$. □

Next, in order to obtain the compactness of the solution set, we will consider the integral equation of the form

$$u(t) = \varphi(t) + \int_a^t T(t,s)k(s,u(s))dg(s), \quad t \in [a,T], \tag{3.2}$$

where $a \in [0,T)$ and $\varphi \in G(I,X)$.

Lemma 3.3. *Assume that for every $u \in G([0,a],X)$, $k(\cdot,u)$ is $L^1(dg,[0,a],X)$-integrable, $\{T(t,s)\}_{(t,s)\in\triangle}$ is compact. Suppose, in addition, that*

(i) *for any compact subset $\bar{D} \subset G([0,a],X)$, there exists $\delta > 0$ and $L_{\bar{D}} \in L^1(dg,[0,T],R^+)$ and $0 < \int_a^t L_{\bar{D}}(s)dg(s) < 1$ such that $\|k(t,u_1) - k(t,u_2)\| \leq L_{\bar{D}}(t)\|u_1 - u_2\|$, for a.e. $t \in [0,T]$, each $u_1,u_2 \in B_\delta(\bar{D})$;*

(ii) *there exists $r_1(t) \in L^1(dg,[0,T],R^+)$ such that $\|k(t,u)\| \leq r_1(t)(\bar{c} + \|u\|)$, for a.e. $t \in [0,T]$ and each $u \in G([0,a],X)$, where $\bar{c}$ is arbitrary, but fixed.*

Then, Problem (3.2) *admits a unique solution for every $\varphi \in G(I,X)$. Moreover, the solution to* (3.2) *depends continuously on φ.*

Proof. The proof process is divided into several steps. Set the operator W defined by

$$(Wu)(t) = \varphi(t) + \int_a^t T(t,s)k(s,u(s))dg(s), \quad t \in [a,T],$$

there exists a ξ arbitrarily close to a such that $\rho \geq \sup_{t\in[a,\xi]}|\varphi(t)| + (\rho + \bar{c})\|r_1\|_{L[a,\xi]}$. Let

$$B_\rho(\xi) = \left\{u \in G([0,\xi],X);\ \sup_{t\in[0,\xi]}|u(t)| \leq \rho\right\}.$$

Step I. We will prove $W(B_\rho(\xi)) \subset B_\rho(\xi)$. According to assumptions, we obtain

$$\|(Wu)(t)\| = \left\|\varphi(t) + \int_a^t T(t,s)k(s,u(s))dg(s)\right\| \leq \sup_{t\in[0,\xi]}|\varphi(t)| + (\bar{c} + \rho)\|r_1\|_{L^1_{(g,[a,\xi])}} \leq \rho.$$

Therefore, $W(B_\rho(\xi)) \subset B_\rho(\xi)$.

Step II. We will show that the operator $W(B_\rho(\xi))$ is an equiregulated family of functions on $[0,\xi]$.

For $t_0 \in [0,\xi)$, we have

$$\begin{aligned}\|(Wu)(t) - (Wu)(t_0^+)\| &\leq \|\varphi(t) - \varphi(t_0^+)\| + \int_a^{t_0^+}\|(T(t,s) - T(t_0^+,s))k(s,u(s))\|dg(s) + \int_{t_0^+}^t \|T(t,s)k(s,u(s))\|dg(s)\\ &:= A_1 + A_2 + A_3,\end{aligned}$$

where

$$A_1 = \|\varphi(t) - \varphi(t_0^+)\|,$$

$$A_2 = \int_a^{t_0^+}\|(T(t,s) - T(t_0^+,s))k(s,u(s))\|dg(s),$$

$$A_3 = \int_{t_0^+}^t \|T(t,s)k(s,u(s))\|dg(s).$$

The combination of compactness of $\{T(t,s)\}_{(t,s)\in\triangle}$ and strongly continuity of $T(t,s)$ in the uniform operator topology, and applying dominated convergence theorem, we can derive that A_1 and A_2 tend to zero independently of u as $t\to t_0^+$. Let $\omega(t)=\int_0^t r_1(s)dg(s)$, from Lemma 2.1, $\omega(t)$ is regulated function on $[0,\xi]$. Thus, $A_3=\int_{t_0^+}^t\|T(t,s)k(s,u(s))\|dg(s)\le(\bar{c}+\rho)\int_{t_0^+}^t r_1(s)\mathrm{d}g(s)\le(\bar{c}+\rho)(\omega(t)-\omega(t_0^+))$. So, when $t\to t_0^+$, we have $A_3\to 0$ also independently of u.

Similar to the aforementioned analysis, we can also derive that $\|(Wu)(t)-(Wu)(t_0^-)\|\to 0$, as $t_0^-\to t$, for every $t\in[0,\xi]$. So $W(B_\rho(\xi))$ is equiregulated on $[0,\xi]$ from Definition 2.2.

Step III. We will prove that the operator $W:B_\rho(\xi)\to B_\rho(\xi)$ is continuous. To this end, let $\{u_n\}_{n=1}^\infty\subset B_\rho(\xi)\subset G([0,\xi],X)$ be a sequence such that $\lim_{n\to\infty}u_n=u$ in $B_\rho(\xi)$.

$$\|(Wu_n)(t)-(Wu)(t)\|\le\int_a^t L_D(s)\|u_n-u\|dg(s)\to 0,\quad \text{as } n\to\infty.$$

In addition, the same analysis as Step II demonstrates that $\{Wu_n\}_{n=1}^\infty$ is equiregulated on $[0,\xi]$. This property combined with Lemma 2.2 implies that $\{Wu_n\}$ converges uniformly to $\{Wu\}$ as $n\to\infty$, namely,

$$\|(Wu_n)-(Wu)\|_\infty\le\sup_{t\in[0,\xi]}\|(Wu_n)(t)-(Wu)(t)\|\to 0,\quad \text{as } n\to\infty.$$

Therefore, the operator $W:B_\rho(\xi)\to B_\rho(\xi)$ is continuous.

Step IV. We demonstrate that the operator $W:B_\rho(\xi)\to B_\rho(\xi)$ is compact. To prove this, we show that $V(t)=\{(Wu)(t):u\in B_\rho(\xi)\}$ is relatively compact in X, for $t\in[0,\xi]$. It is easy to see that for every $u\in B_\rho(\xi)$, $\tilde{u}\in V(t)$, such that

$$\tilde{u}(t)=\varphi(t)+\int_a^t T(t,s)k(s,u(s))dg(s).$$

For an arbitrary $\varepsilon\in(0,t)$, define an operator $I_\varepsilon:V(t)\to X$ by

$$(I_\varepsilon\tilde{u})(t)=\varphi(t)+T(t,t-\varepsilon)\int_a^{t-\varepsilon}T(t-\varepsilon,s)k(s,u(s))dg(s).$$

From the compactness of $\{T(t,s)\}_{(t,s)\in\triangle}$ and $\int_a^{t-\varepsilon}T(t-\varepsilon,s)k(s,u(s))dg(s)\le(\bar{c}+\rho)\|r_1\|_{L^1_{g,[a,t]}}$, we obtain that the set $V_\varepsilon(t)=\{(I_\varepsilon\tilde{u})(t):\tilde{u}\in V(t)\}$ is relatively compact in X for each $\varepsilon\in(0,t)$. Moreover, it follows

$$|(\tilde{u})(t)-(I_\varepsilon\tilde{u})(t)|\le\left|\int_{t-\varepsilon}^t T(t,s)k(s,u(s))dg(s)\right|\le(\bar{c}+\rho)\int_{t-\varepsilon}^t r_1(s)dg(s)\to 0,\quad \varepsilon\to 0.$$

Therefore, there is a relatively compact set arbitrary close to the set $V(t)$. Thus, the set $V(t)$ is also relatively compact in X. According to Lemma 2.3, we know that $W:B_\rho(\xi)\to B_\rho(\xi)$ is a compact operator. And by the Schauder fixed point theorem, we obtain that operator W has a fixed point $u\in B_\rho(\xi)$, which is local solution to equation (3.2).

Step V. We prove that the solution is unique. Let u_1 be another local solution to equation (3.2). Then,

$$|u(t)-u_1(t)|\le\left|\int_a^t T(t,s)(k(s,u(s))-k(s,u_1(s)))dg(s)\right|\le\|u-u_1\|\int_a^t L_D(s)dg(s).$$

Namely, $\int_a^t L_D(s)dg(s)\ge 1$ for $\in[0,\xi]$, which is contradicts our assumption.

Next, we continue the solution for $t\ge\xi$. For $t\in[\xi,\xi_1]$, where $\xi<\xi_1$, we say that a function $\breve{u}(t)$ is a continuation of $u(t)$ to the interval $[\xi,\xi_1]$, if

(a) $\breve{u}(t)\in G([0,\xi_1],X)$, and

(b) $\breve{u}(t)=\varphi(t)+\int_\xi^t T(t,s)k(s,\breve{u}(s))dg(s)$.

By the observation that if we define a new function $v(t)$ on $[0,\xi_1]$ by setting

$$v(t)=\begin{cases}u(t), & t\in[0,\xi],\\ \breve{u}(t), & t\in[\xi,\xi_1],\end{cases}$$

$v(t)$ is a mild solution to (3.2) on $[0,\xi_1]$. The existence and uniqueness of the mild continuation $\breve{u}(t)$ are demonstrated exactly as earlier with only some minor changes, so omitted. Repeating this procedure and by the *a priori* estimate of the solution, one continues the solution till the time $\xi_m=\xi_{\max}$, where $[0,\xi_{\max}]$ is the maximum interval of the existence and uniqueness of a solution, and ú denotes the solution on the interval $[0,\xi_{\max}]$. We prove $\xi_{\max}=T$. If this is not the case, then $\xi_{\max}<T$. Put

$$\bar{\varphi}(t)=\varphi(t)+\int_a^{\xi_{\max}}T(t,s)k(s,\acute{u}(s))dg(s),$$

with $\bar{\varphi}(t)\in G([\xi_{\max},T],X)$. We consider the following integral equation:

$$u(t)=\bar{\varphi}(t)+\int_{\xi_{\max}}^{t}T(t,s)k(s,u(s))dg(s).$$

One can use the previous arguments to extend the solution beyond $\xi_{\max}$, which is a contradiction.

Step VI. Let $\varphi_n\to\varphi_0$ in $G(I,X)$ as $n\to\infty$, and u_n be the solution to equation (3.2) with the perturbation φ_n, i.e.,

$$u_n(t)=\varphi_n(t)+\int_a^t T(t,s)k(s,u_n(s))dg(s),\quad \text{for } t\in[0,T]. \tag{3.3}$$

Obviously, from the compactness of $\{T(t,s)\}_{(t,s)\in\triangle}$, it follows that the set

$$\left\{\int_a^t T(t,s)k(s,u_n(s))dg(s):n\geq 1\right\}$$

is relatively compact in $G([a,T],X)$. Now, we should prove the set

$$\left\{u_n(t)-\int_a^t T(t,s)k(s,u_n(s))dg(s):n\geq 1\right\}$$

is relatively compact in $G([a,T],X)$, we only prove that $\lim_{n\to\infty}u_n$ exists in $G([a,T],X)$. On the contrary, if $\lim_{n\to\infty}u_n$ does not exist in $G([a,T],X)$, then for any $n\in\mathbb{N}$, we have n_1, and n_2 with $n_1,n_2>n$ such that $\|u_{n_1}-u_{n_2}\|_{[0,T]}>\varepsilon_0$ (ε_0 is a constant), i.e., there exists t_* such that

$$\|u_{n_1}(t_*)-u_{n_2}(t_*)\|_{[0,T]}>\varepsilon_0.$$

Let $\hat{u}_n(t)=u_n(t)-\int_0^t T(t,s)k(s,u_n(s))dg(s)$, we can estimate

$$\begin{aligned}|\hat{u}_{n_1}(t_*)-\hat{u}_{n_2}(t_*)|&\geq|u_{n_1}(t_*)-u_{n_2}(t_*)|-\left|\int_0^{t_*}T(t_*,s)(k(s,u_{n_1}(s))-k(s,u_{n_2}(s)))\right|dg(s)\\&\geq\varepsilon_0-\int_0^{t_*}L_D(s)dg(s)\varepsilon_0\\&=\left(1-\int_0^{t_*}L_D(s)dg(s)\right)\varepsilon_0,\end{aligned}$$

which contradicts the compactness of $\hat{u}_n$ in $G([a,T],X)$. Hence, $\lim_{n\to\infty}u_n$ converges in $G([a,T],X)$ and $u_n\to u$. Therefore, taking the limit in (3.3) as $n\to\infty$, one finds, again by (H2) and Lebesgue-dominated convergence theorem, that u is the solution to equation (3.2) with the perturbation φ_0. This completes the proof. □

Theorem 3.1. *If the assumptions* (*H*0)–(*H*3) *hold, in addition, suppose that* $\{T(t,s)\}_{(t,s)\in\triangle}$ *is compact in X, then the solution set of system* (1.3) *has at least one mild solution with*

$$r \geq \max\left\{\frac{\|u_0\| + c\int_0^T a(s)dg(s)}{(1-k-bc\int_0^T dg(s))}, \bar{M}\right\} \quad \text{and} \quad \left(1-k-bc\int_0^T dg(s)\right) > 0.$$

Proof. We denote the operator $N_T : G(I,X) \to P(G(I,X))$,

$$N_T(u)(t) = \{v(t) \in G(I,X) : v(t) \in S_T(\mathrm{Sel}_F(u)), t \in I, u \in G(I,X)\},$$

$$S_T(f)(t) = \left\{T(t,0)[u_0 + H(\sigma(u),u)] + \int_0^t T(t,s)f(s)dg(s), t \in I\right\}.$$

Let set

$$\bar{B}_r = \{u \in G(I,X) : \|u\|_\infty \leq r\} \subset G(I,X).$$

We seek for solutions in $\bar{B}_r$. It is clear that we will obtain the result if we shows that N_T admits a fixed point in $\bar{B}_r$. The proof process is divided into some steps.

Step I. We prove that $N_T(\bar{B}_r) \subset \bar{B}_r$. Since S is compact, it is bounded, and let us say by some $c > 0$. For every $u \in \bar{B}_r$, by Lemma 3.2, $\mathrm{Sel}_F(u) \neq \varnothing$, taking $f \in \mathrm{Sel}_F(u)$ with $u \in \bar{B}_r$, it follows that

$$v(t) = T(t,0)[u_0 + H(\sigma(u),u)] + \int_0^t T(t,s)f(s)dg(s),$$

for all $t \in [0,T]$. Then,

$$\begin{aligned}\|v(t)\| &\leq \|u_0\| + k\|u\| + \int_0^T (a(s) + b\|u(s)\|)cdg(s)\\ &\leq \|u_0\| + kr + \int_0^T (a(s) + br)cdg(s)\\ &\leq r.\end{aligned}$$

Hence, $\|v(t)\|_\infty = \sup_{t\in I}\|v(t)\| \leq r$, for every $v \in N_T(u)$, namely, $N_T(\bar{B}_r) \subset \bar{B}_r$.

Step II. In order to achieve the u.s.c. of N_T on $\bar{B}_r$, applying Lemma 2.7, it is sufficient to show that is quasi-compact and closed.

Let $B_r \subset \bar{B}_r$ be a bounded set of $G(I,X)$. We will prove that for each $t \in I$, $V(t) = \{N_T(u)(t) : u \in \bar{B}_r\}$ is relatively compact in X.

Obviously, for $t = 0$, $V(0) = \{u_0 + H(\sigma(u),u)\}$ is relatively compact in X. Let $t \in [0,b]$ be fixed, and for $u \in \bar{B}_r$ and $v(t) \in V(t)$, there exists $f \in \mathrm{Sel}_F(u)$ such that

$$v(t) = T(t,0)[u_0 + H(\sigma(u),u)] + \int_0^t T(t,s)f(s)dg(s).$$

For an arbitrary $\varepsilon \in (0,t)$, define an operator $J_\varepsilon : V(t) \to X$ by

$$J_\varepsilon v(t) = T(t,0)[u_0 + H(\sigma(u),u)] + T(t,t-\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon,s)f(s)dg(s).$$

From the compactness of $\{T(t,s)\}_{(t,s)\in\triangle}$, we obtain that the set $V_\varepsilon(t) = \{J_\varepsilon v(t) : v(t) \in V(t)\}$ is relatively compact in X for each $\varepsilon \in (0,t)$. Moreover, it follows

$$v(t) - J_\varepsilon v(t) \leq \left| \int_{t-\varepsilon}^{t} T(t,s)f(s)dg(s) \right| \leq \int_{t-\varepsilon}^{t} (a(s) + br)cdg(s).$$

Therefore, there is a relatively compact set arbitrarily close to the set $V(t)$. Thus, the set $V(t)$ is also relatively compact in X, which yields that $V(t) = \{N_T(u)(t) : u \in \bar{B}_r\}$ is relatively compact in X for each $t \in I$.

Thus, we obtain $N_T(B_r)$ is relatively compact, which implies that N_T is quasi-compact. Next, we will show that the operator $N_T(B_r)$ is an equiregulated family of functions on $[0, T]$.

For $t_0 \in [0, T)$, we have

$$\begin{aligned} &\|(N_T u)(t) - (N_T u)(t_0^+)\| \\ &\leq \|T(t,0) - T(t_0^+,0)\|\|u_0\| + \|T(t,0)H(\sigma(u),u) - T(t_0^+,0)H(\sigma(u(t_0^+)), u(t_0^+))\| \\ &\quad + \int_0^{t_0^+} \|(T(t,s) - T(t_0^+,s))f(s)\|dg(s) + \int_{t_0^+}^{t} \|T(t,s)\|f(s)dg(s) \\ &:= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \|T(t,0) - T(t_0^+,0)\|\|u_0\|, \\ I_2 &= \|T(t,0)H(\sigma(u),u) - T(t_0^+,0)H(\sigma(u(t_0^+)), u(t_0^+))\|, \\ I_3 &= \int_0^{t_0^+} \|(T(t,s) - T(t_0^+,s))f(s)\|dg(s) + \int_{t_0^+}^{t} \|T(t,s)\|f(s)dg(s), \\ I_4 &= \int_{t_0^+}^{t} \|T(t,s)\|f(s)dg(s). \end{aligned}$$

The combination of compactness of $\{T(t,s)\}_{(t,s)\in\triangle}$, strongly continuity of $T(t,s)$ in the uniform operator topology and from (H3), applying dominated convergence theorem, we can derive that I_1, I_2, and I_3 tend to zero independently of u as $t \to t_0^+$. Let $\bar{\omega}(t) = \int_0^t r_1(s)dg(s)$, from Lemma 2.1, $\bar{\omega}(t)$ is regulated function on $[0, T]$. Thus,

$$\begin{aligned} I_4 = \int_{t_0^+}^{t} \|T(t,s)\|f(s)dg(s) &\leq cbr\int_{t_0^+}^{t} dg(s) + c\int_{t_0^+}^{t} a(s)\mathrm{d}g(s) \\ &= cbr\int_{t_0^+}^{t} dg(s) + c(\bar{\omega}(t) - \bar{\omega}(t_0^+)). \end{aligned}$$

So, when $t \to t_0^+$, we have $I_4 \to 0$ also independently of u. Therefore, from Lemma 2.1, N_T is u.s.c. on $\bar{B}_r$.

Next, we will prove that N_T is closed on $\bar{B}_r$. To this aim, let u_n be a sequence of elements of N_T, which converges to a regulated function u and shows that u is also the element of N_T. There exists a sequence of measurable functions $\{f_n(t)\}_n \in (a(t) + b\|u(t)\|)S$ for every $t \in [0, T]$ and for each n:

$$u_n(t) = T(t,0)[u_0 + H(\sigma(u_n), u_n)] + \int_0^t T(t,s)f_n(s)dg(s) \quad \text{for all } t \in [0, T].$$

Because the sequence $(f_n)_n$ satisfies the assumptions of Lemma 2.3, there exists a subsequence $(f_{n_k})_k$, which is $\sigma(L^1(dg, X), L^\infty(dg, X_{w^*}^*))$-weakly convergent to some integrable w.r.t. g function f, and it can be seen that for a fixed $t \in [0, T]$, the operator

$$f(\cdot) \in L^1(dg, I, X) \mapsto T(t,\cdot)f(\cdot) \in L^1(dg, I, X)$$

is linear and continuous, since

$$\|T(t,\cdot)f(\cdot)\|_{L^1_g} \le \int_0^t \|T(t,s)f(s)\|dg(s) \le \int_0^T \|f(s)\|dg(s) = \|f\|_{L^1_g}.$$

For any continuous linear functional on $L^1(\mu, X)$ for a finite measure, μ can be represented by an element of $L^\infty(\mu, X^*_{w^*})$, where $X^*_{w^*}$ denotes the space of all weak equivalence classes of X^*-valued, X-weakly measurable functions $\hbar : [0,T] \to X^*$ such that they are essentially bounded. Therefore, taking in particular functions of the form $\hbar(s) = x^*\chi_{[0,t]}(s)$ (for some $x^* \in X^*$), one obtains

$$\begin{aligned}\int_0^t \langle x^*, T(t,s)f_{n_k}(s)\rangle \mathrm{d}g(s) &= \int_0^T \hbar(s)T(t,s)f_{n_k}(s)dg(s) \to \int_0^T \hbar(s)T(t,s)f(s)dg(s)\\ &= \int_0^t \langle x^*, T(t,s)f(s)\rangle \mathrm{d}g(s),\end{aligned}$$

and so, the sequence $\left(\int_0^t T(t,s)f_{n_k}(s)dg(s)\right)_k$ is weakly convergent to $\int_0^t T(t-s)f(s)dg(s)$. Then,

$$u_n(t) \to u(t) = T(t,0)[u_0 + H(\sigma(u),u)] + \int_0^t T(t,s)f(s)dg(s).$$

Hence, N_T is closed on $\bar{B}_r$.

Step III. Let us now prove N_T is contractible on $\bar{B}_r$. First, we can define the following operators:

$$\Gamma : G(I,X) \to G(I,X), (\Gamma u)(t) = T(t,0)[u_0 + H(\sigma(u),u)],$$

$$S_1 : L^1(dg, I, X) \to G(I,X), (S_1 f)(t) = \int_0^t T(t,s)f(s)dg(s).$$

For $u \in \bar{B}_r$, fix $f^* \in \mathrm{Sel}_F(u)$ and put $u^* = \Gamma(u) + S_1(f^*)$. Define a function $h : [0,1] \times N_T(u) \to N_T(u)$ by setting

$$h(\lambda, v)(t) = \begin{cases} v(t), & t \in [0, \lambda T],\\ u(t;\lambda, v), & t \in (\lambda T, T],\end{cases}$$

for each $(\lambda, v) \in [0,1] \times N(u)$, where $u(t;\lambda,v)$ is the unique solution of the equation in the form

$$\begin{cases} \mathrm{d}u(t) \in A(t)u(t)\mathrm{d}t + f^*(t)dg(t), & t \in [\lambda T, T],\\ u(t) = v(\lambda T), & t \in [0, \lambda T].\end{cases}$$

Clearly, $h(\lambda, v) \in N_T(u)$, for each $(\lambda, v) \in [0,1] \times N_T(u)$. In fact, for each $v \in N_T(u)$, there exists $\tilde{f} \in \mathrm{Sel}_F(u)$, such that

$$v = \Gamma(u) + S_1(\tilde{f}).$$

Put

$$\hat{f}(t) = \tilde{f}(t)\chi_{[0,\lambda T]}(t) + f^*(t)\chi_{[\lambda T, T]}(t).$$

It is clearly that $\hat{f} \in \mathrm{Sel}_F(h)$. Also, it is readily checked that $\Gamma(u) + S_1(\tilde{f}) = v$, for all $t \in [0, \lambda T]$, $\Gamma(u) + S_1(\tilde{f}) = u(t;\lambda, v)$, for all $t \in (\lambda T, T]$, and hence, $h(\lambda, v) = \Gamma(u) + S_1(\tilde{f})$, i.e. $h(\lambda, v) \in N_T(u)$. Note that

$$h(0, v) = u^*, h(1, v) = v.$$

To show that h is continuous, given $(\lambda_m, v_m) \in [0,1] \times N_T(u)$ be such that $(\lambda_m, v_m) \to (\lambda, v)$ as $m \to \infty$. Then,

$$h(\lambda_m, v_m)(t) = \begin{cases} v_m(t), & t \in [0, \lambda T],\\ u(t;\lambda_m, v_m), & t \in (\lambda T, T].\end{cases}$$

We shall prove that $h(\lambda_m, v_m)(t) \to h(\lambda, v)(t)$ as $m \to \infty$. Without loss of generality, we assume that $\lambda_m \le \lambda$, for $(\lambda_m, v_m) \in [0,1] \times N_T(u)$, there exists $f_m \in \mathrm{Sel}_F(u)$, such that $h(\lambda_m, v_m) = \Gamma(u) + S_1(f_m)$.

Then, we have that for $0 \le t \le \lambda T$,

$$|h(\lambda_m, v_m) - h(\lambda, v)| = |v_m(t) - v(t)| \to 0.$$

If $\lambda_m T \le t \le \lambda T$,

$$|h(\lambda_m, v_m) - h(\lambda, v)| = |u(t; \lambda_m, v_m) - v(t)| \le |u(t; \lambda_m, v_m) - v_m(t)| + |v_m(t) - v(t)|,$$

which tends to 0 as $n \to \infty$, since $u(t; \lambda_m, v_m) \to v_m(t)$ as $t \to \lambda_m T$.

If $\lambda T \le t \le T$,

$$\|h(\lambda_m, v_m) - h(\lambda, v)\|_{[\lambda T, T]} = \sup_{t \in [\lambda T, T]} |u(t; \lambda_m, v_m) - u(t; \lambda, v)| \to 0.$$

Since the function f^* satisfies the conditions in Lemma 3.3 due to Lemma 3.1 (ii) and (v), we know that equation (3.2) has a unique solution for every $u(t) \in G([0, \lambda T], X)$ and the solution to (3.2) depends continuously on (λ, u).

Finally, an application of Theorem 2.3 yields that N_T has at least one point, which is solution of the nonlocal Cauchy Problem (1.3) on compact interval I. □

Now, let Σ_T^F denote the set of all mild solutions for inclusion (1.3) on compact interval $[0, T]$. We will present the following characterization.

Lemma 3.4. *Let the assumptions in* Theorem 3.1 *hold. Then,* $\Sigma_T^F = \mathrm{Fix}(N_T)$ *and* Σ_T^F *is compact in* $G(I, X)$.

Proof. Obviously, $\Sigma_T^F \supset \mathrm{Fix}(N_T)$. To prove the reverse inclusion, we take $u \in \Sigma_T^F$, and there exists $f \in \mathrm{Sel}_F(u)$, such that

$$u(t) = T(t, 0)[u_0 + H(\sigma(u), u)] + \int_0^t T(t, s)f(s)dg(s), \quad \text{for all } t \in I,$$

then,

$$\begin{aligned} \|u(t)\| &\le \|u_0 + H(\sigma(u), u)\| + \int_0^t \|f(s)\|dg(s) \\ &\le \|u_0\| + k\|u\| + \int_0^t [a(s) + b\|u(s)\|]cdg(s) \\ &= \|u_0\| + k\|u\| + \int_0^t a(s)cdg(s) + \int_0^t bc\|u(s)\|dg(s). \end{aligned}$$

Let $M_1(t) = \|u_0\| + k\|u\| + \int_0^t a(s)cdg(s)$, and we can write

$$\|u(t)\| \le M_1(t) + \int_0^t bc\|u(s)\|dg(s).$$

As a consequence of the Gronwall-type lemma (see reference [46]), we obtain

$$\|u(t)\| \le M_1(t) + bc\int_0^t M_1(s)\exp(g(t) - g(s))dg(s).$$

The function $M(t) = M_1(t) + bc\int_0^t M_1(s)\exp(g(t) - g(s))dg(s)$ is LS-integrable w.r.t. g and regulated, therefore, bounded by some $\bar{M} > 0$, and so $\|u(t)\| \le \bar{M} \le r$. Based on the aforementioned considerations, we have $\Sigma_T^F = \mathrm{Fix}(N_T)$. Moreover, as in the proof of Theorem 3.1, $\bar{B}_r$ is compact in $G(I, X)$ and N_T is closed; from this, we see that $\mathrm{Fix}(N_T)$ is a compact set in $\bar{B}_r$, so is Σ_T^F. □

Theorem 3.2. *Let the hypotheses in Theorem 3.1 be satisfied. Then, Σ_T^F is an R_δ-set for each $t \in I$.*

Proof. To this aim, let us consider the following inclusion:

$$\begin{cases} du(t) \in A(t)u(t)dt + F_n(t)dg(t), & t \in I \\ u(0) = u_0 + H(\sigma(u), u), \end{cases} \tag{3.4}$$

where $F_n : [0, T] \times G(I, X) \to P_{cl,cv}(X)$ are established in Lemma 3.1. Let $\Sigma_T^{F_n}$ denote the set of all mild solutions for inclusion (3.4). Note Lemma 3.1(ii) and (vi), and one can see from Theorem 3.1 that each set $\Sigma_T^{F_n}$ is nonempty and compact in $G(I, X)$ for each $n \geq 1$. Moreover, by Lemma 3.1(i), we have

$$\Sigma_T^F \subset \cdots \subset \Sigma_T^{F_n} \subset \Sigma_T^{F_{n-1}} \subset \cdots \subset \Sigma_T^{F_2} \subset \Sigma_T^{F_1}.$$

We prove that $\Sigma_T^F = \bigcap_{n\geq 1} \Sigma_T^{F_n}$. Note that $\Sigma_T^F \subset \bigcap_{n\geq 1} \Sigma_T^{F_n}$. To prove the reverse inclusion, we take $u \in \bigcap_{n\geq 1} \Sigma_T^{F_n}$, and there exists $\{f_n\} \subset L^1(dg, I, X)$, such that $f_n \in \mathrm{Sel}_F(u)$,

$$\|f_n(t)\| \leq \alpha(t)(3 + \|u\|), n \geq 1, \quad \text{for a.e. } t \in I,$$

from Lemma 3.1(ii). According to the reflexivity of the space X and Lemma 3.1, we have the existence of a subsequence, denoted as the sequence, such that $f_n \in L^1(dg, I, X)$. By Mazur's theorem, there exists a sequence of convex combinations convergent in the norm of $L^1(dg, I, X)$, i.e., for every $n \geq 1$, $\tilde{f}_n \in co\{f_k; k \geq n\}$, and $\tilde{f}_n(t) \to f(t)$ for a.e. $t \in I$.

Denote by J the set of all $t \in I$ such that $\tilde{f}_n(t) \to f(t)$ in X and $f_n(t) \in F_n(t)$ for all $n \geq 1$. According to Lemma 3.1(iii), we know that there exists $E \subset I$ with $\mathrm{mes}(E) = 0$ such that for each $t \in (I \backslash E) \bigcap J$, and $x^* \in X^*$, $\varepsilon > 0$,

$$\langle x^*, f(t)\rangle \in co\{\langle x^*, f_{n_k}(t)\rangle : k \geq n\} \subset \langle x^*, F_n(t)\rangle \subset \langle x^*, F(t)\rangle + (-\varepsilon, \varepsilon).$$

Therefore, we obtain that $\langle x^*, f(t)\rangle \in \langle x^*, F(t)\rangle$ for each $x^* \in X^*$ and $t \in (I\backslash E)\bigcap J$. Since F has convex and closed values, we conclude that $f(t) \in F(t)$ for each $t \in (I\backslash E)\bigcap J$, which implies that $f \in \mathrm{Sel}_F(u)$. Moreover, since

$$u(t) = T(t, 0)[u_0 + H(\sigma(u_n), u_n)] + \int_0^t T(t, s)f_n(s)dg(s), \quad \text{for all } t \in I,$$

as in the proof of Theorem 3.1, the sequence $\int_0^t T(t, s)f_n(s)dg(s)$ is weakly convergent to $\int_0^t T(t, s)f(s)dg(s)$, whence

$$u(t) = T(t, 0)[u_0 + H(\sigma(u), u)] + \int_0^t T(t, s)f(s)dg(s), \quad \text{for all } t \in I,$$

which implies that $u \in \Sigma_T^F$ and $\Sigma_T^F = \bigcap_{n\geq 1} \Sigma_T^{F_n}$ as desired.

Finally, we will prove that $\Sigma_T^{F_n}$ is contractible for each $n \geq 1$. From Lemma 3.1(v), let g_n be the selection of F_n and $g_n(t, \cdot)$ be continuous for a.e. $t \in I$. At the same time, $\Omega_n := \{u(t) : t \in I, u \in \Sigma_T^{F_n}\}$ is a relatively compact set in X, since $\Sigma_T^{F_n}$ is compact in $G(I, X)$. Therefore, by Lemma 3.1(v), there exists a neighborhood V of $\bar{\Omega}_n$ and a constant $C_V > 0$ such that $\|g_n(t, u)\| \leq \alpha(t)(3 + \|u\|)$, for a.e. $t \in I$ and each $u \in X$.

Next, we will consider a trivial variant of an argument from Theorem 3.1,

$$\begin{cases} du(t) = A(t)u(t)dt + g_n(t, u(t))dg(t), & t \in [\lambda T, T], \\ u(t) = v(t), & t \in [0, \lambda T], \end{cases} \tag{3.5}$$

for each $s \in I$ and $v \in G(I, X)$. Since the functions g_n satisfy the conditions in Lemma 3.3 due to Lemma 3.1(ii) and (v), by Lemma 3.3, we know that equation (3.5) has a unique solution for every $u(t) \in G(I, X)$. Moreover, the solution to (3.5) depends continuously on (λ, v), denoted by $u(t; \lambda, v)$. Define a function $\tilde{h} : [0, 1] \times \Sigma_T^{F_n} \to \Sigma_T^{F_n}$ by setting

$$\tilde{h}(\lambda, v)(t) = \begin{cases} v(t), & t \in [0, \lambda T], \\ u(t;\, \lambda, v), & t \in (\lambda T, T], \end{cases}$$

for each $(\lambda, v) \in [0, 1] \times \Sigma_T^{F_n}$. It follows, as in the proof of Theorem 3.1 we can show that $\tilde{h}(\lambda, v) \in \Sigma_T^{F_n}$ for each $(\lambda, v) \in [0, 1] \times \Sigma_T^{F_n}$ and $\tilde{h}(0, v) = u(t;\, 0, v)$, $\tilde{h}(1, v) = v$ for each $v \in \Sigma_T^{F_n}$.

At the next step, show that $\tilde{h}$ is continuous. As seen earlier, one consider a sequence $\{(\lambda_m, v_m)\} \subset [0, 1] \times \Sigma_T^{F_n}$ such that $(\lambda_m, v_m) \to (\lambda, v)$ in $[0, 1] \times \Sigma_T^{F_n}$ as $m \to \infty$. Then,

$$\tilde{h}(\lambda_m, v_m)(t) = \begin{cases} v_m(t), & t \in [0, \lambda T], \\ u(t;\, \lambda_m, v_m), & t \in (\lambda T, T]. \end{cases}$$

Without loss of generality, we assume that $\lambda_m \le \lambda$, for $(\lambda_m, v_m) \in [0, 1] \times \Sigma_T^{F_n}$, since the remaining cases can be treated in a similar way. For simplicity in presentation, one can put $\theta_m(t) = \|\tilde{h}(\lambda_m, v_m) - \tilde{h}(\lambda, v)\|$ for $t \in I$, and we are going to show that $\|\theta_m(t)\| \to 0$ as $m \to \infty$.

Then, we have that for $0 \le t \le \lambda T$,

$$\|\theta_m(t)\| = |v_m(t) - v(t)| \to 0.$$

If $\lambda_m T \le t \le \lambda T$,

$$\|\theta_m(t)\| = |u(t;\, \lambda_m, v_m) - v(t)| \le |u(t;\, \lambda_m, v_m) - v_m(t)| + |v_m(t) - v(t)|,$$

which tends to 0 as $n \to \infty$, since $u(t;\, \lambda_m, v_m) \to v_m(t)$ as $t \to \lambda_m T$.

If $\lambda T \le t \le T$,

$$\|\theta_m(t)\|_{[\lambda T, T]} = \sup_{t \in [\lambda T, T]} |u(t;\, \lambda_m, v_m) - u(t;\, \lambda, v)| \to 0.$$

Since the functions $g_n(t, u(t))$ satisfy the conditions in Lemma 3.3 due to Lemma 3.1(ii) and (v), by Lemma 3.3 , we know that equation (3.5) has a unique solution for every $u(t) \in G([0, \lambda T], X)$ and the solution to (3.5) depends continuously on (λ, u).

Accordingly, our result follows. Hence, we conclude that $\Sigma_T^{F_n}$ is contractible, and thus, Σ_T^F is an R_δ-set. □

3.2 Topological structure on non-compact intervals

We consider the set $\triangle_\infty = \{(t, s) \in [0, \infty) \times [0, \infty) | 0 \le s \le t\}$ and the evolution system $\{T(t, s)\}_{(t,s) \in \triangle_\infty}$ is compact in X, and the $F : I_\infty \times G(I_\infty, X) \to P_{cl,cv}(X)$ such that

$(H0)$ $^\infty\{A(t)\}_{t \in [0,\infty)}$ is a family of linear not necessarily bounded operators $(A(t) : D(A) \subset X \to X, t \in [0, \infty)$, $D(A)$ a dense subset of X not depending on $t)$ generating an evolution operator $T : \triangle_\infty \to L(X)$, where $L(X)$ is the space of all bounded linear operators in X.

$(H1)$ $^\infty F(t, \cdot)$ is weakly u.s.c. for a.e. $t \in I_\infty$, and $F(\cdot, u)$ has strongly measurable selection for every $u \in G(I_\infty, X)$.

$(H2)$ $^\infty$ There exists a constant $b > 0$, a function $a(t) \in L^1(dg, I_\infty, R^+)$, and a compact set $S \subset X$ such that $F(t, u) \subseteq a(t)(1 + b\|u\|)S$.

$(H3)$ $^\infty$ The nonlocal function $H(\cdot,\cdot) : [0, \infty) \times G(I_\infty, X) \to X$ and the function $\sigma(\cdot) : G(I_\infty, X) \to [0, \infty)$ are both continuous. $H(\sigma(\cdot), \cdot) : G(I_\infty, X) \to X$ is a compact mapping, and there is a constant $K > 0$ such that, for $u \in G(I_\infty, X)$,

$$\|H(\sigma(u), u)\| \le K\|u\|.$$

For each $u \in G(I_\infty, X)$, let us denote

$$S_m(f)(t) = \left\{ T(t, 0)[u_0 + H(\sigma(u), u)] + \int_0^t T(t, s)f(s)dg(s),\, t \in [0, m] \right\},$$

$$S_\infty(f)(t) = \left\{T(t,0)[u_0 + H(\sigma(u),u)] + \int_0^t T(t,s)f(s)dg(s),\, t \in [0,\infty)\right\},$$

$$\mathrm{Sel}_F^\infty = \{f \in L^1(dg, I_\infty, X) : f(t) \in F(t,u(t)), \text{ for a.e. } t \in I_\infty\}.$$

In the sequel, we assume that $\{G([a,m],X), \pi^p_{a,m}, N(a)\}$ and $\{L(dg,[0,m],X), \dot{\pi}^p_{a,m}, N\backslash\{0\}\}$ are inverse systems, and we have $\{id, S_m\}$ is a mapping from $\{L(dg,[0,m],X), \dot{\pi}^p_{a,m}, N\backslash\{0\}\}$ into $\{G([a,m],X), \pi^p_{a,m}, N\backslash\{0\}\}$. From the observation, we obtain

$$\dot{\pi}^p_m(S_p(f)) = S_m(\dot{\pi}^p_m) \quad \text{for every } f \in L(dg,[0,p],X),\ m \le p,$$

so $\{id, S_m\}$ induces a limit mapping $S_\infty : L(dg,[0,\infty),X) \to G([0,\infty),X)$ such that

$$S_\infty(f)|_{[0,m]} = S_m(f|_{[0,m]}), \quad \text{for each } f \in L(dg,[0,\infty),X),\ m \le N\backslash\{0\}.$$

Lemma 3.5. *Let X be reflexive. Suppose further that F satisfies $(H1)^\infty$ and $(H2)^\infty$. Then, the set $\mathrm{Sel}_F^\infty(u)$ and is nonempty for each $u \in G(I_\infty, X)$.*

Proof. Let $u \in G(I_\infty, X)$. By Lemma 3.2, we can obtain $f_m \in \mathrm{Sel}_F|_{[0,m]}(u|_{[0,m]})$ for each $m \in N\backslash\{0\}$, where $F|_{[0,m]}$ is the restriction of F to $[0,m]$, and it is clearly seen that

$$F|_{[0,m]}(t,u) = F(t,u), \quad \text{on } [0,m] \times G(I_\infty, X).$$

Consider the function $f : [0,\infty)$ defined as

$$f(t) = \sum_{m=1}^{\infty} \chi_{[m-1,m)}(t) f_m(t), \quad t \in [0,\infty),$$

where $\chi_{[m-1,m)}$ denotes the characteristic function of interval $[m-1,m)$. It is not difficult to see that $f \in \mathrm{Sel}_F^\infty(u)$, and it is locally integrable w.r.t. g. □

Let Σ_∞^F stand for the set of all solutions to the inclusion (1.3). In the sequel, we are going to present our main result.

Theorem 3.3. *Let Hypothesis $(H0)^\infty$ hold, and let the multivalued map $F : I_\infty \times G(I_\infty, X) \to P_{cl,cv}(X)$ satisfy $(H1)^\infty$–$(H3)^\infty$. Then, Σ_∞^F is an R_δ-set in $G(I_\infty, X)$.*

Proof. For every $m \in N\backslash\{0\}$, let $N_m : \hat{B}_m \to 2^{\hat{B}_m}$ be a multivalued mapping defined by

$$N_m(u) = S_T(\mathrm{Sel}_F|_{[0,m]}(u)), \quad \text{for each } \hat{B}_m,$$

where $\hat{B}_m := \{u \in G[0,m], X\}$.

Making use of Theorem 3.1 and Lemma 3.3 to $F|_{[0,m]}$, and we can obtain $\mathrm{Fix}(N_m) = \Sigma_m^{F|[0,m]}$ and $\mathrm{Fix}(N_m)$ is nonempty and compact. At the same time, from Theorem 3.3, $\mathrm{Fix}(N_m)$ is an R_δ-set. Then, it is readily checked that $\{\hat{B}_m, \dot{\pi}^p_m, N\backslash\{0\}\}$ is an inverse system and its limit is

$$\tilde{B} := \{u \in G(I_\infty, X)\} = \underleftarrow{\lim}\{\hat{B}_m, \dot{\pi}^p_m, N\backslash\{0\}\}.$$

From the observation that when $p, m \in N$ and $p \ge m$,

$$\mathrm{Sel}_F|_{[0,m]}(u|_{[0,m]}) = \{f|_{[0,m]}; f \in \mathrm{Sel}_F|_{[0,p]}(u)\}, \quad \text{for all } u \in \hat{B}_p.$$

So (id, N_m) is the mapping from $\{\hat{B}_m, \dot{\pi}^p_m, N\backslash\{0\}\}$ into itself. Next, using the fact $\dot{\pi}^p_m(S_p(f)) = S_m(\dot{\pi}^p_m)$, for every $f \in L(dg,[0,p],X)$, we obtain

$$\begin{aligned}\dot{\pi}_m^p(N_p(u)) &= \dot{\pi}_m^p(S_p(\mathrm{Sel}_F\,|_{[0,p]}(u)))\\ &= \{S_m(\dot{\pi}_m^p(f));\, f\in \mathrm{Sel}_F\,|_{[0,p]}(u)\}\\ &= \{S_m(f);\, f\in \mathrm{Sel}_F\,|_{[0,m]}(u\,|_{[0,m]})\}\\ &= N_m(\dot{\pi}_m^p)(u).\end{aligned}$$

Therefore, (id, N_m) induces a limit mapping $N_\infty : \tilde{B} \to 2^{\tilde{B}}$, which is defined by

$$N_\infty(u) = \{v \in \tilde{B};\ v\,|_{[0,m]} = S_m(f\,|_{[0,m]}),\ \text{ for every } m \in N\backslash\{0\},\ f \in L^1(dg, I_\infty, X),\ \text{ and } f(t)\in F(t,u(t)),\ \text{ for } t \in I_\infty\},$$

for each $u \in \tilde{B}$.

At the end of this step, applying Theorem 2.4, we obtain that the solution set $\Sigma_\infty^F = \mathrm{Fix}(N_\infty)$ is an R_δ-set in $G(I_\infty, X)$. □

4 Example

In this section, we give an example to demonstrate the feasibility of our results, and we consider the following fractional differential inclusion of the form:

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x) \in \dfrac{\partial^2}{\partial x^2}u(t,x) - a(t)u(t,x) + \displaystyle\int_0^t b(s)c\left(s, \frac{\partial u(s,x)}{\partial x}\right)dg(s), \quad x\in[0,\pi],\ t\in[0,1],\\ u(0)(x) + \displaystyle\int_\delta^\infty h(s)\int_0^\pi d\left(x, \frac{\partial u(s,y)}{\partial y}\right)\mathrm{d}y\mathrm{d}s = u_0(x),\end{cases} \tag{4.1}$$

where

(A1) The function $a : [0,1] \to R$ is a continuously differentiable function and satisfies $a_{\min} = \min_{t\in[0,1]} a(t) > -1$;

(A2) The function $b(\cdot) \in C([0,1], R)$ satisfies $\int_0^\infty |b(s)|ds < \infty$;

(A3) The multivalued function $c : [0,\infty) \times R \to P_{cl,cv}(R)$ satisfies $c(t,\cdot)$, which is weakly u.s.c. for a.e. $t \in [0,\infty)$, and there is a constant $c_1 > 0$ such that $|c(t,x)| \le c_1(|x|+1)$, for $t\in[0,\infty)$, $x, y \in R$;

(A4) The function $h(\cdot) \in C([0,1], R)$ and satisfies $\int_0^\infty |h(s)|ds < \infty$. The function $d(\cdot,\cdot) \in C^2([0,\pi]\times R)$ satisfies that there is $d_1 > 0$ such that, for $x\in[0,\pi]$, $y \in R$,

$$\max\left\{\left|\frac{\partial d(x,y)}{\partial x}\right|, \left|\frac{\partial^2 d(x,y)}{\partial x^2}\right|\right\} \le d_1|y|.$$

And $\delta \in (0,1)$, let $X = L^2([0,\pi], R)$ with the norm $\|\cdot\|_2$ and inner product $\langle\cdot,\cdot\rangle$. Consider the linear operator B in Hilbert space X defined by

$$Bu \coloneqq \frac{\partial^2}{\partial x^2}u,\ u \in D(B),$$

where

$$D(B) \coloneqq \{u \in L^2([0,\pi],R) | u, \dot{u} \text{ are absolutely, } \ddot{u} \in L^2([0,\pi],R), u(0) = u(\pi) = 0\}.$$

By ([47] Lemma 6.1 in Chapter 7), it is well known that B generates a compact C_0-semigroup in X. Furthermore, B has a discrete spectrum, and its eigenvalues are $-n^2$, $n \in N^+$ with the corresponding normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $0 \le x \le \pi$, $n = 1, 2, \ldots$.

Define the operator $A(t)u = Bu - bu$ with $D(A(t)) = D(B)$. It follows from ([48] Lemma 6.1 in Chapter 7) that operator $A(t)$ satisfies the known conditions of Acquistapace and Terreni [30]. Therefore, $A(t)$ generates a strongly continuous evolution family $\{T(t,s) : 0 \le s \le t \le 1\}$ defined by

$$T(t,s)u = \sum_{n=1}^{\infty} e^{-\left(\int_s^t a(\tau)\mathrm{d}\tau + n^2(t-s)\right)}\langle u, e_n\rangle e_n, \quad 0 \le s \le t \le 1, u \in X.$$

A direct calculation gives

$$\|T(t,s)\|_{L(X)} \le e^{-(1+a_{\min})(t-s)}, \quad 0 \le s \le t \le 1.$$

It is evident $M \coloneqq \sup_{0\le s\le t\le 1}\|T(t,s)\|_{L(X)} = 1$.

Note also from [49] that, for each $t, s \in [0,1]$ with $t > s$, the evolution family $T(t,s)$ is a nuclear operator, which implies the compactness of $T(t,s)$ for $t > s$.

Define $u(t)(x) = u(t,x)$, $F(t,u(t))(x) = \int_0^t b(s)c\left(s, \frac{\partial u(s,x)}{\partial x}\right)dg(s)$, $\sigma(u(t,x)) = h(t)$, and $H(\sigma(u), \sigma(u(t,x)))(x) = \int_\delta^\infty h(s)\int_0^\pi d\left(x, \frac{\partial u(s,y)}{\partial y}\right)\mathrm{d}y\mathrm{d}s$. Then, with this notions, system (4.1) can be rewritten well into the abstract form of system (1.3).

Taking

$$g(t) = \begin{cases} \frac{1}{2}, t \in \left[0, 1 - \frac{1}{2}\right], \\ \cdots, \\ 1 - \frac{1}{n-1}, t \in \left(1 - \frac{1}{n-1}, 1 - \frac{1}{n}\right], \\ \cdots, \\ 1, t = 1. \end{cases}$$

It is evident that $g : [0,1] \to R$ is a left continuous and nondecreasing function on $[0,1]$. It is easy to certify that assumption (H3) for the function H is verified as well due to assumption (A4). Meanwhile, the assumptions (A2) and (A3) imply that function $F : [0,1] \times X \to P_{cl,cv}$ satisfies well the condition (H1). We assume that assumptions $(H2)$ and $(H3)$ are satisfied. From Theorem 3.1, system (4.1) has at least one mild solution on [0, 1].

5 Conclusion

This article pioneers the investigation of the topological structure of the solution set for a class of non-autonomous evolution inclusions driven by measures on the half-line. Initially, we establish that the solution set is nonempty, compact, and an R_δ-set on compact intervals by employing fixed point theorems, and we derive certain sufficient conditions to guarantee the existence and R_δ-property of the solution set. Subsequently, utilizing the limit inverse method alongside our existing results on compact intervals, it is demonstrated that the solution set is nonempty, compact, and an R_δ-set on the half-line as well. The results given in this study developed and extended some previous results. Finally, an illustrative example is given to show the practical usefulness of the analytical results. Furthermore, we will investigate topological structure of solution set of the fractional measure evolution inclusion in the next work.

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