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Research Article

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Topological structure of the solution sets to neutral evolution inclusions driven by measures

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Abstract: This study is concerned with topological structure of the solution sets to evolution inclusions of neutral type involving measures on compact intervals. By using Górniewicz-Lassonde fixed-point theorem, the existence of solutions and the compactness of solution sets for neutral measure differential inclusions are obtained. Second, based on the R_δ-structure equivalence theorem, by constructing a continuous function that can make the solution set homotopic at a single point, the R_δ-type structure of the solution sets of this kind of differential inclusion is obtained.

Keywords: neutral measure evolution inclusions, existence of solutions, R_δ-set, measure of noncompactness, regulated functions

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1 Introduction

When studying the evolution of complex phenomena in real life, one often notices that the measured quantities are usually discontinuous. For example, when the phenomenon under study has discrete disturbances in the process of continuous change, one can find this characteristic. The nature of the solutions of this type of dynamic system is difficult to obtain. Particularly when there are infinite discrete disturbances (i.e., impulses) and the impulse moments are accumulated in the observation time interval, it will be more difficult to study the nature of the solution. The above situation is described as Zeno phenomenon in the theory of hybrid systems [1,2], and most of the work on classical impulsive differential equations [2,3] often avoids this behavior. However, the theory of measure differential equations provides a very convenient tool for dealing with this problem, which is also called differential equations driven by measures [4,5]. For specific measures, such as absolute continuous measure, discrete measure and the sum of an absolutely continuous measure with a discrete one, measure differential equations are transformed into usual differential equations, difference equations, impulse equations, respectively. This provides new light for studying these equation problems. In addition, the time scale dynamic equations can also be regarded as measure differential equations [5].

It is worth noting that in the context of differential equations this method has been known for many years. To the best of our knowledge, the first equation with a measure as a coefficient was considered by Kronig and Penney in 1931 [6] and, in the context of the Riemann-Stieltjes integral, as an integral problem by Atkinson in 1964 [7]. The systematic study for measure-perturbed problems started in the 1970s. It is worth mentioning that Sharma [8], Shendge and Joshi [9], or Dhage and Bellale [10] have done some basic work in measure

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differential equations. Schwabik et al. [11] systematically continued the method of solving certain differential equations (with Perron-Stieltjes integral) through integral equations in 1979. And then, some cases have been studied by Wyderka [12,13]. With the increase in the discussion of complex phenomena, the research on measure differential equations will also continue to increase [14,15].

On the other hand, when dealing with problems in control theory, economics or game theory, we must consider set value functions; therefore, the model may involve multi-value differential equations, i.e., differential inclusion [16,17]. In the latest development of differential inclusion theory, especially the study of measure differential inclusion [18–23], it has gained wide popularity. The reason why it has developed so quickly and widely is that this type of differential inclusion has a general structure, including as special cases differential and difference inclusion, impulsive and hybrid problems [24–26]. A very interesting and valuable course on hybrid inclusions can be found in [27], for instance. Nevertheless, several different approaches, different solutions, and many applications for measure differential equations or inclusions can also be found in the literature.

In [28], by viewing Borel measure as Lebesgue Stieltjes measure and Successive iteration method of approximate solution, Cichon et al. studied the existence of regulated or bounded variation solutions of the following measure differential inclusions

$$\begin{cases} \mathrm{d}x(t) \in G(t, x(t))\mathrm{d}\mu(t), \\ x(0) = x_0, \end{cases}$$

where $G : [0, 1] \times R^d \to \mathcal{P}(R^d)$ is a closed convex valued multivalue mapping and μ is a positive regular Borel measure.

In [29], by using a characterization of the space of regulated functions on a compact interval $[0, 1]$ endowed with the topology of uniform convergence as being isometrically isomorphic to some space of continuous functions, Cichon et al. got several theorems on the existence of regulated selections for multifunctions, and based on this, they investigated the existence of regulated solutions for the general measure differential inclusion as follows:

$$\begin{cases} \mathrm{d}y(t) \in F(t, y(t))\mathrm{d}g(t), \quad t \in [0, 1] \\ y(0) = y_0, \end{cases}$$

where $g : [0, 1] \to R$ is a left continuous nondecreasing function (continuous at 0) and $F : [0, 1] \times R^N \to cl(R^N)$ is a convex valued multifunction with values in an Euclidean space.

In [30], by using the notion of uniformly bounded ε-variations which mixes the supremum norm with the uniformly bounded variation condition and approximating the solutions of a differential problem driven by a rough measure by solutions of similar problems driven by "smoother" measures, Di Piazza et al. developed the existence of bounded variation solutions of the following measure differential inclusions and the continuous dependence of solutions on measures

$$\begin{cases} \mathrm{d}x(t) \in F(t, x(t))\mathrm{d}\mu(t), \quad t \in [0, 1] \\ x(0) = x_0, \end{cases}$$

where μ is the Stieltjes measure associated with a left-continuous non-decreasing function, $F : [0, 1] \times R^d \to \mathcal{P}_{kc}(R^d)$ is a multifunction (i.e., a function having values compact convex subsets of the d-dimensional Euclidean space), and $x_0 \in R^d$.

By using Kakutani-Ky Fan's fixed-point theorem, Cichon and Satco [31] established the existence of mild solutions and the compactness of the solution set of the following differential inclusions involving measures in the space of regulated functions endowed with weak, respectively, strong topologies:

$$\begin{cases} \mathrm{d}u(t) \in \mathrm{A}u(t) + F(t, x(t))\mathrm{d}g(t), \quad t \in [0, 1] \\ u(0) = u_0, \end{cases}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ of contractions on a separable Banach space X and $g : [0, 1] \to R$ is a right-continuous non-decreasing function.

Inspired by the above research, in this work, we will study the following neutral evolution inclusions driven by measures:

$$\begin{cases} d[x(t) - q(t, x(t))] \in Ax(t)\mathrm{d}t + F(t, x(t))\mathrm{d}g(t), \ \ t \in [0, T] \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator. A is the infinitesimal generator of an analytic C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, which is uniformly continuous on $(0, \infty)$. $F : [0, T] \times X \to \mathcal{P}(X)$ is a multivalue mapping with convex, closed values for which $F(t, \cdot)$ is weakly upper semicontinuous (usc) for a.e. $t \in [0, T]$ and $F(\cdot, x)$ has a strongly measurable selection for each $x \in X$, and $x_0 \in X$. $q : [0, T] \times X \to X$ is a given function satisfying some assumptions and $g : [0, T] \to R$ is a right continuous nondecreasing function and $g(0) = 0$.

The contributions and novelties of the current work are as follows:

(1) One of the goals of this study is to discuss the existence of mild solutions for neutral evolution inclusions driven by measures. As far as we know, there are few papers investigating this issue.
(2) Another goal of this study is to establish the R_δ-type topological structure of solution sets for neutral evolution inclusions driven by measures. As far as we know, there are also few papers investigating this issue.
(3) The research methods used are innovative and novel. The proof methods of relative compactness and closure are different from the existing works.

The rest of this study is organized as follows. In Section 2, some notations and preparation are given. In Section 3, a suitable concept on mild solution for our problem is introduced. The existence result of the mild solution and the R_δ-type topological structure of solution sets of system (1.1) are discussed. In Section 4, the research work of this study is summarized.

2 Preliminaries

Throughout the study, unless otherwise specified, we use the following definition, notations, and preliminary facts from multivalued analysis.

For two normed space X, W, let $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : \text{Y is bounded}\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : \text{Y is closed}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : \text{Y is compact}\}$, $\mathcal{P}_{cl,b}(X) = \{Y \in \mathcal{P}(X) : \text{Y is closed and bounded}\}$, $\mathcal{P}_{cl,cv}(X) = \{Y \in \mathcal{P}(X) : \text{Y is closed and convex}\}$, $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : \text{Y is compact and convex}\}$, and $\mathcal{P}_{cp,cl}(X) = \{Y \in \mathcal{P}(X) : \text{Y is compact and closed}\}$.

Definition 2.1. [32] A multivalued map $G : X \to \mathcal{P}(X)$ is closed (convex) valued if $G(x)$ is closed (convex) for all $x \in X$.

Definition 2.2. [32] The multivalued map $G : X \to \mathcal{P}(W)$ is *usc* at $x_0 \in X$ if for every open set $V \subset W$ such that $G(x_0) \subset V$, there exists a neighborhood U of x_0 such that for any $x \in U$, $G(x) \subset V$.

Definition 2.3. [32] A multivalued map $G : X \to \mathcal{P}(W)$ is said to be *usc* on X if $F^{-1}(V) = \{x \in X : F(x) \subset V\}$ is an open set of X for every open $V \subset W$.

Definition 2.4. [32] A multivalued map $G : X \to \mathcal{P}(W)$ is *continuous* if it is both usc and lsc.

Definition 2.5. [33] A multivalued map $G : X \to \mathcal{P}(W)$ is *quasi-compact* if $G(B)$ is relatively compact for each compact set $B \subset W$.

Lemma 2.1. [33] *Let $G : X \to \mathcal{P}_{cp,cl}(W)$ is a closed and quasi-compact multivalued mapping. Then, G is usc.*

Definition 2.6. [29] The $e(C,D)$ is called the excess of C over D if for every $C,D \in \mathcal{P}_{cl,b}(X)$, $e(C,D) = \sup_{x\in C}\text{dist}(x,D) = \inf\{r > 0 : C \subset D + rB\}$, where B is the close unit ball of X.

Definition 2.7. [29] The $h(C,D)$ is said to be *Hausdorff distance* if for every $C,D \in \mathcal{P}_{cl,b}(X)$, $h(C,D) = \max\{e(C,D), e(D,C)\}$. Denote by $|C| = h(C,\{0\})$. Therefore, $(\mathcal{P}_{cl,b}(X), h(\cdot,\cdot))$ is a Banach space.

Definition 2.8. [33] A multivalued map $G : X \to \mathcal{P}(W)$ has a fixed point if $X \cap W \neq \emptyset$, and exists $x \in X \cap W$ such that $x \in G(x)$. The fixed-point set of the multivalued operator G will be denoted by $\mathcal{F}ixG$, i.e., $\mathcal{F}ixG = \{x \in X \cap W : x \in G(x)\}$.

Definition 2.9. [32] Let $I \subset R$ be a compact interval, μ be a Lebesgue measure on I, and W be a Banach space. A multivalued map $G : I \to \mathcal{P}_{cp}(W)$ is said to be *measurable* if for every open subset $V \subset W$ the set $\{t \in I : G(t) \subset V\}$ is measurable.

Definition 2.10. [32] A multivalued map $G : I \to \mathcal{P}_{cp}(W)$ is said to be *strongly measurable* if there exist a sequence $\{G_n\}_{n=1}^{\infty}$ of step multivalued maps such that $h(G_n(t), G(t)) \to 0$ as $n \to \infty$ for μ – a.e. $t \in I$.

Definition 2.11. [33] A multivalued map $G : D \subset X \to \mathcal{P}(W)$ is said to be weakly usc if $G^{-1}(B)$ is closed in D for every weakly closed set $B \subset W$.

Definition 2.12. [33] A multivalued map $G : X \to \mathcal{P}(W)$ is said to be $\varepsilon - \delta$ usc if for every $x_0 \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $G(x) \subset G(x_0) + B_\varepsilon(0)$ for all $x \in B_\delta(x_0)$.

It is noted that usc is stronger than $\varepsilon - \delta$ usc, but for multivalued mappings with compact values, the two concepts coincide.

Lemma 2.2. [33] *Let $G : D \subset X \to \mathcal{P}(W)$ be a multi-valued mapping with weakly compact values. Then,*
(1) *G is weakly usc if G is $\varepsilon - \delta$ usc*
(2) *Suppose further that G has convex values and W is reflexive, then, G is weakly usc if and only if for each sequence $\{(x_n, w_n)\}$ in $D \times W$ such that $x_n \to x$ in X and $w_n \in G(x_n)$, $n \geq 1$, it follows that there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and $w \in G(x)$ such that $w_{n_k} \to w$ weakly in W.*

Lemma 2.3. [33] *Let D be a non-empty, compact, and convex subset of a Banach space and $G : D \to \mathcal{P}_{cp}(D)$ an usc multivalued mapping with contractible values. Then, G has at least one fixed-point.*

Definition 2.13. [34] Assume that S is a bounded set of X. Then, the Kuratowski measure of noncompactness $\beta(\cdot)$ defined on S is

$$\beta(S) = \inf\left\{\delta > 0 : S = \bigcup_{i=1}^{m} S_i \ \text{ with } \ \text{diam}(S_i) \leq \delta, \ i = 1, 2, \ldots, m\right\},$$

where $\text{diam}(S_i)$ denotes the diameter of a set S.

Lemma 2.4. [34] *Let Z, M be bounded sets of X, and $\lambda \in R$. Then,*
(*i*) *$\beta(Z) = 0$ if and only if Z is relatively compact*;
(*ii*) *$Z \subset M$ implies $\beta(Z) \leq \beta(M)$*;
(*iii*) *$\beta(\overline{Z}) = \beta(coZ) = \beta(Z)$, where coZ denotes the convex hull of Z*;
(*iv*) *$\beta(Z \cup M) = \max\{\beta(Z), \beta(M)\}$*;
(*v*) *$\beta(\lambda Z) = |\lambda|\beta(Z)$, where $\lambda Z = \{\lambda z : z \in Z\}$*;
(*vi*) *$\beta(Z + M) \leq \beta(Z) + \beta(M)$, where $Z + M = \{z + m : z \in Z, m \in M\}$*;
(*vii*) *$|\beta(Z) - \beta(M)| \leq 2h(Z, M)$, where $h(Z, M)$ denotes the Hausdorff distance of Z and M.*

Lemma 2.5. [34] *Let X be a Banach space, and $M \subset X$ be a bounded set. Then, there exists a countable subset M_0 of M such that $\beta(M) \le 2\beta(M_0)$.*

Lemma 2.6. [34] *If $\tilde{N} = \{u_n\}_1^\infty \subset C([0,T],X)$ is a countable set in a Banach space X and there is a function $m \in L^1([0,T],R^+)$ such that for every $n \in N$,*

$$\|u_n(s)\| \le m(s), \ \text{a.e. } s \in [0,T],$$

then $\beta(\tilde{N}(s))$ is Lebesgue integrable on $[0,T]$ and

$$\beta\left(\left\{\int_0^t u_n(s)\mathrm{d}s : n \in N\right\}\right) \le 2\int_0^t \beta(\tilde{N}(s))\mathrm{d}s.$$

Throughout, let $L^1(\mathrm{d}g,[0,T],X)$ stand for the Banach space of all $\mathrm{d}g$ – a.e. LS-integrable functions from $[0,T]$ to X, equipped with the norm

$$\|f\|_{L^1} = \int_{[0,T]} \|f(s)\|\mathrm{d}g(s), \ f \in L^1(\mathrm{d}g,[0,T],X).$$

Definition 2.14. [31] Let δ be a gauge. $\Delta = \{([t_{i-1},t_i],c_i), i = 1,2,\dots,n\}$, where $c_i \in [t_{i-1},t_i], i = 1,2,\dots,n$ and $\bigcup_{i=1}^n [t_{i-1},t_i] = [0,T]$, is a partition of $[0,1]$, then Δ is said to be δ-fine partition if $[t_{i-1},t_i] \subset (c_i - \delta(c_i), c_i + \delta(c_i))$, for any $i = 1,2,\dots,n$.

Definition 2.15. [31] A function $f : [0,T] \to X$ is said to be *Kurzweil-Stieltjes* (KS) *integrable* w.r.t. $g : [0,T] \to R$ if there exists a vector $(KS)\int_0^T f(s)\mathrm{d}g(s) \in X$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε satisfying

$$\left\|\sum_{i=1}^n f(c_i)(g(t_i) - g(t_{i-1})) - (KS)\int_0^T f(s)\mathrm{d}g(s)\right\| < \varepsilon$$

for every δ_ε-fine partition.

We point out the following relation:

$$\text{Lebesgue Stieltjes integrability} \Rightarrow KS \ \text{integrability}.$$

Lemma 2.7. [35] *Let $\tilde{H}_0 \subseteq L^1(\mathrm{d}g,[0,T],X)$. Assume that there exists a positive function $\theta \in L^1(\mathrm{d}g,[0,T],X)$ such that $\|\tilde{h}(t)\| \le \theta(t)$ $\mathrm{d}g$ – a.e. for all $\tilde{h} \in \tilde{H}_0$. Then, we have*

$$\beta\left(\int_{[0,T]} \tilde{H}_0(t)\mathrm{d}g(t)\right) \le 4\int_{[0,T]} \beta(\tilde{H}_0(t))\mathrm{d}g(t).$$

Definition 2.16. [33] A subset $\mathcal{A} \subset L^1(\mathrm{d}g,[0,T],X)$ is called integrably bounded if there exists $\varsigma \in L^1(\mathrm{d}g,[0,T],X)$ such that

$$\|\tilde{f}(t)\| \le \varsigma(t), \ \text{a.e. } t \in [0,T]$$

for each $\tilde{f} \in \mathcal{A}$.

Definition 2.17. [33] A subset $\mathcal{A} \subset L^1(\mathrm{d}g,[0,T],X)$ is called uniformly integrable if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for each Borel measurable subset $I \subset [0,T]$ with $m(I) < \delta(\varepsilon)$, we have

$$\int_I \|\tilde{f}(t)\|\mathrm{d}t \le \varepsilon \quad \text{uniformly for } \tilde{f} \in \mathcal{A}.$$

It is clear that if $\mathcal{A} \subset L^1(\mathrm{d}g,[0,T],X)$ is integrably bounded, then $\mathcal{A}$ is uniformly integrable.

Lemma 2.8. [36] *Let* (Ω, Σ, μ) *be a finite measure space and* X *be a reflexive Banach space. A subset* K *of* $L^1(\mu, X)$ *is relatively weakly compact if*

(*i*) K *is bounded*;

(*ii*) K *is uniformly integrable*;

(*iii*) *for each* $E \subset \Sigma$, *the set* $\left\{\int_E f \,\mathrm{d}\mu : f \in K\right\}$ *is relatively weakly compact.*

Definition 2.18. [29] A function $f : [0, T] \to X$ is said to be regulated if there exists the limit $f(t+)$ and $f(s-)$ for any point $t \in [0, 1)$ and $s \in (0, 1]$.

Let $(X, \|\cdot\|)$ be Banach space. Denote by X_w the space X with weak topology. We denote by $G([0, T], X)$ the space of all regulated functions from $[0, T]$ to X is a Banach space when endowed with the topology of uniform convergence, i.e., with the norm

$$\|f\|_\infty = \sup_{t\in[0,T]} \|f(t)\|.$$

The set of discontinuity points of a regulated function is known to be at most countable, and regulated functions are bounded. We have the following inclusions between spaces of functions: $C([0, T]) \subset BV([0, T]) \subset G([0, T]) \subset R([0, T]) \subset M([0, T])$, where $R([0, T])$ denotes the set of Riemann-measurable functions, $BV([0, T])$ denotes the space of functions on $[0, T]$ with bounded variation, and $M([0, T])$ denotes the space of measurable functions on $[0, T]$. All the inclusions are strict.

We provided that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ which are uniformly bounded linear operators on X, $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A.

Definition 2.19. [31] A semigroup $\{T(t)\}_{t\geq 0}$ is said to be uniformly continuous on $(0, \infty)$ if on this set it is continuous with respect to the uniform operator topology.

It is worthwhile to recall that any compact C_0-semigroup is uniformly continuous on $(0, \infty)$, but the converse is not necessarily satisfied.

Under these conditions, let us define the fractional power A^β, $0 < \beta \leq 1$, as a closed linear operator on its domain $D(A^\beta)$. For the analytic semigroup $\{T(t)\}_{t\geq 0}$, the following properties will be used:

(i) there is a $M \geq 1$ such that $M := \sup_{t\geq 0}\|T(t)\| < \infty$;

(ii) for any $\beta \in (0, 1]$, there exists a positive constant C_β such that

$$\|A^\beta T(t)\| \leq \frac{C_\beta}{t^\beta}, \quad 0 < t < T.$$

Note that $A^\beta T(t)x = T(t)A^\beta x$ for $x \in D(A^\beta)$. Then $AT(t)x = A^{1-\beta}T(t)A^\beta x$ for $x \in D(A^\beta)$.

Definition 2.20. [31] A set $\mathcal{A} \subset G([0, T], X)$ is said to be equiregulated if for every $\varepsilon > 0$ and every $t_0 \in [0, T]$ there exists $\delta > 0$ such that

(i) for any $t_0 - \delta < t' < t_0$, $\|x(t') - x(t_0-)\| < \varepsilon$;

(ii) for any $t_0 < t'' < t_0 + \delta$, $\|x(t'') - x(t_0+)\| < \varepsilon$

for all $x \in \mathcal{A}$.

Definition 2.21. [31] The operator-valued function $T : [0, T] \to L(X)$ is said to be *Kurzweil-Stieltjes integrable with respect to* $h : [0, T] \to X$ if there exists $\int_0^T T(t)\mathrm{d}h(t) \in X$ such that for every $\varepsilon > 0$, there exists a gauge δ_ε satisfying

$$\left\| \sum_{i=1}^{n} T(c_i)(h(t_i) - h(t_{i-1})) - \int_0^T T(t)\mathrm{d}h(t) \right\| \leq \varepsilon$$

for every δ_ε-fine partition of $[0, T]$.

In particular, if the gauge δ_ε is constant, we obtain the Riemann-Stieltjes integral.

Lemma 2.9. [31] *A set $\mathcal{K}$ of regulated functions is relatively compact in the space $G([0,T],X)$ if and only if it is equiregulated and for every $t \in [0,T]$, $\{f(t), f \in \mathcal{K}\}$ is relatively compact in X.*

Lemma 2.10. [31] *Let the C_0-semigroup of uniformly bounded $\{T(t)\}_{t\geq 0}$ be uniformly continuous on $(0,\infty)$ and $f\colon [0,T] \to X$ be LS-integrable with respect to g. Then, for every $t \in [0,T]$, $T(t-\cdot)f(\cdot)$ is KS-integrable with respect to g on $[0,t]$ and*

$$\int_0^t T(t-s)f(s)\mathrm{d}g(s) = \int_0^t T(t-s)d\left(\int_0^s f(\tau)\mathrm{d}g(\tau)\right).$$

Definition 2.22. [32] A subset $B \subset X$ is called a retract of X if there exists a continuous function $r: X \to B$, such that $r(x) = x$, for every $x \in B$.

Definition 2.23. [32] A subset $B \subset X$ is called a neighborhood retract of X if there exists an open subset $U \subset X$ such that $B \subset U$ and B is retract of U.

Definition 2.24. [32] Let two subsets $X, Y \subset X$, X be embedded in Y if there exists a homeomorphism $h: X \to Y$ such that $h(X)$ is a closed subset of Y.

Definition 2.25. [32] The space X is an absolute retract (AR-space) if for any space Y and for any embedding $h: X \to Y$, the set $h(X)$ is a retract of Y.

Definition 2.26. [32] The space X is an absolute neighborhood retract (ANR-space) if for any space Y and for any embedding $h: X \to Y$, the set $h(X)$ is a neighborhood retract of Y.

Definition 2.27. [33] A subset $B \subset X$ is said to be contractible if there exists a point $y_0 \in B$ and a continuous function $h: [0,1] \times B \to B$ such that $h(0,y) = y_0$ and $h(1,y) = y$ for every $y \in B$.

Definition 2.28. [32] A subset B of a metric space is called an R_δ-set if there exists a decreasing sequence B_n of compact and contractible sets such that

$$B = \bigcap_{n=1}^{\infty} B_n.$$

Any intersection of decreasing sequence of R_δ-set is R_δ.

Lemma 2.11. [32] *Let X be a complete metric space, β denote the Kuratowski measure of noncompactness in X, and let $\varnothing \neq B \subset X$. Then, the following statements are equivalent*:
(*i*) *B is an R_δ-set*;
(*ii*) *B is an intersection of a decreasing sequence $\{B_n\}$ of closed contractible spaces with $\beta(B_n) \to 0$.*

Lemma 2.12. [31] *Let $\mu = \mu^c + \mu^s$ be a finite Borel measure on $[0,T]$ with $\mu(\{0\}) = 0$; $u: [0,T] \to R$ be regulated and μ-integrable, $\beta: [0,T] \to R^+$ be μ-integrable and $\alpha: [0,T] \to R^+$ be nondecreasing and μ-integrable. Suppose that*

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)\mathrm{d}\mu(s), \;\; t \in [0,T],$$

then

$$u(t) \leq \alpha(t) + \int_0^t \varepsilon(t,s)\beta(s)\alpha(s)\mathrm{d}\mu(s), \;\; t \in [0,T],$$

where

$$\varepsilon(t,s) = e^{\int_s^t \beta(\tau)\mathrm{d}\mu^c(\tau)} \prod_{\tau\in D_\mu\cap[s,t]} (1+\beta(\tau)\Delta\mu(\tau)).$$

3 Structure of the solution sets on compact intervals

In this section, we will study the existence result of the mild solution for system (1). And the R_δ-structure of solution sets for system (1) will be obtained.

For the sake of convenience, put $I = [0, T]$.

We first introduce the following assumptions:

(A_0) A generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ which is uniformly bounded linear operator on X, and it is also a C_0-semigroup of uniformly continuous on $(0, \infty)$. In addition, $D(A)$ is a dense subset of X; and the multivalued map $F : I \times G(I, X) \to \mathcal{P}_{cl,cv}(X)$ satisfies the following:

(F_1) $F(t, \cdot)$ is *weakly usc* for a.e. $t \in I$;

(F_2) $F(\cdot, x)$ has a strongly measurable selection for each $x \in G(I, X)$;

(F_3) There exists a bounded function $\zeta(t) \in L^1(\mathrm{d}g, I, R^+)$, and a compact convex set $S \subset X$ such that

$$F(t, x(t)) \subset (\zeta(t)(1 + \|x(t)\|))S,$$

for a.e. $t \in I$ and $x \in G(I, X)$.

(F_4) There exist a bounded function $\theta \in L^1(\mathrm{d}g, I, R^+)$, such that

$$\beta(F(t, D)) \leq \theta(t) \sup_{0\leq s\leq t} \beta(D(s)),$$

for a.e. $t \in I$, and every bounded $D \subset G(I, X)$, $D(s) = \{f(s) : f \in D\}$.

(Q_1) For the function $q : I \times G(I, X) \to X$, there exist $\beta \in (0, 1)$ such that $q \in D(A^\beta)$ and for each $x \in G(I, X)$, the function $A^\beta q(\cdot, x) : I \to X$ is strongly measurable.

(Q_2) The function $q : I \times G(I, X) \to X$ is continuous, bounded variation and there exists u, v, $L_q > 0$, $A^\beta q(t, \cdot)$ satisfies the condition

$$\|A^\beta q(t, x(t))\| \leq u\|x(t)\| + v,$$

$$\|A^\beta q(t, x_1(t)) - A^\beta q(t, x_2(t))\| \leq L_q\|x_1(t) - x_2(t)\|,$$

for a.e. $t \in I,\ x, x_1, x_2 \in G(I, X)$.

(Q_3) There exist a constant $r > 0$, such that

$$\beta(A^\beta q(t, K)) \leq r \sup_{0\leq s\leq t} \beta(K(s)),$$

for a.e. $t \in I$, and every bounded $K \subset G(I, X)$, $K(s) = \{f(s) : f \in K\}$.

For each $x \in G([0, T], X)$, let us denote

$$\mathrm{Sel}_F(x) = \{f \in L^1(\mathrm{d}g, [0, T], X) : f(t) \in F(t, x(t)), \text{ for a.e. } t \in [0, T]\}.$$

Definition 3.1. A regulated function $x : I \to X$ is called a mild solution on I of inclusion problem (1.1) if $x(0) = x_0$, and there exists a function $f \in \mathrm{Sel}_F(x)$ such that $x(t)$ satisfies the following integral equation

$$x(t) = T(t)[x_0 - q(0, x_0)] + q(t, x(t)) + \int_0^t AT(t-s)q(s, x(s))\mathrm{d}s + \int_0^t T(t-s)f(s)\mathrm{d}g(s)$$

for $t \in I$.

Remark 3.1. Following Lemma 2.10, the last integral in this definition is taken in KS sense and we can write

$$\int_0^t T(t-s)f(s)\mathrm{d}g(s) = \int_0^t T(t-s)d\left(\int_0^s f(\tau)\mathrm{d}g(\tau)\right).$$

Since $\int_0^\cdot f(\tau)\mathrm{d}g(\tau)$ is BV function, by [38, Theorem 3.1], we obtain the regularity and BV-property of $\int_0^\cdot T(t-s)f(s)\mathrm{d}g(s)$. Owing to $\|T(t-s)f(s)\| \le M\|f(s)\|$ for every $s \in [0,t]$, the integral exists in Lebesgue-Stieltjes sense as well.

Lemma 3.1. *Let (F_1)–(F_3) be satisfied and X be reflexive. Then,* Sel_F *is non-empty.*

Proof. We show that $\mathrm{Sel}_F(x) \neq \varnothing$ for each $x \in G(I,X)$. Let $x \in G(I,X)$ and $\{x_n\}_{n\ge 1}$ be a sequence of step functions from I to X such that

$$\sup_{t\in I}\|x_n(t) - x(t)\| \to 0, \quad \text{as } n \to \infty. \tag{3.1}$$

By (F_2), for each $n \ge 1$, $F(\cdot, x_n(\cdot))$ admits a strongly measurable selection $f_n(\cdot)$. Furthermore, by (3.1), there exists $\widetilde{M} > 0,\ \forall n \in N,\ t \in I,\ \|x_n(t)\| \le \widetilde{M}$. By (F_3), we obtain $\{f_n\}$ is integrably bounded in $L^1(\mathrm{d}g, I, X)$. Due to Lemma 2.8, it suffices to prove that (iii) holds. By (F_3), for each $t \in [0,T]$, we have

$$f_n(t) \in F(t, x_n(t)) \subset (\zeta(t)(1 + \|x_n(t)\|))S \subset (\zeta(t)(1+\widetilde{M}))S.$$

Thus, for each $t \in [0,T]$, there exists a subsequence $\{f_{n_k}\}$ which is $\sigma(X,X^*)$-weakly convergent to some integrable (with respect to g) function f. i.e., for every $x^* \in X^*$, we obtain $\langle x^*, f_{n_k}(t)\rangle \to \langle x^*, f(t)\rangle,\ n \to \infty$. Making use of Lebesgue controlled convergence theorem, we have

$$\left\langle x^*, \int_0^t f_{n_k}(s)\mathrm{d}g(s)\right\rangle = \int_0^t \langle x^*, f_{n_k}(s)\rangle \mathrm{d}g(s),$$
$$\to \int_0^t \langle x^*, f(s)\rangle \mathrm{d}g(s) = \langle x^*, \int_0^t f(s)\mathrm{d}g(s)\rangle,\ n \to \infty,$$

and so, for each $t \in [0,T]$, $\{\int_0^t f_n(s)\mathrm{d}g(s)\}_{n\ge1}$ is relatively weakly compact. Hence, by Lemma 2.8, we may assume that $f_n \to f,\ n \to \infty$ weakly in $L^1(\mathrm{d}g, I, X)$. By Mazur's lemma, we obtain a sequence $\tilde{f}_n \in co\{f_k : k \ge n\}$ strongly L^1-convergent: therefore, on a subsequence, dg − a.e. convergent to g. Without loss of generality, we assume that $\tilde{f}_n(t) \to f(t)$, for a.e. $t \in I$. And then, by (F_1), we have $F(s,\cdot) : G_w(I,X) \to \mathcal{P}(G_w(I,X))$ is usc. To obtain the result, it suffices to prove that $f(t) \in F(t, x(t))$, for a.e. $t \in I$. Indeed, let N_0 be a null measure set such that $F(s,\cdot) : G_w(I,X) \to \mathcal{P}(G_w(I,X))$ is usc, $f_n(t) \in F(t, x_n(t))$ and $\tilde{f}_n(t) \to f(t)$ for all $t \in I \backslash N_0$ and $n \in N$.

Given $t_0 \notin N_0$ and provided that $f(t_0) \notin F(t_0, x(t_0))$. Since $F(t_0, x(t_0))$ is closed and convex, by Hahn-Banach theorem, there exists a weakly open convex set V such that $F(t_0, x(t_0)) \subset V$ and $f(t_0) \notin \overline{V}$. Due to $F(t_0,\cdot) : G_w(I,X) \to \mathcal{P}(G_w(I,X))$ is usc, we can find a neighborhood U of $x(t_0)$ such that $F(t_0, x) \subset V$ for all $x \in U$. $x_n(t_0) \rightharpoonup x(t_0),\ n \to \infty$ implies that there exists $N > 0$ satisfying $x_n(t_0) \in U$, for all $n > N$. Thus, we obtain $f_n(t_0) \in F(t_0, x(t_0)) \subset V$, for all $n > N$. By convexity of V and $\tilde{f}_n(t_0) \to f(t_0)$, we have $\tilde{f}_n(t_0) \in V$, for all $n > N$ and $f(t_0) \in \overline{V}$, which is a contradiction. Therefore, we have $f(t) \in F(t, x(t))$ for a.e. $t \in I$. This proof is complete. □

Theorem 3.1. *Let hypothesis (A_0) and (F_1)–(F_4), (Q_1), (Q_2) be satisfied, if*

$$u\|A^{-\beta}\| + \frac{C_{1-\beta}uT^\beta}{\beta} + Mc\int_0^T \zeta(s)\mathrm{d}g(s) < 1$$

and

$$L_q\left(\|A^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta}\right) < 1. \tag{3.2}$$

Then, the inclusion problem (1.1) *has at least one mild solution.*

Proof. Let us consider the operator $\mathcal{N}: G(I,X) \to \mathcal{P}(G(I,X))$ defined by

$$\mathcal{N}(x) = \{w \in G(I,X) : w(t) \in \Phi_T(x)(t),\ \ \forall t \in I\},\ \ x \in G(I,X),$$

where

$$\Phi_T(x)(t) = \left\{T(t)[x_0 - q(0,x_0)] + q(t,x(t)) + \int_0^t AT(t-s)q(s,x(s))\mathrm{d}s + \int_0^t T(t-s)f(s)\mathrm{d}g(s),\ f \in \mathrm{Sel}_F(x)\right\}.$$

Taking $R > 0$ such that

$$\begin{aligned} R \geq \max&\left\{\left(M\|x_0 - q(0,x_0)\| + v\|A^{-\beta}\| + \frac{C_{1-\beta}vT^\beta}{\beta} + Mc\int_0^T \zeta(s)\mathrm{d}g(s)\right)\right.\\ &\times\left\{1 - \left[u\|A^{-\beta}\| + \frac{C_{1-\beta}uT^\beta}{\beta} + Mc\int_0^T\zeta(s)\mathrm{d}g(s)\right]\right\}^{-1},\\ &\frac{M\|x_0 - q(0,x_0)\| + v\|A^{-\beta}\| + \frac{C_{1-\beta}vT^\beta}{\beta} + Mc\int_0^T\zeta(s)\mathrm{d}g(s)}{1 - \left[u\|A^{-\beta}\| + \frac{C_{1-\beta}uT^\beta}{\beta}\right]}\\ &\left.\times\left(1 + Mc\int_0^T \varepsilon(T,s)\zeta(s)\mathrm{d}g(s)\right)\right\}. \end{aligned}$$

We consider the set $\overline{B}_R = \{x \in G(I,X) : \|x\|_\infty \leq R\} \subset G(I,X)$ be a non-empty, compact, and convex subset of $G(I,X)$.

It is clear that we will obtain the result if we show that the operator $\mathcal{N}$ admits a fixed-point in $\overline{B}_R$. Therefore, we will show that the operator $\mathcal{N}$ satisfy all the conditions of Lemma 2.3. For the sake of convenience, we split the proof into several steps.

Step 1. Let us first show that $\mathcal{N}\overline{B}_R \subset \overline{B}_R$. Since S is compact, it is bounded, we say by $c > 0$. Observe that for every $x \in \overline{B}_R$, by Lemma 3.1, $\mathrm{Sel}_F(x) \neq \emptyset$ and for each $w \in \mathcal{N}(x)$, there exists $f \in \mathrm{Sel}_F(x)$, such that

$$w(t) = T(t)[x_0 - q(0,x_0)] + q(t,x(t)) + \int_0^t AT(t-s)q(s,x(s))\mathrm{d}s + \int_0^t T(t-s)f(s)\mathrm{d}g(s),\ \ \forall t \in I.$$

Then, for $t \in I$, we have

$$\begin{aligned} \|w(t)\| &\leq M\|x_0 - q(0,x_0)\| + \|q(t,x(t))\| + \int_0^t \|AT(t-s)q(s,x(s))\|ds + \int_0^t \|T(t-s)f(s)\|\mathrm{d}g(s)\\ &\leq M\|x_0 - q(0,x_0)\| + \|A^{-\beta}\|\ \|A^\beta q(t,x(t))\| + \int_0^t \|A^{1-\beta}T(t-s)\|\ \|A^\beta q(s,x(s))\|ds\\ &\quad + Mc\int_0^t \zeta(s)(1 + \|x(s)\|)\mathrm{d}g(s)\\ &\leq M\|x_0 - q(0,x_0)\| + \|A^{-\beta}\|(u\|x(t)\| + v) + C_{1-\beta}\int_0^t \frac{u\|x(s)\| + v}{(t-s)^{1-\beta}}\mathrm{d}s + Mc\int_0^t \zeta(s)(1 + \|x(s)\|)\mathrm{d}g(s)\\ &\leq M\|x_0 - q(0,x_0)\| + \|A^{-\beta}\|(u\|x\|_\infty + v) + \frac{C_{1-\beta}(u\|x\|_\infty + v)T^\beta}{\beta} + Mc\int_0^t \zeta(s)(1 + \|x\|_\infty)\mathrm{d}g(s)\\ &\leq M\|x_0 - q(0,x_0)\| + \|A^{-\beta}\|(uR + v) + \frac{C_{1-\beta}(uR + v)T^\beta}{\beta} + Mc(1 + R)\int_0^T \zeta(s)\mathrm{d}g(s), \end{aligned}$$

which yields

$$\|w\|_\infty = \sup_{t\in I}\|w(t)\| \le R$$

for all $w \in \mathcal{N}(x)$ and all $x \in \overline{B}_R$. So $\mathcal{N}\overline{B}_R \subset \overline{B}_R$.

Step 2. We process to verify that $\mathcal{N}$ is usc on $\overline{B}_R$. Due to Lemma 2.1, it suffices to prove that $\mathcal{N}$ is quasi-compact and closed. Let us first define a single-valued operator $\Omega : G(I,X) \to G(I,X)$, i.e.,

$$\Omega(x)(t) = T(t)[x_0 - q(0,x_0)] + q(t,x(t)) + \int_0^t AT(t-s)q(s,x(s))\mathrm{d}s,$$

$\forall x \in G(I,X)$, $\forall t \in I$, and the operator $\Upsilon : L^1(\mathrm{d}g, I, X) \to G(I,X)$, i.e.,

$$\Upsilon(f)(t) = \int_0^t T(t-s)f(s)\mathrm{d}g(s), \quad \forall t \in I.$$

Let $B \subset \overline{B}_R$ be a compact set. We will prove that $\mathcal{N}(B)$ is a relatively compact set of $G(I,X)$. Assume that $\{x_n\} \subset B$, $\{w_n\} \subset \mathcal{N}(x_n)$, and so, there exists $\{f_n\} \in \mathrm{Sel}_F(x_n)$ such that

$$w_n(t) = \Omega(x_n)(t) + \Upsilon(f_n)(t), \quad \forall t \in I.$$

Due to the compactness of B, there exists $x \in B$ such that $x_n \to x$. On the one hand, we have the inequalities

$$\begin{aligned}\|\Omega(x_n)(t) - \Omega(x)(t)\| &\le \|q(t,x_n(t)) - q(t,x(t))\| + \left\|\int_0^t AT(t-s)[q(s,x_n(s)) - q(s,x(s))]\mathrm{d}s\right\| \\ &\le L_q\|A^{-\beta}\|\,\|x_n(t) - x(t)\| + L_qC_{1-\beta}\int_0^t \frac{\|x_n(s) - x(s)\|}{(t-s)^{1-\beta}}\mathrm{d}s \\ &\le L_q\|A^{-\beta}\|\,\|x_n - x\|_\infty + L_qC_{1-\beta}\int_0^t \frac{\|x_n - x\|_\infty}{(t-s)^{1-\beta}}\mathrm{d}s \\ &= L_q\left(\|A^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta}\right)\|x_n - x\|_\infty,\end{aligned}$$

for $t \in I$. Then,

$$\begin{aligned}\|\Omega(x_n) - \Omega(x)\| &= \sup_{t\in I}\|\Omega(x_n)(t) - \Omega(x)(t)\| \\ &\le L_q\left(\|A^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta}\right)\|x_n - x\|_\infty \to 0, \quad n \to \infty.\end{aligned}$$

Therefore, $\{\Omega(x_n)\}_{n\ge1}$ is relatively compact in $G(I,X)$. Furthermore, we prove $\{\Upsilon(f_n)\}_{n\ge1}$ is also relatively compact . By (F_4) and Lemma 2.7, for every $t \in I$, we have

$$\begin{aligned}\beta\left(\left\{\int_0^t T(t-s)f_n(s)\mathrm{d}g(s)\right\}_{n\ge1}\right) &\le 4\int_0^t \beta(\{T(t-s)f_n(s)\}_{n\ge1})\mathrm{d}g(s) \\ &\le 4\int_0^t \|T(t-s)\|\beta(\{f_n(s)\}_{n\ge1})\mathrm{d}g(s) \le 4M\int_0^t \theta(s)\sup_{0\le\tau\le s}\beta(\{x_n(\tau)\}_{n\ge1})\mathrm{d}g(s).\end{aligned}$$

Since the compactness of B, $\beta(\{x_n(s)\}_{n\geq 1}) = 0$, $\forall s \in [0, t]$. And so, for each $t \in I$, we obtain $\{\Upsilon(f_n)(t)\}_{n\geq 1}$ is relatively compact. Due to Lemma 2.9, it suffices to prove that $\{\Upsilon f_n\}_{n\geq 1} \subset G(I, X)$ is equiregulated. To see this, fixed $t \in I$, let us show that for every $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ such that for all $h < \delta_\varepsilon$,

$$\|\Upsilon f_n(t+h) - \Upsilon f_n(t) - (m_{f_n}(t+0) - m_{f_n}(t))\| \leq \varepsilon \tag{3.3}$$

$$\|\Upsilon f_n(t) - \Upsilon f_n(t-h) - (m_{f_n}(t) - m_{f_n}(t-0))\| \leq \varepsilon \tag{3.4}$$

for all $n \in N$, where $m_{f_n}(t) = \int_0^t f_n(s)\mathrm{d}g(s)$.

From the arguments above, we have $\|f_n(t)\| \leq c(1+R)\zeta(t) \equiv R_1\zeta(t)$, $\forall n \in N$. When $t < T$,

$$\begin{aligned}
&\|\Upsilon f_n(t+h) - \Upsilon f_n(t) - (m_{f_n}(t+0) - m_{f_n}(t))\| \\
&\leq \Bigg\| (T(h) - I)(\Upsilon f_n(t)) + \int_t^{t+h} \chi_{(t,t+h]}(s)T(t+h-s)f_n(s)\mathrm{d}g(s) + T(h)f_n(t)(g(t+0) - g(t)) \\
&\quad - (m_{f_n}(t+0) - m_{f_n}(t))\Bigg\| \\
&\leq \|(T(h) - I)(\Upsilon f_n(t) + m_{f_n}(t+0) - m_{f_n}(t))\| + \mathrm{var}(m_{f_n}, (t, t+h]) \\
&\leq \|(T(h) - I)(\Upsilon f_n(t) + m_{f_n}(t+0) - m_{f_n}(t))\| + \mathrm{var}(m_{R_1\zeta(t)}, (t, t+h]),
\end{aligned}$$

where $m_{R_1\zeta(t)}(t) = R_1\int_0^t \zeta(s)\mathrm{d}g(s)$.

Since $\{\Upsilon f_n(t)\}_{n\geq 1}$ is relatively compact, $m_{fn}(t+0) - m_{f_n}(t) = f_n(t)(g(t+0) - g(t))$, $\{m_{f_n}(t+0) - m_{f_n}(t) : f_n(t) \in \zeta(t)(1+R)S,\ \forall t \in I,\ n \in N\} \subset (1+R)\zeta(t)(g(t+0) - g(t))S$, and so, the set $\{\Upsilon f_n(t) + m_{f_n}(t+0) - m_{f_n}(t) : f_n(t) \in \zeta(t)(1+R)S,\ \forall t \in I,\ n \in N\}$ is relatively compact. From the arguments above and applying [40, Corollary 1], there exists $\delta_\varepsilon^1 > 0$, such that for all $h < \delta_\varepsilon^1$, $n \in N$,

$$\|(T(h) - I)(\Upsilon f_n(t) + m_{f_n}(t+0) - m_{f_n}(t))\| \leq \frac{\varepsilon}{2}.$$

As $m_{R_1\zeta(t)}$ is a BV function, applying [38, Lemma 2.1], we imply that there exist δ_ε^2, such that for all $h < \delta_\varepsilon^2$,

$$\mathrm{var}(m_{R_1\zeta(t)}, (t, t+h]) \leq \frac{\varepsilon}{2}$$

and so, for all $h < \min\{\delta_\varepsilon^1, \delta_\varepsilon^2\}$, we obtain inequality (3.3).

For inequality (3.3), when $t > 0$, from the arguments above, there exists $\delta > 0$ such that

$$\mathrm{var}(m_{R_1\zeta(t)}, [t-\delta, t)) < \frac{\varepsilon}{8M^2}.$$

Therefore,

$$\begin{aligned}
&\|\Upsilon f_n(t) - \Upsilon f_n(t-h) - (m_{f_n}(t) - m_{f_n}(t-0))\| \\
&\leq \|\Upsilon f_n(t) - T(h)\Upsilon f_n(t-h) - (m_{f_n}(t) - m_{f_n}(t-0))\| + \|T(h)\Upsilon f_n(t-h) - \Upsilon f_n(t-h)\| \\
&\leq \left\| \int_{t-h}^{t} \chi_{[t-h,t)}(s)T(t-s)f_n(s)\mathrm{d}g(s) \right\| + \left\| (T(\delta) - T(\delta-h)) \int_0^{t-\delta} T(t-\delta-s)f_n(s)\mathrm{d}g(s) \right\| \\
&\quad + \left\| (T(h) - I) \int_{t-\delta}^{t-h} \chi_{[t-\delta,t-h)}(s)T(t-h-s)f_n(s)\mathrm{d}g(s) \right\| + \|(T(h) - I)(m_{f_n}(t-h) - m_{f_n}(t-h-0))\| \\
&\leq M\mathrm{var}(m_{f_n}, [t-h, t)) + M\|T(\delta) - T(\delta-h)\|\mathrm{var}(m_{f_n}, [0,1]) + 2M^2\mathrm{var}(m_{f_n}, [t-\delta, t)) \\
&\quad + \|(T(h) - I)(m_{f_n}(t-h) - m_{f_n}(t-h-0))\| \\
&\leq M\mathrm{var}(m_{R_1\zeta(t)}, [t-h, t)) + M\|T(\delta) - T(\delta-h)\|\mathrm{var}(m_{R_1\zeta(t)}, [0,1]) + 2M^2\mathrm{var}(m_{R_1\zeta(t)}, [t-\delta, t)) \\
&\quad + \|(T(h) - I)(m_{f_n}(t-h) - m_{f_n}(t-h-0))\|
\end{aligned}$$

for any $h \in (0, \delta)$. And, there exists $\delta_\varepsilon^3 < \delta$, such that for all $h < \delta_\varepsilon^3$,

$$\|T(\delta) - T(\delta - h)\|\mathrm{var}(m_{R_1\zeta(t)}, [0,1]) < \frac{\varepsilon}{4M}.$$

Meanwhile, since

$$\|m_{f_n}(t_2) - m_{f_n}(t_1)\| \leq \int_{t_1}^{t_2} R_1\zeta(t)\mathrm{d}g(s), \ \forall f_n(t) \in \zeta(t)(1+R)S, \ \forall t \in I, \ n \in N,$$

the set $\{m_{f_n} : f_n(t) \in \zeta(t)(1+R)S, \ \forall t \in I, \ n \in N\}$ is equiregulated. And for each $t \in I, \{m_{f_n}(t) : f_n(t) \in \zeta(t)(1+R)S, \ \forall t \in I, \ n \in N\}$ is relatively compact. By Lemma 2.9, the set

$$\bigcup_{n\geq 1}\{m_{f_n} : f_n(t) \in \zeta(t)(1+R)S, \ \forall t \in I, \ n \in N\}$$

is relatively compact. From the arguments above, there exists $\delta_\varepsilon^4 < \delta$, such that for all $h < \delta_\varepsilon^4$

$$\|(T(h) - I)(m_{f_n}(t-h) - m_{f_n}(t-h-0))\| < \frac{\varepsilon}{4}.$$

Thus, we obtain inequality (3.4).

By applying Lemma 2.9, we obtain the relative compactness of set $\{\Upsilon(f_n)\}_{n\geq 1}$ in $G(I, X)$. Thus, we obtain $\mathcal{N}(B)$ is relatively compact. This in particular implies that $\mathcal{N}$ is quasi-compact.

Next let us show that $\mathcal{N}$ is closed. Let $\{x_n\} \subset \overline{B}_R$ with $x_n \to x$ and $w_n \in \mathcal{N}(x_n)$ with $w_n \to w$, we shall prove that $w \in \mathcal{N}(x)$. By the definition of $\mathcal{N}$, there exists $f_n \in \mathrm{Sel}_F(x_n)$ such that

$$w_n(t) = \Omega(x_n)(t) + \Upsilon(f_n)(t), \ \forall t \in I. \tag{3.5}$$

By Assumption (F_3), let $\Gamma(t) = \zeta(t)(1+R)S \subset G(I,X)$, then for every $t \in I$, it is a compact convex set, and is dg-measurable, such that

$$F(t, x_n(t)) \subset \Gamma(t), \ \forall n \geq 1.$$

Let us now prove that there exists a sequence of convex combinations of $\{f_n\}_{n\geq 1}$ which is pointwisely convergent on I to some function $f \in L^1(\mathrm{d}g, I, X)$. To this purpose, for each $p \in \mathcal{N}$, define the sets

$$E_p = \{t \in I : p - 1 \leq \Gamma(t) \leq p\}.$$

Since the measurability of Γ implies the measurability of $|\Gamma|$, we obtain a countable disjoint family of dg-measurable sets that covers the interval I.

By the proof of Lemma 3.1, one can conclude that $\{f_n\}_{n\geq 1}$ is weakly $L^1(\mathrm{d}g, E_1, X)$-relatively compact and so, we can find a subsequence (not relabeled) weakly $L^1(\mathrm{d}g, E_1, X)$-convergent to some dg-measurable function w_1. By Mazur's theorem, there exists a sequence of convex combinations convergent in the norm of $L^1(\mathrm{d}g, E_1, X)$, thus dg – a.e. on E_1 to w_1. We have got a sequence $\tilde{f}_n^1 \in co\{f_k : k \geq n\}$ that is dg – a.e. convergent on E_1 to w_1.

Next using the same argument as above, we obtain a sequence $\tilde{f}_n^2 \in co\{\tilde{f}_k^1 : k \geq n\}$ (obviously, $\tilde{f}_n^2 \in co\{f_k : k \geq n\}$) that is dg – a.e. convergent on E_2 to some dg-measurable function w_2.

Continuing in the same way and taking, by a diagonal procedure, the sequence $\tilde{f}_n^n$ denoted simply by $\tilde{f}_n$, we obtain that $\tilde{f}_n \in co\{f_k : k \geq n\}$ and it is dg – a.e. convergent on all E_p, therefore, on I, to a dg-measurable function f.

More precisely, for each $n \in N$, one can find $k_n \in N$, $\alpha_i^n > 0$ with $\sum_{i=1}^{k_n}\alpha_i^n = 1$ and $\beta_i^n \in N$ such that

$$\tilde{f}_n = \sum_{i=1}^{k_n}\alpha_i^n f_{n+\beta_i^n}.$$

Since $\|T(t-s)\tilde{f}_n(s)\| \le Mc(1+R)\zeta(s)$ and dominated convergence theorem, we obtain

$$\int_0^t T(t-s)\tilde{f}_n(s)\mathrm{d}g(s) \to \int_0^t T(t-s)f(s)\mathrm{d}g(s), \quad \forall t \in I.$$

The condition (F_1) implies that f is a selection of $F(\cdot, x(\cdot))$. By the assumption (F_1) and $x_n \to x$, $n \to \infty$, we obtain that for every $s \in I$, $F(s,\cdot) : X \to \mathcal{P}(X)$ is weakly usc And thus, for $\forall\ x^* \in X^*$, we have

$$\sup_{f' \in F(s,x(s))} \langle x^*, f'\rangle \ge \overline{\lim_{n\to\infty}} \sup_{f \in F(s,x_n(s))} \langle x^*, f\rangle.$$

Therefore,

$$\sup_{f' \in F(s,x(s))} \langle x^*, f'\rangle \ge \overline{\lim_{n\to\infty}} \sup_{f \in F(s,x_{n+\beta_i^n}(s))} \langle x^*, f\rangle \ge \overline{\lim_{n\to\infty}} \langle x^*, f_{n+\beta_i^n}\rangle.$$

And then, by $\tilde{f}_n \to f$ and the linear properties of x^*, we obtain

$$\sup_{f' \in F(s,x(s))} \langle x^*, f'\rangle = \sum_{i=1}^{k_n} \alpha_i^n \sup_{f \in F(s,x(s))} \langle x^*, f\rangle \ge \overline{\lim_{n\to\infty}} \langle x^*, \sum_{i=1}^{k_n} \alpha_i^n f_{n+\beta_i^n}\rangle = \langle x^*, f\rangle.$$

To establish the result, we prove by contradiction. Let $f(s) \notin F(s, x(s))$, Since $F(s, x(s))$ is closed and convex, by the Hahn-Banach theorem, there exists $x_0^* \in X^*$ and $\alpha \in R$ such that $\langle x_0^*, f(s)\rangle > \alpha > \langle x_0^*, f'\rangle$, $\forall f' \in F(s.\ x(s))$, i.e., $\langle x_0^*, f(s)\rangle > \sup_{f' \in F(s,x(s))}\langle x_0^*, f'\rangle$, which is a contradiction. Therefore, we have $f(s) \in F(s, x(s))$.

Now we can obtain that for each $t \in I$,

$$w(t) = \Omega(x)(t) + \Upsilon(f)(t).$$

Indeed, for each $n \in N$, with the same notations used in the writing of $\tilde{f}_n$,

$$\begin{aligned} w_{n+\beta_i^n}(t) &= \Omega(x_{n+\beta_i^n})(t) + \Upsilon(f_{n+\beta_i^n})(t) \\ &= T(t)[x_0 - q(0,x_0)] + q(t, x_{n+\beta_i^n}(t)) + \int_0^t AT(t-s)q(s, x_{n+\beta_i^n}(s))\mathrm{d}s + \int_0^t T(t-s)f_{n+\beta_i^n}(s)\mathrm{d}g(s), \end{aligned}$$

for $i = 1, \ldots, k_n$, whence

$$\begin{aligned} \sum_{i=1}^{k_n} \alpha_i^n w_{n+\beta_i^n}(t) &= \sum_{i=1}^{k_n} \alpha_i^n \Omega(x_{n+\beta_i^n})(t) + \Upsilon(\tilde{f}_n)(t) \\ &= T(t)[x_0 - q(0,x_0)] + \sum_{i=1}^{k_n} \alpha_i^n q(t, x_{n+\beta_i^n}(t)) + \int_0^t AT(t-s)\left(\sum_{i=1}^{k_n} \alpha_i^n q(s, x_{n+\beta_i^n}(s))\right)\mathrm{d}s \\ &\quad + \int_0^t T(t-s)\tilde{f}_n(s)\mathrm{d}g(s). \end{aligned}$$

Inasmuch as for each $t \in I$, $w_n(t) \to w(t)$ one can write that for each $t \in I$ and $\varepsilon > 0$, one can find $N > 0$ satisfying

$$\|w_n(t) - w(t)\| < \varepsilon, \quad \forall n \ge N,$$

whence

$$\left\|\sum_{i=1}^{k_n} \alpha_i^n w_{n+\beta_i^n}(t) - w(t)\right\| \le \sum_{i=1}^{k_n} \alpha_i^n \|w_{n+\beta_i^n}(t) - w(t)\| < \varepsilon, \quad \forall n \ge N.$$

Since $x_n \to x$, by the continuity of the function q and dominated convergence theorem, we obtain

$$\sum_{i=1}^{k_n} \alpha_i^n q(t, x_{n+\beta_i^n}(t)) \to q(t, x(t)),$$

$$\int_0^t AT(t-s)\left(\sum_{i=1}^{k_n} a_i^n q(s, x_{n+\beta_i^n}(s))\right)\mathrm{d}s \to \int_0^t AT(t-s)q(s,x(s))\mathrm{d}s,$$

and

$$\int_0^t T(t-s)\tilde{f}_n(s)\mathrm{d}g(s) \to \int_0^t T(t-s)f(s)\mathrm{d}g(s).$$

So, by passing to the limit it is clear that

$$w(t) = T(t)[x_0 - q(0,x_0)] + q(t,x(t)) + \int_0^t AT(t-s)q(s,x(s))\mathrm{d}s + \int_0^t T(t-s)f(s)\mathrm{d}g(s)$$

for all $t \in I$, which yields that $w \in \mathcal{N}(x)$. It follows that $\mathcal{N}$ is closed.

Step 3. Now we show that $\mathcal{N}$ has a fixed-point in $\overline{B}_R$. By Lemma 2.3, it suffices to show that $\mathcal{N}$ has compact, contractible values. Due to the closedness and quasi-compactness of $\mathcal{N}$, we obtain that for each $x \in \overline{B}_R$, $\mathcal{N}(x)$ is compact. Next we prove that $\mathcal{N}$ has contractible values. For $x \in \overline{B}_R$, we fixed $f^* \in \mathrm{Sel}_F(x)$ and $w^* = \Omega(x) + \Upsilon(f^*)$.

Define a function $\mathcal{G} : [0,1] \times \mathcal{N}(x) \to \mathcal{N}(x)$ by

$$\mathcal{G}(\lambda, w)(t) = \begin{cases} w(t), & t \in [0, \lambda T], \\ x(t, \lambda T, w(\lambda T), f^*), & t \in (\lambda T, T], \end{cases}$$

for each $(\lambda, w) \in [0,1] \times \mathcal{N}(x)$. Applying Theorem 3.3 in [39], $x(t, \lambda T, w(\lambda T), f^*)$ is the mild solution of the semilinear evolution problem in the form

$$\begin{cases} d[x(t) + q(t,x(t))] = Ax(t)\mathrm{d}t + f^*(t)\mathrm{d}g(t), & t \in [\lambda T, T], \\ x(\lambda T) = w(\lambda T). \end{cases} \tag{3.6}$$

Moreover, we point out that the mild solution to (3.6) is unique. In fact, if x_1 and x_2 are two solutions of (3.6), then we have

$$\|x_1 - x_2\|_\infty \le L_q\left(\|A^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta}\right)\|x_1 - x_2\|_\infty.$$

By condition (3.2), we see that $x_1 \equiv x_2$.

Next we shall show that $\mathcal{G}(\lambda, w) \in \mathcal{N}(x)$ for each $(\lambda, w) \in [0,1] \times \mathcal{N}(x)$. It is to be noted that for each $w \in \mathcal{N}(x)$, there exists $\hat{f} \in \mathrm{Sel}_F(x)$ such that $w = \Omega(x) + \Upsilon(\hat{f})$. Let

$$\tilde{f}(t) = \hat{f}\chi_{[0,\lambda T]}(t) + f^*(t)\chi_{(\lambda T, T]}(t), \quad \forall t \in I.$$

It is clear that $\tilde{f}(t) \in \mathrm{Sel}_F(x)$. Also, it is easy to check that $w = \Omega(x) + \Upsilon(\tilde{f})$, $\forall t \in [0, \lambda T]$, and $x(t, \lambda T, w(\lambda T), f^*) = \Omega(x) + \Upsilon(\tilde{f})$, $\forall t \in (\lambda T, T]$, which yields

$$\mathcal{G}(\lambda, w)(t) = \Omega(x)(t) + \Upsilon(\tilde{f})(t), \quad \forall t \in I.$$

And so, $\mathcal{G}(\lambda, w) \in \mathcal{N}(x)$.

It is clear that

$$\mathcal{G}(0, w) = w^* \;\text{ and }\; \mathcal{G}(1, w) = w \quad \text{for every } w \in \mathcal{N}(x).$$

Below, we shall prove that $\mathcal{G}$ is continuous. For $(\lambda_1, w_1), (\lambda_2, w_2) \in [0,1] \times \mathcal{N}(x)$, with $\lambda_1 \le \lambda_2$, there exists $f_1, f_2 \in \mathrm{Sel}_F(x)$ such that

$$\mathcal{G}(\lambda_1, w_1) = \Omega(x) + \Upsilon(f_1) \;\text{ and }\; \mathcal{G}(\lambda_2, w_2) = \Omega(x) + \Upsilon(f_2).$$

Thus, for $0 \le s \le t \le T$, we obtain

$$\|\mathcal{G}(\lambda_1, w_1)(t) - \mathcal{G}(\lambda_2, w_2)(t)\| \le \|\mathcal{G}(\lambda_1, w_1)(s) - \mathcal{G}(\lambda_2, w_2)(s)| + M\int_s^t \|f_1(\tau) - f_2(\tau)\| \mathrm{d}g(\tau).$$

Since $f_1(t) = f_2(t)$ *for* $t \in [\lambda_2 T, T]$, for all $t \in [\lambda_2 T, T]$, we have

$$\begin{aligned}
&\|\mathcal{G}(\lambda_1, w_1)(t) - \mathcal{G}(\lambda_2, w_2)(t)\| \\
&\quad \le \|\mathcal{G}(\lambda_1, w_1)(\lambda_2 T) - \mathcal{G}(\lambda_2, w_2)(\lambda_2 T)\| + M\int_{\lambda_2 T}^{t} \|f_1(\tau) - f_2(\tau)\| \mathrm{d}g(\tau) \\
&\quad = \|\mathcal{G}(\lambda_1, w_1)(\lambda_2 T) - \mathcal{G}(\lambda_2, w_2)(\lambda_2 T)\| \\
&\quad \le \|\mathcal{G}(\lambda_1, w_1)(\lambda_1 T) - \mathcal{G}(\lambda_2, w_2)(\lambda_1 T)\| + M\int_{\lambda_1 T}^{\lambda_2 T} \|f_1(\tau) - f_2(\tau)\| \mathrm{d}g(\tau).
\end{aligned}$$

For all $t \in [\lambda_1 T, \lambda_2 T]$, we have

$$\begin{aligned}
&\|\mathcal{G}(\lambda_1, w_1)(t) - \mathcal{G}(\lambda_2, w_2)(t)\| \\
&\quad \le \|\mathcal{G}(\lambda_1, w_1)(\lambda_1 T) - \mathcal{G}(\lambda_2, w_2)(\lambda_1 T)\| + M\int_{\lambda_1 T}^{t} \|f_1(\tau) - f_2(\tau)\| \mathrm{d}g(\tau) \\
&\quad \le \|\mathcal{G}(\lambda_1, w_1)(\lambda_1 T) - \mathcal{G}(\lambda_2, w_2)(\lambda_1 T)\| + M\int_{\lambda_1 T}^{\lambda_2 T} \|f_1(\tau) - f_2(\tau)\| \mathrm{d}g(\tau).
\end{aligned}$$

For all $t \in [0, \lambda_1 T]$, we have $\|\mathcal{G}(\lambda_1, w_1)(t) - \mathcal{G}(\lambda_2, w_2)(t)\| \le \|w_1 - w_2\|_\infty$. From the above argument, we have

$$\|\mathcal{G}(\lambda_1, w_1) - \mathcal{G}(\lambda_2, w_2)\|_\infty \le \|w_1 - w_2\|_\infty + 2Mc(1 + \|x\|_\infty)\int_{\lambda_1 T}^{\lambda_2 T} \zeta(\tau)\mathrm{d}g(\tau) \to 0$$

as $(\lambda_2, w_2) \to (\lambda_1, w_1)$. Therefore, $\mathcal{G}$ is continuous.

Finally, by Lemma 2.3, we yield that $\mathcal{N}$ has at least one fixed-point in $\overline{B}_R$, which is a mild solution of the inclusion problem (1.1). This completes the proof. □

In the following, let us define a set $\mathcal{S}_T^F$, the solution set of the inclusion problem (1.1), i.e.,

$$\mathcal{S}_T^F = \{x \in G(I, X) : x \text{ is the mild solution of } (1.1) \text{ and } x(0) = x_0\}.$$

Denote by $\mathcal{F}ix(\mathcal{N})$, the fixed-point set of $\mathcal{N}$ on $\overline{B}_R$, see Theorem 3.1.

Lemma 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Then,* $\mathcal{S}_T^F = \mathcal{F}ix(\mathcal{N})$, *and* $\mathcal{S}_T^F$ *is compact in* $G(I, X)$.

Proof. We first show that $\mathcal{S}_T^F = \mathcal{F}ix(\mathcal{N})$. It is clear that $\mathcal{F}ix(\mathcal{N}) \subset \mathcal{S}_T^F$. Thus, it will be sufficient to prove that for each $x \in \mathcal{S}_T^F$, we have $x \in \overline{B}_R$. Taking $x \in \mathcal{S}_T^F$, there exists $f \in \mathrm{Sel}_F(x)$ such that

$$x(t) = T(t)[x_0 - q(0, x_0)] + q(t, x(t)) + \int_0^t AT(t-s)q(s, x(s))\mathrm{d}s + \int_0^t T(t-s)f(s)\mathrm{d}g(s), \quad \forall t \in I.$$

Then, for $t \in I$, we have

$$\begin{aligned}
\|x(t)\| \le{}& M\|x_0 - q(0, x_0)\| + \|A^{-\beta}\| \, \|A^\beta q(t, x(t))\| + \int_0^t \|A^{1-\beta}T(t-s)\| \, \|A^\beta q(s, x(s))\| ds \\
&+ Mc\int_0^t \zeta(s)(1 + \|x(s)\|)\mathrm{d}g(s)
\end{aligned}$$

$$\leq M\|x_0 - q(0,x_0)\| + \|A^{-\beta}\|(u\|x(t)\| + v) + C_{1-\beta}\int_0^t \frac{u\|x(s)\| + v}{(t-s)^{1-\beta}}\mathrm{d}s + Mc\int_0^t \zeta(s)(1 + \|x(s)\|)\mathrm{d}g(s)$$

$$\leq M\|x_0 - q(0,x_0)\| + u\|A^{-\beta}\| \sup_{0\leq\theta\leq t}\|x(\theta)\| + v\|A^{-\beta}\| + \frac{C_{1-\beta}(u\sup_{0\leq\theta\leq t}\|x(\theta)\| + v)T^{\beta}}{\beta}$$

$$+ Mc\int_0^T \zeta(s)\mathrm{d}g(s) + Mc\int_0^t \zeta(s)\sup_{0\leq\theta\leq s}\|x(\theta)\|\mathrm{d}g(s).$$

The right-hand side is an increasing function in t, so we have the same estimate for all $0 \leq \theta \leq t$. Therefore,

$$\sup_{0\leq\theta\leq t}\|x(\theta)\| \leq M\|x_0 - q(0,x_0)\| + u\|A^{-\beta}\| \sup_{0\leq\theta\leq t}\|x(\theta)\| + v\|A^{-\beta}\| + Mc\int_0^T \zeta(s)\mathrm{d}g(s)$$

$$+ \frac{C_{1-\beta}(u\sup_{0\leq\theta\leq t}\|x(\theta)\| + v)T^{\beta}}{\beta} + Mc\int_0^t \zeta(s)\sup_{0\leq\theta\leq s}\|x(\theta)\|\mathrm{d}g(s).$$

By calculations, we obtain

$$\sup_{0\leq\theta\leq t}\|x(\theta)\| \leq \frac{M\|x_0 - q(0,x_0)\| + v\|A^{-\beta}\| + \frac{C_{1-\beta}vT^{\beta}}{\beta} + Mc\int_0^T\zeta(s)\mathrm{d}g(s)}{1 - \left[u\|A^{-\beta}\| + \frac{C_{1-\beta}uT^{\beta}}{\beta}\right]} + Mc\int_0^t \zeta(s)\sup_{0\leq\theta\leq s}\|x(\theta)\|\mathrm{d}g(s).$$

Since the regularity and dg-integrability of x, $\sup_{0\leq\theta\leq t}\|x(\theta)\|$ are regulated and dg-integrable. Thus, by the generalized Gronwall's inequality of Lemma 2.12, we obtain that for each $t \in I$,

$$\begin{aligned}\sup_{0\leq\theta\leq t}\|x(\theta)\| &\leq \frac{M\|x_0 - q(0,x_0)\| + v\|A^{-\beta}\| + \frac{C_{1-\beta}vT^{\beta}}{\beta} + Mc\int_0^T\zeta(s)\mathrm{d}g(s)}{1 - \left[u\|A^{-\beta}\| + \frac{C_{1-\beta}uT^{\beta}}{\beta}\right]} \\ &\quad\times\left(1 + Mc\int_0^t \varepsilon(t,s)\zeta(s)\mathrm{d}g(s)\right) \\ &\leq \frac{M\|x_0 - q(0,x_0)\| + v\|A^{-\beta}\| + \frac{C_{1-\beta}vT^{\beta}}{\beta} + Mc\int_0^T\zeta(s)\mathrm{d}g(s)}{1 - \left[u\|A^{-\beta}\| + \frac{C_{1-\beta}uT^{\beta}}{\beta}\right]} \\ &\quad\times\left(1 + Mc\int_0^T \varepsilon(T,s)\zeta(s)\mathrm{d}g(s)\right) \\ &\leq R.\end{aligned}$$

And so, we obtain $\|x\|_\infty \leq R$, which yields that $x \in \overline{B}_R$. From the above argument, we obtain $\mathcal{S}_T^F = \mathcal{F}ix(\mathcal{N})$.

Furthermore, by the proof of Theorem 3.1, we have $\overline{B}_R$ is compact in $G(I,X)$ and $\mathcal{N}$ is closed, from this we see that $\mathcal{F}ix(\mathcal{N})$ is a compact set in $\overline{B}_R$, so is $\mathcal{S}_T^F$. This completes the proof. □

The next result for the multivalued mapping F is based on Lemma 3.3 from [33]. Although the original theorem in [33] is formulated for the space of $C([-\tau,0],X)$, the result still holds in $G(I,X)$.

Lemma 3.3. [33] *Suppose that F satisfies the hypotheses (F_1)–(F_3), then there exists a sequence of multivalued mapping $\{F_n\}$ with $\{F_n\} : I \times G(I,X) \to \mathcal{P}_{cl,cv}(X)$ such that*

(1) $F(t,x) \subset \ldots \subset F_{n+1}(t,x) \subset F_n(t,x) \subset \ldots \subset \overline{co}(F(t, B_{\frac{3}{3^n}}(x)))$, $n \geq 1$, *for each* $t \in I$, $x \in G(I,X)$;

(2) $\|F_n(t,x)\| \leq \zeta(t)(3 + \|x\|)$, $n \geq 1$, *for a.e.* $t \in I$ *and each* $x \in G(I,X)$;

(3) *There exists* $E \subset I$ *with* $\mathrm{mes}(E) = 0$ *such that for each* $x^* \in X^*$, $\varepsilon > 0$, *and* $(t, x) \in I \backslash E \times G(I, X)$, *there exists* $N > 0$ *such that for all* $n \in N$

$$x^*(F_n(t, x)) \subset x^*(F(t, x)) + (-\varepsilon, \varepsilon);$$

(4) $F_n(t, \cdot) : G(I, X) \to \mathcal{P}_{cl,cv}(X)$ *is continuous for a.e.* $t \in I$ *with respect to the Hausdorff metric for each* $n \geq 1$;

(5) *For each* $n \geq 1$, *there exists a selection* $g_n : I \times G(I, X) \to X$ of F_n *such that* $g_n(\cdot, x)$ *is strongly measurable for each* $x \in G(I, X)$ *and for any compact subset* $\mathcal{D} \subset G(I, X)$, *there exists constants* $C_V > 0$ *and* $\delta > 0$ *for which the estimate*

$$\|g_n(t, x_1(t)) - g_n(t, x_2(t))\| \leq C_V \zeta(t)\|x_1(t) - x_2(t)\| \tag{3.7}$$

holds for a.e. $t \in I$ *and each* $x_1, x_2 \in G(I, X)$ *with* $V \coloneqq \mathcal{D} + B_\delta(0)$;

(6) F_n *verifies condition* (F_1), (F_2) *with* F_n *instead of* F *for each* $n \geq 1$, *provided that* X *is reflexive.*

The following result is the main result in this section.

Theorem 3.2. *Let the conditions in Theorem* 3.1 *and* (Q_3) *be satisfied. If*

$$\|A^{-\beta}\|r + \frac{2rC_{1-\beta}T^\beta}{\beta} < 1,$$

Then, $\mathcal{S}_T^F$ *is an* R_δ*-set.*

Proof. We shall verify this result in several steps.

Step 1. We first define a multivalue mapping $\overline{F} : I \times G(I, X) \to \mathcal{P}_{cl,cv}(X)$, i.e.,

$$\overline{F}(t, x) = \begin{cases} F(t, x), & t \in I, \ \|x\|_\infty \leq R, \\ F\left(t, \dfrac{Rx}{\|x\|_\infty}\right), & t \in I, \ \|x\|_\infty > R. \end{cases}$$

Next let us define a function $r : G(I, X) \to \overline{B}_R(0)$, $r(x) = \frac{Rx}{\|x\|_\infty}$ for each $\|x\|_\infty > R$ and $r(x) = x$ for each $\|x\|_\infty \leq R$, which is a continuous retraction. Thus, $\overline{F}(t, x) = F(t, r(x))$, $\overline{F}$ and F have the same measurability and continuity. Meanwhile, we have the following inclusion relations:

$$\overline{F}(t, x) = F(t, x) \subset \zeta(t)(1 + R)S,$$

hold for every $t \in I$, $\|x\|_\infty \leq R$, and

$$\overline{F}(t, x) = F\left(t, \frac{Rx}{\|x\|_\infty}\right) \subset \zeta(t)\left(1 + \left\|\frac{Rx}{\|x\|_\infty}\right\|\right)S \subset \zeta(t)(1 + R)S,$$

for every $t \in I$, $\|x\|_\infty > R$. So we have

$$\overline{F}(t, x) \subset \zeta(t)(1 + R)S \equiv \psi(t)S, \ \psi(t) \in L^1(\mathrm{d}g, I, R^+).$$

Now we consider an inclusion problem with a multivalued mapping $\overline{F}$

$$\begin{cases} \mathrm{d}[x(t) + q(t, x(t))] \in Ax(t)\mathrm{d}t + \overline{F}(t, x(t))\mathrm{d}g(t), t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{3.8}$$

Let $\mathcal{S}_T^{\overline{F}}$ be the solution set of problem (3.8). If $x \in \mathcal{S}_T^F$, by Lemma 3.2, we obtain $\|x\|_\infty \leq R$ and then $x \in \mathcal{S}_T^{\overline{F}}$. Of course, the converse is also true. Therefore, we have $\mathcal{S}_T^{\overline{F}} = \mathcal{S}_T^F$. Consequently, we can assume from now on, without loss of generality, that

$$F(t, x) \subset \psi(t)S, \ \psi(t) \in L^1(\mathrm{d}g, I, R^+).$$

Step 2. Now we consider the inclusion problem with the multivalued mapping F_n which is the approximate sequence established in Lemma 3.3.

$$\begin{cases} d[x(t) + q(t, x(t))] \in Ax(t)dt + F_n(t, x(t))dg(t), t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{3.9}$$

Let $\mathcal{S}_T^{F_n}$ denote the solution set of problem (3.9). By Lemma 3.3 (ii) and (vi) and the same argument as in Theorem 3.1 and Lemma 3.2, we say that $\mathcal{S}_T^{F_n}$ is non-empty and compact in $G(I, X)$. In the following, we shall show that each sequence $\{x_n\}$ such that $x_n \in \mathcal{S}_T^{F_n}$ for all $n \geq 1$ is relatively compact.

Since $x_n \in \mathcal{S}_T^{F_n}$, there exists $f_n \in \mathrm{Sel}_{F_n}(x_n)$ such that

$$x_n(t) = T(t)[x_0 - q(0, x_0)] + q(t, x_n(t)) + \int_0^t AT(t - s)q(s, x_n(s))ds + \int_0^t T(t - s)f_n(s)dg(s), \quad \forall t \in I, \ n \geq 1.$$

It is known that for every bounded linear operator $T : X \to X$, we have the property $\beta(TS) \leq \|T|\beta(S)$ for every $S \in B(X)$. From this we have

$$\begin{aligned} \beta(\{q(t, x_n(t))\}_{n\geq1}) &= \beta(\{A^{-\beta}A^{\beta}q(t, x_n(t))\}_{n\geq1}) \\ &\leq \|A^{-\beta}\|\beta(\{A^{\beta}q(t, x_n(t))\}_{n\geq1}) \\ &\leq \|A^{-\beta}\|r \sup_{0\leq s\leq t} \beta(\{x_n(s)\}_{n\geq1}) \\ &\leq \|A^{-\beta}\|r\bar{\rho}(t), \end{aligned}$$

where $\bar{\rho}(t) = \sup_{0\leq s\leq t}\beta(\{x_n(s)\}_{n\geq1})$. We have

$$\begin{aligned} \beta\left(\left\{\int_0^t AT(t - s)q(s, x_n(s))ds\right\}_{n\geq1}\right) &\leq 2\int_0^t \beta(\{AT(t - s)q(s, x_n(s))\}_{n\geq1})ds \\ &\leq 2\int_0^t \|A^{1-\beta}T(t - s)\|\beta(\{A^{\beta}q(s, x_n(s))\}_{n\geq1})ds \\ &\leq 2C_{1-\beta}\int_0^t \frac{r\bar{\rho}(s)}{(t - s)^{1-\beta}}ds. \end{aligned}$$

For any $m \geq 1$, we obtain

$$\begin{aligned} \beta(\{f_n(s)\}_{n\geq1}) &= \beta(\{f_n(s)\}_{n\geq m}) \\ &\leq \beta(F(\{s\}, B(\{x_n\}_{n\geq m}, 3^{1-m}))) \\ &\leq \theta(s) \sup_{0\leq\sigma\leq s} \beta(B(\{x_n(s)\}_{n\geq m}, 3^{1-m})) \\ &\leq \theta(s)(\bar{\rho}(s) + 3^{1-m}). \end{aligned}$$

Therefore, by Lemma 2.7

$$\begin{aligned} \beta\left(\int_0^t T(t - s)f_n(s)dg(s)\right) &\leq 4\int_0^t \beta(\{T(t - s)f_n(s)\}_{n\geq1})dg(s) \\ &\leq 4\int_0^t \|T(t - s)\|\beta(\{f_n(s)\}_{n\geq1})dg(s) \\ &\leq 4M\int_0^t \beta(\{f_n(s)\}_{n\geq1})dg(s) \\ &\leq 4M\int_0^t \theta(s)(\bar{\rho}(s) + 3^{1-m})dg(s). \end{aligned}$$

Now

$$\beta(\{x_n(t)\}_{n\geq 1}) = \beta\left(\left\{T(t)[x_0 - q(0,x_0)] + q(t,x_n(t)) + \int_0^t AT(t-s)q(s,x_n(s))\mathrm{d}s + \int_0^t T(t-s)f_n(s)\mathrm{d}g(s)\right\}_{n\geq 1}\right)$$
$$\leq \|A^{-\beta}\|r\bar{\rho}(t) + 2rC_{1-\beta}\int_0^t \frac{\bar{\rho}(s)}{(t-s)^{1-\beta}}\mathrm{d}s + 4M\int_0^t \theta(s)(\bar{\rho}(s) + 3^{1-m})\mathrm{d}g(s).$$

Let $m \to \infty$, we obtain

$$\bar{\rho}(t) \leq \|A^{-\beta}\|r\bar{\rho}(t) + 2rC_{1-\beta}\int_0^t \frac{\bar{\rho}(s)}{(t-s)^{1-\beta}}\mathrm{d}s + 4M\int_0^t \theta(s)\bar{\rho}(s)\mathrm{d}g(s)$$
$$\leq \|A^{-\beta}\|r\bar{\rho}(t) + \frac{2rC_{1-\beta}T^{\beta}}{\beta}\bar{\rho}(t) + 4M\int_0^t \theta(s)\bar{\rho}(s)\mathrm{d}g(s).$$

Then, we have

$$\bar{\rho}(t) \leq \frac{4M}{1 - \left[\|A^{-\beta}\|r + \frac{2rC_{1-\beta}T^{\beta}}{\beta}\right]}\int_0^t \theta(s)\bar{\rho}(s)\mathrm{d}g(s).$$

By Lemma 2.12, we obtain $\bar{\rho}(t) = 0,\ \forall t \in I$ and, as a consequence, $\beta(\{x_n(t)\}_{n\geq 1}) = 0,\ \forall t \in I$. This also implies that $\beta(\{f_n(t)\}_{n\geq 1}) = 0,\ \forall t \in I$. From this we have that for each $t \in I$, $\{x_n(t)\}_{n\geq 1}$ and $\{f_n(t)\}_{n\geq 1}$ are relatively compact.

Due to Lemma 2.9, it suffices to prove that $\{x_n\}_{n\geq 1}$ is equiregulated. To see this, fixed $t \in I$, let us show that for every $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ such that for all $h < \delta_\varepsilon$,

$$\|x_n(t+h) - x_n(t) - (m_{f_n}(t+0) - m_{f_n}(t))\| \leq \varepsilon, \tag{3.10}$$
$$\|x_n(t) - x_n(t-h) - (m_{f_n}(t) - m_{f_n}(t-0))\| \leq \varepsilon \tag{3.11}$$

for all $n \in N$, where $m_{f_n}(t) = \int_0^t f_n(s)\mathrm{d}g(s)$.

From the arguments above, we have $\|f_n(t)\| \leq \psi(t)c,\ \forall n \in N$. When $t < T$,

$$\|x_n(t+h) - x_n(t) - (m_{f_n}(t+0) - m_{f_n}(t))\|$$
$$\leq \Bigg\|(T(h)-I)\left(T(t)[x_0 - q(0,x_0)] + \int_0^t AT(t-s)q(s,x_n(s))\mathrm{d}s + \int_0^t T(t-s)f_n(s)\mathrm{d}g(s)\right)$$
$$+ \int_t^{t+h} \chi_{(t,t+h]}(s)T(t+h-s)f_n(s)\mathrm{d}g(s) + q(t+h, x_n(t+h)) - q(t,x_n(t))$$
$$+ \int_t^{t+h} AT(t-s)q(s,x_n(s))\mathrm{d}s + T(h)f_n(t)(g(t+0) - g(t)) - (m_{f_n}(t+0) - m_{f_n}(t))\Bigg\|$$
$$\leq \left\|(T(h)-I)\left(T(t)[x_0 - q(0,x_0)] + \int_0^t AT(t-s)q(s,x_n(s))\mathrm{d}s + \int_0^t T(t-s)f_n(s)\mathrm{d}g(s) + m_{f_n}(t+0) - m_{f_n}(t)\right)\right\|$$
$$+ \|q(t+h,x_n(t+h)) - q(t,x_n(t))\| + \left\|\int_t^{t+h} AT(t-s)q(s,x_n(s))\mathrm{d}s\right\| + M\mathrm{var}(m_{f_n}, (t,t+h])$$
$$\leq \left\|(T(h)-I)\left(T(t)[x_0 - q(0,x_0)] + \int_0^t AT(t-s)q(s,x_n(s))\mathrm{d}s + \int_0^t T(t-s)f_n(s)\mathrm{d}g(s) + m_{f_n}(t+0) - m_{f_n}(t)\right)\right\|$$
$$+ \|q(t+h,x_n(t+h)) - q(t,x_n(t))\| + \left\|\int_t^{t+h} AT(t-s)q(s,x_n(s))\mathrm{d}s\right\|$$
$$+ M\mathrm{var}(m_{\psi c}, (t,t+h]) = I_1 + I_2 + I_3 + I_4,$$

where $m_{\psi c}(t) = \int_0^t \psi(s) c \mathrm{d}g(s)$.

From the arguments above, we have $\{\int_0^t AT(t-s)q(s, x_n(s))\mathrm{d}s\}_{n\geq 1}$, $\{\int_0^t T(t-s)f_n(s)\mathrm{d}g(s)\}_{n\geq 1}$ are relatively compact, while $m_{fn}(t+0) - m_{f_n}(t) = f_n(t)(g(t+0) - g(t))$, $\{m_{f_n}(t+0) - m_{f_n}(t) : f_n(t) \in K,\ \forall t \in I,\ n \in N\} \subset \psi(t)(g(t+0) - g(t))S$ and so, the set $\{T(t)[x_0 - q(0, x_0)] + \int_0^t AT(t-s)q(s, x_n(s))\mathrm{d}s + \int_0^t T(t-s)f_n(s)\mathrm{d}g(s) + m_{f_n}(t+0) - m_{f_n}(t) : f_n(t) \in \psi(t)S,\ \forall t \in I,\ n \in N\}$ is relatively compact. By the uniform continuity of C_0-semigroup $T(t)$, $(t > 0)$ and applying [40, Corollary 1], let $h \to 0$, we obtain $I_1 \to 0$ independent of x_n. As $m_{\psi c}$ is a BV function, applying [38, Lemma 2.1], let $h \to 0$, we obtain $I_4 \to 0$. By the condition $(Q_1)_T$, $(Q_2)_T$, $I = A^{-\beta}A^{\beta}$, and $AT(t-s)q(s, x_n(s)) = (A^{1-\beta}T(t-s))(A^{\beta}q(s, x_n(s)))$, we obtain $I_2, I_3 \to 0$. And so, we have inequality (3.10).

For inequality (3.11), when $t > 0$, From the arguments above, there exists $\delta > 0$ such that

$$\mathrm{var}(m_{\psi c}, [t-\delta, t)) < \frac{\varepsilon}{16M^2}. \tag{3.12}$$

Therefore,

$$\begin{aligned}
&\|x_n(t) - x_n(t-h) - (m_{f_n}(t) - m_{f_n}(t-0))\| \\
&\quad\leq \|x_n(t) - T(h)x_n(t-h) - (m_{f_n}(t) - m_{f_n}(t-0))\| + \|T(h)x_n(t-h) - x_n(t-h)\| \\
&\quad\leq \left\| \int_{t-h}^{t} AT(t-s)q(s, x_n(s))\mathrm{d}s \right\| + \left\| \int_{t-h}^{t} \chi_{[t-h,t)}(s)T(t-s)f_n(s)\mathrm{d}g(s) \right\| \\
&\qquad + \|q(t, x_n(t)) - q(t-h, x_n(t-h))\| + \left\| (T(\delta) - T(\delta-h)) \int_0^{t-\delta} T(t-\delta-s)f_n(s)\mathrm{d}g(s) \right\| \\
&\qquad + \left\| (T(h) - I) \int_{t-\delta}^{t-h} \chi_{[t-\delta,t-h)}(s)T(t-h-s)f_n(s)\mathrm{d}g(s) \right\| \\
&\qquad + \|(T(h) - I)(m_{f_n}(t-h) - m_{f_n}(t-h-0))\| + \|(T(h) - I)[x_0 - q(0, x_0)]\| \\
&\qquad + \left\| (T(h) - I) \int_0^{t-h} AT(t-s)q(s, x_n(s))\mathrm{d}s \right\| \\
&\quad\leq \left\| \int_{t-h}^{t} AT(t-s)q(s, x_n(s))\mathrm{d}s \right\| + M\mathrm{var}(m_{f_n}, [t-h, t)) + \|q(t, x_n(t)) - q(t-h, x_n(t-h))\| \\
&\qquad + M\|T(\delta) - T(\delta-h)\|\mathrm{var}(m_{f_n}, [0,1]) + 2M^2\mathrm{var}(m_{f_n}, [t-\delta, t)) + \|(T(h) - I)[x_0 - q(0, x_0)]\| \\
&\qquad + \|(T(h) - I)(m_{f_n}(t-h) - m_{f_n}(t-h-0))\| + \left\| (T(h) - I) \int_0^{t-h} AT(t-s)q(s, x_n(s))\mathrm{d}s \right\| \\
&\quad\leq \left\| \int_{t-h}^{t} AT(t-s)q(s, x_n(s))\mathrm{d}s \right\| + M\mathrm{var}(m_{\psi c}, [t-h, t)) + \|q(t, x_n(t)) - q(t-h, x_n(t-h))\| \\
&\qquad + M\|T(\delta) - T(\delta-h)\|\mathrm{var}(m_{\psi c}, [0,1]) + 2M^2\mathrm{var}(m_{\psi c}, [t-\delta, t)) + \|(T(h) - I)[x_0 - q(0, x_0)]\| \\
&\qquad + \|(T(h) - I)(m_{f_n}(t-h) - m_{f_n}(t-h-0))\| + \left\| (T(h) - I) \int_0^{t-h} AT(t-s)q(s, x_n(s))\mathrm{d}s \right\| \\
&\quad= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8,
\end{aligned}$$

for any $h \in (0, \delta)$. From the above argument, there exists $\delta_\varepsilon^1 < \delta$, for $h < \delta_\varepsilon^1$, we obtain $I_1, I_3, I_4, I_6, I_8 \leq \frac{\varepsilon}{8}$.

Meanwhile, since

$$\|m_{f_n}(t_2) - m_{f_n}(t_1)\| \leq \int_{t_1}^{t_2} \psi(s) c \mathrm{d}g(s),\ \ \forall f_n(t) \in \psi(t)S,\ \ \forall t \in I,\ \ n \in N,$$

the set $\{m_{f_n} : f_n(t) \in \psi(t)S,\ \forall t \in I,\ n \in N\}$ is equiregulated. And for each $t \in I, \{m_{f_n}(t) : f_n(t) \in \psi(t)S,$ $\forall\ t \in I,\ n \in N\}$ is relatively compact. By Lemma 2.9, the set

$$\bigcup_{n\geq 1}\{m_{f_n} : f_n(t) \in \psi(t)S,\ \forall t \in I,\ n \in N\}$$

is relatively compact. It means that there exists $\delta_\varepsilon^2 < \delta$, for $h < \delta_\varepsilon^2$, we obtain $I_7 \leq \frac{\varepsilon}{8}$. By formula (3.12), we obtain $I_5 \leq \frac{\varepsilon}{8}$, $I_2 \leq \frac{\varepsilon}{16M}$. And so, we have inequality (3.11).

By applying Lemma 2.9, we obtain the relative compactness of set $\{x_n\}_{n\geq 1}$. Therefore, it has a subsequence (not relabelled) uniformly convergent to a function $x \in G(I, X)$.

Step 3. Next we show that $x \in \mathcal{S}_T^F$.

By Lemma 3.3 (i) and Step 1, let $\Gamma(t) = \psi(t)S \subset G(I, X)$, then for every $t \in I$, it is a compact convex set, and is dg-measurable, such that

$$F_n(t, x_n(t)) \subset \Gamma(t),\ \ \forall n \geq 1.$$

Now performing an argument similar to that of the proof of the closeness of $\mathcal{N}$ in Theorem 3.1, we can obtain a sequence $\tilde{f}_n \in co\{f_k : k \geq n\}$ and it is dg – a.e. convergent on I, to a dg-measurable function f.

Since $\|T(t-s)\tilde{f}_n(s)\| \leq Mc\psi(s)$ and dominated convergence theorem, we obtain

$$\int_0^t T(t-s)\tilde{f}_n(s)\mathrm{d}g(s) \to \int_0^t T(t-s)f(s)\mathrm{d}g(s),\ \ \forall t \in I.$$

Since $B_{\frac{3}{3^n}}(x_n) \to x,\ n \to \infty$, there exists N such that $B_{\frac{3}{3^n}}(x_n) \subset B_{\frac{3}{3^m}}(x),\ n > N$. Note that for every $n \in N$, $f_n(s) \in F_n(s, x_n(s))$ and $\tilde{f}_n \in co\{f_k : k \geq n\}$, by Lemma 3.3 (i), it comes that

$$\tilde{f}_n(s) \subset co\left(\bigcup_{k\geq n} F_k(t, x_k(s))\right) \subset \bigcup_{k\geq n}\overline{co}\left(F\left(s, B_{\frac{3}{3^k}}(x_k(s))\right)\right) \subset \overline{co}\left(F\left(s, B_{\frac{3}{3^m}}(x(s))\right)\right)$$

for $n > N$. Hence, $f(s) = \lim_{n\to\infty}\tilde{f}_n(s) \in \overline{co}\left(F\left(s, B_{\frac{3}{3^m}}(x(s))\right)\right)$, then let $m \to \infty$, we obtain $f(s) \in \overline{co}(F(s, x(s))) = F(s, x(s))$. Using the same argument as the proof of the closeness of $\mathcal{N}$ in Theorem 3.1, we can obtain that for each $t \in I$,

$$x(t) = T(t)[x_0 - q(0, x_0)] + q(t, x(t)) + \int_0^t AT(t-s)q(s, x(s))\mathrm{d}s + \int_0^t T(t-s)f(s)\mathrm{d}g(s).$$

Therefore, we have $x \in \mathcal{S}_T^F$.

Step 4. From Step 3 it follows that $\sup\{d(x, \mathcal{S}_T^F) : x \in \mathcal{S}_T^{F_n}\} \to 0,\ n \to \infty$. Thus, we have $\sup\{d(x, \mathcal{S}_T^F) : x \in \overline{\mathcal{S}_T^{F_n}}\} \to 0,\ n \to \infty$. Hence, since $\mathcal{S}_T^F$ is compact, $\mathcal{S}_T^{F_{n+1}} \subset \mathcal{S}_T^{F_n}$, and $\beta(\mathcal{S}_T^{F_n}) = \beta(\overline{\mathcal{S}_T^{F_n}}) \to 0,\ n \to \infty$, we obtain $\mathcal{S}_T^F = \bigcap_{n=1}^\infty \mathcal{S}_T^{F_n}$.

Step 5. Finally, in order to show that $\mathcal{S}_T^F$ is a R_δ-set, it is sufficient to verify that $\mathcal{S}_T^{F_n}$ is contractible for each $n \geq 1$. By Lemma 3.3 (v), we obtain that g_n is the selection of F_n and $g_n(t, \cdot)$ is continuous for $t \in I$. Since $\mathcal{S}_T^{F_n}$ is compact, we have $\mathcal{D}_n = \{x(t) : t \in I, x \in \mathcal{S}_T^{F_n}\}$ is a relatively compact set in X. Thus, again by Lemma 3.3 (v), there exists a neighborhood V of $\overline{\mathcal{D}_n}$, satisfying formula (3.7). Furthermore, it is clear that g_n satisfies

$$\|g_n(t, x(t))\| \leq \zeta(t)(3 + \|x(t)\|) \tag{3.13}$$

for $t \in I$ and each $x \in G(I, X)$. Now, performing a trivial variant of an argument from Theorem 3.1, we obtain the existence of mild solutions of the inclusion problem of the form

$$\begin{cases} d[x(t) + q(t, x(t))] = Ax(t)\mathrm{d}t + g_n(t, x(t))\mathrm{d}g(t), t \in [s, T], \\ x(s) = w(s), \end{cases} \tag{3.14}$$

for each $s \in I$ and $w \in G(I, X)$.

Moreover, we point out that the mild solution to (3.14) is unique. In fact, if x_1 and x_2 are two solutions of (3.14), then there exists a neighborhood V' related to x_1, x_2 such that

$$\begin{aligned}&\sup_{\theta\in[s,t]}\|x_1(\theta)-x_2(\theta)\|\\ &\le L_q\left(\|A^{-\beta}\|+\frac{C_{1-\beta}T^{\beta}}{\beta}\right)\sup_{\theta\in[s,t]}\|x_1(\theta)-x_2(\theta)\|+M\int_s^t\zeta(\tau)\|g_n(\tau,x_1(\tau))-g_n(\tau,x_2(\tau))\|\mathrm{d}g(\tau)\\ &\le L_q\left(\|A^{-\beta}\|+\frac{C_{1-\beta}T^{\beta}}{\beta}\right)\sup_{\theta\in[s,t]}\|x_1(\theta)-x_2(\theta)\|+MC_{V'}\int_s^t\zeta(\tau)\sup_{\theta\in[s,\tau]}\|x_1(\theta)-x_2(\theta)\|\mathrm{d}g(\tau).\end{aligned}$$

By calculations, we obtain

$$\sup_{\theta\in[s,t]}\|x_1(\theta)-x_2(\theta)\|\le\frac{MC_{V'}}{1-L_q\left(\|A^{-\beta}\|+\frac{C_{1-\beta}T^{\beta}}{\beta}\right)}\int_s^t\zeta(\tau)\sup_{\theta\in[s,\tau]}\|x_1(\theta)-x_2(\theta)\|\mathrm{d}g(\tau),$$

for every $t\in[s,T]$. Therefore, by Lemma 2.12 we see that $x_1\equiv x_2$.

We denote by $x(\cdot,s,w)$, the unique mild solution of the inclusion problem (3.14) corresponding to $s\in I$ and $w\in G(I,X)$. Define a function $\tilde{\mathcal{H}}:I\times\mathcal{S}_T^{F_n}\to\mathcal{S}_T^{F_n}$ by setting

$$\tilde{\mathcal{H}}(\lambda,w)(t)=\begin{cases}w(t), & t\in[0,\lambda T],\\ x(t,\lambda T,w(\lambda T)), & t\in(\lambda T,T],\end{cases}$$

for each $(\lambda,w)\in I\times\mathcal{S}_T^{F_n}$. Using the same argument as the proof in Theorem 3.1, we can obtain that $\tilde{\mathcal{H}}(\lambda,w)\in\mathcal{S}_T^{F_n}$ for each $(\lambda,w)\in I\times\mathcal{S}_T^{F_n}$, and

$$\tilde{\mathcal{H}}(0,w)=x(\cdot,0,x_0)\ \text{ and }\ \tilde{\mathcal{H}}(1,w)=w,\ \text{ for each }\ w\in\mathcal{S}_T^{F_n}.$$

Next we show that $\tilde{\mathcal{H}}$ is continuous. Let a sequence $\{(\lambda_m,w_m)\}_{m\ge1}\subset I\times\mathcal{S}_T^{F_n}$ satisfy $(\lambda_m,w_m)\to(\lambda,w),\ k\to\infty$, we shall show that

$$\|\tilde{\mathcal{H}}(\lambda_m,w_m)-\tilde{\mathcal{H}}(\lambda,w)\|_\infty\to0,\ m\to\infty.$$

Without loss of generality, we provided that $\lambda_m\le\lambda$. For simplicity in presentation, we denote $\eta_m=\tilde{\mathcal{H}}(\lambda_m,w_m),\ m\ge1$, and $\eta=\tilde{\mathcal{H}}(\lambda,w)$.

Since $\eta_m\in\mathcal{S}_T^{F_n}$, there exists $f_m^n\in\mathrm{Sel}_{F_n}(\eta_m)$ and $f^n\in\mathrm{Sel}_{F_n}(\eta)$ such that

$$\eta_m(t)=T(t)[x_0-q(0,x_0)]+q(t,\eta_m(t))+\int_0^t AT(t-s)q(s,\eta_m(s))\mathrm{d}s+\int_0^t T(t-s)f_m^n(s)\mathrm{d}g(s),\ \ \forall t\in I,$$

$$\eta(t)=T(t)[x_0-q(0,x_0)]+q(t,\eta(t))+\int_0^t AT(t-s)q(s,\eta(s))\mathrm{d}s+\int_0^t T(t-s)f^n(s)\mathrm{d}g(s),\ \ \forall t\in I,$$

respectively. Then, we have that for $0\le s\le t\le T$,

$$\begin{aligned}\|\eta_m(t)-\eta(t)\|\le{}&\|\eta_m(s)-\eta(s)\|+\|q(t,\eta_m(t))-q(t,\eta(t))\|+\|q(s,\eta_m(s))-q(s,\eta(s))\|\\ &+\left\|\int_0^s A(T(t-\tau)-T(s-\tau))(q(\tau,\eta_m(\tau))-q(\tau,\eta(\tau)))\mathrm{d}\tau\right\|\\ &+\left\|\int_0^s (T(t-\tau)-T(s-\tau))(f_m^n(\tau)-f^n(\tau))\mathrm{d}g(\tau)\right\|\\ &+\left\|\int_s^t AT(t-\tau)(q(\tau,\eta_m(\tau))-q(\tau,\eta(\tau)))\mathrm{d}\tau\right\|\\ &+\left\|\int_s^t T(t-\tau)(f_m^n(\tau)-f^n(\tau))\mathrm{d}g(\tau)\right\|.\end{aligned}$$

From Lemma (3.3) (v) it follows that for each $t \in [\lambda T, T]$,

$$\begin{aligned}\|\eta_m(t) - \eta(t)\| \le{}& \|\eta_m(\lambda T) - \eta(\lambda T)\| + \|q(t, \eta_m(t)) - q(t, \eta(t))\| + \|q(\lambda T, \eta_m(\lambda T)) - q(\lambda T, \eta(\lambda T))\| \\
&+ \left\| \int_0^{\lambda T} A(T(t-\tau) - T(\lambda T - \tau))(q(\tau, \eta_m(\tau)) - q(\tau, \eta(\tau)))\mathrm{d}\tau \right\| \\
&+ \left\| \int_0^{\lambda T} (T(t-\tau) - T(\lambda T - \tau))(g_n(\tau, \eta_m(\tau)) - g_n(\tau, \eta(\tau)))\mathrm{d}g(\tau) \right\| \\
&+ \left\| \int_{\lambda T}^{t} AT(t-\tau)(q(\tau, \eta_m(\tau)) - q(\tau, \eta(\tau)))\mathrm{d}\tau \right\| \\
&+ \left\| \int_{\lambda T}^{t} T(t-\tau)(g_n(\tau, \eta_m(\tau)) - g_n(\tau, \eta(\tau)))\mathrm{d}g(\tau) \right\| \\
\le{}& (1 + L_q\|A^{-\beta}\|)\|\eta_m(\lambda T) - \eta(\lambda T)\| + L_q\|A^{-\beta}\|\|\eta_m(t) - \eta(t)\| \\
&+ 2ML_q\|A^{1-\beta}\| \int_0^{\lambda T} \|\eta_m(\tau) - \eta(\tau)\|\mathrm{d}\tau + 2M \int_0^{\lambda T} \|g_n(\tau, \eta_m(\tau)) - g_n(\tau, \eta(\tau))\|\mathrm{d}g(\tau) \\
&+ L_q C_{1-\beta} \int_{\lambda T}^{t} \frac{\|\eta_m(\tau) - \eta(\tau)\|}{(t-\tau)^{1-\beta}}\mathrm{d}\tau + MC_V \int_{\lambda T}^{t} \zeta(\tau)\|\eta_m(\tau) - \eta(\tau)\|\mathrm{d}g(\tau).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\sup_{\theta \in [\lambda T, t]} \|\eta_m(\theta) - \eta(\theta)\| \\
&\le \frac{1 + L_q\|A^{-\beta}\|}{1 - \left[L_q\|A^{-\beta}\| + \frac{L_q C_{1-\beta} T^\beta}{\beta}\right]} \|\eta_m(\lambda T) - \eta(\lambda T)\| \\
&+ \frac{1}{1 - \left[L_q\|A^{-\beta}\| + \frac{L_q C_{1-\beta} T^\beta}{\beta}\right]} \left(2ML_q\|A^{1-\beta}\| \int_0^{\lambda T} \|\eta_m(\tau) - \eta(\tau)\|\mathrm{d}\tau + 2M \int_0^{\lambda T} \|g_n(\tau, \eta_m(\tau)) - g_n(\tau, \eta(\tau))\|\mathrm{d}g(\tau)\right) \\
&+ \frac{MC_V}{1 - \left[L_q\|A^{-\beta}\| + \frac{L_q C_{1-\beta} T^\beta}{\beta}\right]} \int_{\lambda T}^{t} \zeta(\tau) \left(\sup_{\theta \in [\lambda T, \tau]} \|\eta_m(\theta) - \eta(\theta)\| \right) \mathrm{d}g(\tau).\end{aligned}$$

Then, by Lemma 2.12, we yield

$$\begin{aligned}&\sup_{\theta \in [\lambda T, t]} \|\eta_m(\theta) - \eta(\theta)\| \\
&\le \left(\frac{1 + L_q\|A^{-\beta}\|}{1 - \left[L_q\|A^{-\beta}\| + \frac{L_q C_{1-\beta} T^\beta}{\beta}\right]} \|\eta_m(\lambda T) - \eta(\lambda T)\| + \frac{2ML_q\|A^{1-\beta}\|}{1 - \left[L_q\|A^{-\beta}\| + \frac{L_q C_{1-\beta} T^\beta}{\beta}\right]} \int_0^{\lambda T} \|\eta_m(\tau) - \eta(\tau)\|\mathrm{d}\tau \right. \\
&\quad \left. + \frac{2M}{1 - \left[L_q\|A^{-\beta}\| + \frac{L_q C_{1-\beta} T^\beta}{\beta}\right]} \int_0^{\lambda T} \|g_n(\tau, \eta_m(\tau)) - g_n(\tau, \eta(\tau))\|\mathrm{d}g(\tau) \right) \\
&\quad \times \left(1 + \frac{MC_V}{1 - \left[L_q\|A^{-\beta}\| + \frac{L_q C_{1-\beta} T^\beta}{\beta}\right]} \int_{\lambda T}^{t} \zeta(\tau)\varepsilon(t, \tau)\mathrm{d}g(\tau) \right)\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{1+L_q\|A^{-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\|\eta_m(\lambda T)-\eta(\lambda T)\|\right.\\
&\quad+\frac{2ML_q\|A^{1-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\left(\int_0^{\lambda_m T}\|w_m(\tau)-w(\tau)\|\mathrm{d}\tau+\int_{\lambda_m T}^{\lambda T}\|\eta_m(\tau)-\eta(\tau)\|\mathrm{d}\tau\right)\\
&\quad\left.+\frac{2MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\left(\int_0^{\lambda_m T}\|\zeta(\tau)\|\|w_m(\tau)-w(\tau)\|\mathrm{d}g(\tau)+\int_{\lambda_m T}^{\lambda T}\|\zeta(\tau)\|\|\eta_m(\tau)-\eta(\tau)\|\mathrm{d}g(\tau)\right)\right]\\
&\quad\times\left(1+\frac{MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int_{\lambda T}^{t}\zeta(\tau)\varepsilon(t,\tau)\mathrm{d}g(\tau)\right)
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&\leq \left[\frac{1+L_q\|A^{-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\|\eta_m(\lambda T)-\eta(\lambda T)\|\right.\\
&\quad+\frac{2ML_q\|A^{1-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\left(\int_0^{\lambda_m T}\|w_m(\tau)-w(\tau)\|\mathrm{d}\tau+\int_{\lambda_m T}^{\lambda T}\|\eta_m(\tau)-\eta(\tau)\|\mathrm{d}\tau\right)\\
&\quad\left.+\frac{2MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\left(\int_0^{\lambda_m T}\|\zeta(\tau)\|\|w_m(\tau)-w(\tau)\|\mathrm{d}g(\tau)+\int_{\lambda_m T}^{\lambda T}\|\zeta(\tau)\|\|\eta_m(\tau)-\eta(\tau)\|\mathrm{d}g(\tau)\right)\right]\\
&\quad\times\left(1+\frac{MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int_{\lambda T}^{t}\zeta(\tau)\varepsilon(t,\tau)\mathrm{d}g(\tau)\right)
\end{aligned}$$

for $t\in[\lambda T,T]$. Also, we note that $\mathcal{S}_T^{F_n}$ is compact, we can find a constant $\tilde{M}>0$ satisfying $\|x\|_\infty\leq\tilde{M}$ hold for all $x\in\mathcal{S}_T^{F_n}$, by (3.13), we imply that for every $t\in[\lambda_m T,\lambda T]$,

$$\begin{aligned}
&\sup_{\theta\in[\lambda_m T,t]}\|\eta_m(\theta)-\eta(\theta)\|\\
&\leq\frac{1+L_q\|A^{-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\|w_m(\lambda_m T)-w(\lambda_m T)\|\\
&\quad+\frac{2ML_q\|A^{1-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int_0^{\lambda_m T}\|w_m(\tau)-w(\tau)\|\mathrm{d}\tau\\
&\quad+\frac{2MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int_0^{\lambda_m T}\|\zeta(\tau)\|\|w_m(\tau)-w(\tau)\|\mathrm{d}g(\tau)\\
&\quad+\frac{M(6+2\tilde{M})}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int_{\lambda_m T}^{t}\zeta(\tau)\mathrm{d}g(\tau).
\end{aligned}\tag{3.16}$$

It is clear that

$$\|\eta_m(t)-\eta(t)\|=\|w_m(t)-w(t)\|,\quad\text{for } t\in[0,\lambda_m T].\tag{3.17}$$

By formula (3.16), we yield

$$\begin{aligned}\|\eta_m(\lambda T)-\eta(\lambda T)\|\le&\left(\frac{1+L_q\|A^{-\beta}\|+2\lambda_m ML_qT\|A^{1-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}+\frac{2MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int\limits_0^{\lambda_m T}\|\zeta(\tau)\|\mathrm{d}g(\tau)\right)\\ &\|w_m-w\|_\infty+\frac{M(6+2\tilde{M})}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int\limits_{\lambda_m T}^{\lambda T}\zeta(\tau)\mathrm{d}g(\tau).\end{aligned}\tag{3.18}$$

By (3.15)–(3.18), we have

$$\begin{aligned}\|\eta_m(t)-\eta(t)\|\le&\left(1+\frac{1+L_q\|A^{-\beta}\|+2\lambda_m ML_qT\|A^{1-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}+\frac{2MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int\limits_0^{\lambda_m T}\|\zeta(\tau)\|\mathrm{d}g(\tau)\right)\|w_m-w\|_\infty\\ &+\frac{M(6+2\tilde{M})}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int\limits_{\lambda_m T}^{\lambda T}\zeta(\tau)\mathrm{d}g(\tau)+\left\{\frac{1+L_q\|A^{-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\right.\\ &\times\left[\left(\frac{1+L_q\|A^{-\beta}\|+2\lambda_m ML_qT\|A^{1-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}+\frac{2MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int\limits_0^{\lambda_m T}\|\zeta(\tau)\|\mathrm{d}g(\tau)\right)\|w_m-w\|_\infty\right.\\ &\left.+\frac{M(6+2\tilde{M})}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int\limits_{\lambda_m T}^{\lambda T}\zeta(\tau)\mathrm{d}g(\tau)\right]+\frac{2ML_q\|A^{1-\beta}\|}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\\ &\times\left(\lambda_m T\|w_m-w\|_\infty+\int\limits_{\lambda_m T}^{\lambda T}\|\eta_m(\tau)-\eta(\tau)\|\mathrm{d}\tau\right)\\ &\left.+\frac{2MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\left(\|w_m-w\|_\infty\int\limits_0^{\lambda_m T}\|\zeta(\tau)\|\mathrm{d}g(\tau)+\int\limits_{\lambda_m T}^{\lambda T}\|\zeta(\tau)\|\|\eta_m(\tau)-\eta(\tau)\|\mathrm{d}g(\tau)\right)\right\}\\ &\times\left(1+\frac{MC_V}{1-\left[L_q\|A^{-\beta}\|+\frac{L_qC_{1-\beta}T^{\beta}}{\beta}\right]}\int\limits_{\lambda T}^{T}\zeta(\tau)\varepsilon(t,\tau)\mathrm{d}g(\tau)\right),\end{aligned}$$

for every $t\in I$. Now, we can prove that

$$\|\eta_m-\eta\|_\infty\to 0,\ \ m\to\infty.$$

Therefore, we conclude that $\mathcal{S}_T^{F_n}$ is contractible, and thus $\mathcal{S}_T^F$ is an R_δ-set. This proof is complete. □

In fact, the following two kinds of classical evolution inclusions are special cases of neutral measure evolution inclusions studied by the authors of this study.

Corollary 3.1. *When g is an absolutely continuous function, system* (1.1) *is transformed into the following neutral evolution inclusion Cauchy problem*:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}[x(t)-q(t,x(t))]\in Ax(t)\mathrm{d}t+G(t,x(t)),\ \ t\in[0,T],\\ x(0)=x_0,\end{cases}\tag{3.19}$$

where $G(t,x(t)) = F(t,x(t))g'(t)$. If the assumption of Theorem 3.2 *hold, then the solution set of the inclusion problem* (3.19) *is a R_δ-set, and the R_δ-type structure of the solution set of such systems was studied by Zhou* [37] *in 2016*.

Corollary 3.2. *When $g = t + \sum_i \chi_{\{t_i\}}$, $t_i \in I$, $i = 1, 2, \dots, p$, and the multivalued mapping $F(t,x)$ has the following structure, i.e.,*

$$F(t,x) = \begin{cases} G(t,x), & t \neq t_i, \; i = 1, 2, \dots, p, \\ I_i(x(t)), & t = t_i, \; i = 1, 2, \dots, p. \end{cases}$$

It is clear that g is non-decreasing, but discontinuous, and discontinuous set is $\{t_i, \; i = 1, 2, \dots, p\}$. Meanwhile, since the discontinuous set $\{t_i, \; i = 1, 2, \dots, p\}$ is finite, the continuity properties of G and F are essentially the same. Therefore, system (1.1) is transformed into the following neutral impulsive evolution inclusion:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}[x(t) - q(t,x(t))] \in Ax(t)\mathrm{d}t + G(t,x(t)), \; t \in [0,T], \; t \neq t_i, \; i = 1, 2, \dots, p, \\ x(t_i^+) = x(t_i) + I_i(x(t_i)), \; i = 1, 2, \dots, p, \\ x(0) = x_0. \end{cases} \tag{3.20}$$

If the assumption of Theorem 3.2 hold, then the solution set of the inclusion problem (3.20) is a R_δ-set, and when $q(t,x) = 0$, the R_δ-type structure of the solution set of the impulsive evolution inclusion problem (3.20) was studied by Gabor et al. [32] in 2011.

4 Conclusion

Nowadays, we need to study more natural phenomena to gain more abilities for modeling, such as Zeno phenomenon. When discrete perturbations occur on a finite set of moments, the theory of impulsive differential equations offers the necessary tools, but for dealing with infinitely many abrupt changes (i.e., Zeno phenomenon) more refined methods are needed. An effective method to address such problems rely on measure differential equation and differential inclusion. Therefore, their importance has become more and more apparent to researchers. In this work, we have developed the existence theory, the compactness of solution sets, and R_δ-type structure of the solution set for neutral semilinear measure differential inclusions. We apply Górniewicz-Lassonde fixed-point theorem and R_δ-structure equivalence thesis, i.e., Hyman theorem to establish the desired results. Eventually, we give two corollaries, which are two special cases of the main results, and they have been studied by some researchers.

The work accomplished in this study is new and the research on the topological structure of the solution set of ordinary differential inclusions is extended to measure differential inclusions. For future works, one can extend the research on the topological structure of the solution set to more general differential inclusions, such as Sobolev-type measure differential inclusions, measure evolution inclusions with fractional derivatives, and so on.

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