Research Article

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Involvement of three successive fractional derivatives in a system of pantograph equations and studying the existence solution and MLU stability

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Abstract: Developing a model of fractional differential systems and studying the existence and stability of a solution is considebly one of the most important topics in the field of analysis. Therefore, this manuscript was dedicated to deriving a new type of fractional system that arises from the combination of three sequential fractional derivatives with fractional pantograph equations. Also, the fixed-point technique was used to evaluate the existence and uniqueness of solutions to the supposed hybrid model. Furthermore, stability results for the intended system in the sense of the Mittag-Leffler-Ulam have been investigated. Ultimately, an illustrative example has been highlighted in order to reinforce the theoretical results and suggest applications for this article.

Keywords: Gronwall's inequalities, fixed-point techniques, fractional derivatives, stability analysis, evaluation metrics

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Abbreviations

CFD	MLU-Caputo fractional derivative
DE	MLU-differential equation
FP	MLU-fixed point
FDE	fractional differential equation
MLUH	MLU-Hyers
MLUHR	MLU-Hyers-Rassias
MLU	Mittag-Leffler-Ulam
RL	MLU-Riemann-Liouville
SFPE	MLU-sequential fractional pantograph equation

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1 Introduction

Research on the analysis of differential equations (DEs) of fractional order has recently gained prominence and interest. Fractional derivatives are an excellent tool for studying memory and the inherited properties of diverse materials and processes. Fractional derivatives are frequently used to replicate the dynamics of real-world processes and phenomena, where previous behavior impacts the current state. Significant efforts have been made to advance the theoretical and application aspects of fractional differential equations (FDEs) in a variety of disciplines, including physics, chemistry, biology, and engineering sciences. Many contributions have been made by researchers about the existence and uniqueness of solutions to many classes of DEs. We urge the reader refer to previous studies [1–13] for more details.

In recent years, there has been a surge of interest in the application of novel analysis techniques to a variety of mathematical models, including delay FDEs. These equations are critical to understanding dynamic systems with delays and memory effects. One important component is the investigation of mild solutions, which provide a powerful framework for understanding the long-term behavior of delay FDEs. These solutions take into consideration both the initial conditions and the influence of previous states. For instance, the fractional Langevin equation that appears in the classical Brownian motion theory is modeled as fractional integro-differential equations [14]. For more details, see [15–17].

In 1960, Kalman proposed the fundamental concept of mathematical control theory, controllability. In general, controllability refers to the ability to employ a correct control function to steer the state of a control dynamical equation from an arbitrary initial state to the desirable terminal state. Many writers [18–26] have explored the controllability results. Furthermore, the investigation of controllability results on temporal scales is a new topic with minimal evidence [27,28].

The existence and uniqueness problems of FDEs are studied using fixed point (FP) theory. Over the course of a century, FP theory began to take shape and quickly advanced. Because of its applications, FP theory is highly respected and is still being investigated. Furthermore, this theory applies to a broad variety of spaces, such as Sobolev spaces, metric spaces, and abstract spaces. Because of this quality, FP theory is extremely beneficial for comprehending a wide variety of practical science problems represented by fractional ordinary, partial, and DEs [29–33].

On the other hand, stability theory is frequently applied to dynamical issues. Lyapunov and Mittag-Leffler stabilities have been established with great success for ordinary classical fractional calculus problems as well as exponential varieties of stabilities. The required attention has recently been given to the Hyer-Ulam [34] stability. Some stability and existence results from the FP approach were looked into in [35]. Furthermore, many writers discussed the stability analysis and existence theory for FDEs. For more details, see [36–40] and references therein, as several authors have recently developed an interest in the Ulam and Mittag-Leffler-Ulam (MLU) stability concerns.

The pantograph equation is a form of functional DE with a proportional delay. It occurs in a variety of pure and applied mathematics domains, including electrodynamics, control systems, number theory, probability, and quantum mechanics. Because of its significance, several academics have recently concentrated on the fractional pantograph equation and produced valuable contributions in this area. Balachandran et al. [41] investigated the existence of solutions to nonlinear fractional pantograph equations. Vivek et al. [42] established the existence and uniqueness of results for nonlinear neutral pantograph equations with generalized fractional derivatives. Several noteworthy findings on this subject have been reached using various types of fractional operators [43–49].

Now, the pantograph equation takes the following analogous form:

$$\theta'(\ell) = P\theta(\ell) + Q\theta(\upsilon\ell), \quad \theta(\ell) = \theta_0, \quad \ell \in [0, T], \quad T > 0,$$

where P, Q, and υ are the real constants such that $Q \neq 0$ and $\upsilon \in (0, 1)$. Academics [48,49] have researched several variations of the aforementioned pantograph as well as the multi-pantograph equation in the following:

$$\theta'(\ell) = P\theta(\ell) + \sum_{j=1}^{n} \delta_j(\ell)\theta(\upsilon_j\ell) + k(\ell), \quad \ell \geq 0,$$

under the condition $\theta(0) = \kappa$, where $P, \kappa \in \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers), $\delta_j(\ell)$, and $k(\ell)$ are the analytical functions, and $\upsilon_j \in (0,1)$.

The authors of [48] investigated the non-linear neutral pantograph equation as follows:

$$\theta'(\ell) = \phi(\ell, \theta(\ell), \theta(\upsilon\ell), \theta'(\upsilon\ell)), \quad \theta(\ell) = \theta_0, \quad \ell \in [0, T], \quad \upsilon \in (0,1).$$

The author recently assessed the following difficulty for the pantograph type in [41]:

$${}^{\mathrm{C}}D^{\varkappa}\theta(\ell) = \phi(\ell, \theta(\ell), \varpi(\upsilon\ell)), \quad \theta(\ell) = \theta_0, \quad \ell \in [0, T], \quad \upsilon, \varkappa \in (0,1),$$

where ${}^{\mathrm{C}}D^{\varkappa}$ refers to the Caputo fractional derivative (CFD).

Continuing with the aforementioned contributions, in this manuscript, we suggest the following model, the so-called sequential fractional pantograph equations (SFPEs):

$$\begin{cases} {}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}\theta(\ell))) = P\phi(\ell, \theta(\ell), \theta(\upsilon_1\ell), \varpi(\ell)) + QI^{\vartheta}[\psi(\ell, \theta(\ell), \theta(\upsilon_2\ell), \varpi(\ell))], \\ {}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}\varpi(\ell))) = P\phi(\ell, \theta(\ell), \varpi(\ell), \varpi(\upsilon_1\ell)) + QI^{\vartheta}[\psi(\ell, \theta(\ell), \varpi(\ell), \varpi(\upsilon_2\ell))], \\ \ell \in U = [0,1], \quad \omega, \varkappa, \varrho \in (0,1], \quad P, Q \in \mathbb{R}, \quad \vartheta \geq 0, \quad \upsilon_1, \upsilon_2 \in (0,1), \\ \theta(\ell) = 0, \quad \rho_1\theta(1) - \rho_2\theta(\alpha) = \sigma(\theta), \quad {}^{\mathrm{C}}D^{\varrho}\theta(0) = \beta, \quad \rho_1, \rho_2, \beta \in \mathbb{R}, \quad \alpha \in (0,1), \\ \varpi(\ell) = 0, \quad \rho_1\varpi(1) - \rho_2\varpi(\alpha) = \sigma(\varpi), \quad {}^{\mathrm{C}}D^{\varrho}\varpi(0) = \beta, \quad \rho_1 \neq \rho_2\alpha^{\omega+\varkappa+\varrho-1}, \end{cases} \tag{1.1}$$

where ${}^{\mathrm{RL}}D^{\omega}$, and ${}^{\mathrm{C}}D^{a}$, $a \in \{\varkappa, \varrho\}$ represent the fractional derivatives of Riemann-Liouville (RL) and Caputo, respectively. Also, $\phi, \psi : U \times \mathbb{R}^3 \to \mathbb{R}$, $\sigma : C(U, \mathbb{R}) \to \mathbb{R}$ are the continuous functions.

Due to its importance in many applied fields, it is interesting to study the fractional model of the pantograph equations. Such a model can be considered suitable to be applied when the corresponding process occurs through strongly anomalous media. In this article, we apply the FP technique to study the existence and uniqueness of solutions to the SFPEs (1.1). Furthermore, stability results for the considerable problem in the sense of the MLU have been investigated. Finally, an illustrative example has been highlighted in order to reinforce the theoretical results and suggest applications for this article.

2 Preliminary work

We need some basic definitions and properties of fractional calculus that are used in this article. So, this section is devoted to presenting some definitions and earlier works that will help the reader understand our text and will assist us in overcoming the challenges that we will experience in proofreading.

Definition 2.1. [5] The RL fractional integral operator of order ω is defined by

$$I^{\omega}\theta(\ell) = \frac{1}{\Gamma(\omega)}\int_0^{\ell}(\ell - \hbar)^{\omega-1}\theta(\hbar)\mathrm{d}\hbar, \tag{2.1}$$

while the CFD is defined by

$${}^{\mathrm{C}}D^{\omega}\theta(\ell) = \frac{1}{\Gamma(n-r)}\int_0^{\ell}(\ell - \hbar)^{n-r-1}\theta^{n}(\hbar)\mathrm{d}\hbar, \quad n = [\omega] + 1,$$

where $\tau > 0$, the function θ defined on $L^1(U)$ and $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. [5] Assume that $\theta : (0, \infty) \to \mathbb{R}$ is a continuous function, and the RL fractional derivative of order ω is described as

$${}^{\mathrm{RL}}D^{\omega}\theta(\ell) = \frac{1}{\Gamma(n-\omega)}\left(\frac{\mathrm{d}}{\mathrm{d}\ell}\right)^{n}\int_0^{\ell}(\ell - \hbar)^{n-\omega-1}\theta(\hbar)\mathrm{d}\hbar, \quad n = [\omega] + 1.$$

Lemma 2.3. [5,50] *Let* $s, t > 0$ *and* $\theta \in L^1(U)$. *Then,* $I^sI^t\theta(\ell) = I^{s+t}\theta(\ell)$ *and* $D^sI^s\theta(\ell) = \theta(\ell)$.

Lemma 2.4. [5,50] *Let* $s > t > 0$ *and* $\theta \in L^1(U)$. *Then,* $D^tI^s\theta(\ell) = I^{s-t}\theta(\ell)$.

Lemma 2.5. [50] *Assume that* $\omega \in C(0,1) \cap L^1(0,1)$ *and* ${}^{\mathrm{RL}}D^{\omega}\theta \in C(0,1) \cap L^1(0,1)$, *we obtain*

$$I^{\omega}({}^{\mathrm{RL}}D^{\omega}\theta(\ell)) = \theta(\ell) + \sum_{i=1}^{n} l_i\ell^{\omega-i},$$

where $l_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *and* $n = [\omega] + 1$.

Lemma 2.6. [50] *Let* $s > 0$. *Then,*

$$I^{s}({}^{\mathrm{C}}D^{s}\theta(\ell)) = \theta(\ell) + \sum_{i=0}^{n-1} l_i\ell^{i},$$

for some $l_i \in \mathbb{R}$, $i \geq 0$, $n = [s] + 1$.

We need a singular type for the following Gronwall inequality:

Lemma 2.7. [51] *For* $0 \leq \ell < 1$ *and the constants* $\varsigma_i > 0$, $(i = 1, 2, \ldots, n)$. *Assume that* D_i $(i = 1, 2, \ldots, n)$ *are bounded and monotonic increasing on* $[0, 1)$ *and S and Z are non-negative and continuous functions. If*

$$S(\ell) \leq Z(\ell) + \sum_{i=0}^{n} D_i(\ell)\int_0^{\ell}(\ell - \hbar)^{\varsigma_i-1}S(\hbar)\mathrm{d}\hbar,$$

then

$$S(\ell) \leq Z(\ell) + \sum_{m=1}^{\infty}\left(\sum_{1',2',\ldots,m'=1}^{n}\frac{\prod_{i=1}^{m}D_{i'}(\ell)\Gamma(\varsigma_{i'})}{\Gamma\left(\sum_{i=1}^{m}\varsigma_{i'}\right)}\int_0^{\ell}(\ell - \hbar)^{\sum_{i=1}^{m}\varsigma_{i'}-1}Z(\hbar)\mathrm{d}\hbar\right).$$

Remark 2.8. If $n = 2$, $\varsigma_1, \varsigma_2 > 0$, $D_1, D_2 \geq 0$, and $Z(\ell)$, $S(\ell)$ are non-negative and locally integrable on $[0, 1)$ with

$$S(\ell) \leq Z(\ell) + D_1\int_0^{\ell}(\ell - \hbar)^{\varsigma_1-1}S(\hbar)\mathrm{d}\hbar + D_2\int_0^{\ell}(\ell - \hbar)^{\varsigma_2-1}S(\hbar)\mathrm{d}\hbar,$$

then

$$S(\ell) \leq Z(\ell) + \sum_{m=1}^{\infty}\left(\frac{(D_1\Gamma(\varsigma_1))^m}{\Gamma(m\varsigma_1)}\int_0^{\ell}(\ell - \hbar)^{m\varsigma_1-1}Z(\hbar)\mathrm{d}\hbar + \frac{(D_2\Gamma(\varsigma_2))^m}{\Gamma(m\varsigma_2)}\int_0^{\ell}(\ell - \hbar)^{m\varsigma_2-1}Z(\hbar)\mathrm{d}\hbar\right).$$

Remark 2.9. Via the assumptions of Remark 2.8, consider that $Z(\ell)$ is a non-decreasing function on $[0, 1)$. Then, we obtain

$$S(\ell) \leq Z(\ell)\ (E_{\varsigma_1}[D_1\Gamma(\varsigma_1)\ell^{\varsigma_1}] + E_{\varsigma_2}[D_2\Gamma(\varsigma_2)\ell^{\varsigma_2}]),$$

where E_{ς} refers to the Mittag-Leffler function [52], which is defined by $E_{\varsigma}(\ell) = \sum_{m=1}^{\infty}\frac{\ell^{\varsigma}}{\Gamma(m\varsigma+1)}$, $\ell \in \mathbb{C}$.

The following auxiliary lemma is also necessary:

Lemma 2.10. *Assume that* $\Xi_{\theta,\varpi}(\ell), \Xi_{\varpi,\theta}(\ell) \in C([0,1], \mathbb{R})$. *The solution to the problem*

$$\begin{cases} {}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}\theta(\ell))) = \Xi_{\theta,\varpi}(\ell), \quad \ell \in [0,1], \quad \omega, \varkappa, \varrho \in (0,1], \quad P, Q \in \mathbb{R}, \\ {}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}\theta(\ell))) = \Xi_{\varpi,\theta}(\ell), \quad \ell \in [0,1], \quad \omega, \varkappa, \varrho \in (0,1], \quad P, Q \in \mathbb{R}, \\ \theta(\ell) = 0, \quad \rho_1\theta(1) - \rho_2\theta(a) = \sigma(\theta), \quad {}^{\mathrm{C}}D^{\varrho}\theta(0) = 0, \\ \varpi(\ell) = 0, \quad \rho_1\varpi(1) - \rho_2\varpi(a) = \sigma(\varpi), \quad {}^{\mathrm{C}}D^{\varrho}\varpi(0) = 0, \end{cases} \tag{2.2}$$

takes the form

$$\begin{aligned}\theta(\ell) = {} & \frac{1}{\Gamma(\omega+\varkappa+\varrho)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}\,\Xi_{\theta,\varpi}(\hbar)\mathrm{d}\hbar + \frac{\ell^{\omega+\varkappa+\varrho-1}}{\rho_1-\rho_2 a^{\omega+\varkappa+\varrho-1}} \\ & \times\left(\frac{\rho_2}{\Gamma(\omega+\varkappa+\varrho)}\int_0^a (a-\hbar)^{\omega+\varkappa+\varrho-1}\,\Xi_{\theta,\varpi}(\hbar)\mathrm{d}\hbar - \frac{\rho_1}{\Gamma(\omega+\varkappa+\varrho)}\int_0^1 (1-\hbar)^{\omega+\varkappa+\varrho-1}\,\Xi_{\theta,\varpi}(\hbar)\mathrm{d}\hbar\right) \\ & + \frac{\ell^{\omega+\varkappa+\varrho-1}}{\rho_1-\rho_2 a^{\omega+\varkappa+\varrho-1}}\sigma(\theta)\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\varpi(\ell) = {} & \frac{1}{\Gamma(\omega+\varkappa+\varrho)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}\,\Xi_{\varpi,\theta}(\hbar)\mathrm{d}\hbar + \frac{\ell^{\omega+\varkappa+\varrho-1}}{\rho_1-\rho_2 a^{\omega+\varkappa+\varrho-1}} \\ & \times\left(\frac{\rho_2}{\Gamma(\omega+\varkappa+\varrho)}\int_0^a (a-\hbar)^{\omega+\varkappa+\varrho-1}\,\Xi_{\varpi,\theta}(\hbar)\mathrm{d}\hbar - \frac{\rho_1}{\Gamma(\omega+\varkappa+\varrho)}\int_0^1 (1-\hbar)^{\omega+\varkappa+\varrho-1}\,\Xi_{\varpi,\theta}(\hbar)\mathrm{d}\hbar\right) \\ & + \frac{\ell^{\omega+\varkappa+\varrho-1}}{\rho_1-\rho_2 a^{\omega+\varkappa+\varrho-1}}\sigma(\varpi).\end{aligned} \tag{2.4}$$

Proof. Taking the RL fractional integral of order ω on both sides of the first equation of (2.2) and using Lemmas 2.3 and 2.5, we obtain

$$ {}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho})\theta(\ell) = I^{\omega}\,\Xi_{\theta,\varpi}(\ell) + a_1\ell^{\omega-1}, \quad a_1\in\mathbb{R}. \tag{2.5}$$

Again, taking the RL fractional integral of order $\varkappa$ on both sides of (2.5) and applying Lemma 2.6, one can write

$$ {}^{\mathrm{C}}D^{\varrho}\theta(\ell) = I^{\omega+\varkappa}\,\Xi_{\theta,\varpi}(\ell) + \frac{\Gamma(\omega)a_1}{\Gamma(\omega+\varkappa)}a_1\ell^{\omega+\varkappa-1} + a_2, \quad a_2\in\mathbb{R}. \tag{2.6}$$

On both sides of (2.5), taking the RL fractional integral of order ϱ, we obtain that

$$\theta(\ell) = I^{\omega+\varkappa+\varrho}\,\Xi_{\theta,\varpi}(\ell) + \frac{\Gamma(\omega)}{\Gamma(\omega+\varkappa+\varrho)}a_1\ell^{\omega+\varkappa+\varrho-1} + \frac{a_2}{\Gamma(\varrho+1)}\ell^{\varrho} + a_3, \quad a_3\in\mathbb{R}. \tag{2.7}$$

Similarly, we have

$$\varpi(\ell) = I^{\omega+\varkappa+\varrho}\,\Xi_{\varpi,\theta}(\ell) + \frac{\Gamma(\omega)}{\Gamma(\omega+\varkappa+\varrho)}a_1\ell^{\omega+\varkappa+\varrho-1} + \frac{a_2}{\Gamma(\varrho+1)}\ell^{\varrho} + a_{3,} \quad a_3\in\mathbb{R}. \tag{2.8}$$

From the third and fourth conditions given in (2.2), we have

$$a_1 = \frac{\Gamma(\omega+\varkappa+\varrho)}{(\rho_1-\rho_2 a^{\omega+\varkappa+\varrho-1})\Gamma(\omega)}(\sigma(\theta) + \rho_2 I^{\omega+\varkappa+\varrho}\,\Xi_{\theta,\varpi}(a) - \rho_2 I^{\omega+\varkappa+\varrho}\,\Xi_{\theta,\varpi}(1)), \quad a_2 = a_3 = 0, \tag{2.9}$$

and

$$a_1 = \frac{\Gamma(\omega+\varkappa+\varrho)}{(\rho_1-\rho_2 a^{\omega+\varkappa+\varrho-1})\Gamma(\omega)}(\sigma(\varpi) + \rho_2 I^{\omega+\varkappa+\varrho}\,\Xi_{\varpi,\theta}(a) - \rho_2 I^{\omega+\varkappa+\varrho}\,\Xi_{\varpi,\theta}(1)), \quad a_2 = a_3 = 0. \tag{2.10}$$

Substituting (2.9) into (2.7) and substituting (2.10) into (2.8), we obtain (2.3) and (2.4), respectively. This completes the proof. □

3 Existence results

In this section, we use the FP approach to present the necessary and sufficient conditions for studying the existence and uniqueness of a solution to the SFPEs (1.1). Therefore, we need to define a space as follows:

Assume that $\Theta = C(U, \mathbb{R})$ is the space of all continuous functions on the interval U. Obviously, this space is a Banach space under the norm $\|\theta\| = \sup_{\ell\in U}\{|\theta(\ell)|\}$. Moreover, the product space $\Theta = \Theta_1 \times \Theta_1$ is also a Banach space with the norm $\|(\varpi, \theta)\|_{\Theta} = \|(\varpi)\|_{\Theta_1} + \|(\theta)\|_{\Theta_2}$.

According to Lemma 2.10, define the operator $\mho : \Theta \to \Theta$ by

$$\begin{aligned}
\mho(\theta, \varpi)(\ell) = {} & \frac{P}{\Gamma(\omega + \varkappa + \varrho)}\int_0^{\ell}(\ell - \hbar)^{\omega+\varkappa+\varrho-1}\phi(\hbar, \theta(\hbar), \theta(\upsilon_1\hbar), \varpi(\hbar))\mathrm{d}\hbar \\
& + \frac{Q}{\Gamma(\omega + \varkappa + \varrho + \vartheta)}\int_0^{\ell}(\ell - \hbar)^{\omega+\varkappa+\varrho+\vartheta-1}\psi(\hbar, \theta(\hbar), \theta(\upsilon_2\hbar), \varpi(\hbar))\mathrm{d}\hbar \\
& + \frac{\ell^{\omega+\varkappa+\varrho-1}}{\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}}\left(\frac{\rho_2 P}{\Gamma(\omega + \varkappa + \varrho)}\int_0^{\alpha}(\alpha - \hbar)^{\omega+\varkappa+\varrho-1}\phi(\hbar, \theta(\hbar), \theta(\upsilon_1\hbar), \varpi(\hbar))\mathrm{d}\hbar\right. \\
& + \frac{\rho_2 Q}{\Gamma(\omega + \varkappa + \varrho + \vartheta)}\int_0^{\alpha}(\alpha - \hbar)^{\omega+\varkappa+\varrho+\vartheta-1}\psi(\hbar, \theta(\hbar), \theta(\upsilon_2\hbar), \varpi(\hbar))\mathrm{d}\hbar \\
& - \frac{\rho_1 P}{\Gamma(\omega + \varkappa + \varrho)}\int_0^{1}(1 - \hbar)^{\omega+\varkappa+\varrho-1}\phi(\hbar, \theta(\hbar), \theta(\upsilon_1\hbar), \varpi(\hbar))\mathrm{d}\hbar \\
& \left. - \frac{\rho_1 Q}{\Gamma(\omega + \varkappa + \varrho + \vartheta)}\int_0^{1}(1 - \hbar)^{\omega+\varkappa+\varrho+\vartheta-1}\psi(\hbar, \theta(\hbar), \theta(\upsilon_2\hbar), \varpi(\hbar))\mathrm{d}\hbar\right) \\
& + \frac{\ell^{\omega+\varkappa+\varrho-1}}{\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}}\sigma(\varpi),
\end{aligned} \tag{3.1}$$

where $\upsilon_1, \upsilon_2 \in (0, 1)$. For convenience, we consider the following assumptions:

(A_1) For the continuous functions $\phi, \psi : U \times \mathbb{R}^3 \to \mathbb{R}$, there exist constants $G_\phi, G_\psi > 0$ such that for $\ell \in U$, $\theta_i, \varpi_i \in \mathbb{R}$, $i = 1, 2, 3$, we have

$$|\phi(\ell, \theta_1, \theta_2, \theta_3) - \phi(\ell, \varpi_1, \varpi_2, \varpi_3)| \le \frac{G_\phi}{2}(|\theta_1 - \varpi_1| + |\theta_2 - \varpi_2| + |\theta_3 - \varpi_3|),$$
$$|\psi(\ell, \theta_1, \theta_2, \theta_3) - \psi(\ell, \varpi_1, \varpi_2, \varpi_3)| \le \frac{G_\psi}{2}(|\theta_1 - \varpi_1| + |\theta_2 - \varpi_2| + |\theta_3 - \varpi_3|).$$

(A_2) The function $\sigma : C(U, \mathbb{R}) \to \mathbb{R}$ is continuous with $\sigma(0) = 0$, and there exists a positive constant ξ such that

$$|\sigma(\theta) - \sigma(\varpi)| \le \xi|\theta - \varpi|, \quad \theta, \varpi \in \Theta.$$

Also, we consider

$$\begin{aligned}
\Lambda &= \frac{|P|}{\Gamma(\omega + \varkappa + \varrho + 1)}\left(1 + \frac{1}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}[|\rho_2|\alpha^{\omega+\varkappa+\varrho} + |\rho_1|]\right), \\
\Lambda^* &= \frac{|Q|}{\Gamma(\omega + \varkappa + \varrho + \vartheta + 1)}\left(1 + \frac{1}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}[|\rho_2|\alpha^{\omega+\varkappa+\varrho+\vartheta} + |\rho_1|]\right).
\end{aligned} \tag{3.2}$$

Now, we can present the first main result of this part as follows:

Theorem 3.1. *Under assumptions* (A_1) *and* (A_2), *problem* (1.1) *has a unique solution on* U, *provided that the following condition holds*:

$$3G(\Lambda + \Lambda^*) < 1 - \frac{\xi}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|},$$

where $G = \max\{G_\phi, G_\psi\}$ *and* Λ, Λ^* *are described in* (3.2).

Proof. Let us consider $\varDelta = \max\{\varDelta_i, i = 1, 2\}$, where $\varDelta_i$ are the finite numbers given by $\varDelta_1 = \sup_{\ell \in U}|\phi(\ell, 0, 0, 0)|$ and $\varDelta_2 = \sup_{\ell \in U}|\psi(\ell, 0, 0, 0)|$. Put

$$\frac{\varDelta(\Lambda + \Lambda^*)}{1 - \left(3G(\Lambda + \Lambda^*) + \frac{\xi}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}\right)} \leq f, \quad f > 0.$$

Now, we claim that $VS_f \subset S_f$, where $S_f = \{\theta \in \Theta : \|\theta\| \leq f\}$. For $\theta, \varpi \in S_f$ and using assumptions (A_1) and (A_2) for all $\ell \in U$, we obtain

$$\begin{aligned} |\phi(\ell, \theta(\ell), \theta(\upsilon_1\ell), \varpi(\ell))| &\leq |\phi(\ell, \theta(\ell), \theta(\upsilon_1\ell), \varpi(\ell)) - \phi(\ell, 0, 0, 0)| + |\phi(\ell, 0, 0, 0)| \\ &\leq G_\phi(2\|\theta\| + \|\varpi\|) + \varDelta_1 = 2G_\phi\|\theta\| + G_\phi\|\varpi\| + \varDelta_1 \\ &= 3G_\phi f + \varDelta_1. \end{aligned} \tag{3.3}$$

Similarly,

$$\phi(\ell, \theta(\ell), \varpi(\ell), \varpi(\upsilon_1\ell)) \leq G_\phi(\|\theta\| + 2\|\varpi\|) + \varDelta_1 = 3G_\phi f + \varDelta_1. \tag{3.4}$$

Also,

$$\begin{aligned} |\psi(\ell, \theta(\ell), \theta(\upsilon_2\ell), \varpi(\ell))| &\leq |\psi(\ell, \theta(\ell), \theta(\upsilon_1\ell), \varpi(\ell)) - \psi(\ell, 0, 0, 0)| + |\psi(\ell, 0, 0, 0)| \\ &\leq G_\psi(2\|\theta\| + \|\varpi\|) + \varDelta_1 = 2G_\psi\|\theta\| + G_\psi\|\varpi\| + \varDelta_2 \\ &\leq 3G_\psi f + \varDelta_2. \end{aligned} \tag{3.5}$$

Analogously,

$$|\psi(\ell, \theta(\ell), \varpi(\ell), \varpi(\upsilon_2\ell))| \leq G_\psi(\|\theta\| + 2\|\varpi\|) + \varDelta_1 \leq 3G_\psi f + \varDelta_2. \tag{3.6}$$

Moreover,

$$|\sigma(\theta)| \leq \xi\|\theta\| \leq \xi f \quad \text{and} \quad |\sigma(\varpi)| \leq \xi\|\varpi\| \leq \xi f. \tag{3.7}$$

Taking supremum on both sides of (3.1), for $\upsilon_1, \upsilon_2 \in (0, 1)$, we have

$$\begin{aligned} \|\mho(\theta, \varpi)\|_{\Theta_1} \leq \sup_{\ell \in U}\Bigg\{ &\frac{|P|}{\Gamma(\omega + \varkappa + \varrho)}\int_0^\ell (\ell - \hbar)^{\omega+\varkappa+\varrho-1}|\phi(\hbar, \theta(\hbar), \theta(\upsilon_1\hbar), \varpi(\hbar))|d\hbar \\ &+ \frac{|Q|}{\Gamma(\omega + \varkappa + \varrho + \vartheta)}\int_0^\ell (\ell - \hbar)^{\omega+\varkappa+\varrho+\vartheta-1}|\psi(\hbar, \theta(\hbar), \theta(\upsilon_2\hbar), \varpi(\hbar))|d\hbar \\ &+ \frac{\ell^{\omega+\varkappa+\varrho-1}}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\Bigg(\frac{|\rho_2||P|}{\Gamma(\omega + \varkappa + \varrho)}\int_0^\alpha (\alpha - \hbar)^{\omega+\varkappa+\varrho-1}|\phi(\hbar, \theta(\hbar), \theta(\upsilon_1\hbar), \varpi(\hbar))|d\hbar \\ &+ \frac{|\rho_2||Q|}{\Gamma(\omega + \varkappa + \varrho + \vartheta)}\int_0^\alpha (\alpha - \hbar)^{\omega+\varkappa+\varrho+\vartheta-1}|\psi(\hbar, \theta(\hbar), \theta(\upsilon_2\hbar), \varpi(\hbar))|d\hbar \\ &+ \frac{|\rho_1||P|}{\Gamma(\omega + \varkappa + \varrho)}\int_0^1 (1 - \hbar)^{\omega+\varkappa+\varrho-1}|\phi(\hbar, \theta(\hbar), \theta(\upsilon_1\hbar), \varpi(\hbar))|d\hbar \\ &+ \frac{|\rho_1||Q|}{\Gamma(\omega + \varkappa + \varrho + \vartheta)}\int_0^1 (1 - \hbar)^{\omega+\varkappa+\varrho+\vartheta-1}|\psi(\hbar, \theta(\hbar), \theta(\upsilon_2\hbar), \varpi(\hbar))|d\hbar\Bigg) \\ &+ \frac{\ell^{\omega+\varkappa+\varrho-1}}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}|\sigma(\varpi)|\Bigg\}. \end{aligned}$$

According to estimates (3.3), (3.5), and (3.7), for $\ell \in U$, we obtain

$$
\begin{aligned}
\|\mho(\theta,\varpi)\|_{\Theta_1} \le{}& \frac{|P|\ell^{\omega+\varkappa+\varrho}(3G_\phi f+\Delta_1)}{\Gamma(\omega+\varkappa+\varrho+1)} + \frac{|Q|(3G_\psi f+\Delta_2)\ell^{\omega+\varkappa+\varrho+\vartheta}}{\Gamma(\omega+\varkappa+\varrho+\vartheta+1)}\\
&+ \frac{\ell^{\omega+\varkappa+\varrho-1}}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\left(\frac{|\rho_2||P|(3G_\phi f+\Delta_1)\alpha^{\omega+\varkappa+\varrho}}{\Gamma(\omega+\varkappa+\varrho+1)} + \frac{|\rho_2||Q|(3G_\psi f+\Delta_2)\alpha^{\omega+\varkappa+\varrho+\vartheta}}{\Gamma(\omega+\varkappa+\varrho+\vartheta+1)}\right.\\
&\left.+ \frac{|\rho_1||P|(3G_\phi f+\Delta_1)}{\Gamma(\omega+\varkappa+\varrho+1)} + \frac{|\rho_1||Q|(3G_\psi f+\Delta_2)}{\Gamma(\omega+\varkappa+\varrho+\vartheta+1)}\right) + \frac{\ell^{\omega+\varkappa+\varrho-1}\xi f}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\\
\le{}& \frac{|P|(3G_\phi f+\Delta_1)}{\Gamma(\omega+\varkappa+\varrho+1)}\left(1+\frac{1}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}(|\rho_2|\alpha^{\omega+\varkappa+\varrho}+|\rho_1|)\right)\\
&+ \frac{|Q|(3G_\psi f+\Delta_2)}{\Gamma(\omega+\varkappa+\varrho+\vartheta+1)}\left(1+\frac{1}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}(|\rho_2|\alpha^{\omega+\varkappa+\varrho+\vartheta}+|\rho_1|)\right)\\
&+ \frac{\xi f}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\mho(\theta,\varpi)\|_{\Theta_1} &\le (3Gf+\Delta)\Lambda + (3Gf+\Delta)\Lambda^* + \frac{\xi f}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\\
&= \left(3G(\Lambda+\Lambda^*) + \frac{\xi}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)f + \Delta(\Lambda+\Lambda^*) \le f.
\end{aligned}
\tag{3.8}
$$

Similarly, from estimates (3.4), (3.6), and (3.7), for $\ell \in U$, we have $\|\mho(\varpi,\theta)\|_{\Theta_2} \le f$. Hence, $\mho S_f \subset S_f$.

Now, for $\theta, \varpi, \widetilde{\theta}, \widetilde{\varpi} \in S_f$, and for $\upsilon_1, \upsilon_2 \in (0,1)$, one has

$$
\begin{aligned}
&\|\mho(\theta,\varpi) - \mho(\widetilde{\theta},\widetilde{\varpi})\|_{\Theta_1}\\
&\le \sup_{\ell\in U}\left\{\frac{|P|}{\Gamma(\omega+\varkappa+\varrho)}\int_0^\ell(\ell-\hbar)^{\omega+\varkappa+\varrho-1}|\phi(\hbar,\theta(\hbar),\theta(\upsilon_1\hbar),\varpi(\hbar)) - \phi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_1\hbar),\widetilde{\varpi}(\hbar))|\mathrm{d}\hbar\right.\\
&+ \frac{|Q|}{\Gamma(\omega+\varkappa+\varrho+\vartheta)}\int_0^\ell(\ell-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1}|\psi(\hbar,\theta(\hbar),\theta(\upsilon_2\hbar),\varpi(\hbar)) - \psi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_2\hbar),\widetilde{\varpi}(\hbar))|\mathrm{d}\hbar\\
&+ \frac{\ell^{\omega+\varkappa+\varrho-1}}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\left(\frac{|\rho_2||P|}{\Gamma(\omega+\varkappa+\varrho)}\right.\\
&\times \int_0^\alpha(\alpha-\hbar)^{\omega+\varkappa+\varrho-1}|\phi(\hbar,\theta(\hbar),\theta(\upsilon_1\hbar),\varpi(\hbar)) - \phi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_1\hbar),\widetilde{\varpi}(\hbar))|\mathrm{d}\hbar\\
&+ \frac{|\rho_2||Q|}{\Gamma(\omega+\varkappa+\varrho+\vartheta)}\int_0^\alpha(\alpha-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1}|\psi(\hbar,\theta(\hbar),\theta(\upsilon_2\hbar),\varpi(\hbar)) - \psi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_2\hbar),\widetilde{\varpi}(\hbar))|\mathrm{d}\hbar\\
&+ \frac{|\rho_1||P|}{\Gamma(\omega+\varkappa+\varrho)}\int_0^1(1-\hbar)^{\omega+\varkappa+\varrho-1}|\phi(\hbar,\theta(\hbar),\theta(\upsilon_1\hbar),\varpi(\hbar)) - \phi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_1\hbar),\widetilde{\varpi}(\hbar))|\mathrm{d}\hbar\\
&\left.+ \frac{|\rho_1||Q|}{\Gamma(\omega+\varkappa+\varrho+\vartheta)}\int_0^1(1-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1}|\psi(\hbar,\theta(\hbar),\theta(\upsilon_2\hbar),\varpi(\hbar)) - \psi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_2\hbar),\widetilde{\varpi}(\hbar))|\mathrm{d}\hbar\right)\\
&\left.+ \frac{\ell^{\omega+\varkappa+\varrho-1}}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}|\sigma(\theta)-\sigma(\widetilde{\theta})|\right\}.
\end{aligned}
\tag{3.9}
$$

Clearly, for $\upsilon_1, \upsilon_2 \in (0,1)$, we obtain

$$\begin{aligned}|\phi(\hbar,\theta(\hbar),\theta(\upsilon_1\hbar),\varpi(\hbar))-\phi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_1\hbar),\widetilde{\varpi}(\hbar))| &\le \frac{G_\phi}{2}(2\|\theta-\widetilde{\theta}\|+\|\varpi-\widetilde{\varpi}\|)\\ &\le G_\phi(\|\theta-\widetilde{\theta}\|+\|\varpi-\widetilde{\varpi}\|),\\ |\psi(\hbar,\theta(\hbar),\theta(\upsilon_2\hbar),\varpi(\hbar))-\psi(\hbar,\widetilde{\theta}(\hbar),\widetilde{\theta}(\upsilon_2\hbar),\widetilde{\varpi}(\hbar))| &\le \frac{G_\varpi}{2}(2\|\theta-\widetilde{\theta}\|+\|\varpi-\widetilde{\varpi}\|)\\ &\le G_\varpi(\|\theta-\widetilde{\theta}\|+\|\varpi-\widetilde{\varpi}\|).\end{aligned} \tag{3.10}$$

Substituting (3.8) and (3.10) into (3.9), we have

$$\|\mho(\theta,\varpi)-\mho(\widetilde{\theta},\widetilde{\varpi})\|_{\Theta_1} \le \left(3G(\Lambda+\Lambda^*)+\frac{\xi}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)(\|\theta-\widetilde{\theta}\|_{\Theta_1}+\|\varpi-\widetilde{\varpi}\|_{\Theta_2}).$$

Similarly,

$$\|\mho(\varpi,\theta)-\mho(\widetilde{\varpi},\widetilde{\theta})\|_{\Theta_2} \le \left(3G(\Lambda+\Lambda^*)+\frac{\xi}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)(\|\varpi-\widetilde{\varpi}\|_{\Theta_2}+\|\theta-\widetilde{\theta}\|_{\Theta_1}).$$

Hence,

$$\|\mho(\theta,\varpi)-\mho(\widetilde{\theta},\widetilde{\varpi})\|_{\Theta} \le \left(3G(\Lambda+\Lambda^*)+\frac{\xi}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)\|(\theta,\varpi)-(\widetilde{\theta},\widetilde{\varpi})\|_{\Theta}.$$

Since $3G(\Lambda+\Lambda^*)+\frac{\xi}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}<1$, $\mho$ is a contraction mapping. According to Banach's FP theorem, the operator $\mho$ has a unique FP that corresponds to a unique solution to the SFPEs (1.1). □

The second main result came from the Leray-Schauder alternative.

Lemma 3.2. [41] (Leray-Schauder alternative) *Assume that W is a non-empty set and the operator $O: W \to W$ is completely continuous. Suppose that $\Xi(O)=\{e\in W: e=kO(e)$ for some $k\in(0,1)\}$. Then, either $\Xi(O)$ is bounded or O admits at least one FP.*

The following two hypotheses are required to achieve the next result:

(A_3) For the continuous functions $\phi,\psi: U\times\mathbb{R}^3\to\mathbb{R}$, there exist positive constants $b_i, t_i, i=1,2,3$, and $b_0, t_0>0$ such that for $\ell\in U$, $\theta_i\in\mathbb{R}$, $i=1,2,3$, we have

$$|\phi(\ell,\theta_1,\theta_2,\theta_3)| \le b_0+b_1|\theta_1|+b_2|\theta_2|+b_3|\theta_3|$$

and

$$|\psi(\ell,\theta_1,\theta_2,\theta_3)| \le t_0+t_1|\theta_1|+t_2|\theta_2|+t_3|\theta_3|.$$

(A_4) The function $\sigma: C(U,\mathbb{R})\to\mathbb{R}$ is continuous with $\sigma(0)=0$, and there exists a positive constant ξ^* such that

$$|\sigma(\theta)|\le\xi^*\|\theta\| \quad \text{and} \quad |\sigma(\varpi)|\le\xi^*\|\varpi\|, \quad \theta,\varpi\in\Theta.$$

Theorem 3.3. *According to hypotheses* (A_3) *and* (A_4), *problem* (1.1) *has at least one solution on U, provided that following condition is true*:

$$\sum_{i=1}^{3} b_i\Lambda+\sum_{i=1}^{3} t_i\Lambda^*+\frac{\xi^*}{|\rho_1-\rho_2\alpha^{\omega+\varkappa+\varrho-1}|}<1.$$

Proof. First, we show that the operator $\mho : \Theta \to \Theta$ is completely continuous. The continuity of ϕ, ψ, and σ leading to $\mho$ is continuous. Suppose that ϕ, ψ, and $\sigma \subset \Theta$ are bounded. Then, there are constants $D_1, D_2, M > 0$ so that $|\phi(\ell, \theta_1, \theta_2, \theta_3)| \le D_1$, $|\psi(\ell, \theta_1, \theta_2, \theta_3)| \le D_2$, $|\sigma(\theta)| \le M$ and $|\sigma(\varpi)| \le M$, for all $\theta, \theta_1, \theta_2, \theta_3, \varpi \in \Theta$.

Now, according to our assumptions (A_3) and (A_4), for $\theta, \varpi \in \Theta$, one has

$$\begin{aligned}\|\mho(\theta, \varpi)\|_{\Theta_1} \le{}& \frac{|P|D_1}{\Gamma(\omega + \varkappa + \varrho + 1)}\left(1 + \frac{1}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}(|\rho_2|\alpha^{\omega+\varkappa+\varrho} + |\rho_1|)\right)\\ &+ \frac{|Q|D_2}{\Gamma(\omega + \varkappa + \varrho + \vartheta + 1)}\left(1 + \frac{1}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}(|\rho_2|\alpha^{\omega+\varkappa+\varrho+\vartheta} + |\rho_1|)\right)\\ &+ \frac{M}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|},\end{aligned}$$

which implies that

$$\|\mho(\theta, \varpi)\|_{\Theta_1} \le \Lambda D_1 + \Lambda^* D_2 + \frac{M}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}.$$

Analogously,

$$\|\mho(\varpi, \theta)\|_{\Theta_2} \le \Lambda D_1 + \Lambda^* D_2 + \frac{M}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}.$$

Hence, $\mho$ is uniformly bounded on Θ. Next, we show that $\mho$ is equicontinuous on Θ. For this, assume that $\ell_1, \ell_2 \in U$ with $\ell_1 < \ell_2$. Then, we have

$$\begin{aligned}\|\mho(\theta, \varpi)(\ell_1) - \mho(\theta, \varpi)(\ell_2)\|_{\Theta_1} \le{}& \frac{|P|D_1}{\Gamma(\omega + \varkappa + \varrho + 1)}((\ell_2 - \ell_1)^{\omega+\varkappa+\varrho} + |\ell_2^{\omega+\varkappa+\varrho} - \ell_1^{\omega+\varkappa+\varrho}|)\\ &+ \frac{|Q|D_2}{\Gamma(\omega + \varkappa + \varrho + \vartheta + 1)}((\ell_2 - \ell_1)^{\omega+\varkappa+\varrho+\vartheta} + |\ell_2^{\omega+\varkappa+\varrho+\vartheta} - \ell_1^{\omega+\varkappa+\varrho+\vartheta}|)\\ &+ \frac{|\ell_2^{\omega+\varkappa+\varrho-1} - \ell_1^{\omega+\varkappa+\varrho-1}|}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}\left(\frac{|P||\rho_2|\alpha^{\omega+\varkappa+\varrho}}{\Gamma(\omega + \varkappa + \varrho + 1)} + \frac{|Q||\rho_2|\alpha^{\omega+\varkappa+\varrho+\vartheta}}{\Gamma(\omega + \varkappa + \varrho + \vartheta + 1)}\right.\\ &\left.+ \frac{|P||\rho_1|}{\Gamma(\omega + \varkappa + \varrho + 1)} + \frac{|Q||\rho_1|}{\Gamma(\omega + \varkappa + \varrho + \vartheta + 1)}\right) + M.\end{aligned}$$

It follows that

$$\|\mho(\theta, \varpi)(\ell_1) - \mho(\theta, \varpi)(\ell_2)\|_{\Theta_1} \to 0, \quad \text{as } \ell_1 \to \ell_2.$$

Similarly,

$$\|\mho(\varpi, \theta)(\ell_1) - \mho(\varpi, \theta)(\ell_2)\|_{\Theta_2} \to 0, \quad \text{as } \ell_1 \to \ell_2.$$

Therefore, $\mho$ is equicontinuous on Θ. Using the Arzelá-Ascoli theorem, $\mho$ is completely continuous.

Ultimately, we prove that the set $\Xi(\mho) = \{e \in \Theta : e = k\mho(e) \text{ for some } k \in (0, 1)\}$ is bounded. Suppose that $\theta, \varpi \in \Xi$, then for all $\ell \in U$, we obtain that

$$\begin{aligned}|\theta(\ell)| \le{}& \frac{[b_0 + (b_1 + b_2 + b_3)\|\theta\|]|P|}{\Gamma(\omega + \varkappa + \varrho + 1)}\left(1 + \frac{1}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}(|\rho_2|\alpha^{\omega+\varkappa+\varrho} + |\rho_1|)\right)\\ &+ \frac{[t_0 + (t_1 + t_2 + t_3)\|\theta\|]|Q|}{\Gamma(\omega + \varkappa + \varrho + \vartheta + 1)}\left(1 + \frac{1}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}(|\rho_2|\alpha^{\omega+\varkappa+\varrho+\vartheta} + |\rho_1|)\right)\\ &+ \frac{\xi}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}\|\theta\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|\theta\|_{\Theta_1} &\le [b_0 + (b_1 + b_2 + b_3)\|\theta\|]\Lambda + [t_0 + (t_1 + t_2 + t_3)\|\theta\|]\Lambda^* + \frac{\xi}{|\rho_1 - \rho_2 \alpha^{\omega+\varkappa+\varrho-1}|}\|\theta\|_{\Theta_1}\\ &= \left(\sum_{i=1}^{3} b_i\Lambda + \sum_{i=1}^{3} t_i\Lambda^* + \frac{\xi^*}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)\|\theta\|_{\Theta_1} + b_0\Lambda + t_0\Lambda^*.\end{aligned}$$

Consequently,

$$\|\theta\|_{\Theta_1} \le \frac{b_0\Lambda + t_0\Lambda^*}{1 - \left(\sum_{i=1}^{3} b_i\Lambda + \sum_{i=1}^{3} t_i\Lambda^* + \frac{\xi^*}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)}.$$

In the same manner, we conclude that

$$\|\varpi\|_{\Theta_2} \le \frac{b_0\Lambda + t_0\Lambda^*}{1 - \left(\sum_{i=1}^{3} b_i\Lambda + \sum_{i=1}^{3} t_i\Lambda^* + \frac{\xi^*}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)}.$$

Hence,

$$\|(\theta, \varpi)\|_{\Theta} \le \frac{2(b_0\Lambda + t_0\Lambda^*)}{1 - \left(\sum_{i=1}^{3} b_i\Lambda + \sum_{i=1}^{3} t_i\Lambda^* + \frac{\xi^*}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}\right)}.$$

Therefore, Ξ is bounded. Thanks to Lemma 3.2, the operator $\mho$ has at least one FP. Hence, problem (1.1) has at least one solution on U. □

4 Stability results

The stability of the solution is critical in physical problems because the mathematical equations describing the problem will be unable to predict the future with any degree of accuracy if small deviations from the mathematical model caused by inevitable measurement errors do not also have a correspondingly small effect on the solution. The aforementioned factors are critical from a numerical and optimization standpoint. This is because most nonlinear problems in fractional calculus and applied analysis are extremely difficult to solve. In this case, approximate solutions that are close to the true solution of the relevant problem are required. Stability of solutions is required in the aforementioned situation. In such cases, one requires approximations that are close to the true solution of the relevant problem. The stability of the solutions is required in the given scenario. In this part, we study the MLU stability for our proposed system.

Now, for $\ell \in [0, 1]$, and for $v_1, v_2 \in (0, 1)$, consider the following inequalities:

$$\begin{cases} |{}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}z(\ell))) - P\phi(\ell, z(\ell), z(v_1\ell), y(\ell)) - QI^{\vartheta}[\psi(\ell, z(\ell), z(v_2\ell), y(\ell))]| \le \dfrac{\mu}{2},\\ |{}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}y(\ell))) - P\phi(\ell, z(\ell), y(\ell), y(v_1\ell)) - QI^{\vartheta}[\psi(\ell, z(\ell), y(\ell), y(v_2\ell))]| \le \dfrac{\mu}{2}, \end{cases} \tag{4.1}$$

and

$$\begin{cases} |{}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}z(\ell))) - P\phi(\ell, z(\ell), z(v_1\ell), y(\ell)) - QI^{\vartheta}[\psi(\ell, z(\ell), z(v_2\ell), y(\ell))]| \le \dfrac{\mu}{2}\lambda(\ell),\\ |{}^{\mathrm{RL}}D^{\omega}({}^{\mathrm{C}}D^{\varkappa}({}^{\mathrm{C}}D^{\varrho}y(\ell))) - P\phi(\ell, z(\ell), y(\ell), y(v_1\ell)) - QI^{\vartheta}[\psi(\ell, z(\ell), y(\ell), y(v_2\ell))]| \le \dfrac{\mu}{2}\lambda(\ell), \end{cases} \tag{4.2}$$

where $\mu > 0$ and $\lambda : U \to \mathbb{R}_+$ is a continuous function.

Definition 4.1. We say that the solution of SFPEs (1.1) is MLU-Hyers (MLUH) stable with respect to $H_{\omega+\varkappa+\varrho}$, if there exists a real number u, and for all $\mu > 0$, there exist a solution $z, y \in \Theta$ for inequalities (4.1) and a solution $\theta, \varpi \in \Theta$ to system (1.1) such that

$$\|(z, y) - (\theta, \varpi)\|_{\Theta} \leq u\mu E_{\omega+\varkappa+\varrho}(\ell), \quad \ell \in U.$$

Definition 4.2. We say that the solution of SFPEs (1.1) is MLU-Hyers-Rassias (MLUHR) stable with respect to $\lambda H_{\omega+\varkappa+\varrho}$, if there exists a real number u_λ, and for all $\mu > 0$, there exist a solution $z, y \in \Theta$ for inequalities (4.2) and a solution $\theta, \varpi \in \Theta$ to problem (1.1) such that

$$\|(z, y) - (\theta, \varpi)\|_{\Theta} \leq u_\lambda \mu \lambda(\ell) E_{\omega+\varkappa+\varrho}(\ell), \quad \ell \in U.$$

Remark 4.3. The functions $z, y \in \Theta$ are a solution of (4.1), if there exist $j, j^* \in \Theta$ (which depend on z and y) such that

$$|j(\ell)| \leq \mu, |j^*(\ell)| \leq \mu,$$

and

$$\begin{aligned} &{}^{RL}D^{\omega}({}^{C}D^{\varkappa}({}^{C}D^{\varrho}z(\ell))) - P\phi(\ell, z(\ell), z(\upsilon_1\ell), y(\ell)) + QI^{\vartheta}[\psi(\ell, z(\ell), z(\upsilon_2\ell), y(\ell))] = j(\ell), \\ &{}^{RL}D^{\omega}({}^{C}D^{\varkappa}({}^{C}D^{\varrho}y(\ell))) - P\phi(\ell, z(\ell), y(\ell), y(\upsilon_1\ell)) + QI^{\vartheta}[\psi(\ell, z(\ell), y(\ell), y(\upsilon_2\ell))] = j^*(\ell), \end{aligned}$$

for all $\ell \in U$, and all $\upsilon_1, \upsilon_2 \in (0, 1)$.

Theorem 4.4. *The solution to SFPEs* (1.1) *is MLUH stable if the assumptions* (A_1) *and* (A_2) *are satisfied.*

Proof. Let $z, y \in \Theta$ be a solution to (4.1) and $\theta, \varpi \in \Theta$ be a solution to problem (1.1). Using Lemma 2.10, we obtain

$$\theta(\ell) = I^{\omega+\varkappa+\varrho}\, \Xi_{\theta,\varpi}(\ell) + \frac{\Gamma(\omega)a_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} + \frac{a_2\ell^{\varrho}}{\Gamma(\varrho+1)} + a_3, \quad a_i \in \mathbb{R}, \quad i = 1, 2, 3,$$

and

$$\varpi(\ell) = I^{\omega+\varkappa+\varrho}\, \Xi_{\varpi,\theta}(\ell) + \frac{\Gamma(\omega)a_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} + \frac{a_2\ell^{\varrho}}{\Gamma(\varrho+1)} + a_3, \quad a_i \in \mathbb{R}, \quad i = 1, 2, 3.$$

We obtain by integrating inequality (4.1) that

$$\begin{aligned} &\left| z(\ell) - I^{\omega+\varkappa+\varrho}\, \Xi_{z,y}(\ell) - \frac{\Gamma(\omega)c_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} - \frac{c_2\ell^{\varrho}}{\Gamma(\varrho+1)} - c_3 \right| \\ &\leq \frac{\mu\ell^{\omega+\varkappa+\varrho}}{2\Gamma(\omega+\varkappa+\varrho+1)} \leq \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, 3, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\left| y(\ell) - I^{\omega+\varkappa+\varrho}\, \Xi_{y,z}(\ell) - \frac{\Gamma(\omega)c_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} - \frac{c_2\ell^{\varrho}}{\Gamma(\varrho+1)} - c_3 \right| \\ &\leq \frac{\mu\ell^{\omega+\varkappa+\varrho}}{2\Gamma(\omega+\varkappa+\varrho+1)} \leq \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, 3, \end{aligned} \tag{4.4}$$

where

$$\Xi_{z,y}(\ell) = P\phi(\ell, z(\ell), z(\upsilon_1\ell), y(\ell)) + QI^{\vartheta}[\psi(\ell, z(\ell), z(\upsilon_2\ell), y(\ell))]$$

and

$$\Xi_{y,z} = P\phi(\ell, z(\ell), y(\ell), y(\upsilon_1\ell)) + QI^{\vartheta}[\psi(\ell, z(\ell), y(\ell), y(\upsilon_2\ell))].$$

From the conditions $z(0) = \theta(0)$, $z(\upsilon) = \theta(\upsilon)$, $y(0) = \varpi(0)$, $y(\upsilon) = \varpi(\upsilon)$, ${}^{C}D^{\varrho}(z(0)) = {}^{C}D^{\varrho}(\theta(0))$, and ${}^{C}D^{\varrho}(y(0)) = {}^{C}D^{\varrho}(\varpi(0))$, we obtain $a_1 = c_1$, $a_2 = c_2$, and $a_3 = c_3$.

Now, for all $\ell \in U$, we have

$$\begin{aligned} |z(\ell) - \theta(\ell)| \leq & \Bigg| z(\ell) - I^{\omega+\varkappa+\varrho}\ \Xi_{\varpi,\theta}(\ell) - \frac{\Gamma(\omega)a_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} \\ & - \frac{a_2\ell^{\varrho}}{\Gamma(\varrho+1)} - a_3 + I^{\omega+\varkappa+\varrho}[\Xi_{\varpi,\theta}(\ell) - \Xi_{y,z}(\ell)] \Bigg| \\ \leq & \left| z(\ell) - I^{\omega+\varkappa+\varrho}\ \Xi_{\varpi,\theta}(\ell) - \frac{\Gamma(\omega)a_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} - \frac{a_2\ell^{\varrho}}{\Gamma(\varrho+1)} - a_3 \right| \\ & + |I^{\omega+\varkappa+\varrho}[\Xi_{y,z}(\ell) - \Xi_{\varpi,\theta}(\ell)]|. \end{aligned} \tag{4.5}$$

Since, for $\upsilon_1, \upsilon_2 \in (0, 1)$,

$$\begin{aligned} & |I^{\omega+\varkappa+\varrho}[\Xi_{y,z}(\ell) - \Xi_{\varpi,\theta}(\ell)]| \\ & \leq \frac{|P|}{\Gamma(\omega+\varkappa+\varrho)} \int_0^{\ell} (\ell-\hbar)^{\omega+\varkappa+\varrho-1} |\phi(\hbar, y(\hbar), y(\upsilon_1\hbar), z(\hbar)) - \phi(\hbar, \theta(\hbar), \theta(\upsilon_1\hbar), \varpi(\hbar))| \mathrm{d}\hbar \\ & + \frac{|Q|}{\Gamma(\omega+\varkappa+\varrho+\vartheta)} \int_0^{\ell} (\ell-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1} |\psi(\hbar, y(\hbar), y(\upsilon_2\hbar), z(\hbar)) - \psi(\hbar, \theta(\hbar), \theta(\upsilon_2\hbar), \varpi(\hbar))| \mathrm{d}\hbar. \end{aligned}$$

Then, by (A_1), one has

$$\begin{aligned} & |I^{\omega+\varkappa+\varrho}[\Xi_{y,z}(\ell) - \Xi_{\varpi,\theta}(\ell)]| \\ & \leq \frac{|P|G_\phi}{\Gamma(\omega+\varkappa+\varrho)} \int_0^{\ell} (\ell-\hbar)^{\omega+\varkappa+\varrho-1} (|y(\hbar) - \theta(\hbar)| + |z(\hbar) - \varpi(\hbar)|) \mathrm{d}\hbar \\ & + \frac{|Q|G_\psi}{\Gamma(\omega+\varkappa+\varrho+\vartheta)} \int_0^{\ell} (\ell-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1} (|y(\hbar) - \theta(\hbar)| + |z(\hbar) - \varpi(\hbar)|) \mathrm{d}\hbar. \end{aligned} \tag{4.6}$$

Substituting (4.3) and (4.6) into (4.5), we have

$$\begin{aligned} & |z(\ell) - \theta(\ell)|_{\Theta_1} \\ & \leq \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)} + \frac{|P|G_\phi}{2\Gamma(\omega+\varkappa+\varrho)} \int_0^{\ell} (\ell-\hbar)^{\omega+\varkappa+\varrho-1} (|y(\hbar) - \theta(\hbar)|_{\Theta_1} + |z(\hbar) - \varpi(\hbar)|_{\Theta_2}) \mathrm{d}\hbar \\ & + \frac{|Q|G_\psi}{2\Gamma(\omega+\varkappa+\varrho+\vartheta)} \int_0^{\ell} (\ell-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1} (|y(\hbar) - \theta(\hbar)|_{\Theta_1} + |z(\hbar) - \varpi(\hbar)|_{\Theta_2}) \mathrm{d}\hbar. \end{aligned}$$

From Remarks 2.8 and 2.9, we obtain

$$\|z(\ell) - \theta(\ell)\|_{\Theta_1} \leq \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)} (E_{\omega+\varkappa+\varrho}[|P|G_\phi \ell^{\omega+\varkappa+\varrho}] + E_{\omega+\varkappa+\varrho+\vartheta}[|Q|G_\phi \ell^{\omega+\varkappa+\varrho+\vartheta}]). \tag{4.7}$$

Repeating the same steps in (4.4), we have

$$\|y(\ell) - \varpi(\ell)\|_{\Theta_2} \leq \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)} (E_{\omega+\varkappa+\varrho}[|P|G_\phi \ell^{\omega+\varkappa+\varrho}] + E_{\omega+\varkappa+\varrho+\vartheta}[|Q|G_\phi \ell^{\omega+\varkappa+\varrho+\vartheta}]). \tag{4.8}$$

Combining (4.7) and (4.8), we obtain that

$$\begin{aligned} & \|(z(\ell), y(\ell)) - (\theta(\ell), \varpi(\ell))\|_{\Theta} \\ & \leq |z(\ell) - \theta(\ell)|_{\Theta_1} + |y(\ell) - \varpi(\ell)|_{\Theta_2} \\ & \leq \frac{\mu}{\Gamma(\omega+\varkappa+\varrho+1)} (E_{\omega+\varkappa+\varrho}[|P|G_\phi \ell^{\omega+\varkappa+\varrho}] + E_{\omega+\varkappa+\varrho+\vartheta}[|Q|G_\phi \ell^{\omega+\varkappa+\varrho+\vartheta}]) \\ & = u\mu(E_{\omega+\varkappa+\varrho}[|P|G_\phi \ell^{\omega+\varkappa+\varrho}] + E_{\omega+\varkappa+\varrho+\vartheta}[|Q|G_\phi \ell^{\omega+\varkappa+\varrho+\vartheta}]), \end{aligned}$$

where $u = \frac{1}{\Gamma(\omega+\varkappa+\varrho+\vartheta+1)}$. Consequently, the solution of SFPEs (1.1) is MLUH stable. □

Theorem 4.5. *Via the assumptions* (A_1) *and* (A_2), *the solution of SFPEs* (1.1) *is MLUHR stable with respect to* $\lambda H_{\omega+\varkappa+\varrho}$, *provided that there exists* $u_\lambda > 0$ *such that*

$$\frac{1}{\Gamma(\omega+\varkappa+\varrho)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}\lambda(\hbar)\mathrm{d}\hbar \le u_\lambda \lambda(\ell), \quad \ell \in U, \tag{4.9}$$

where $\lambda \in C(U, \mathbb{R}_+)$ *is an increasing function.*

Proof. Integrating inequalities (4.2), we obtain

$$\begin{aligned}&\left|z(\ell) - I^{\omega+\varkappa+\varrho}\,\Xi_{z,y}(\ell) - \frac{\Gamma(\omega)c_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} - \frac{c_2\ell^{\varrho}}{\Gamma(\varrho+1)} - c_3\right|\\ &\quad\le \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}\lambda(\hbar)\mathrm{d}\hbar\end{aligned}$$

and

$$\begin{aligned}&\left|y(\ell) - I^{\omega+\varkappa+\varrho}\,\Xi_{y,z}(\ell) - \frac{\Gamma(\omega)c_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} - \frac{c_2\ell^{\varrho}}{\Gamma(\varrho+1)} - c_3\right|\\ &\quad\le \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}\lambda(\hbar)\mathrm{d}\hbar,\end{aligned}$$

where $z, y \in \Theta$ are a solution to (4.2). Assume that $\theta, \varpi \in \Theta$ are a solution to proposed problem (1.1), and then, we have

$$\theta(\ell) = I^{\omega+\varkappa+\varrho}\,\Xi_{\theta,\varpi}(\ell) + \frac{\Gamma(\omega)a_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} + \frac{a_2\ell^{\varrho}}{\Gamma(\varrho+1)} + a_3$$

and

$$\varpi(\ell) = I^{\omega+\varkappa+\varrho}\,\Xi_{\varpi,\theta}(\ell) + \frac{\Gamma(\omega)a_1\ell^{\omega+\varkappa+\varrho-1}}{\Gamma(\omega+\varkappa+\varrho)} + \frac{a_2\ell^{\varrho}}{\Gamma(\varrho+1)} + a_3.$$

Similar to the proof of Theorem 4.4, we have

$$\begin{aligned}&|z(\ell)-\theta(\ell)|_{\Theta_1}\\ &\le \frac{\mu}{2\Gamma(\omega+\varkappa+\varrho+1)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}\lambda(\hbar)\mathrm{d}\hbar\\ &\quad+\frac{|P|G_\phi}{2\Gamma(\omega+\varkappa+\varrho)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}(|y(\hbar)-\theta(\hbar)|_{\Theta_1}+|z(\hbar)-\varpi(\hbar)|_{\Theta_2})\mathrm{d}\hbar\\ &\quad+\frac{|Q|G_\psi}{2\Gamma(\omega+\varkappa+\varrho+\vartheta)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1}(|y(\hbar)-\theta(\hbar)|_{\Theta_1}+|z(\hbar)-\varpi(\hbar)|_{\Theta_2})\mathrm{d}\hbar.\end{aligned}$$

Using (4.9), we obtain that

$$\begin{aligned}&|z(\ell)-\theta(\ell)|_{\Theta_1}\\ &\le \frac{\mu}{2}u_\lambda\lambda(\ell)+\frac{|P|G_\phi}{2\Gamma(\omega+\varkappa+\varrho)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho-1}(|y(\hbar)-\theta(\hbar)|_{\Theta_1}+|z(\hbar)-\varpi(\hbar)|_{\Theta_2})\mathrm{d}\hbar\\ &\quad+\frac{|Q|G_\psi}{2\Gamma(\omega+\varkappa+\varrho+\vartheta)}\int_0^\ell (\ell-\hbar)^{\omega+\varkappa+\varrho+\vartheta-1}(|y(\hbar)-\theta(\hbar)|_{\Theta_1}+|z(\hbar)-\varpi(\hbar)|_{\Theta_2})\mathrm{d}\hbar.\end{aligned}$$

Applying Remarks 2.8 and 2.9, we obtain

$$\|z-\theta\|_{\Theta_1} \leq \frac{\mu}{2} u_\lambda \lambda(\ell)(E_{\omega+\varkappa+\varrho}[|P|G_\phi \ell^{\omega+\varkappa+\varrho}] + E_{\omega+\varkappa+\varrho+\vartheta}[|Q|G_\phi \ell^{\omega+\varkappa+\varrho+\vartheta}]). \tag{4.10}$$

Analogously, we have

$$\|y-\varpi\|_{\Theta_2} \leq \frac{\mu}{2} u_\lambda \lambda(\ell)(E_{\omega+\varkappa+\varrho}[|P|G_\phi \ell^{\omega+\varkappa+\varrho}] + E_{\omega+\varkappa+\varrho+\vartheta}[|Q|G_\phi \ell^{\omega+\varkappa+\varrho+\vartheta}]). \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$\begin{aligned}\|(z,y)-(\theta,\varpi)\|_{\Theta} &\leq \|z-\theta\|_{\Theta_1} + \|y-\varpi\|_{\Theta_2}\\ &\leq \mu u_\lambda \lambda(\ell)(E_{\omega+\varkappa+\varrho}[|P|G_\phi \ell^{\omega+\varkappa+\varrho}] + E_{\omega+\varkappa+\varrho+\vartheta}[|Q|G_\phi \ell^{\omega+\varkappa+\varrho+\vartheta}]).\end{aligned}$$

Hence, the solution to the considerable problem (1.1) is MLUHR stable. □

5 Application

There is no doubt that providing numerical examples as special cases of the subject under study strengthens the model and clarifies each parameter and its significance. It also enhances theoretical results and validates the validity of assumptions utilized to reach the intended goal. Therefore, in this section, we give the following example as a special model of problem (1.1):

Example 5.1. Consider the following fractional pantograph problem:

$$\begin{cases} {}^{RL}D^{\frac{5}{6}}\left({}^{C}D^{\frac{1}{3}}\left({}^{C}D^{\frac{3}{4}}\theta(\ell)\right)\right) = \dfrac{e^{-3}}{39\pi^3}\left(\dfrac{1}{6} + \dfrac{e^{-3\ell}}{55\pi e^3}\sin\theta(\ell) + \dfrac{e^{-3\ell}}{55\pi e^3}\cos\theta\left(\dfrac{11}{12}\ell\right) + \dfrac{e^{-3\ell}}{55\pi e^3}\sin\varpi(\ell)\right)\\ \quad + \dfrac{1}{49e^3} I^{\frac{5}{2}}\left[\dfrac{11}{13} + \dfrac{1}{\sqrt{(50e^2)^2 + l^2}}\theta(\ell) + \dfrac{e^{-\ell}}{50e^2 + l^2}\sin\theta\left(\dfrac{13}{12}\ell\right) + \dfrac{e^{-\ell}}{50e^2 + l^2}\cos\varpi(\ell)\right],\\ {}^{RL}D^{\frac{5}{6}}\left({}^{C}D^{\frac{1}{3}}\left({}^{C}D^{\frac{3}{4}}\varpi(\ell)\right)\right) = \dfrac{e^{-3}}{39\pi^3}\left(\dfrac{1}{6} + \dfrac{e^{-3\ell}}{55\pi e^3}\sin\theta(\ell) + \dfrac{e^{-3\ell}}{55\pi e^3}\sin(\varpi\ell) + \dfrac{e^{-3\ell}}{55\pi e^3}\cos\varpi\left(\dfrac{11}{12}\ell\right)\right)\\ \quad + \dfrac{1}{49e^3} I^{\frac{5}{2}}\left[\dfrac{11}{13} + \dfrac{1}{\sqrt{(50e^2)^2 + l^2}}\theta(\ell) + \dfrac{e^{-\ell}}{50e^2 + l^2}\sin\varpi(\ell) + \dfrac{e^{-\ell}}{50e^2 + l^2}\cos\varpi\left(\dfrac{13}{12}\ell\right)\right],\\ \theta(\ell) = 0, \quad \dfrac{5}{4}\theta(1) - \dfrac{4}{3}\theta\left(\dfrac{6}{7}\right) = \dfrac{1}{55}\theta(\ell), \quad {}^{C}D^{\varrho}\theta(0) = \dfrac{\sqrt{3}}{4},\\ \varpi(\ell) = 0, \quad \dfrac{5}{4}\varpi(1) - \dfrac{4}{3}\varpi\left(\dfrac{6}{7}\right) = \dfrac{1}{55}\varpi(\ell), \quad {}^{C}D^{\varrho}\varpi(0) = \dfrac{\sqrt{3}}{4}, \end{cases} \tag{5.1}$$

and the following fractional inequalities:

$$\begin{cases} \left|{}^{RL}D^{\frac{5}{6}}\left({}^{C}D^{\frac{1}{3}}\left({}^{C}D^{\frac{3}{4}}z(\ell)\right)\right) - \dfrac{e^{-3}}{39\pi^3}\phi(\ell, z(\ell), z(\upsilon_1\ell), y(\ell)) - \dfrac{1}{49e^3} I^{\frac{5}{2}}[\psi(\ell, z(\ell), z(\upsilon_2\ell), y(\ell))]\right| \leq \dfrac{\mu}{2},\\ \left|{}^{RL}D^{\frac{5}{6}}\left({}^{C}D^{\frac{1}{3}}\left({}^{C}D^{\frac{3}{4}}y(\ell)\right)\right) - \dfrac{e^{-3}}{39\pi^3}\phi(\ell, z(\ell), y(\ell), y(\upsilon_1\ell)) - \dfrac{1}{49e^3} I^{\frac{5}{2}}[\psi(\ell, z(\ell), y(\ell), y(\upsilon_2\ell))]\right| \leq \dfrac{\mu}{2}, \end{cases}$$

and

$$\begin{cases} \left|{}^{RL}D^{\frac{5}{6}}\left({}^{C}D^{\frac{1}{3}}\left({}^{C}D^{\frac{3}{4}}z(\ell)\right)\right) - \dfrac{e^{-3}}{39\pi^3}\phi(\ell, z(\ell), z(\upsilon_1\ell), y(\ell)) - \dfrac{1}{49e^3} I^{\frac{5}{2}}[\psi(\ell, z(\ell), z(\upsilon_2\ell), y(\ell))]\right| \leq \dfrac{\mu}{2}\lambda(\ell),\\ \left|{}^{RL}D^{\frac{5}{6}}\left({}^{C}D^{\frac{1}{3}}\left({}^{C}D^{\frac{3}{4}}y(\ell)\right)\right) - \dfrac{e^{-3}}{39\pi^3}\phi(\ell, z(\ell), y(\ell), y(\upsilon_1\ell)) - \dfrac{1}{49e^3} I^{\frac{5}{2}}[\psi(\ell, z(\ell), y(\ell), y(\upsilon_2\ell))]\right| \leq \dfrac{\mu}{2}\lambda(\ell), \end{cases}$$

where

$$\phi(\ell, z(\ell), z(\upsilon_1\ell), y(\ell)) = \frac{1}{6} + \frac{e^{-3\ell}}{55\pi e^3}\sin\theta(\ell) + \frac{e^{-3\ell}}{55\pi e^3}\cos\theta\left(\frac{11}{12}\ell\right) + \frac{e^{-3\ell}}{55\pi e^3}\sin\varpi(\ell),$$
$$\phi(\ell, z(\ell), y(\ell), y(\upsilon_1\ell)) = \frac{1}{6} + \frac{e^{-3\ell}}{55\pi e^3}\sin\theta(\ell) + \frac{e^{-3\ell}}{55\pi e^3}\sin(\varpi\ell) + \frac{e^{-3\ell}}{55\pi e^3}\cos\varpi\left(\frac{11}{12}\ell\right),$$
$$\psi(\ell, z(\ell), z(\upsilon_2\ell), y(\ell)) = \frac{11}{13} + \frac{1}{\sqrt{(50e^2)^2 + l^2}}\theta(\ell) + \frac{e^{-\ell}}{50e^2 + l^2}\sin\theta\left(\frac{13}{12}\ell\right) + \frac{e^{-\ell}}{50e^2 + l^2}\cos\varpi(\ell),$$
$$\psi(\ell, z(\ell), y(\ell), y(\upsilon_2\ell)) = \frac{11}{13} + \frac{1}{\sqrt{(50e^2)^2 + l^2}}\theta(\ell) + \frac{e^{-\ell}}{50e^2 + l^2}\sin\varpi(\ell) + \frac{e^{-\ell}}{50e^2 + l^2}\cos\varpi\left(\frac{13}{12}\ell\right),$$

$\omega = \frac{5}{6}$, $\varkappa = \frac{1}{3}$, $\varrho = \frac{3}{4}$, $\vartheta = 2.5$, $P = \frac{e^{-3}}{39\pi^3}$, $Q = \frac{1}{49e^3}$, $\sigma(\theta) = \frac{1}{55}\theta(\ell)$, $\sigma(\varpi) = \frac{1}{55}\varpi(\ell)$, $\rho_1 = \frac{5}{4}$, $\rho_2 = \frac{4}{3}$, $\upsilon_1 = \frac{11}{12}$, $\upsilon_2 = \frac{13}{12}$, $\alpha = \frac{6}{7}$, $\beta = \frac{\sqrt{3}}{4}$, and $\rho_1 \neq \rho_2\alpha^{\omega+\varkappa+\varrho-1}$.

For $\ell \in [0, 1]$, $\theta_i, \varpi_i \in \mathbb{R}$, $i = 1, 2, 3$, we have

$$|\phi(\ell, \theta_1(\ell), \theta_2(\ell), \theta_3(\upsilon\ell)) - \phi(\ell, \varpi_1(\ell), \varpi_2(\ell), \varpi_3(\upsilon\ell))| \le \frac{1}{110\pi e^3}(|\theta_1 - \varpi_1| + |\theta_2 - \varpi_2| + |\theta_3 - \varpi_3|),$$
$$|\psi(\ell, \theta_1(\ell), \theta_2(\ell), \theta_3(\upsilon\ell)) - \psi(\ell, \varpi_1(\ell), \varpi_2(\ell), \varpi_3(\upsilon\ell))| \le \frac{1}{100e^2}(|\theta_1 - \varpi_1| + |\theta_2 - \varpi_2| + |\theta_3 - \varpi_3|),$$

and

$$|\sigma(\theta_1) - \sigma(\varpi_1)| \le \frac{1}{55}|\theta_1 - \varpi_1|.$$

Therefore, assumptions (A_1) and (A_2) are satisfied with $G_\phi = \frac{1}{55\pi e^3}$, $G_\psi = \frac{1}{50e^2}$, and $\xi = \frac{1}{55}$, respectively. Using the provided data, it is discoverable that

$$G = \max\{G_\psi, G_\phi\} = \frac{1}{50e^2} \quad \text{and} \quad \rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1} = 9.236713958 \times 10^{-2}.$$

After routine calculations, we found that $\Lambda = 7.128062804 \times 10^{-4}$ and $\Lambda^* = 7.0972085 \times 10^{-4}$. Hence, the hypothesis

$$3G(\Lambda + \Lambda^*) = 9.54264412 \times 10^{-3} < 0.7997420678 = 1 - \frac{\xi}{|\rho_1 - \rho_2\alpha^{\omega+\varkappa+\varrho-1}|}$$

is fulfilled. Therefore, the considerable problem (4.11) has a solution on U based on Theorem 3.1. Moreover, the solution to system (4.11) is MLUH stable with

$$|z(\ell) - \theta(\ell)| + |y(\ell) - \varpi(\ell)| \le \frac{\mu}{\Gamma\left(\frac{35}{12}\right)}\left(E_{\frac{23}{12}}\left[\frac{e^{-3}}{2{,}145\pi^4e^3}\ell^{\frac{23}{12}}\right] + E_{\frac{53}{12}}\left[\frac{1}{2{,}695e^6\pi}\ell^{\frac{53}{12}}\right]\right), \quad \ell \in [0, 1].$$

Now, let $\lambda(\ell) = \ell^2$, then by (2.1), one can write

$$I^{\omega+\varkappa+\varrho}\lambda(\ell) = I^{\frac{5}{6}+\frac{1}{3}+\frac{3}{4}}(\ell^2) = \frac{3}{16\Gamma\left(\frac{16}{3}\right)}\ell^{2+\frac{5}{6}+\frac{1}{3}+\frac{3}{4}} \le \frac{3}{16\Gamma\left(\frac{16}{3}\right)}\ell^2 = u_\lambda\mu\lambda(\ell).$$

Hence, condition (4.9) is fulfilled with $\lambda(\ell) = \ell^2$ and $u_\lambda = \frac{3}{16\Gamma\left(\frac{16}{3}\right)}$. According to Theorem 4.5, the solution to problem (4.11) is MLUHR stable with

$$|z(\ell) - \theta(\ell)| + |y(\ell) - \varpi(\ell)| \le \frac{\mu\ell^2}{\Gamma\left(\frac{47}{12}\right)}\left(E_{\frac{23}{12}}\left[\frac{e^{-3}}{2{,}145\pi^4e^3}\ell^{\frac{23}{12}}\right] + E_{\frac{53}{12}}\left[\frac{1}{2{,}695e^6\pi}\ell^{\frac{53}{12}}\right]\right), \quad \ell \in [0, 1].$$

6 Conclusion and future work

A pantograph is a mechanical connection that is used to replicate and scale a drawing or image. It is made up of a sequence of interconnected parallelograms that can keep their shape even as they change size. The pantograph was invented in the seventeenth century and has since been utilized for a wide range of purposes. The pantograph can be mathematically represented by a set of equations known as the pantograph equations. These equations define the relationships between the various lengths and angles of the parallelograms that comprise the pantograph. The pantograph has several applications in engineering, including the design of machines and mechanisms that require exact scaling and copying of movements. So, in this manuscript, we used the FP tool to give necessary and sufficient conditions for the existence and uniqueness of the solution for a system of SFPEs intervening in three sequential fractional derivatives. Also, we studied the MLU stability of the considerable system via generalized singular Gronwall's inequality. Finally, we provided an illustrative example to support the obtained results. As future work, our results can be reflected on DEs with arbitrary fractional order, linear and nonlinear fractional integro-differential systems, Hadamard fractional derivatives, and many fractional operators.

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