Research Article

Arunachalam Selvam, Sriramulu Sabarinathan, and Arumugam Ponmana Selvan*

Mittag-Leffler-Hyers-Ulam stability for a first- and second-order nonlinear differential equations using Fourier transform

https://doi.org/10.1515/dema-2024-0033
received November 7, 2021; accepted May 24, 2024

Abstract: In this article, we apply the Fourier transform to prove the Hyers-Ulam and Hyers-Ulam-Rassias stability for the first- and second-order nonlinear differential equations with initial conditions. Additionally, we extend the results to investigate the Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability of these differential equations using the proposed method.

Keywords: Fourier transform, Hyers-Ulam and Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability, nonlinear differential equations

MSC 2020: 34K20, 26D10, 39B82, 34A40, 39A30

1 Introduction

The study of stability for various functional equations was invented by a famous mathematician, Ulam [1], in 1940. He raised the question concerning the stability of functional equations: "Give conditions for a linear function near an approximately linear function to exist." The first solution was brilliantly answered to the problem of Ulam for Cauchy additive functional equations based on Banach spaces by Hyers [2] in 1941. Rassias [3] later provided a generalized solution to Ulam's question for approximately linear mappings in 1978. During these days, numerous mathematicians have contributed to the advancement of Ulam's problem, addressing various functional equations in different spaces and directions [4–15].

Ulam's problem has recently been extended by replacing functional equations with differential equations in the following form:

$$\zeta(g, \varphi, \varphi', \varphi'', \ldots, \varphi^{(n)}) = 0,$$

which is said to have Hyers-Ulam stability if for a given $\varepsilon > 0$ and a function φ such that

$$|\zeta(g, \varphi, \varphi', \varphi'', \ldots, \varphi^{(n)})| \le \varepsilon,$$

there exists some φ_a of the differential equation such that $|\varphi(s) - \varphi_a(s)| \le H(\varepsilon)$ and

$$\lim_{\varepsilon \to 0} H(\varepsilon) = 0.$$

*** Corresponding author: Arumugam Ponmana Selvan,** Department of Mathematics, Rajalakshmi Engineering College (Autonomous), Thandalam, Chennai - 602 105, Tamil Nadu, India, e-mail: selvaharry@yahoo.com
Arunachalam Selvam: Department of Mathematics, College of Engineering and Technology, SRM Institute of Science & Technology, Kattankulthur – 603 203, Tamil Nadu, India, e-mail: selvamamaths@gmail.com
Sriramulu Sabarinathan: Department of Mathematics, College of Engineering and Technology, SRM Institute of Science & Technology, Kattankulthur – 603 203, Tamil Nadu, India, e-mail: ssabarimaths@gmail.com

Oblaza seems to be the first author to investigate the Hyers-Ulam stability of linear differential equations [16,17]. Then, Alsina and Ger [18] published their work, which exhibits the Hyers-Ulam stability of the linear differential equation $\psi'(\ell) = \lambda\psi(\ell)$. Jung derived the general setting for Hyers-Ulam stability of the first-order linear differential equations in [19–22].

The generalized Hyers-Ulam stability of higher-order linear differential equations using the Laplace transform method was proved by Alqifiary and Jung [23] in 2014. In [24], the authors proved the Mittag-Leffler-Hyers-Ulam stability of the first- and second-order linear differential equations using the Fourier transform method [25,26]. The Hyers-Ulam stability of differential equations is being studied right now [27–36], and the research is still going on [37,38].

In [39], the authors investigated the existence, uniqueness, and Hyers-Ulam stability of the solution derived for the coupled system of fractional differential equations in Caputo sense with nonlinear p-Laplacian operator. In 2018, by nonlinear Leray-Schauder-type alternative and Banach's fixed point theorem, they investigated the existence and uniqueness of solutions and also proved the Hyers-Ulam stability for the coupled system of fractional differential equations with the nonlinear p-Laplacian operator and Riemann-Liouville integral boundary conditions in [40]. In [41], the authors studied the general class of nonlinear singular fractional differential equations with p-Laplacian for the existence and uniqueness of a positive solution and the Hyers-Ulam stability [42–44].

Recently, the Ulam stability of the various types of differential equations of first-order, second-order, and higher-orders are investigated using Laplace transforms in [45,46], Aboodth transform in [47,48], and Mahgoub transform in [49–51].

Motivated and inspired by the existing literature, the main goal of this article is to implement the Fourier transform technique, we establish the Hyers-Ulam and Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam, and Mittag-Leffler-Hyers-Ulam-Rassias stability of the following first- and second-order nonlinear differential equations:

$$u\varphi'(\ell) + v\varphi(\ell) = n(\ell, \varphi(\ell)), \tag{1}$$

$$\varphi(0) = \alpha_0, \tag{2}$$

and

$$u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) = n(\ell, \varphi(\ell), \varphi'(\ell)), \tag{3}$$

$$\varphi(0) = \alpha_0, \varphi'(0) = \alpha_1, \tag{4}$$

for any $\ell > 0$. Here, u, v, and w represent the given constants in $\mathbb{F}$, $\varphi(\ell)$ is a continuously differentiable function, and $n(\ell, \varphi(\ell))$ and $n(\ell, \varphi(\ell), \varphi'(\ell))$ denote the given nonlinear functions.

2 Preliminaries

In this section, we refer to a few basic definitions, notations, and theorems that are useful throughout this article to prove the main results. Throughout this article, $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Definition 2.1. For every function $p : (0, \infty) \to \mathbb{F}$ of exponential order, we define the Fourier transform of p by

$$\mathcal{F}\{p(\ell)\} = P(s) = \int_{-\infty}^{\infty} p(\ell)e^{-i\ell s}\mathrm{d}\ell.$$

Then, at the point of continuity of p, we have

$$p(\ell) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(s)e^{-i\ell s}\mathrm{d}s,$$

which is said to be an inverse Fourier transform, and it is denoted by $\mathcal{F}^{-1}$. We also introduce the convolution of two functions.

Definition 2.2. Let p and q be two functions both Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of s to which the Lebesgue integral

$$r(s) = \int_{-\infty}^{\infty} p(\ell)q(s-\ell)\mathrm{d}\ell$$

exists. This integral defines a function r on S, which is said to be a convolution of p and q. We also denote $r = p * q$ for convolution.

Definition 2.3. A Mittag-Leffler function of one parameter is defined as

$$E_\nu(\ell) = \sum_{k=0}^{\infty} \frac{\ell^k}{\Gamma(\nu k + 1)},$$

where $\ell, \nu \in \mathbb{C}$ and $Re(\nu) > 0$. If we put $\nu = 1$, then the aforementioned equation becomes

$$E_1(\ell) = \sum_{k=0}^{\infty} \frac{\ell^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\ell^k}{k} = e^\ell,$$

where $E_\nu(\ell)$ is the Mittag-Leffler function of one parameter.

Definition 2.4. Generalization by $E_\nu(\ell)$ is defined in the function

$$E_{\nu,\vartheta}(\ell) = \sum_{k=0}^{\infty} \frac{\ell^k}{\Gamma(\nu k + \vartheta)},$$

where $\ell, \nu, \vartheta \in \mathbb{C}$, $Re(\nu) > 0$ and $Re(\vartheta) > 0$.

Definition 2.5. Let $\varphi : (0, \infty) \to \mathbb{F}$ be a continuously differentiable function that satisfies

$$|u\varphi'(\ell) + \nu\varphi(\ell) - n(\ell, \varphi(\ell))| \leq \varepsilon,$$

for any $\ell > 0$. An initial value problem (IVP) (1) with initial condition (2) is Hyers-Ulam stable; if for all $\varepsilon > 0$, then there exist $H > 0$ and $\psi : (0, \infty) \to \mathbb{F}$, which is satisfies the nonlinear differential equation (1) such that $|\varphi(\ell) - \psi(\ell)| \leq H\varepsilon$, for any $\ell > 0$, where H is called Hyers-Ulam stability constant of (1).

Definition 2.6. Let $\varphi : (0, \infty) \to \mathbb{F}$ be a continuously differentiable function; if for each $\varepsilon > 0$, there exists $\phi : (0, \infty) \to (0, \infty)$, which satisfies the inequality

$$|u\varphi'(\ell) + \nu\varphi(\ell) - n(\ell, \varphi(\ell))| \leq \phi(\ell)\varepsilon,$$

for any $\ell > 0$. An IVP (1) with initial condition (2) is Hyers-Ulam-Rassias stable; then, there exists $H > 0$ and $\psi : (0, \infty) \to \mathbb{F}$, which satisfies the nonlinear differential equation (1) such that $|\varphi(\ell) - \psi(\ell)| \leq H\phi(\ell)\varepsilon$, for any $\ell > 0$, where H is called Hyers-Ulam-Rassias stability constant of (1).

Definition 2.7. Let $\varphi : (0, \infty) \to \mathbb{F}$ be a continuously differentiable function that satisfies

$$|u\varphi'(\ell) + \nu\varphi(\ell) - n(\ell, \varphi(\ell))| \leq \varepsilon E_\nu(\ell^\nu),$$

where $E_\nu(\ell^\nu)$ is a Mittag-Leffler function, for any $\ell > 0$. An IVP (1) is Mittag-Leffler-Hyers-Ulam stable; if for all $\varepsilon > 0$, then there exist $H > 0$ and $\psi : (0,\infty) \to \mathbb{F}$, which is satisfies the nonlinear differential equation (1) such that $|\varphi(\ell) - \psi(\ell)| \le H\varepsilon E_\nu(\ell^\nu)$, for any $\ell > 0$, where H is called Mittag-Leffler-Hyers-Ulam stability constant of (1).

Definition 2.8. Let $\varphi : (0,\infty) \to \mathbb{F}$ be a continuously differentiable function, if there exist $\phi : (0,\infty) \to (0,\infty)$, which satisfies

$$|u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell))| \le \phi(\ell)\varepsilon E_\nu(\ell^\nu),$$

where $E_\nu(\ell^\nu)$ is a Mittag-Leffler function, for any $\ell > 0$. An IVP (1) is Mittag-Leffler-Hyers-Ulam-Rassias stable; if for all $\varepsilon > 0$, then there exist $H > 0$ and $\psi : (0,\infty) \to \mathbb{F}$, which is satisfies the nonlinear differential equation (1) such that $|\varphi(\ell) - \psi(\ell)| \le H\phi(\ell)\varepsilon E_\nu(\ell^\nu)$, for any $\ell > 0$, where H is called Mittag-Leffler-Hyers-Ulam-Rassias stability constant of (1).

Definition 2.9. Let $\varphi : (0,\infty) \to \mathbb{F}$ be a twice continuously differentiable function that satisfies $|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \le \varepsilon$, for any $\ell > 0$. A nonlinear differential equation (3) with initial condition (4) is Hyers-Ulam stable, then for every $\varepsilon > 0$, there exist $H > 0$ and $\psi : (0,\infty) \to \mathbb{F}$, which satisfies the nonlinear differential equation (3) such that $|\varphi(\ell) - \psi(\ell)| \le H\varepsilon$, for any $\ell > 0$, where H is called Hyers-Ulam stability constant of (3).

Definition 2.10. Let $\varphi : (0,\infty) \to \mathbb{F}$ be a twice continuously differentiable function, if there exist $\phi : (0,\infty) \to (0,\infty)$, which satisfies

$$|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \le \phi(\ell)\varepsilon,$$

for any $\ell > 0$. An IVP (3) with initial condition (4) is Hyers-Ulam-Rassias stable, and then for all $\varepsilon > 0$, there exist $H > 0$ and $\psi : (0,\infty) \to \mathbb{F}$, which satisfies the nonlinear differential equation (3) such that $|\varphi(\ell) - \psi(\ell)| \le H\phi(\ell)\varepsilon$, for any $\ell > 0$, where H is called Hyers-Ulam-Rassias stability constant of (3).

Definition 2.11. Let $\varphi : (0,\infty) \to \mathbb{F}$ be a twice continuously differentiable function that satisfies

$$|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \le \varepsilon E_\nu(\ell^\nu),$$

where $E_\nu(\ell^\nu)$ is a Mittag-Leffler function, for any $\ell > 0$. An IVP (3) with initial condition (4) is Mittag-Leffler-Hyers-Ulam stable; if for all $\varepsilon > 0$, then there exist $H > 0$ and $\psi : (0,\infty) \to \mathbb{F}$, which satisfies the nonlinear differential equation (3) such that $|\varphi(\ell) - \psi(\ell)| \le H\varepsilon E_\nu(\ell^\nu)$, for any $\ell > 0$, where H is called Mittag-Leffler-Hyers-Ulam stability constant of (3).

Definition 2.12. Let $\varphi : (0,\infty) \to \mathbb{F}$ be a twice continuously differentiable function, if there exist $\phi : (0,\infty) \to (0,\infty)$, which satisfies

$$|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \le \phi(\ell)\varepsilon E_\nu(\ell^\nu),$$

where $E_\nu(\ell^\nu)$ is a Mittag-Leffler function, for any $\ell > 0$. An IVP (3) with initial condition (4) has Mittag-Leffler-Hyers-Ulam-Rassias stability; if for each $\varepsilon > 0$, then there exist $H > 0$ and $\psi : (0,\infty) \to \mathbb{F}$, which satisfies the nonlinear differential equation (3) such that $|\varphi(\ell) - \psi(\ell)| \le H\phi(\ell)\varepsilon E_\nu(\ell^\nu)$, for any $\ell > 0$, where H is called Mittag-Leffler-Hyers-Ulam-Rassias stability constant of (3).

3 Ulam stability of first-order nonlinear IVP

In the following theorems, we will prove various types of Ulam stability of the first-order nonlinear equation (1) with initial conditions (2).

Theorem 3.1. *Let u and v be constants in $\mathbb{F}$. For every $\varepsilon > 0$, there is a constant $H > 0$ such that $\varphi : (0, \infty) \to \mathbb{F}$, which is a continuously differentiable function, and it satisfies the inequality*

$$|u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell))| \leq \varepsilon, \tag{5}$$

for any $\ell > 0$, there exists $\psi : (0, \infty) \to \mathbb{F}$ satisfying (1) such that $|\varphi(\ell) - \psi(\ell)| \leq H\varepsilon$, for any $\ell > 0$.

Proof. Assume that $\varphi : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function that satisfies inequality (5). Let us choose a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell)), \tag{6}$$

for any $\ell > 0$. In view of (5), we have $|p(\ell)| \leq \varepsilon$. Applying Fourier transform to (6), we obtain that

$$\begin{aligned} \mathcal{F}\{p(\ell)\} &= \mathcal{F}\{u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell))\} \\ &= u\mathcal{F}\{\varphi'(\ell)\} + v\mathcal{F}\{\varphi(\ell)\} - \mathcal{F}\{n(\ell, \varphi(\ell))\} \\ P(\xi) &= (v - i\xi u)\Phi(\xi) - \mathcal{F}\{n(\ell, \varphi(\ell))\} \end{aligned}$$

$$\Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell))\}}{v - i\xi u}. \tag{7}$$

Thus,

$$\mathcal{F}\{\varphi(\ell)\} = \Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell))\}}{v - i\xi u}. \tag{8}$$

Now, choosing $Q(\xi) = \mathcal{F}\{q(\ell)\} = \frac{1}{v - i\xi u}$, then we have

$$\mathcal{F}\{q(\ell)\} = \frac{1}{v - i\xi u} \Rightarrow q(\ell) = \mathcal{F}^{-1}\left\{\frac{1}{v - i\xi u}\right\}.$$

We will set the solution

$$\psi(\ell) = e^{-\frac{v}{u}\ell} + \{n(\ell, \psi(\ell))\} * q(\ell).$$

Then, using Fourier transform on $\psi(\ell)$, we will have

$$\mathcal{F}\{\psi(\ell)\} = \Psi(\xi) = \int_{-\infty}^{\infty} e^{-\frac{v}{u}\ell} e^{is\ell} \mathrm{d}\ell + \frac{\mathcal{F}\{n(\ell, \psi(\ell))\}}{v - i\xi u},$$

$$\Psi(\xi) = \frac{\mathcal{F}\{n(\ell, \psi(\ell))\}}{v - i\xi u}, \tag{9}$$

where $\int_{-\infty}^{\infty} e^{-\frac{v}{u}\ell} e^{is\ell} \mathrm{d}\ell = 0$. Now,

$$\mathcal{F}\{u\psi'(\ell) + v\psi(\ell)\} = -i\xi u\Psi(\xi) + v\Psi(\xi) = \mathcal{F}\{n(\ell, \psi(\ell))\},$$

we have $\mathcal{F}\{u\psi'(\ell) + v\psi(\ell)\} = \mathcal{F}\{n(\ell, \psi(\ell))\}$. Since $\mathcal{F}$ is linear and one-to-one operator, $u\psi'(\ell) + v\psi(\ell) = n(\ell, \psi(\ell))$. Hence, $\psi(\ell)$ is a solution of the differential equation (1). Now, using (8) and (9),

$$\begin{aligned} \mathcal{F}\{\varphi(\ell)\} - \mathcal{F}\{\psi(\ell)\} &= \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell))\}}{v - i\xi u} - \frac{\mathcal{F}\{n(\ell, \psi(\ell))\}}{v - (i\xi u)} \\ &= \frac{P(\xi)}{v - i\xi u} = P(\xi) \times Q(\xi) \\ \Rightarrow \quad \mathcal{F}\{\varphi(\ell) - \psi(\ell)\} &= \mathcal{F}\{p(\ell) \times q(\ell)\}. \end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one linear operator, $\varphi(\ell) - \psi(\ell) = p(\ell) \times q(\ell)$. Taking modulus on both sides $|p(\ell)| \leq \varepsilon$, we have

$$|\varphi(\ell) - \psi(\ell)| = |p(\ell) \times q(\ell)| = \left|\int_0^\ell p(s)q(\ell - s)\mathrm{d}s\right|,$$

$$\leq |p(\ell)|\left|\int_0^\ell q(\ell - s)\mathrm{d}s\right| \leq \varepsilon\left|\int_0^\ell q(\ell - s)\mathrm{d}s\right| \leq H\varepsilon,$$

for any $\ell > 0$, where $H = |\int_0^\ell q(\ell - s)\mathrm{d}s|$ and the integral exists for each value of ℓ. Therefore, the nonlinear differential equation (1) is Hyers-Ulam stable. □

Corollary 3.2. *Let u and v be constants in $\mathbb{F}$. For every $\varepsilon > 0$, there exists a real constant $H > 0$ such that $\varphi : (0, \infty) \to \mathbb{F}$ being a continuously differentiable function with $\phi : (0, \infty) \to (0, \infty)$ being an integrable function that satisfies the following inequality*:

$$|u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell))| \leq \phi(\ell)\varepsilon, \tag{10}$$

for any $\ell > 0$, there exists some $\psi : (0, \infty) \to \mathbb{F}$ of (1) *such that $|\varphi(\ell) - \psi(\ell)| \leq H\phi(\ell)\varepsilon$, for any $\ell > 0$.*

Proof. Choosing a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell)), \tag{11}$$

for any $\ell > 0$. Assume that $\varphi : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function that satisfies inequality (10). In view of (10), we have $|p(\ell)| \leq \phi(\ell)\varepsilon$. Using Fourier transform to equation (11), we have

$$\Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell))\}}{v - i\xi u}. \tag{12}$$

Using the same technique in Theorem 3.1, we can easily reach the rest of the proof. Also, one can easily reach at

$$|\varphi(\ell) - \psi(\ell)| = |p(\ell) \times q(\ell)| = \left|\int_0^\ell p(s)q(\ell - s)\mathrm{d}s\right|,$$

$$\leq |p(\ell)|\left|\int_0^\ell q(\ell - s)\mathrm{d}s\right| \leq \phi(\ell)\varepsilon\left|\int_0^\ell q(\ell - s)\mathrm{d}s\right| \leq H\phi(\ell)\varepsilon,$$

for any $\ell > 0$, where $H = |\int_0^\ell q(\ell - s)\mathrm{d}s|$ and the integral exists for each value of ℓ. Therefore, the nonlinear differential equation (1) is Hyers-Ulam-Rassias stable. □

Now, we will establish the Mittag-Leffler-Hyers-Ulam stability of the differential equation (1).

Theorem 3.3. *Let u and v be the constants in $\mathbb{F}$. For all $\varepsilon > 0$ and a real constant $H > 0$ such that $\varphi : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function that satisfies*

$$|u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell))| \leq \varepsilon E_v(\ell^v), \tag{13}$$

where $E_v(\ell^v)$ is a Mittag-Leffler function; for any $\ell > 0$, there exists some $\psi : (0, \infty) \to \mathbb{F}$, which satisfies the differential equation (1) *such that $|\varphi(\ell) - \psi(\ell)| \leq H\varepsilon E_v(\ell^v)$, for any $\ell > 0$.*

Proof. Assume that $\varphi : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function that satisfies inequality (13). Let us define a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell)), \tag{14}$$

for any $\ell > 0$. In view of (13), we have $|p(\ell)| \le \varepsilon E_v(\ell^v)$. Taking Fourier transform to equation (14), we have

$$\mathcal{F}\{\varphi(\ell)\} = \Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell))\}}{v - i\xi u}. \tag{15}$$

Now, taking $Q(\xi) = \mathcal{F}\{q(\ell)\} = \frac{1}{(v - i\xi u)}$, then we have

$$\mathcal{F}\{q(\ell)\} = \frac{1}{v - i\xi u} \Rightarrow q(\ell) = \mathcal{F}^{-1}\left\{\frac{1}{v - i\xi u}\right\}.$$

We define a solution

$$\psi(\ell) = e^{-\frac{v}{u}\ell} + \{n(\ell, \psi(\ell))\} * q(\ell).$$

Then, by applying Fourier transform on both sides, we obtain

$$\mathcal{F}\{\psi(\ell)\} = \Psi(\xi) = \int_{-\infty}^{\infty} e^{-\frac{v}{u}\ell} e^{is\ell} \mathrm{d}\ell + \frac{\mathcal{F}\{n(\ell, \psi(\ell))\}}{v - i\xi u},$$

$$\Psi(\xi) = \frac{\mathcal{F}\{n(\ell, \psi(\ell))\}}{v - i\xi u}, \tag{16}$$

where $\int_{-\infty}^{\infty} e^{-\frac{v}{u}\ell} e^{is\ell} \mathrm{d}\ell = 0$. Now,

$$\mathcal{F}\{u\psi'(\ell) + v\psi(\ell)\} = -i\xi u\Psi(\xi) + v\Psi(\xi) = (v - i\xi u)\Psi(\xi) = \mathcal{F}\{n(\ell, \psi(\ell))\}.$$

We have $\mathcal{F}\{u\psi'(\ell) + v\psi(\ell)\} = \mathcal{F}\{n(\ell, \psi(\ell))\}$. Since $\mathcal{F}$ is linear and one-to-one operator,

$$u\psi'(\ell) + v\psi(\ell) = n(\ell, \psi(\ell)).$$

Hence, $\psi(\ell)$ is a solution to the differential equation (1). Now, using (15) and (16),

$$\begin{aligned}\mathcal{F}\{\varphi(\ell)\} - \mathcal{F}\{\psi(\ell)\} &= \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell))\}}{v - i\xi u} - \frac{\mathcal{F}\{n(\ell, \psi(\ell))\}}{v - (i\xi u)} \\ &= P(\xi)\frac{1}{v - i\xi u} = P(\xi) \times Q(\xi) \\ \Rightarrow \mathcal{F}\{\varphi(\ell) - \psi(\ell)\} &= \mathcal{F}\{p(\ell) \times q(\ell)\}.\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one linear operator, $\varphi(\ell) - \psi(\ell) = p(\ell) \times q(\ell)$. Taking modulus on both sides and using the inequality $|p(\ell)| \le \varepsilon E_v(\ell^v)$, we have

$$\begin{aligned}|\varphi(\ell) - \psi(\ell)| = |p(\ell) \times q(\ell)| &= \left|\int_0^{\ell} p(s)q(\ell - s)\mathrm{d}s\right|, \\ &\le |p(\ell)|\left|\int_0^{\ell} q(\ell - s)\mathrm{d}s\right| \le E_v(\ell^v)\varepsilon\left|\int_0^{\ell} q(\ell - s)\mathrm{d}s\right| \le H\varepsilon E_v(\ell^v),\end{aligned}$$

for any $\ell > 0$, where $H = |\int_0^{\ell} q(\ell - s)\mathrm{d}s|$ and the integral exists for each value of ℓ. Therefore, the nonlinear differential equation (1) has the Mittag-Leffler-Hyers-Ulam stability. □

Corollary 3.4. *Let u and v be the constants in $\mathbb{F}$. For every $\varepsilon > 0$ and $H > 0$ such that let $\varphi : (0, \infty) \to \mathbb{F}$ be a continuously differentiable function $\phi : (0, \infty) \to (0, \infty)$, which satisfies*

$$|u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell))| \le \phi(\ell)\varepsilon E_v(\ell^v), \tag{17}$$

where $E_v(\ell^v)$ is a Mittag-Leffler function, for any $\ell > 0$, there exists some $\psi : (0, \infty) \to \mathbb{F}$ of (1) such that $|\varphi(\ell) - \psi(\ell)| \le H\phi(\ell)\varepsilon E_v(\ell^v)$, for any $\ell > 0$.

Proof. We define a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi'(\ell) + v\varphi(\ell) - n(\ell, \varphi(\ell)), \tag{18}$$

for any $\ell > 0$. We have $|p(\ell)| \leq \phi(\ell)\varepsilon E_\nu(\ell^\nu)$. Using the same technique in Theorem 3.3, we will have

$$\Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell))\}}{\nu - i\xi u}, \tag{19}$$

and using the inequality $|p(\ell)| \leq \phi(\ell)\varepsilon E_\nu(\ell^\nu)$, we have

$$|\varphi(\ell) - \psi(\ell)| = |p(\ell) \times q(\ell)| \leq H\phi(\ell)\varepsilon E_\nu(\ell^\nu),$$

for any $\ell > 0$, where $H = |\int_0^\ell q(\ell - s)\mathrm{d}s|$ and the integral exists for each value of ℓ. Therefore, the nonlinear IVP (1) has Mittag-Leffler-Hyers-Ulam-Rassias stability. □

4 Ulam stability of second-order nonlinear IVP

In the following theorems, we will investigate various types of Ulam stability of the second-order nonlinear equation (3) with initial conditions (4).

Theorem 4.1. *Let u, v, and w are constants in* $\mathbb{F}$. *For every* $\varepsilon > 0$ *and a real constant* $H > 0$ *such that* $\varphi : (0, \infty) \to \mathbb{F}$ *is a twice continuously differentiable function that satisfies the inequality*

$$|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \leq \varepsilon, \tag{20}$$

for any $\ell > 0$, *there exists some* $\psi : (0, \infty) \to \mathbb{F}$, *which satisfies the IVP* (3) *such that* $|\varphi(\ell) - \psi(\ell)| \leq H\varepsilon$, *for any* $\ell > 0$.

Proof. Let $\varphi : (0, \infty) \to \mathbb{F}$ be a twice continuously differentiable function that satisfies inequality (20). We define a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell)), \tag{21}$$

for any $\ell > 0$. Thus, $|p(\ell)| \leq \varepsilon$. Using Fourier transform to (21),

$$\mathcal{F}\{p(\ell)\} = u\mathcal{F}\{\varphi''(\ell)\} + v\mathcal{F}\{\varphi'(\ell)\} + w\mathcal{F}\{\varphi(\ell)\} - \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}$$
$$P(\xi) = (-u\xi^2 - i\xi v + w)\Phi(\xi) - \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}$$

$$\Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}}{-u\xi^2 - i\xi v + w}. \tag{22}$$

Since for every u, v and w are the constants in $\mathbb{F}$, there exists a constant $\lambda, \mu \in \mathbb{F}$ such that $\lambda + \mu = \frac{v}{u}$ and $\lambda\mu = \frac{w}{u}$ where $\lambda \neq \mu$,

$$-u\xi^2 - i\xi v + w = u(i\xi - \lambda)(i\xi - \mu).$$

Thus,

$$\mathcal{F}\{\varphi(\ell)\} = \Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)}. \tag{23}$$

Now, taking $Q(\xi) = \mathcal{F}\{q(\ell)\} = \frac{1}{u(i\xi - \lambda)(i\xi - \mu)}$, then we have

$$q(\ell) = \mathcal{F}^{-1}\left\{\frac{1}{u(i\xi - \lambda)(i\xi - \mu)}\right\}.$$

We set

$$\psi(\ell) = \frac{\lambda e^{-\lambda\ell} - \mu e^{-\mu\ell}}{\lambda - \mu} + \{n(\ell, \psi(\ell), \psi'(\ell))\} * q(\ell).$$

Then, Fourier transform to $\psi(\ell)$, we obtain

$$\mathcal{F}\{\psi(\ell)\} = \Psi(\xi) = \int_{-\infty}^{\infty} \frac{\lambda e^{-\lambda\ell} - \mu e^{-\mu\ell}}{\lambda - \mu} e^{is\ell} \mathrm{d}\ell + \frac{\mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)},$$

$$\Psi(\xi) = \frac{\mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)}, \tag{24}$$

where $\int_{-\infty}^{\infty} \frac{\lambda e^{-\lambda\ell} - \mu e^{-\mu\ell}}{\lambda - \mu} e^{is\ell} \mathrm{d}\ell = 0$. Now,

$$\begin{aligned}\mathcal{F}\{u\psi''(\ell) + v\psi'(\ell) + w\psi(\ell)\} &= (-i\xi)^2 u\Psi(\xi) + (-i\xi)v\Psi(\xi) + w\Psi(\xi)\\ &= (-\xi^2 u - i\xi v + w)\Psi(\xi) = \mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}.\end{aligned}$$

We have $\mathcal{F}\{u\psi''(\ell) + v\psi'(\ell) + w\psi(\ell)\} = \mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}$ because $\mathcal{F}$ is one-to-one and linear operator; thus, $u\psi''(\ell) + v\psi'(\ell) + w\psi(\ell) = n(\ell, \psi(\ell), \psi'(\ell))$. Hence, $\psi(\ell)$ a solution to the differential equation (3). Now, using (23) and (24),

$$\begin{aligned}\mathcal{F}\{\varphi(\ell)\} - \mathcal{F}\{\psi(\ell)\} &= \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)} - \frac{\mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)}\\ &= \frac{P(\xi)}{u(i\xi - \lambda)(i\xi - \mu)} = P(\xi) \times Q(\xi)\\ \Rightarrow\ \mathcal{F}\{\varphi(\ell) - \psi(\ell)\} &= \mathcal{F}\{p(\ell) \times q(\ell)\}.\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one linear operator, $\varphi(\ell) - \psi(\ell) = p(\ell) \times q(\ell)$. Taking modulus on both sides and using the inequality $|p(\ell)| \le \varepsilon$, we have

$$\begin{aligned}|\varphi(\ell) - \psi(\ell)| = |p(\ell) \times q(\ell)| &= \left|\int_0^{\ell} p(s)q(\ell - s)\mathrm{d}s\right|,\\ &\le |p(\ell)| \left|\int_0^{\ell} q(\ell - s)\mathrm{d}s\right| \le \varepsilon \left|\int_0^{\ell} q(\ell - s)\mathrm{d}s\right| \le H\varepsilon,\end{aligned}$$

for any $\ell > 0$, where $H = |\int_0^{\ell} q(\ell - s)\mathrm{d}s\,|$ and the integral exists for each value of ℓ. Therefore, the nonlinear differential equation (3) has Hyers-Ulam stability. □

Corollary 4.2. *Let u, v, and w be the constants in* $\mathbb{F}$. *For every* $\varepsilon > 0$, *and a real constant* $H > 0$ *such that let* $\varphi : (0, \infty) \to \mathbb{F}$ *be a twice continuously differentiable function* $\phi : (0, \infty) \to (0, \infty)$, *which satisfies the inequality*

$$|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \le \phi(\ell)\varepsilon, \tag{25}$$

for any $\ell > 0$, *there exists some* $\psi : (0, \infty) \to \mathbb{F}$ *of the differential equation* (3) *such that* $|\varphi(\ell) - \psi(\ell)| \le H\phi(\ell)\varepsilon$, *for any* $\ell > 0$.

Proof. Let us choose a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell)), \tag{26}$$

for any $\ell > 0$. Let $\varphi : (0, \infty) \to \mathbb{F}$ be a twice continuously differentiable function that satisfies inequality (30). So, $|p(\ell)| \le \phi(\ell)\varepsilon$. Taking Fourier transform to (25), we have

$$\Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}}{-u\xi^2 - i\xi v + w}. \tag{27}$$

Using the same technique in Theorem 4.1, we will have

$$|\varphi(\ell) - \psi(\ell)| = \left|\int_0^{\ell} p(s)q(\ell - s)\mathrm{d}s\right| \le |p(\ell)| \left|\int_0^{\ell} q(\ell - s)\mathrm{d}s\right| \le \phi(\ell)\varepsilon \left|\int_0^{\ell} q(\ell - s)\mathrm{d}s\right| \le H\phi(\ell)\varepsilon.$$

Using the inequality $|p(\ell)| \leq \phi(\ell)\varepsilon$, for any $\ell > 0$, where $H = |\int_0^{\ell} q(\ell - s)\mathrm{d}s|$ and the integral exists for each value of ℓ. Therefore, the nonlinear differential equation (3) has the Hyers-Ulam-Rassias stability. □

Theorem 4.3. *Let u, v, and w be constants in* $\mathbb{F}$. *For every* $\varepsilon > 0$, *there is a real constant* $H > 0$ *such that* $\varphi : (0, \infty) \to \mathbb{F}$ *is a twice continuously differentiable function that satisfies*

$$|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \leq \varepsilon E_v(\ell^v), \tag{28}$$

where $E_v(\ell^v)$ *is a Mittag-Leffler function for all* $\ell > 0$, *then there exists some* $\psi : (0, \infty) \to \mathbb{F}$ *satisfying* (3) *such that* $|\varphi(\ell) - \psi(\ell)| \leq H\varepsilon E_v(\ell^v)$.

Proof. Let us define a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell)), \tag{29}$$

for any $\ell > 0$. Then, in view of (28), we obtain $|p(\ell)| \leq \varepsilon E_v(\ell^v)$. Applying Fourier transform to equation (29), we obtain that

$$\Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}}{-u\xi^2 - i\xi v + w}. \tag{30}$$

Since for every u, v and w are the constants in $\mathbb{F}$, there exists a constant $\lambda, \mu \in \mathbb{F}$ such that $\lambda + \mu = \frac{v}{u}$ and $\lambda\mu = \frac{w}{u}$ where $\lambda \neq \mu$, then,

$$-u\xi^2 - i\xi v + w = u(i\xi - \lambda)(i\xi - \mu).$$

Thus,

$$\mathcal{F}\{\varphi(\ell)\} = \Phi(\xi) = \frac{P(\xi) + \mathcal{F}\{n(\ell, \varphi(\ell), \varphi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)}. \tag{31}$$

Now, taking $Q(\xi) = \mathcal{F}\{q(\ell)\} = \frac{1}{u(i\xi - \lambda)(i\xi - \mu)}$, then we have

$$q(\ell) = \mathcal{F}^{-1}\left\{\frac{1}{u(i\xi - \lambda)(i\xi - \mu)}\right\}.$$

Setting

$$\psi(\ell) = \frac{\lambda e^{-\lambda\ell} - \mu e^{-\mu\ell}}{\lambda - \mu} + \{n(\ell, \psi(\ell), \psi'(\ell))\} * q(\ell),$$

then Fourier transform on both sides, we obtain

$$\mathcal{F}\{\psi(\ell)\} = \Psi(\xi) = \int_{-\infty}^{\infty} \frac{\lambda e^{-\lambda\ell} - \mu e^{-\mu\ell}}{\lambda - \mu} e^{is\ell}\mathrm{d}\ell + \frac{\mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)},$$

$$\Psi(\xi) = \frac{\mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}}{u(i\xi - \lambda)(i\xi - \mu)}, \tag{32}$$

where $\int_{-\infty}^{\infty} \frac{\lambda e^{-\lambda\ell} - \mu e^{-\mu\ell}}{\lambda - \mu} e^{is\ell}\mathrm{d}\ell = 0$. Now, we can easily see that

$$\mathcal{F}\{u\psi''(\ell) + v\psi'(\ell) + w\psi(\ell)\} = \mathcal{F}\{n(\ell, \psi(\ell), \psi'(\ell))\}.$$

Since $\mathcal{F}$ is linear and one-to-one operator, $u\psi''(\ell) + v\psi'(\ell) + w\psi(\ell) = n(\ell, \psi(\ell), \psi'(\ell))$. Hence, $\psi(\ell)$ a solution of the IVP (3). Now, using (31) and (32)

$$\mathcal{F}\{\varphi(\ell) - \psi(\ell)\} = \mathcal{F}\{p(\ell) \times q(\ell)\}.$$

Since the operator $\mathcal{F}$ is one-to-one linear operator, so $\varphi(\ell) - \psi(\ell) = p(\ell) \times q(\ell)$. Using the inequality $|p(\ell)| \le \varepsilon E_v(\ell^v)$ and simplifying, we obtain

$$|\varphi(\ell) - \psi(\ell)| \le \varepsilon E_v(\ell^v)\left|\int_0^\ell q(\ell - s)\mathrm{d}s\right| \le H\varepsilon E_v(\ell^v),$$

for any $\ell > 0$, where $H = |\int_0^\ell q(\ell - s)\mathrm{d}s|$ and the integral exists for each value of ℓ. Therefore, the second-order nonlinear IVP (3) has Mittag-Leffler-Hyers-Ulam stability. □

Corollary 4.4. *Let u, v, and w be the constants in* $\mathbb{F}$. *For every* $\varepsilon > 0$ *and* $H > 0$ *such that let* $\varphi : (0, \infty) \to \mathbb{F}$ *be a twice continuously differentiable function* $\phi : (0, \infty) \to (0, \infty)$, *which satisfies*

$$|u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell))| \le \phi(\ell)\varepsilon E_v(\ell^v), \tag{33}$$

where $E_v(\ell^v)$ *is a Mittag-Leffler function, for any* $\ell > 0$, *there exists some* $\psi : (0, \infty) \to \mathbb{F}$ *of* (3) *such that*

$$|\varphi(\ell) - \psi(\ell)| \le H\phi(t)\varepsilon E_v(\ell^v),$$

for any $\ell > 0$.

Proof. Choosing a function $p : (0, \infty) \to \mathbb{F}$ as follows:

$$p(\ell) = u\varphi''(\ell) + v\varphi'(\ell) + w\varphi(\ell) - n(\ell, \varphi(\ell), \varphi'(\ell)), \tag{34}$$

for any $\ell > 0$. So, $|p(\ell)| \le \phi(\ell)\varepsilon E_v(\ell^v)$. Using the same technique in Theorem 4.3, we have

$$|\varphi(\ell) - \psi(\ell)| \le H\phi(t)\varepsilon E_v(\ell^v),$$

for each $\ell > 0$. Therefore, the nonlinear differential equation (3) has Mittag-Leffler-Hyers-Ulam-Rassias stability. □

5 Conclusion

In this article, we have established the various types of Hyers-Ulam stabilities and Mittag-Leffler-Hyers-Ulam stability of first- and second-order nonlinear differential equations with initial conditions using the Fourier transform method.

Acknowledgements: Not applicable.

Funding information: Not applicable.

Author contributions: The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Conflict of interest: The authors declare that they have no competing interests.

Data availability statement: Not applicable.

References

[1] S. M. Ulam, *Problem in Modern Mathematics*, John Wiley & Sons, New York, 1964.
[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27** (1941), no. 4, 222–224, DOI: https://doi.org/10.1073/pnas.27.4.222.

[3] T. M. Rassias, *On the stability of the linear mappings in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300, DOI: https://doi.org/10.2307/2042795.

[4] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), no. 1–2, 64–66, DOI: https://doi.org/10.2969/jmsj/00210064.

[5] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), no. 4, 223–237, DOI: https://doi.org/10.1090/S0002-9904-1951-09511-7.

[6] N. Brillouet-Belluot, J. Brzdek, and K. Cieplinski, *On some recent developments in Ulam's type stability*, Abs. Appl. Anal. **2012** (2012), 41, DOI: https://doi.org/10.1155/2012/716936.

[7] M. Burger, N. Ozawa, and A. Thom, *On Ulam stability*, Israel J. Math **193** (2013), 109–129, DOI: https://doi.org/10.1007/s11856-012-0050-z.

[8] L. Cădariu, L. Găvruţa, and P. Găvruţa, *Fixed points and generalized Hyers-Ulam stability*, Abs. Appl. Anal. **2012** (2012), 10, DOI: https://doi.org/10.1155/2012/712743.

[9] S. M. Jung, *Hyers-Ulam-Rassias stability of functional equation in nonlinear analysis*, Optimization Application, Springer, New York, 2011.

[10] M. Almahalebi, A. Chahbi, and S. Kabbaj, *A Fixed point approach to the stability of a bicubic functional equations in 2-Banach spaces*, Palestine J. Math. **5** (2016), no. 2, 220–227.

[11] A. Ponmana Selvan, S. Sabarinathan, and A. Selvam, *Approximate solution of the special type differential equation of higher-order using Taylor's series*, J. Math Comp. Sci. **27** (2022), no. 2, 131–141, DOI: https://doi.org/10.22436/jmcs.027.02.04.

[12] R. Murali, J. M. Rassias, and V. Vithya, *The general solution and stability of nonadecic functional equation in matrix normed spaces*, Malaya J. Matematik **5** (2017), 416–427, DOI: https://doi.org/10.26637/mjm502/020.

[13] K. Ravi, J. M. Rassias, and B. V. Senthil Kumar, *Ulam-Hyers stability of undecic functional equation in quasi-beta-normed spaces fixed point method*, Tbilisi Math. Sci. **9** (2016), no. 2, 83–103, DOI: https://doi.org/10.1515/tmj-2016-0022.

[14] J. M. Rassias, *On approximately of approximately linear mappings by linear mappings*, *J. Functional Anal*. **46** (1982), no. 1, 126–130, DOI: https://doi.org/10.1016/0022-1236(82)90048-9.

[15] J. Huang, S. M. Jung, and Y. Li, *On Hyers-Ulam stability of nonlinear differential equations*, Bull. Korean Math. Soc. **52** (2015), no. 2, 685–697. DOI: http://dx.doi.org/10.4134/BKMS.2015.52.2.685.

[16] M. Obloza, *Hyers stability of the linear differential equation*, Rocznik Naukowo-Dydaktyczny, Prace Matematyczne **13** (1993), 259–270.

[17] M. Obloza, *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Naukowo-Dydaktyczny, Prace Matematyczne **14** (1997), 141–146, http://hdl.handle.net/11716/8211.

[18] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequl. Appl. **2** (1998), 373–380.

[19] S. M. Jung, *Hyers-Ulam stability of linear differential equation of first-order*, Appl. Math. Lett. **17** (2004), no. 10, 1135–1140, DOI: https://doi.org/10.1016/j.aml.2003.11.004.

[20] S. M. Jung, *Hyers-Ulam stability of linear differential equations of first-order (III)*, J. Math. Anal. Appl. **311** (2005), no. 1, 139–146, DOI: https://doi.org/10.1016/j.jmaa.2005.02.025.

[21] S. M. Jung, *Hyers-Ulam stability of linear differential equations of first-order (II)*, Appl. Math. Lett. **19** (2006), no. 9, 854–858, DOI: https://doi.org/10.1016/j.aml.2005.11.004.

[22] S. M. Jung, *Hyers-Ulam stability of a system of first-order linear differential equations with constant coefficients*, J. Math. Anal. Appl. **320** (2006), 549–561, DOI: https://doi.org/10.1016/j.jmaa.2005.07.032.

[23] Q. H. Alqifiary and S. M. Jung, *Laplace transform and generalized Hyers-Ulam stability of linear differential equations*, Electron. J. Differential Equations **2014** (2014), 1–11. https://ejde.math.txstate.edu/.

[24] J. M. Rassias, R. Murali, and A. Ponmana Selvan, *Mittag-Leffler-Hyers-Ulam stability of linear differential equations using Fourier transforms*, J. Comp. Anal. Appl. **29** (2021), no. 1, 68–85.

[25] R. Murali and A. Ponmana Selvan, *Fourier transforms and Ulam stabilities of linear differential equations*, Front. Funct. Equ. Anal. Inequalit. **2019** (2019), 195–217, DOI: https://doi.org/10.1007/978-3-030-28950-8_12.

[26] R. Murali, A. Ponmana Selvan, and C. Park, *Ulam stability of linear differential equations using Fourier transform*, AIMS Math. **5** (2019), no. 2, 766–780. DOI: http://dx.doi.org/10.3934/math.2020052.

[27] Y. Li and Y. Shen, *Hyers-Ulam stability of linear differential equations of second-order*, Appl. Math. Lett. **23** (2010), no. 3, 306–309, DOI: https://doi.org/10.1016/j.aml.2009.09.020.

[28] I. Fakunle and P. O. Arawomo, *Hyers-Ulam stability of certain class of nonlinear second-order differential equations*, Global J. Pure Appl. Math. **11** (2018), no. 1, 55–65. https://api.semanticscholar.org/CorpusID:198938167.

[29] J. Xue, *Hyers-Ulam stability of linear differential equations of second-order with constant coefficient*, Italian J. Pure Appl. Math. **32** (2014), 419–424.

[30] M. N. Qarawani, *Hyers-Ulam stability of linear and nonlinear differential equation of second-order*, Int. J. Appl. Math. Res. **1** (2012), no. 4, 422–432.

[31] M. N. Qarawani, *Hyers-Ulam stability of a generalized second-order nonlinear differential equation*, Appl. Math. **3** (2012), no. 12, 1857–1861. DOI: http://dx.doi.org/10.4236/am.2012.312252.

[32] S. M. Jung, *Approximate solution of a linear differential equation of third order*, Bull. Malaysian Math. Soc. Ser 2 **35** 2012, no. 4, 1063–1073.

[33] P. Găvruţa, S. M. Jung, and Y. Li, *Hyers-Ulam stability for second-order linear differential equations with boundary conditions*, Electron. J. Differential Equations **2011** (2011), 1–5.

[34] V. Kalvandi, N. Eghbali, and J. M. Rassias, *Mittag-Leffler-Hyers-Ulam stability of fractional differential equations of second-order*, J. Math. Extension **13** (2019), 1–15.

[35] R. Murali and A. Ponmana Selvan, *Hyers-Ulam stability of nth order linear differential equation*, Proyecciones: J. Math. **38** (2019), no. 3, 553–566.

[36] R. Murali and A. Ponmana Selvan, *Hyers-Ulam stability of a free and forced vibrations*, Kragujevac J. Math. **44** (2020), no. 2, 299–312.

[37] A. Zada, S. Faisal, and Y. Li, *Hyers-Ulam-Rassias stability of non-linear delay differential equations*, J. Nonlinear Sci. Appl. **10** (2017), 504–510, DOI: http://dx.doi.org/10.22436/jnsa.010.02.15.

[38] J. H. Huang and Y. Li, *Hyers-Ulam stability of delay differential equations of first-order*, Mathematische Nachrichten **289** (2016), no. 1, 60–66.

[39] H. Khan, Y. Li, W. Chen, D. Baleanu, and A. Khan, *Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator*, Boundary Value Problems **2017** (2017), 157, DOI: http://dx.doi.org/10.1186/s13661-017-0878-6.

[40] H. Khan, W. Chen, A. Khan, T. S. Khan, and Q. M. Al-Madlal, *Hyers-Ulam stability and existence criteria for coupled fractional differential equations involving p-Laplacian operator*, Adv. Differential Equations **2018** (2018), 455, DOI: https://doi.org/10.1186/s13662-018-1899-x.

[41] H. Khan, T. A. Jawad, M. Muhammad Aslam, R. A. Khan, and A. Khan, *Existence of positive solution and Hyers-Ulam stability for a nonlinear singular-delay-fractional differential equation*, Adv. Differential Equations **2019** (2019), 104, DOI: https://doi.org/10.1186/s13662-019-2054-z.

[42] H. Khan, K. Alam, H. Gulzar, S. Etemad, and S. Rezapour, *A case study of fractal-fractional tuberculosis model in China: Existence and stability theories along with numerical simulations*, Math. Comput. Simulat. **198** (2022), 455–473, DOI: https://doi.org/10.1016/j.matcom.2022.03.009.

[43] H. Khan, J. F. Gomez-Aguilar, A. Alkhazzan, and A. Khan, *A fractional order HIV-TB coinfection model with nonsingular Mittag-Leffler law*, Math. Methods Appl. Sci. **43** (2020), no. 6, 1–21, DOI: https://doi.org/10.1002/mma.6155.

[44] H. Khan, W. Chen, and H. Sun, *Analysis of positive solution and Hyers-Ulam stability for a class of singular fractional differential equations with p-Laplacian in Banach space*, Math. Methods Appl. Sci. **41** (2018), no. 9, 1–11, DOI: https://doi.org/10.1002/mma.4835.

[45] R. Murali and A. Ponmana Selvan, *Mittag-Leffler-Hyers-Ulam stability of a linear differential equations of first-order using Laplace transforms*, Canadian J. Appl. Math. **2** (2020), no. 2, 47–59.

[46] A. Ponmana Selvan, G. Ganapathy, M. Saravanan, and R. Veerasivaji, *Laplace transform and Hyers-Ulam stability of differential equation for logistic growth in a population model*, Commun. Korean Math. Soc. **38** (2023), no. 4, 1163–1173.

[47] R. Murali, A. Ponmana Selvan, C. Park, and J. R. Lee, *Aboodh transform and the stability of second-order linear differential equations*, Adv. Differential Equations **2021** (2021), 296, DOI: https://doi.org/10.1186/s13662-021-03451-4.

[48] R. Murali, A. Ponmana Selvan, S. Baskaran, C. Park, and J. R. Lee, *Hyers-Ulam stability of first-order linear differential equations using Aboodh transform*, J. Inequal. Appl. **2021** (2021), 133, DOI: https://doi.org/10.1186/s13660-021-02670-3.

[49] S. Sabarinathan, D. Muralidharan, and A. Ponmana Selvan, *Application of Mahgoub integral transform to Bessel's differential equations*, Commun. Math. Appl. **12** (2021), no. 4, 919–930, DOI: http://doi.org/10.26713/cma.v12i4.1645.

[50] S. M. Jung, A. Ponmana Selvan, and R. Murali, *Mahgoub transform and Hyers-Ulam stability of first-order linear differential equations*, J. Math. Inequal. **15** (2021), no. 3, 1201–1218, DOI: http://dx.doi.org/10.7153/jmi-2021-15-80.

[51] R. Murali, A. Ponmana Selvan, and S. Baskaran, *Stability of linear differential equation higher-order using Mahgoub transforms*, J. Math. Comput. sci. **30** (2023), 1–9, DOI: https://doi.org/10.22436/jmcs.030.01.01.