

Research Article

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Some applications and maximum principles for multi-term time-space fractional parabolic Monge-Ampère equation

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Abstract: This study first establishes several maximum and minimum principles involving the nonlocal Monge-Ampère operator and the multi-term time-space fractional Caputo-Fabrizio derivative. Based on the maximum principle established above, on the one hand, we show that a family of multi-term time-space fractional parabolic Monge-Ampère equations has at most one solution; on the other hand, we establish some comparison principles of linear and nonlinear multi-term time-space fractional parabolic Monge-Ampère equations.

Keywords: nonlocal Monge-Ampère operator, multi-term time-space parabolic equations, fractional Caputo-Fabrizio derivative, maximum and minimum principles

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1 Introduction

As we all know, the famous Monge-Ampère operator is a fully nonlinear, degenerate operator. It is derived from several areas of analysis and geometry, in which it is often used to describe the prescribed Gaussian curvature, optimal transportation, affine geometry, etc. The original representation of this operator consists of prescribing the determinant of the Hessian of a convex function ω , denoted as $\det(D^2\omega)$. For its recent research progress, see [1–5] (including its generalized k -Hessian problem).

Very recently, Caffarelli and Charro [6] successfully modified the classic Monge-Ampère operator to the nonlocal case, which is named as the nonlocal Monge-Ampère operator defined by

$$D_s\omega(\xi) = \inf \left\{ P.V. \int_{\mathbb{R}^N} \frac{\omega(y) - \omega(\xi)}{A^{-1} |\xi - y|^{N+2s}} dy \right\},$$

where A satisfies $A > 0$ and $\det A = 1$, here $A > 0$ implies A is positive definite square matrix.

In 2023, Wang et al. [7] studied the asymptotic radial solution of a class of parabolic tempered fractional Laplacian problems with logarithmic nonlinearity

$$\frac{\partial z(x, t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} z(x, t) = az(x, t) \ln|z(x, t)| + 1|^p,$$

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where $(\Delta + \lambda)^{\frac{\beta}{2}}$ is the tempered fractional Laplacian operator. Based on some new asymptotic maximum principles and asymptotic narrow region principles for antisymmetric functions, authors derived some properties of asymptotic radial solution to the above parabolic tempered fractional Laplacian problem in a unit ball. Inspired by the above two papers, in this work, we study a family of multi-term time-space fractional parabolic Monge-Ampère equations:

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CFF}} D_t^{\alpha_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) = G(\xi, \tau, \omega), & (\xi, \tau) \in \Sigma \times [0, +\infty), \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, +\infty), \\ \omega(\xi, 0) = g(\xi), & \xi \in \Sigma, \end{cases} \quad (1.1)$$

where $\Sigma \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain in $\mathbb{R}^N$. D_s is a nonlocal Monge-Ampère operator defined by

$$D_s \omega(\xi, \tau) = \inf \left\{ P.V. \int_{\mathbb{R}^N} \frac{\omega(y, \tau) - \omega(\xi, \tau)}{A^{-1} |\xi - y|^{N+2s}} dy \right\},$$

where A satisfies $A > 0$ and $\det A = 1$. ${}^{\text{CFF}}D_t^{\alpha_i}$ is the fractional Caputo-Fabrizio derivative [8,9] of order $0 < \alpha_i < 1$ ($i = 1, 2, \dots, n$) defined by

$${}^{\text{CFF}}D_t^{\alpha_i} f(\xi, \tau) = \frac{1}{1 - \alpha_i} \int_0^\tau \exp \left(-\frac{\alpha_i}{1 - \alpha_i} (\tau - s) \right) \frac{\partial f}{\partial s}(\xi, s) ds, \quad t \geq 0,$$

and $\mu_i \geq 0$, $\sum_{i=1}^n \mu_i > 0$ ($i = 1, 2, \dots, n$).

Fractional calculus, as a powerful supplement to classical calculus, has received great attention and carried forward. From the germination of fractional calculus to its vigorous growth, there have been many variants, which serve different theoretical and practical needs. The fractional Caputo-Fabrizio derivative is one of these brilliant achievements. In 2018, Atanacković *et al.* [10] analyzed the Caputo-Fabrizio fractional derivative in classical and distributional settings. In 2020, Baleanu *et al.* [11] proposed a new fractional model for human liver involving fractional Caputo-Fabrizio derivative with the exponential kernel. For some developments of fractional calculus, please refer to the literature [12–16].

Maximum principle is a useful tool to study the fractional partial differential equations without certain knowledge of solutions. In 2020, Kirane and Torebek [17] applied the maximum principles to obtain the properties of solutions for the space-time fractional diffusion and pseudo-parabolic equations with Caputo and Riemann-Liouville time-fractional derivatives. In 2020, Zeng *et al.* [18] established the maximum principles for multi-term space-time fractional diffusion equations with generalized variable-order fractional Caputo derivatives and variable-order fractional Riesz-Caputo derivatives. Other investigations of the maximum principle can be found in [19,20].

In this paper, we focus our attention on the maximum principle for equation (1.1). We emphasize that the introduction of the nonlocal Monge-Ampère operator and the multi-term time-space fractional Caputo-Fabrizio derivative bring the main difficulties to prove our main result (Theorem 2.2). To handle these difficulties, we first obtain inequality (2.3) by using Theorem 2.1. Then, we prove inequality (2.4) for nonlocal Monge-Ampère operator. Finally, we obtain our main result (Theorem 2.2). Our goal is to give several maximum and minimum principles involving the nonlocal Monge-Ampère operator and the multi-term time-space fractional Caputo-Fabrizio derivative, and to show some selected applications.

2 Maximum and minimum principles

In this section, we will give several maximum and minimum principles of multi-term time-space fractional parabolic Monge-Ampère equations (1.1). To successfully accomplish this task, we need an important theorem from [21], in which we take $(2 - \alpha)M(\alpha) = 2$.

Theorem 2.1. Let a function $f \in C^1([0, D])$ attain its maximum over the interval $[0, D]$ at the point $\tau_0 \in (0, D]$. Then, for any $0 < \alpha < 1$, the fractional Caputo-Fabrizio derivative has the following property:

$${}^{\text{CFF}}D^\alpha f(\tau_0) \geq \frac{1}{1-\alpha} \exp\left(-\frac{\alpha\tau_0}{1-\alpha}\right)[f(\tau_0) - f(0)]. \quad (2.1)$$

Next we will apply inequality (2.1) to deduce several maximum principles.

Theorem 2.2. Let $\omega \in C^{2,1}(\Sigma \times (0, D))$ and $\omega \in C(\bar{\Sigma} \times [0, D])$. If ω satisfies

$$\begin{cases} \sum_{i=0}^n \mu_i {}^{\text{CFF}}D_\tau^{\alpha_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) + h(\xi, \tau) \omega(\xi, \tau) \geq 0, & (\xi, \tau) \in \Sigma \times (0, D], \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, D], \\ \omega(\xi, 0) \geq 0, & \xi \in \Sigma, \end{cases} \quad (2.2)$$

with $h(\xi, \tau) \geq 0$, then $\omega(\xi, \tau) \geq 0$, $(\xi, \tau) \in \bar{\Sigma} \times [0, D]$.

Proof. Assume that the conclusion is not true. Since the condition $\omega(\xi, \tau) = 0$, $(\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, D]$, and $\omega(\xi, 0) \geq 0$, $\xi \in \Sigma$, there exists a point $(\xi_0, \tau_0) \in \Sigma \times (0, D]$, such that

$$\omega(\xi_0, \tau_0) = \min_{(\xi, \tau) \in \bar{\Sigma} \times [0, D]} \omega(\xi, \tau) < 0.$$

By Theorem 2.1, one obtains

$$\sum_{i=0}^n \mu_i {}^{\text{CFF}}D_\tau^{\alpha_i} \omega(\xi, \tau) \leq \sum_{i=1}^n \frac{\mu_i}{1-\alpha_i} \exp\left(-\frac{\alpha_i \tau_0}{1-\alpha_i}\right) [\omega(\xi_0, \tau_0) - \omega(\xi_0, 0)] \leq 0. \quad (2.3)$$

Since $\omega(\xi, \tau_0) \in C^2(\Sigma) \cap C(\bar{\Sigma})$ and $\omega(\xi, \tau_0) = 0$, $\xi \in \mathbb{R}^N \setminus \Sigma$, fix $\varepsilon > 0$, arbitrarily, we have

$$\begin{aligned} D_s \omega(\xi_0, \tau_0) &= \inf \left\{ P.V. \int_{\mathbb{R}^N} \frac{\omega(y, \tau_0) - \omega(\xi_0, \tau_0)}{A^{-1} |y - \xi_0|^{N+2s}} dy \right\} = \inf \left\{ \int_{\mathbb{R}^N \setminus \Sigma} \frac{\omega(y, \tau_0) - \omega(\xi_0, \tau_0)}{A^{-1} |y - \xi_0|^{N+2s}} dy \right. \\ &\quad \left. + P.V. \int_{\Sigma} \frac{\omega(y, \tau_0) - \omega(\xi_0, \tau_0)}{A^{-1} |y - \xi_0|^{N+2s}} dy \right\} \geq \int_{\mathbb{R}^N \setminus \Sigma} \frac{\omega(y, \tau_0) - \omega(\xi_0, \tau_0)}{A^{-1} |y - \xi_0|^{N+2s}} dy - \varepsilon. \end{aligned} \quad (2.4)$$

Let $\varepsilon \rightarrow 0$, by $\omega(\xi, \tau) = 0$, $(\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, +\infty)$, and $\omega(\xi_0, \tau_0) < 0$, we have

$$D_s \omega(\xi_0, \tau_0) > 0. \quad (2.5)$$

In fact, if $D_s \omega(\xi_0, \tau_0) = 0$, then $\omega(\xi, \tau_0) = 0$, which contradicts with $\omega(\xi_0, \tau_0) < 0$. So, (2.5) holds.

Applying inequalities (2.3), (2.5), and $h(\xi, \tau) \geq 0$, we obtain

$$\sum_{i=0}^n \mu_i {}^{\text{CFF}}D_\tau^{\alpha_i} \omega(\xi, \tau) - D_s \omega(\xi_0, \tau_0) + h(\xi_0, \tau_0) \omega(\xi_0, \tau_0) < 0,$$

which contradicts with (2.2). Therefore, it holds $\omega(\xi, \tau) \geq 0$, $(\xi, \tau) \in \bar{\Sigma} \times [0, D]$.

The proof is complete. □

By a similar process, we can obtain

Theorem 2.3. Let $\omega \in C^{2,1}(\Sigma \times (0, D))$ and $\omega \in C(\bar{\Sigma} \times [0, D])$. If ω satisfies

$$\begin{cases} \sum_{i=0}^n \mu_i {}^{\text{CFF}}D_\tau^{\alpha_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) + h(\xi, \tau) \omega(\xi, \tau) \leq 0, & (\xi, \tau) \in \Sigma \times (0, D], \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, D], \\ \omega(\xi, 0) \leq 0, & \xi \in \Sigma, \end{cases} \quad (2.6)$$

with $h(\xi, \tau) \leq 0$, then $\omega(\xi, \tau) \leq 0$, $(\xi, \tau) \in \bar{\Sigma} \times [0, D]$.

Based on Theorems 2.2 and 2.3, the following two theorems are available immediately, respectively.

Theorem 2.4. Let $\omega \in C^{2,1}(\Sigma \times (0, D)) \cap C(\bar{\Sigma} \times [0, D])$, if ω satisfies (2.2) with $h(\xi, \tau) \geq 0$, then

$$\min_{(\xi, \tau) \in \bar{\Sigma} \times [0, D]} \omega(\xi, \tau) = \min_{(\xi, \tau) \in (\Sigma \times \{0\}) \cup (\partial \Sigma \times [0, D])} \omega(\xi, \tau).$$

Theorem 2.5. Let $\omega \in C^{2,1}(\Sigma \times (0, D)) \cap C(\bar{\Sigma} \times [0, D])$, if ω satisfies (2.6) with $h(\xi, \tau) \leq 0$, then

$$\max_{(\xi, \tau) \in \bar{\Sigma} \times [0, D]} \omega(\xi, \tau) = \max_{(\xi, \tau) \in (\Sigma \times \{0\}) \cup (\partial \Sigma \times [0, D])} \omega(\xi, \tau).$$

3 Some applications of maximum principles

In this section, some applications of maximum and minimum principles are established.

Consider a family of linear multi-term time-space fractional parabolic Monge-Ampère equations:

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CF}} D_\tau^{a_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) + h(\xi, \tau) \omega(\xi, \tau) = f(\xi, \tau), & (\xi, \tau) \in \Sigma \times (0, +\infty), \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, +\infty), \\ \omega(\xi, 0) = g(\xi), & \xi \in \Sigma, \end{cases} \quad (3.1)$$

where functions $h(\xi, \tau)$ and $f(\xi, \tau)$ are bounded on $\Sigma \times [0, D]$.

Theorem 3.1. Let $\omega \in C^{2,1}(\Sigma \times (0, D)) \cap C(\bar{\Sigma} \times [0, D])$, if $h(\xi, \tau) \geq 0$, $(\xi, \tau) \in \Sigma \times [0, D]$, the linear multi-term time-space fractional parabolic Monge-Ampère equation (3.1) have at most one solution on $\bar{\Sigma} \times [0, D]$.

Proof. We will apply a proof by contradiction. Assume $\omega_1, \omega_2 \in C^{2,1}(\Sigma \times (0, D)) \cap C(\bar{\Sigma} \times [0, D])$ are two solutions of the linear multi-term time-space fractional parabolic Monge-Ampère equations (3.1).

Put $\omega = \omega_1 - \omega_2$, then we have

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CF}} D_\tau^{a_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) + h(\xi, \tau) \omega(\xi, \tau) = 0, & (\xi, \tau) \in \Sigma \times (0, D], \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, D], \\ \omega(\xi, 0) = 0, & \xi \in \Sigma. \end{cases}$$

Theorems 2.2 and 2.3 lead to $\omega \geq 0$ and $\omega \leq 0$, respectively. So $\omega = 0$, which implies $\omega_1 = \omega_2$. Thus, the conclusion of Theorem 3.1 is proved. $\square$

Next we consider a family of multi-term time-space nonlinear fractional parabolic Monge-Ampère equations

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CF}} D_\tau^{a_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) + h(\xi, \tau) \omega(\xi, \tau) = G(\xi, \tau, \omega), & (\xi, \tau) \in \Sigma \times (0, +\infty), \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, +\infty), \\ \omega(\xi, 0) = g(\xi), & \xi \in \Sigma. \end{cases} \quad (3.2)$$

Theorem 3.2. Let $G(\xi, \tau, \omega)$ be a smooth function. If $\frac{\partial G}{\partial \omega} - h(\xi, \tau) \leq 0$, $(\xi, \tau) \in \Sigma \times [0, D]$, then nonlinear multi-term time-space fractional parabolic Monge-Ampère equations (3.2) have at most one solution on $\bar{\Sigma} \times [0, D]$.

Proof. We shall employ the same method that was used in the proof of Theorem 3.1.

Let $\omega_1, \omega_2 \in C^{2,1}(\Sigma \times (0, D)) \cap C(\bar{\Sigma} \times [0, D])$ be the two solutions of nonlinear multi-term time-space fractional parabolic Monge-Ampère equations (3.2) on $\bar{\Sigma} \times [0, D]$. By using the auxiliary function $\omega = \omega_1 - \omega_2$, we have

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CFF}} D_\tau^{a_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) + h(\xi, \tau) \omega(\xi, \tau) \\ = G(\xi, \tau, \omega_1) - G(\xi, \tau, \omega_2), & (\xi, \tau) \in \Sigma \times (0, +\infty), \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, +\infty), \\ \omega(\xi, 0) = g(\xi), & \xi \in \Sigma. \end{cases}$$

Because G is a smooth function, it holds that

$$G(\xi, \tau, \omega_1) - G(\xi, \tau, \omega_2) = \frac{\partial G}{\partial \omega}(\zeta)(\omega_1 - \omega_2) = \frac{\partial G}{\partial \omega}(\xi, \tau, \zeta)\omega,$$

where $\zeta = (1 - \lambda)\omega_1 + \lambda\omega_2$, for $\lambda \in [0, 1]$. It is obvious that ζ and $\frac{\partial G}{\partial \omega}(\xi, \tau, \zeta)$ depend on the variables ξ and τ .

Since $\frac{\partial G}{\partial \omega} - h(\xi, \tau) \leq 0$, $(\xi, \tau) \in \Sigma \times [0, D]$, the rest of the proof is similar to that of Theorem 3.1. $\square$

Example 3.1. We consider the following multi-term time-space nonlinear fractional parabolic Monge-Ampère equation:

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CFF}} D_\tau^{a_i} \omega(\xi, \tau) - D_s \omega(\xi, \tau) + \xi^2 \omega(\xi, \tau) = \xi^2 \tau^2 \exp^{-\omega}, & (\xi, \tau) \in \Sigma \times (0, +\infty), \\ \omega(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, +\infty), \\ \omega(\xi, 0) = \xi^3, & \xi \in \Sigma. \end{cases} \quad (3.3)$$

Owing to $h(\xi, \tau) = \xi^2$, $G(\xi, \tau, \omega) = \xi^2 \tau^2 \exp^{-\omega}$, $\frac{\partial G}{\partial \omega} - h(\xi, \tau) = -\xi^2 \tau^2 \exp^{-\zeta} - \xi^2 \leq 0$, where $\zeta = (1 - \lambda)\omega_1 + \lambda\omega_2$, for $\lambda \in [0, 1]$. According to Theorem 3.2, the multi-term time-space nonlinear fractional parabolic Monge-Ampère equation (3.3) has at most one solution on $\bar{\Sigma} \times [0, D]$.

Corollary 3.1. In Theorem 3.2, let $h(\xi, \tau) \geq 0$. If $F(\xi, \tau, \omega)$ is non-increasing with respect to ω , then the nonlinear multi-term time-space fractional parabolic Monge-Ampère equations (3.2) have at most one solution.

An important application of maximum and minimum principles is that derive some comparison principles of linear and nonlinear multi-term time-space fractional parabolic Monge-Ampère equations.

Theorem 3.3. Let $\omega, v \in C^{2,1}(\Sigma \times (0, D)) \cap C(\bar{\Sigma} \times [0, D])$ satisfy

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CFF}} D_\tau^{a_i} \omega - D_s \omega + h(\xi, \tau) \omega - G(\xi, \tau, \omega) \\ \geq \sum_{i=0}^n \mu_i^{\text{CFF}} D_\tau^{a_i} v - D_s v + h(\xi, \tau) v - G(\xi, \tau, v), & (\xi, \tau) \in \Sigma \times [0, D], \\ \omega(\xi, \tau) = v(\xi, \tau), & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, D], \\ \omega(\xi, 0) \geq v(\xi, 0), & \xi \in \Sigma. \end{cases} \quad (3.4)$$

If there exists a bounded function $g \in \bar{\Sigma} \times [0, D]$ with $g + h \geq 0$, such that

$$G(\xi, \tau, \omega) - G(\xi, \tau, v) \geq -g(\xi, \tau)(\omega - v), \quad (\xi, \tau) \in \bar{\Sigma} \times [0, D],$$

then the inequality $\omega(\xi, \tau) \geq v(\xi, \tau)$, $(\xi, \tau) \in \bar{\Sigma} \times [0, D]$ holds.

Proof. Let us introduce an auxiliary function $\sigma = \omega - v$. Thus, we have

$$\begin{cases} \sum_{i=0}^n \mu_i^{\text{CF}} D_{\tau}^{\alpha_i} \sigma - D_s \sigma + (h + g)(\xi, \tau) \sigma \geq 0, & (\xi, \tau) \in \Sigma \times [0, D], \\ \sigma(\xi, \tau) = 0, & (\xi, \tau) \in (\mathbb{R}^N \setminus \Sigma) \times [0, D], \\ \sigma(\xi, 0) \geq 0, & \xi \in \Sigma. \end{cases}$$

It follows from Theorem 2.2 that $\sigma(\xi, \tau) \geq 0$, i.e., $\omega(\xi, \tau) \geq v(\xi, \tau)$, $(\xi, \tau) \in \bar{\Sigma} \times [0, D]$. $\square$

4 Conclusion

The multi-term time-space fractional parabolic Monge-Ampère equations with the fractional Caputo-Fabrizio derivative are considered in this study. New maximum principles of the nonlocal Monge-Ampère operator and the multi-term time-space fractional Caputo-Fabrizio derivative are derived. Based on the maximum principle, some comparison principles and the properties of the solution of multi-term time-space fractional parabolic Monge-Ampère equations are proved.

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