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Research Article

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Uniqueness of meromorphic functions concerning small functions and derivatives-differences

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Abstract: In this article, we study the unicity of meromorphic functions concerning small functions and derivatives-differences. The results obtained in this article extend and improve some results of Chen et al. [*Uniqueness problems on difference operators of meromorphic functions*] and Chen and Huang [*Uniqueness of meromorphic functions concerning their derivatives and shifts with partially shared values*].

Keywords: meromorphic functions, derivatives, differences, small functions, unicity

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1 Introduction and main results

In this article, meromorphic means meromorphic in the complex plane except where explicitly stated otherwise. We use the basic notations and definitions of Nevanlinna's value distribution theory as presented in [1–4]: $m(r,f)$, $N(r,f)$, $T(r,f),\ldots$. In particular, $S(r,f)$ denotes a function with the property $S(r,f)=o(T(r,f))$ as $r\to\infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E \mathrm{d}r/r<\infty$.

Let f be a nonconstant meromorphic function. We define the order of f by

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log^+ T(r,f)}{\log r}.$$

A meromorphic function a is called a small function of f if it satisfies $T(r,a)=S(r,f)$. Let a be a small function of f. We define

$$\lambda(f-a)=\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{f-a}\right)}{\log r}.$$

If $\lambda(f-a)<\rho(f)$ for $\rho(f)>0$ and $N\left(r,\frac{1}{f-a}\right)=O(\log r)$ for $\rho(f)=0$, then a is called a Borel exceptional function of f. If a is a constant, then a is called a Borel exceptional value of f. And we define

$$\delta(a,f)=\underline{\lim_{r\to\infty}}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}=1-\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

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$\delta(a,f)$ is the deficiency of a with respect to f. It is obvious that $0 \le \delta(a,f) \le 1$. If $\delta(a,f) > 0$, then a is called a deficient function of f. If a is a constant, then a is called a deficient value of f. In this article, deficiency possible outside of an exceptional set E with finite logarithmic measure.

Let f and g be two meromorphic functions, and let a be a small function of both f and g. We say that f and g share a CM(IM) if $f - a$ and $g - a$ have the same zeros counting multiplicities (ignoring multiplicities). Moreover, we denote by $N(r,a)$ the counting function for common zeros of both $f - a$ and $g - a$ with the same multiplicities and the multiplicity is counted. If

$$N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{g-a}\right) - 2N(r,a) \le S(r,f) + S(r,g),$$

then we call that f and g share a CM almost. Set $E(a,f) = \{z|f - a = 0\}$, where a zero with multiplicity m is counted m times.

For a nonzero finite complex constant η, we define the difference operators of f as $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $\Delta_\eta^n f(z) = \Delta_\eta(\Delta_\eta^{n-1} f(z))$, where $n \ge 2$ is an integer.

Any polynomial is determined by its zero points except for a non-constant factor, but it is not valid for transcendental entire or meromorphic functions. Therefore, how to uniquely determine a meromorphic function is interesting and complex. Thus, the uniqueness of meromorphic function became an important part of Nevanlinna's value distribution theory.

In 1926, Nevanlinna [4] proved the following result, which has come to be known as Nevanlinna's five values theorem.

Theorem A. *Let f and g be two nonconstant meromorphic functions, and let a_j $(j = 1, 2, 3, 4, 5)$ be five distinct values (one may be ∞). If f and g share a_j $(j = 1, 2, 3, 4, 5)$ IM, then $f \equiv g$.*

It is natural to ask whether or not Nevanlinna's five values theorem is still valid for small functions. In 2000, Li and Qiao [5] proved:

Theorem B. *Let f and g be two nonconstant meromorphic functions, and let a_j $(j = 1, 2, 3, 4, 5)$ be five distinct small functions of both f and g. If f and g share a_j $(j = 1, 2, 3, 4, 5)$ IM, then $f \equiv g$.*

Recently, the uniqueness in difference analogs of meromorphic functions has become a subject of some interests (see [6–14]). Chen et al. [6], Farissi et al. [10] proved:

Theorem C. [10] *Let f be a nonconstant entire function of finite order, let $\eta(\neq 0)$ be a finite complex constant, and let $a(\not\equiv 0)$ be an entire small function of f satisfying $\Delta_\eta a \equiv 0$. If f, $\Delta_\eta f$, and $\Delta_\eta^2 f$ share a CM, then $f \equiv \Delta_\eta f$.*

Chen et al. [8] considered what would obtained if the conditions f, $\Delta_\eta f$, and $\Delta_\eta^2 f$ share a CM were replaced with $\Delta_\eta f$ and $\Delta_\eta^2 f$ share a CM in Theorem C and proved:

Theorem D. *Let f be a transcendental entire function of finite order with $\lambda(f - a) < \rho(f)$, where a is an entire small function of f satisfying $\rho(a) < 1$, and let $\eta(\neq 0)$ be a finite complex constant such that $\Delta_\eta^2 f \not\equiv 0$. If $\Delta_\eta^2 f$ and $\Delta_\eta f$ share b CM, where $b(\not\equiv \Delta_\eta a)$ is a small function of f satisfying $\rho(b) < 1$, then $f(z) = a(z) + Be^{Az}$, where A and B are two nonzero constants.*

According to the aforementioned theorems, we naturally pose the following problem.

Problem 1. Whether $\rho(a) < 1$ and $\rho(b) < 1$ can be deleted or not in Theorem D?

In this article, we give a positive answer to Problem 1. Using different methods, we prove the following result.

Theorem 1. *Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a, and let* $\eta(\neq 0)$ *be a finite complex constant such that* $\Delta_\eta^2 f \not\equiv 0$. *If* $\Delta_\eta^2 f$ *and* $\Delta_\eta f$ *share b CM, where* $b(\not\equiv \Delta_\eta a)$ *is a small function of* f, *then* $f(z) = a(z) + Be^{Az}$, *where A and B are two nonzero constants.*

Remark 1. If $\lambda(f - a) < \rho(f)$, then a is a Borel exceptional function of f. Hence, Theorem 1 deletes the conditions $\rho(a) < 1$ and $\rho(b) < 1$ from Theorem D.

Remark 2. If $b \equiv \Delta_\eta a$, we have proved in [13] that $f(z) = a(z) + Be^{Az}$, where A and B are two nonzero constants, and $a(z)$ is reduced to a constant a. Naturally, we consider the general case where b is any small function. Therefore, we need to prove the case of $b \not\equiv \Delta_\eta a$, which is Theorem 1 in this article.

Example 1. Let $\eta(\neq 0)$ be a finite complex constant, and let $f = a + Be^{Az}$, where a, A, and B are the complex numbers such that $e^{A\eta} = 2$. Then, we obtain that a is a Borel exceptional value of f, and $\Delta_\eta^2 f \equiv \Delta_\eta f$. Thus, $\Delta_\eta^2 f$ and $\Delta_\eta f$ share b CM, where $b(\neq 0)$ is a constant.

To study the uniqueness of meromorphic functions with its derivatives is a very important problem in the uniqueness theory. Rubel and Yang [15], Mues and Steinmetz [16], and Gundersen [17] considered about it and obtained the following result.

Theorem E. *Let f be a nonconstant meromorphic (entire) function. If f and* f' *share two distinct finite values a and b CM(IM), then* $f \equiv f'$.

Frank and Weißenborn [18] and Frank and Ohlenroth [19] and Li and Yang [20] considered whether the aforementioned result is valid or not if f' is changed to $f^{(k)}$ and improved Theorem E as follows.

Theorem F. *Let f be a nonconstant meromorphic (entire) function, and let k be a positive integer. If f and* $f^{(k)}$ *share two distinct finite values a, b CM(IM), then* $f \equiv f^{(k)}$.

In 2011, Heittokangas et al. [11] started to consider meromorphic functions sharing values with its shifts and proved:

Theorem G. *Let f be a nonconstant entire function of finite order, and let* $\eta(\neq 0)$ *be a finite complex constant. If* $f(z)$ *and* $f(z + \eta)$ *share two distinct finite values a and b IM, then* $f(z) \equiv f(z + \eta)$.

Combining Theorem E with Theorem G, it is natural to consider the uniqueness of the shift or difference operator of a meromorphic function f with its derivative. In 2020, Qi and Yang [14] proved:

Theorem H. *Let f be a nonconstant meromorphic function of finite order, and let a and η be two nonzero finite complex values. If* $f'(z)$ *and* $f(z + \eta)$ *share a CM and satisfy* $E(0, f(z + \eta)) \subset E(0, f'(z))$ *and* $E(\infty, f'(z)) \subset E(\infty, f(z + \eta))$, *then* $f'(z) \equiv f(z + \eta)$.

Theorem I. *Let f be a nonconstant entire function of finite order, and let a and η be two nonzero finite complex values. If* $f'(z)$ *and* $f(z + \eta)$ *share 0 CM and a IM, then* $f'(z) \equiv f(z + \eta)$.

In 2022, Chen and Huang [9] proved the following results.

Theorem J. *Let f be a nonconstant meromorphic function of finite order, let a and η be two nonzero finite complex values, and let k be a positive integer. If* $f^{(k)}(z)$ *and* $f(z + \eta)$ *share a IM, and satisfy* $E(0, f(z + \eta)) \subset E(0, f^{(k)}(z))$ *and* $E(\infty, f^{(k)}(z)) \subset E(\infty, f(z + \eta))$. *If* $N\left(r, \frac{1}{f}\right) = S(r, f)$, *then* $f^{(k)}(z) \equiv f(z + \eta)$.

Theorem K. *Let f be a nonconstant meromorphic function of finite order, let a and η be two nonzero finite complex values, and let k be a positive integer. If $f^{(k)}(z)$ and $f(z+\eta)$ share a CM and satisfy $E(0, f(z+\eta)) \subset E(0, f^{(k)}(z))$ and $E(\infty, f^{(k)}(z)) \subset E(\infty, f(z+\eta))$, then $f^{(k)}(z) \equiv f(z+\eta)$.*

According to the aforementioned theorems, we naturally pose the following problem.

Problem 2. Whether $N\left(r, \frac{1}{f}\right) = S(r, f)$ can be deleted or not in Theorem J or Theorem K is valid or not for $f^{(k)}(z)$ and $f(z+\eta)$ share a IM? And whether Theorems J and K are valid or not for a small function a?

In this article, we give a positive answer to Problem 2 and prove the following result.

Theorem 2. *Let f be a nonconstant meromorphic function of finite order, let $a(\not\equiv 0)$ be a small function of f, let $\eta(\neq 0)$ be a finite complex constant, and let k be a positive integer. If $f^{(k)}(z)$ and $f(z+\eta)$ share a IM and satisfy $E(0, f(z+\eta)) \subset E(0, f^{(k)}(z))$ and $E(\infty, f^{(k)}(z)) \subset E(\infty, f(z+\eta))$, then $f^{(k)}(z) \equiv f(z+\eta)$.*

Example 2. Let $f = \sin z$, let $\eta = \frac{\pi}{2}$, and let $k = 5$. Then, we have $f^{(k)}(z) \equiv f(z+\eta) = \cos z$.

2 Lemmas

In order to prove our results, we need the following lemmas. Lemma 3 is the difference analogs of the logarithmic derivative lemma (Lemma 8), which played an important role in the development of Nevanlinna's value distribution theory of difference. Lemma 9 is the case of small function of Nevanlinna's second fundamental theorem. Lemma 10 is the Littlewood inequality, which is often used in discussions related to Miloux inequality.

Lemma 1. [12] *Let f be a nonconstant entire function of finite order. If a is a Borel exceptional entire small function with respect to f, then $\delta(a, f) = 1$.*

Lemma 2. [21] *Let f and g be two nonconstant meromorphic functions satisfying*

$$\delta(0, f) = \delta(\infty, f) = 1 \quad \text{and} \quad \delta(0, g) = \delta(\infty, g) = 1.$$

If f and g share 1 CM almost, then either $f \equiv g$ or $fg \equiv 1$.

Lemma 3. [22–24] *Let $\eta(\neq 0)$ be a finite complex constant, and let f be a nonconstant meromorphic function of finite order $\rho(f)$. Then,*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = S(r, f), \quad m\left(r, \frac{f(z)}{f(z+\eta)}\right) = S(r, f),$$

and for any $\varepsilon > 0$,

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O(r^{\rho(f)-1+\varepsilon}).$$

Lemma 4. [7] *Let f be a transcendental meromorphic function of finite order with two Borel exceptional values a, ∞, and let $\eta(\neq 0)$ be a finite complex constant such that $\Delta_\eta f \not\equiv 0$. If f and $\Delta_\eta f$ share a, ∞ CM, then $a = 0$, $f(z) = e^{Az+B}$, where $A(\neq 0)$ and B are two complex numbers.*

Lemma 5. [22] *Let $\eta(\neq 0)$ be a finite complex constant, and let f be a nonconstant meromorphic function of finite order. Then, we have*

$$N(r, f(z+\eta)) = N(r, f(z)) + S(r, f) \quad \text{and} \quad \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) = \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Lemma 6. [13] *Let f be a transcendental meromorphic function of finite order, let a be a small function with respect to f, and let $\eta(\neq 0)$ be a finite complex constant such that $\Delta_\eta^n f \not\equiv 0$, where n is a positive integer. If $\delta(a,f)=1$, $\delta(\infty,f)=1$, then*
(1) $T(r,\Delta_\eta^n f)=T(r,f)+S(r,f)$,
(2) $\delta(\Delta_\eta^n a,\Delta_\eta^n f)=\delta(\infty,\Delta_\eta^n f)=1$.

Lemma 7. [13] *Let η, c, and d be three nonzero finite complex constants, and let f be a meromorphic function of finite order. If $f(z+\eta)=cf(z)$, then either $T(r,f)\geq dr$ for sufficiently large r or f is a constant.*

Lemma 8. [1,4] *Let f be a nonconstant meromorphic function, and let k be a positive integer. Then,*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f).$$

Lemma 9. [1,4] *Let f be a nonconstant meromorphic function, and let a and b be two distinct small functions with respect to f. Then,*

$$T(r,f)\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+S(r,f).$$

Lemma 10. [1,3,4] *Let f be a nonconstant meromorphic function, let $n\geq 2$ be an integer, and let $a_1,a_2,\dots,a_n$ be distinct small functions with respect to f. Then,*

$$m\left(r,\frac{1}{f-a_1}\right)+\dots+m\left(r,\frac{1}{f-a_n}\right)\leq m\left(r,\frac{1}{f-a_1}+\dots+\frac{1}{f-a_n}\right)+S(r,f).$$

3 Proof of Theorem 1

First, we assume $\rho(f)>0$. Suppose, on the contrary, that $\rho(f)=0$.

Set $F(z)=f(z)-a(z)$. Since a is a Borel exceptional entire small function of f, we obtain

$$N\left(r,\frac{1}{F}\right)=N\left(r,\frac{1}{f-a}\right)=O(\log r).$$

Hence, F has finitely many zeros. Thus, we assume that $z_1,z_2,\dots,z_n$ are zeros of F, where n is a positive integer.

From $\rho(f)=0$, we deduce $\frac{F}{(z-z_1)(z-z_2)\dots(z-z_n)}=e^h$, where h is a constant. It follows that $F(z)=c(z-z_1)(z-z_2)\dots(z-z_n)$, where c is a nonzero constant, which contradicts with f is a transcendental entire function. Hence, $\rho(f)>0$.

Obviously, $\delta(\infty,f)=1$. Since a is a Borel exceptional entire small function of f, by Lemma 1, we obtain $\delta(a,f)=1$.

It follows from Lemma 6 that

$$\delta(\Delta_\eta a,\Delta_\eta f)=1,\quad \delta(\Delta_\eta^2 a,\Delta_\eta^2 f)=1,\tag{3.1}$$

$$\delta(\infty,\Delta_\eta f)=1,\quad\text{and}\quad \delta(\infty,\Delta_\eta^2 f)=1.\tag{3.2}$$

We claim $b\not\equiv\Delta_\eta^2 a$. Otherwise, suppose $b\equiv\Delta_\eta^2 a$. Since $\Delta_\eta^2 f$ and $\Delta_\eta f$ share b CM, then by (3.1), (3.2), Lemmas 6 and 9, we have

$$\begin{aligned}T(r,f)=T(r,\Delta_\eta f)+S(r,f)&\leq\overline{N}(r,\Delta_\eta f)+\overline{N}\left(r,\frac{1}{\Delta_\eta f-\Delta_\eta a}\right)+\overline{N}\left(r,\frac{1}{\Delta_\eta f-b}\right)+S(r,f)\\&\leq\overline{N}\left(r,\frac{1}{\Delta_\eta^2 f-\Delta_\eta^2 a}\right)+S(r,f)\leq S(r,f),\end{aligned}$$

which is a contradiction.

Set

$$F_1 = \frac{\Delta_\eta f - \Delta_\eta a}{b - \Delta_\eta a} \quad \text{and} \quad F_2 = \frac{\Delta_\eta^2 f - \Delta_\eta^2 a}{b - \Delta_\eta^2 a}. \tag{3.3}$$

Since $\Delta_\eta^2 f$ and $\Delta_\eta f$ share b CM, we deduce that F_1 and F_2 share 1 CM almost. By (3.1) and (3.2), we obtain

$$\delta(0, F_1) = 1, \quad \delta(0, F_2) = 1, \tag{3.4}$$

$$\delta(\infty, F_1) = 1, \quad \text{and} \quad \delta(\infty, F_2) = 1. \tag{3.5}$$

Hence, by Lemma 2, we have either $F_1 \equiv F_2$ or $F_1F_2 \equiv 1$.

If $F_1F_2 \equiv 1$, by (3.3), we obtain

$$(\Delta_\eta f - \Delta_\eta a)(\Delta_\eta^2 f - \Delta_\eta^2 a) = (b - \Delta_\eta a)(b - \Delta_\eta^2 a). \tag{3.6}$$

By (3.6), $\delta(a, f) = 1$, Lemma 3, and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} 2T(r,f) &= T\left(r, \frac{1}{(f-a)^2}\right) + S(r,f) \\ &= m\left(r, \frac{1}{(f-a)^2}\right) + S(r,f) \\ &\le m\left(r, \frac{1}{(b - \Delta_\eta a)(b - \Delta_\eta^2 a)}\right) + m\left(r, \frac{\Delta_\eta f - \Delta_\eta a}{f - a}\right) \\ &\quad + m\left(r, \frac{\Delta_\eta^2 f - \Delta_\eta^2 a}{f - a}\right) + S(r,f) \le S(r,f), \end{aligned}$$

which is a contradiction. Thus, we obtain $F_1 \equiv F_2$, i.e., $\frac{\Delta_\eta f - \Delta_\eta a}{b - \Delta_\eta a} \equiv \frac{\Delta_\eta^2 f - \Delta_\eta^2 a}{b - \Delta_\eta^2 a}$.

It follows

$$\frac{\Delta_\eta^2 f - b}{\Delta_\eta f - b} = \frac{b - \Delta_\eta^2 a}{b - \Delta_\eta a}. \tag{3.7}$$

Since f is a transcendental entire function of finite order, by $\Delta_\eta^2 f$ and $\Delta_\eta f$ sharing b CM, we have

$$\frac{\Delta_\eta^2 f - b}{\Delta_\eta f - b} = e^\alpha, \tag{3.8}$$

where α is a polynomial with $\deg\alpha \le \rho(f)$.

It follows from (3.7), (3.8), and $F_1 \equiv F_2$ that

$$\frac{\Delta_\eta^2 f - \Delta_\eta^2 a}{\Delta_\eta f - \Delta_\eta a} = \frac{b - \Delta_\eta^2 a}{b - \Delta_\eta a} = e^\alpha, \tag{3.9}$$

and e^α is a small function of f.

By (3.9), we obtain

$$\Delta_\eta^2 f - \Delta_\eta^2 a = e^\alpha(\Delta_\eta f - \Delta_\eta a). \tag{3.10}$$

Set

$$G = \Delta_\eta f - \Delta_\eta a.$$

By (3.10), we have

$$\Delta_\eta G = e^\alpha G.$$

Hence, we obtain G and $\Delta_\eta G$ that share 0 and ∞ CM.

It follows from (3.1) and (3.2) that

$$\delta(0, G) = 1, \quad \delta(0, \Delta_\eta G) = 1, \tag{3.11}$$

$$\delta(\infty, G) = 1, \quad \text{and} \quad \delta(\infty, \Delta_\eta G) = 1. \tag{3.12}$$

Since $\delta(a, f) = 1$ and $\delta(\infty, f) = 1$, by Lemma 6, we obtain

$$T(r, G) = T(r, f) + S(r, f). \tag{3.13}$$

It follows from a is a Borel exceptional function of f and $\rho(f) > 0$ that $\lambda(f - a) < \rho(f)$. Hence, we have

$$\overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r} < \rho(f). \tag{3.14}$$

By Lemma 6 and Nevanlinna's first fundamental theorem, we obtain

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\le m\left(r, \frac{1}{\Delta_\eta(f-a)}\right) + m\left(r, \frac{\Delta_\eta(f-a)}{f-a}\right), \\ T(r, f-a) - N\left(r, \frac{1}{f-a}\right) &\le T(r, \Delta_\eta(f-a)) - N\left(r, \frac{1}{\Delta_\eta(f-a)}\right) + S(r, f), \\ N\left(r, \frac{1}{\Delta_\eta(f-a)}\right) &\le N\left(r, \frac{1}{f-a}\right) + T(r, \Delta_\eta(f-a)) - T(r, f-a) + S(r, f) \\ &\le N\left(r, \frac{1}{f-a}\right) + S(r, f). \end{aligned} \tag{3.15}$$

By Lemma 3, set $\varepsilon = \frac{1}{2}$, and we have

$$S(r, f) \le M r^{\rho(f) - \frac{1}{2}}, \tag{3.16}$$

where M is a positive complex constant.

It follows from (3.14) that

$$N\left(r, \frac{1}{f-a}\right) < r^{\frac{\rho(f)+\lambda(f-a)}{2}}. \tag{3.17}$$

Combining (3.16) with (3.17), we obtain

$$N\left(r, \frac{1}{f-a}\right) + S(r, f) < (1 + M) r^{M_1}, \tag{3.18}$$

where $M_1 = \max\left\{\rho(f) - \frac{1}{2}, \frac{\rho(f) + \lambda(f-a)}{2}\right\}$.

It follows from (3.15) and (3.18) that

$$\frac{\log^+ N\left(r, \frac{1}{\Delta_\eta(f-a)}\right)}{\log r} \le \frac{\log(1 + M) r^{M_1}}{\log r} \le M_1 + \frac{\log(1 + M)}{\log r}.$$

Hence, we obtain

$$\overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{G}\right)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{\Delta_\eta(f-a)}\right)}{\log r} \le M_1 < \rho(f). \tag{3.19}$$

From (3.13) and (3.19), we deduce that 0 is a Borel exceptional value of G.

By Lemma 4, we have $G(z) = e^{Az+B_1}$, where $A(\neq 0)$ and B_1 are two constants, i.e.,

$$[f(z+\eta) - a(z+\eta)] - [f(z) - a(z)] = e^{Az+B_1}. \tag{3.20}$$

By Hadamard's factorization theorem, we obtain

$$f(z) - a(z) = \beta(z)e^{p(z)}, \tag{3.21}$$

where β is an entire function such that $\rho(\beta) = \lambda(\beta) < \rho(f)$, and p is a nonconstant polynomial with $\deg p = \rho(f)$. It follows

$$T(r, \beta) = S(r, e^p). \tag{3.22}$$

By (3.20) and (3.21), we have

$$\beta(z+\eta)e^{p(z+\eta)} - \beta(z)e^{p(z)} = e^{Az+B_1}. \tag{3.23}$$

In the following sections, we consider two cases.

Case 1. $\deg p \geq 2$. By (3.23), we obtain

$$\frac{\beta(z+\eta)}{e^{Az+B_1}}e^{p(z+\eta)} - \frac{\beta(z)}{e^{Az+B_1}}e^{p(z)} \equiv 1. \tag{3.24}$$

Obviously, $T(r, e^{Az+B_1}) = S(r, e^p)$. It follows from (3.22), (3.24), and Nevanlinna's second fundamental theorem that

$$\begin{aligned}
T(r, e^p) &\leq T\left(r, \frac{\beta}{e^{Az+B_1}}e^p\right) + S(r, e^p)\\
&\leq \overline{N}\left(r, \frac{\beta}{e^{Az+B_1}}e^p\right) + \overline{N}\left(r, \frac{1}{\frac{\beta}{e^{Az+B_1}}e^p}\right) + \overline{N}\left(r, \frac{1}{\frac{\beta}{e^{Az+B_1}}e^p + 1}\right) + S\left(r, \frac{\beta}{e^{Az+B_1}}e^p\right)\\
&\leq S(r, e^p),
\end{aligned}$$

which is a contradiction.

Case 2. $\deg p = 1$. Set $p(z) = mz + n$, where $m(\neq 0)$ and n are two complex numbers. Next, we consider two subcases.

Case 2.1. $A \neq m$. Thus, by (3.23), we have

$$c_1\beta(z+\eta)e^{(m-A)z} + c_2\beta(z)e^{(m-A)z} \equiv 1, \tag{3.25}$$

where $c_1 = e^{m\eta+n-B_1}$, $c_2 = -e^{n-B_1}$.

Obviously, $T(r, \beta) = S(r, e^{(m-A)z})$. It follows from (3.25) and Lemma 9 that

$$\begin{aligned}
T(r, e^{(m-A)z}) &\leq T(r, c_2\beta e^{(m-A)z}) + S(r, e^{(m-A)z})\\
&\leq \overline{N}(r, c_2\beta e^{(m-A)z}) + \overline{N}\left(r, \frac{1}{c_2\beta e^{(m-A)z}}\right) + \overline{N}\left(r, \frac{1}{c_2\beta e^{(m-A)z} - 1}\right) + S(r, c_2\beta e^{(m-A)z}) \leq S(r, e^{(m-A)z}),
\end{aligned}$$

which is a contradiction.

Case 2.2. $A = m$. Thus, by (3.23), we obtain

$$c_1\beta(z+\eta) + c_2\beta(z) \equiv 1, \tag{3.26}$$

where $c_1 = e^{m\eta+n-B_1}$ and $c_2 = -e^{n-B_1}$.

Now, we consider two subcases.

Case 2.2.1. $c_1 + c_2 = 0$. Hence,

$$e^{m\eta+n-B_1} - e^{n-B_1} = e^{n-B_1}(e^{m\eta} - 1) = 0.$$

It follows $e^{m\eta} = 1$.

By Lemma 7 and $\rho(\beta) < \rho(f) = 1$, we deduce

$$\beta(z) = \frac{z}{\eta c_1} + c_3,$$

where c_3 is a constant.

Hence, $f(z) = a(z) + \left(\frac{z}{\eta c_1} + c_3\right)e^{mz+n}$. Then, by $e^{m\eta} = 1$, we have

$$\begin{aligned}\Delta_\eta f &= \Delta_\eta a + \left(\frac{z+\eta}{\eta c_1} + c_3\right)e^{m(z+\eta)+n} - \left(\frac{z}{\eta c_1} + c_3\right)e^{mz+n}\\ &= \Delta_\eta a + \left(\frac{z}{\eta c_1} + c_3 + \frac{1}{c_1}\right)e^{mz+n}e^{m\eta} - \left(\frac{z}{\eta c_1} + c_3\right)e^{mz+n}\\ &= \Delta_\eta a + \frac{1}{c_1}e^{mz+n}\end{aligned}$$

and

$$\begin{aligned}\Delta_\eta^2 f = \Delta_\eta(\Delta_\eta f) &= \Delta_\eta\left(\Delta_\eta a + \frac{1}{c_1}e^{mz+n}\right)\\ &= \Delta_\eta^2 a + \frac{1}{c_1}e^{m(z+\eta)+n} - \frac{1}{c_1}e^{mz+n}\\ &= \Delta_\eta^2 a + \frac{1}{c_1}e^{mz+n}e^{m\eta} - \frac{1}{c_1}e^{mz+n}\\ &= \Delta_\eta^2 a,\end{aligned}$$

which is a contradiction.

Case 2.2.2. $c_1 + c_2 \neq 0$.

By Lemma 7 and $\rho(\beta) < \rho(f) = 1$, we deduce

$$\beta(z) = c,$$

where c is a nonzero constant. It follows $f(z) = a(z) + ce^{mz+n}$. Therefore, we have

$$f(z) = a(z) + Be^{Az},$$

where A and B are two nonzero constants.

This completes the proof of Theorem 1.

4 Proof of Theorem 2

Set

$$\varphi(z) = \frac{f^{(k)}(z)}{f(z+\eta)}. \tag{4.1}$$

It follows from Lemmas 3 and 8 that

$$m(r,\varphi) = S(r,f). \tag{4.2}$$

By $E(0, f(z+\eta)) \subset E(0, f^{(k)}(z))$, $E(\infty, f^{(k)}(z)) \subset E(\infty, f(z+\eta))$, and (4.1), we deduce that φ is an entire function. Thus, by (4.2), we have

$$T(r,\varphi) = m(r,\varphi) = S(r,f). \tag{4.3}$$

By (4.1), we obtain

$$f^{(k)}(z) = \varphi(z)f(z+\eta). \tag{4.4}$$

We claim $\varphi \equiv 1$. Suppose, on the contrary, that $\varphi \not\equiv 1$. Then, we have

$$\overline{N}\left(r, \frac{1}{f^{(k)}(z) - a(z)}\right) = \overline{N}\left(r, \frac{1}{f(z+\eta) - a(z)}\right) \le N\left(r, \frac{1}{\varphi(z) - 1}\right) + S(r,f) \\ \le T(r,\varphi) + S(r,f) \le S(r,f). \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\overline{N}\left(r, \frac{1}{f^{(k)}(z) - \varphi(z)a(z)}\right) = \overline{N}\left(r, \frac{1}{f(z+\eta) - a(z)}\right) + S(r,f) \le S(r,f). \tag{4.6}$$

Since $E(\infty, f^{(k)}(z)) \subset E(\infty, f(z+\eta))$, by Lemma 5, we obtain

$$N(r, f^{(k)}(z)) \le N(r, f(z+\eta)),$$
$$N(r,f) + k\overline{N}(r,f) \le N(r,f) + S(r,f).$$

It follows

$$\overline{N}(r,f) = S(r,f). \tag{4.7}$$

Now, we consider two cases.

Case 1. $a(z) \not\equiv a^{(k)}(z-\eta)$. Set

$$\Delta(f^{(k)}) = \begin{vmatrix} f^{(k)}(z) & a^{(k)}(z-\eta) & a(z) \\ f^{(k+1)}(z) & a^{(k+1)}(z-\eta) & a'(z) \\ f^{(k+2)}(z) & a^{(k+2)}(z-\eta) & a''(z) \end{vmatrix}.$$

Next, we consider two subcases.

Case 1.1. $\Delta(f^{(k)}) \not\equiv 0$. Then by (4.7), Lemmas 8 and 10, and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} & m\left(r, \frac{1}{f(z) - a(z-\eta)}\right) + m\left(r, \frac{1}{f^{(k)}(z) - a(z)}\right) \\ \le\ & m\left(r, \frac{f^{(k)}(z) - a^{(k)}(z-\eta)}{f(z) - a(z-\eta)}\right) + m\left(r, \frac{1}{f^{(k)}(z) - a^{(k)}(z-\eta)}\right) + m\left(r, \frac{1}{f^{(k)}(z) - a(z)}\right) + S(r,f) \\ \le\ & m\left(r, \frac{1}{f^{(k)}(z) - a^{(k)}(z-\eta)} + \frac{1}{f^{(k)}(z) - a(z)}\right) + S(r,f) \\ \le\ & m\left(r, \frac{\Delta(f^{(k)})}{f^{(k)}(z) - a^{(k)}(z-\eta)} + \frac{\Delta(f^{(k)})}{f^{(k)}(z) - a(z)}\right) + m\left(r, \frac{1}{\Delta(f^{(k)})}\right) + S(r,f) \\ \le\ & T(r, \Delta(f^{(k)})) - N\left(r, \frac{1}{\Delta(f^{(k)})}\right) + S(r,f) \\ \le\ & m(r, \Delta(f^{(k)})) + N(r, \Delta(f^{(k)})) - N\left(r, \frac{1}{\Delta(f^{(k)})}\right) + S(r,f) \\ \le\ & m(r, f^{(k)}(z)) + N(r, f^{(k)}(z)) + 2\overline{N}(r, f(z)) - N\left(r, \frac{1}{\Delta(f^{(k)})}\right) + S(r,f) \\ \le\ & T(r, f^{(k)}(z)) - N\left(r, \frac{1}{\Delta(f^{(k)})}\right) + S(r,f). \end{aligned} \tag{4.8}$$

By (4.8) and Nevanlinna's first fundamental theorem, we obtain

$$T(r,f) \le N\left(r, \frac{1}{f(z) - a(z-\eta)}\right) + N\left(r, \frac{1}{f^{(k)}(z) - a(z)}\right) - N\left(r, \frac{1}{\Delta(f^{(k)})}\right) + S(r,f). \tag{4.9}$$

From the properties of the determinant, we have

$$\begin{aligned}\Delta(f^{(k)}) &= \begin{vmatrix} [f(z)-a(z-\eta)]^{(k)} & a^{(k)}(z-\eta) & a(z) \\ [f(z)-a(z-\eta)]^{(k+1)} & a^{(k+1)}(z-\eta) & a'(z) \\ [f(z)-a(z-\eta)]^{(k+2)} & a^{(k+2)}(z-\eta) & a''(z) \end{vmatrix} \\ &= \begin{vmatrix} f^{(k)}(z)-a(z) & a^{(k)}(z-\eta) & a(z) \\ [f^{(k)}(z)-a(z)]' & a^{(k+1)}(z-\eta) & a'(z) \\ [f^{(k)}(z)-a(z)]'' & a^{(k+2)}(z-\eta) & a''(z) \end{vmatrix}.\end{aligned} \tag{4.10}$$

By (4.10), we know that if z_0 is a zero point of $f(z)-a(z-\eta)$ with multiplicity $l(\geq k+3)$, then z_0 must be a zero point of $\Delta(f^{(k)})$ with multiplicity $l-(k+2)$. Similarly, if z_0 is a zero point of $f^{(k)}(z)-a(z)$ with multiplicity $l(\geq 3)$, then z_0 must be a zero point of $\Delta(f^{(k)})$ with multiplicity $l-2$.

It follows from (4.5), (4.9), (4.10), and Lemma 5 that

$$\begin{aligned}T(r,f) &\leq (k+2)\overline{N}\left(r,\frac{1}{f(z)-a(z-\eta)}\right)+2\overline{N}\left(r,\frac{1}{f^{(k)}(z)-a(z)}\right)+S(r,f)\\ &\leq (k+2)\overline{N}\left(r,\frac{1}{f(z+\eta)-a(z)}\right)+S(r,f)\leq S(r,f),\end{aligned}$$

which is a contradiction.

Case 1.2. $\Delta(f^{(k)})\equiv 0$. Then, we have $\Delta_1(f^{(k)})\not\equiv 0$, where

$$\Delta_1(f^{(k)})=\begin{vmatrix} f^{(k)}(z) & a^{(k)}(z-\eta) \\ f^{(k+1)}(z) & a^{(k+1)}(z-\eta)\end{vmatrix}.$$

By $\Delta(f^{(k)})\equiv 0$, we deduce that

$$\Delta_1(f^{(k)})=c\begin{vmatrix} f^{(k)}(z) & a(z) \\ f^{(k+1)}(z) & a'(z)\end{vmatrix},$$

where c is a nonzero constant.

It follows from (4.7), Lemmas 8 and 10, and Nevanlinna's first fundamental theorem that

$$\begin{aligned}&m\left(r,\frac{1}{f(z)-a(z-\eta)}\right)+m\left(r,\frac{1}{f^{(k)}(z)-a(z)}\right)\\ \leq\;& m\left(r,\frac{f^{(k)}(z)-a^{(k)}(z-\eta)}{f(z)-a(z-\eta)}\right)+m\left(r,\frac{1}{f^{(k)}(z)-a^{(k)}(z-\eta)}\right)+m\left(r,\frac{1}{f^{(k)}(z)-a(z)}\right)+S(r,f)\\ \leq\;& m\left(r,\frac{1}{f^{(k)}(z)-a^{(k)}(z-\eta)}+\frac{1}{f^{(k)}(z)-a(z)}\right)+S(r,f)\\ \leq\;& m\left(r,\frac{\Delta_1(f^{(k)})}{f^{(k)}(z)-a^{(k)}(z-\eta)}+\frac{\Delta_1(f^{(k)})}{f^{(k)}(z)-a(z)}\right)+m\left(r,\frac{1}{\Delta_1(f^{(k)})}\right)+S(r,f)\\ \leq\;& T(r,\Delta_1(f^{(k)}))-N\left(r,\frac{1}{\Delta_1(f^{(k)})}\right)+S(r,f)\\ \leq\;& m(r,\Delta_1(f^{(k)}))+N(r,\Delta_1(f^{(k)}))-N\left(r,\frac{1}{\Delta_1(f^{(k)})}\right)+S(r,f)\\ \leq\;& m(r,f^{(k)}(z))+N(r,f^{(k)}(z))+\overline{N}(r,f(z))-N\left(r,\frac{1}{\Delta_1(f^{(k)})}\right)+S(r,f)\\ \leq\;& T(r,f^{(k)}(z))-N\left(r,\frac{1}{\Delta_1(f^{(k)})}\right)+S(r,f).\end{aligned} \tag{4.11}$$

By (4.11) and Nevanlinna's first fundamental theorem, we obtain

$$T(r,f) \le N\left(r, \frac{1}{f(z) - a(z-\eta)}\right) + N\left(r, \frac{1}{f^{(k)}(z) - a(z)}\right) - N\left(r, \frac{1}{\Delta_1(f^{(k)})}\right) + S(r,f). \tag{4.12}$$

From the properties of the determinant, we have

$$\Delta_1(f^{(k)}) = \begin{vmatrix} [f(z) - a(z-\eta)]^{(k)} & a^{(k)}(z-\eta) \\ [f(z) - a(z-\eta)]^{(k+1)} & a^{(k+1)}(z-\eta) \end{vmatrix} = c\begin{vmatrix} f^{(k)}(z) - a(z) & a(z) \\ [f^{(k)}(z) - a(z)]' & a'(z) \end{vmatrix}. \tag{4.13}$$

By (4.13), we know that if z_0 is a zero point of $f(z) - a(z-\eta)$ with multiplicity $l(\ge k+2)$, then z_0 must be a zero point of $\Delta_1(f^{(k)})$ with multiplicity $l - (k+1)$. Similarly, if z_0 is a zero point of $f^{(k)}(z) - a(z)$ with multiplicity $l(\ge 2)$, then z_0 must be a zero point of $\Delta_1(f^{(k)})$ with multiplicity $l-1$.

It follows from (4.5), (4.12), (4.13), and Lemma 5 that

$$\begin{aligned} T(r,f) &\le (k+1)\overline{N}\left(r, \frac{1}{f(z) - a(z-\eta)}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}(z) - a(z)}\right) + S(r,f) \\ &\le (k+1)\overline{N}\left(r, \frac{1}{f(z+\eta) - a(z)}\right) + S(r,f) \le S(r,f), \end{aligned}$$

which is a contradiction.

Case 2. $a(z) \equiv a^{(k)}(z-\eta)$. Then, by $\varphi(z) \not\equiv 1$, we have $\varphi(z)a(z) \not\equiv a^{(k)}(z-\eta)$. Set

$$\Delta(f^{(k)}) = \begin{vmatrix} f^{(k)}(z) & a^{(k)}(z-\eta) & \varphi(z)a(z) \\ f^{(k+1)}(z) & a^{(k+1)}(z-\eta) & (\varphi(z)a(z))' \\ f^{(k+2)}(z) & a^{(k+2)}(z-\eta) & (\varphi(z)a(z))'' \end{vmatrix}.$$

Next, using the same argument as used in Case 1, we obtain $T(r,f) \le S(r,f)$, which is a contradiction.

Thus, $\varphi(z) \equiv 1$. Therefore, Theorem 2 is proved.

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