Research Article

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Cramer's rule for a class of coupled Sylvester commutative quaternion matrix equations

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Abstract: In this article, based on the real representation and Kronecker product, Cramer's rule for a class of coupled Sylvester commutative quaternion matrix equations is studied and its expression is obtained. The proposed algorithm is very simple and convenient because it only involves real operations. Some numerical examples are provided to illustrate the feasibility of the proposed algorithm.

Keywords: commutative quaternion, Sylvester matrix equation, Cramer's rule

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1 Introduction

Quaternion, first proposed by William Rowan Hamilton in 1843, is an associative and noncommutative division algebra. Nowadays, quaternions are not only a part of contemporary mathematics, but also widely and massively used in computer graphics, control theory, quantum physics, signal and color image processing, and so on [1–6].

Many people have studied Cramer's rule for matrix equation over quaternion skew-field: Song and Yu mainly studied Cramer's rule for general solutions of two classes of constrained quaternion matrix equations [7]; Kyrchei studied explicit determinantal representation formulas of solutions (analogs of Cramer's rule) to the quaternion two-sided generalized Sylvester matrix equation [8]; Rehman et al. studied Cramer's rule for the generalized Sylvester quaternion matrix equation with some constrictions [9]; Kyrchei studied Cramer's rule for right and left quaternionic systems of linear equations [10]. Using the theory of column determinant and row determinant, Song et al. established the determinant representation of generalized inverse over the quaternion skew field [11].

However, Serge first studied the problem of commutative quaternion algebra in 1892, that is, the problem of quaternion whose multiplication satisfies the commutativity, opening up a new field of quaternion research [12]. But this set of commutative quaternion contains zero-divisor and isotropic elements [13,14]. Some results are obtained in the study of commutative quaternion algebras. For example, Catoni et al. elaborated detailedly on commutative quaternion [15]; Catoni et al. classified commutative quaternion into elliptic, parabolic, hyperbolic, and special types [16]; Ren et al. studied three real representations of a generalized Segre quaternion matrix [17]; Chen and Wang studied some practical necessary and sufficient conditions for the existence of an η-(anti-)Hermitian solution to a constrained Sylvester-type generalized commutative quaternion

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matrix equation [18]; Kösal derived the expressions of the minimal norm least-squares solution for the reduced biquaternion matrix equation $\mathbf{AX} = \mathbf{B}$ and the Moore-Penrose generalized inverse [19]; Euler and De Moivre's formulas for fundamental matrices of commutative quaternion are obtained in [20]; Eren and Kösal developed a general method to solve the general linear elliptic quaternionic matrix equations in [21].

While there are few existing studies on the commutative quaternion matrix equation of Cramer's rule, so this study investigates it.

In this study, we consider the coupled Sylvester commutative quaternion matrix equations

$$\begin{cases} \mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{C}_1\mathbf{Y}\mathbf{D}_1 = \mathbf{E}_1, \\ \mathbf{A}_2\mathbf{X}\mathbf{B}_2 + \mathbf{C}_2\mathbf{Y}\mathbf{D}_2 = \mathbf{E}_2, \end{cases} \tag{1.1}$$

where $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$, and $\mathbf{E}_i \in \mathbb{Q}_c^{n\times n}(\alpha)$ $(i = 1, 2)$ are the given commutative quaternion matrices, $\mathbf{X} \in \mathbb{Q}_c^{n\times n}(\alpha)$ and $\mathbf{Y} \in \mathbb{Q}_c^{n\times n}(\alpha)$ are unknown commutative quaternion matrices. Inspired by the aforementioned work, and keeping the wide application of commutative quaternion matrix, we consider Cramer's rule to solve the unique solution of coupled Sylvester commutative quaternion matrix equations (1.1).

The remained of this article is organized as follows. In Section 2, we recall some preliminary results with regard to the commutative quaternion matrix. In Section 3, based on the real representation of commutative quaternion matrix, the properties of Kronecker product, and the vec operator, we can convert the matrix equations (1.1) into the corresponding the real matrix equation, and then, the unique solution as well as the expression is established. In Section 4, we provide a numerical algorithm for computing the unique solution of the commutative quaternion matrix equations (1.1). And some numerical examples are presented to verify the feasibility of our proposed algorithm. Finally, we give some conclusions in Section 5.

2 Preliminaries

In this section, we recall some preliminary results that will be used in the following discussion. First, we introduce the definition of commutative quaternion.

Definition 2.1. [16] A commutative quaternion $\mathbf{q} \in \mathbb{Q}_c(\alpha)$ is expressed as

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$, and three imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the following multiplication rules.

1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	α	$\mathbf{k}$	$\alpha\mathbf{j}$
$\mathbf{j}$	$\mathbf{k}$	1	$\mathbf{i}$
$\mathbf{k}$	$\alpha\mathbf{j}$	$\mathbf{i}$	α

In particular, according to the value of $\alpha \in \mathbb{R}$, the commutative quaternion is named as follows:

(1) when $\alpha < 0$, it is called elliptic quaternion, in particular canonical if $\alpha = -1$ (reduced biquaternions);
(2) when $\alpha = 0$, it is called parabolic quaternion;
(3) when $\alpha > 0$, it is called hyperbolic quaternion, in particular canonical, if $\alpha = 1$.

We remark that three imaginary units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{k}^2 = \alpha, \mathbf{j}^2 = 1, \mathbf{ij} = \mathbf{ji} = \mathbf{k}, \mathbf{ik} = \mathbf{ki} = \alpha\mathbf{j}, \mathbf{jk} = \mathbf{kj} = \mathbf{i}.$$

According to Definition 2.1, let $\mathbf{p} = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} \in \mathbb{Q}_c(\alpha)$ and $\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{Q}_c(\alpha)$. The multiplication $\mathbf{pq}$ is

$$\begin{aligned}\mathbf{pq} &= (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})\\ &= (p_0q_0 + \alpha p_1q_1 + p_2q_2 + \alpha p_3q_3) + (p_0q_1 + p_1q_0 + p_2q_3 + p_3q_2)\mathbf{i}\\ &\quad + (p_0q_2 + \alpha p_1q_3 + p_2q_0 + \alpha p_3q_1)\mathbf{j} + (p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0)\mathbf{k}\\ &= (q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) = \mathbf{qp}.\end{aligned}$$

This shows that the commutative quaternion satisfies the commutative property, that is, $\mathbf{pq} = \mathbf{qp}$ in general for $\mathbf{p}, \mathbf{q} \in \mathbb{Q}_c(\alpha)$.

The commutative quaternion skew-field $\mathbb{Q}_c(\alpha)$ is clearly an algebra of rank four over the real number field $\mathbb{R}$. The symbols $\mathbb{R}^{m\times n}$ and $\mathbb{Q}_c^{m\times n}(\alpha)$ stand for the set of all $m \times n$ real matrices and the set of all $m \times n$ commutative quaternion matrices, respectively.

Definition 2.2. [22] A commutative quaternion matrix $\mathbf{A} \in \mathbb{Q}_c^{m\times n}(\alpha)$ is expressed as

$$\mathbf{A} = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k},$$

where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$.

For the commutative quaternion matrix $\mathbf{A} \in \mathbb{Q}_c^{m\times n}(\alpha)$, the symbols $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ stand for the transpose and Moore-Penrose generalized inverse of $\mathbf{A}$, respectively.

We obtain the following definitions of the norm of the commutative quaternion and commutative quaternion matrix.

Definition 2.3. [16] Let $\mathbf{a} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{Q}_c(\alpha)$, and the norm of commutative quaternion is defined as

$$\|\mathbf{a}\| = \sqrt{[(a_0 + a_2)^2 - \alpha(a_1 + a_3)^2][(a_0 - a_2)^2 - \alpha(a_1 - a_3)^2]}.$$

Definition 2.4. [23] For any given $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, and the Frobenius norm of A is defined as

$$\|A\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

Definition 2.5. [24] Let $\mathbf{A} = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{Q}_c^{m\times n}(\alpha)$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$, and the Frobenius norm of the commutative quaternion matrix $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = \sqrt{\|A_0\|^2 + \|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2}.$$

We now turn to recall the real representation of commutative quaternion and commutative quaternion matrix.

Definition 2.6. [22] For any commutative quaternion $\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{Q}_c(\alpha)$, its real representation is defined as

$$\Upsilon_{\mathbf{q}} = \begin{bmatrix} q_0 & \alpha q_1 & q_2 & \alpha q_3\\ q_1 & q_0 & q_3 & q_2\\ q_2 & \alpha q_3 & q_0 & \alpha q_1\\ q_3 & q_2 & q_1 & q_0\end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

For any commutative quaternion matrix $\mathbf{A} = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{Q}_c^{m\times n}(\alpha)$, its real representation matrix is defined as

$$\Upsilon_{\mathbf{A}} = \begin{bmatrix} A_0 & \alpha A_1 & A_2 & \alpha A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & \alpha A_3 & A_0 & \alpha A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix} \in \mathbb{R}^{4m\times 4n}.$$

The definition of real representation establishes a homomorphic relationship between commutative quaternion and its real representation. Thus, mappings $f_{\mathbf{q}} : \mathbf{q} \to \Upsilon_{\mathbf{q}}$ and $f_{\mathbf{A}} : \mathbf{A} \to \Upsilon_{\mathbf{A}}$ are isomorphic mappings of rings $\mathbb{Q}_c(\alpha)$ to $\mathbb{R}^{4\times 4}$ and rings $\mathbb{Q}_c^{m\times n}(\alpha)$ to $\mathbb{R}^{4m\times 4n}$, respectively. In addition, mappings $f_{\Upsilon_{\mathbf{q}}}^{-1} : \Upsilon_{\mathbf{q}} \to \mathbf{q}$ and $f_{\Upsilon_{\mathbf{A}}}^{-1} : \Upsilon_{\mathbf{A}} \to \mathbf{A}$ are their inverse mapping.

Denote the first independent column blocks of $\Upsilon_{\mathbf{A}}$ by

$$\Upsilon_{\mathbf{A}}^c = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}.$$

Next, we investigate some properties for $\Upsilon_{\mathbf{A}}^c$, which will be used in the sequel.

Lemma 2.1. [22] *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{Q}_c^{m\times n}(\alpha)$, $\mathbf{C} \in \mathbb{Q}_c^{n\times p}(\alpha)$, *and* $t \in \mathbb{R}$. *Then, we have*

(1) $\mathbf{A} = \mathbf{B} \Leftrightarrow \Upsilon_{\mathbf{A}} = \Upsilon_{\mathbf{B}} \Leftrightarrow \Upsilon_{\mathbf{A}}^c = \Upsilon_{\mathbf{B}}^c$;

(2) $\Upsilon_{\mathbf{A}+\mathbf{B}} = \Upsilon_{\mathbf{A}} + \Upsilon_{\mathbf{B}}$, $\Upsilon_{t\mathbf{A}} = t\Upsilon_{\mathbf{A}}$, $\Upsilon_{\mathbf{AC}} = \Upsilon_{\mathbf{A}}\Upsilon_{\mathbf{C}}$;

(3) $\Upsilon_{\mathbf{A}+\mathbf{B}}^c = \Upsilon_{\mathbf{A}}^c + \Upsilon_{\mathbf{B}}^c$, $\Upsilon_{t\mathbf{A}}^c = t\Upsilon_{\mathbf{A}}^c$, $\Upsilon_{\mathbf{AC}}^c = \Upsilon_{\mathbf{A}}\Upsilon_{\mathbf{C}}^c$;

(4) $\|\mathbf{A}\| = \|\Upsilon_{\mathbf{A}}^c\|$.

Next, we give the definition of the Kronecker product of the commutative quaternion matrix.

Definition 2.7. For any two matrices $\mathbf{A} = (\mathbf{a}_{ij}) \in \mathbb{Q}_c^{m\times n}(\alpha)$ and $\mathbf{B} \in \mathbb{Q}_c^{p\times q}(\alpha)$, the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} \coloneqq \begin{bmatrix} \mathbf{a}_{11}\mathbf{B} & \mathbf{a}_{12}\mathbf{B} & \cdots & \mathbf{a}_{1n}\mathbf{B} \\ \mathbf{a}_{21}\mathbf{B} & \mathbf{a}_{22}\mathbf{B} & \cdots & \mathbf{a}_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{m1}\mathbf{B} & \mathbf{a}_{m2}\mathbf{B} & \cdots & \mathbf{a}_{mn}\mathbf{B} \end{bmatrix} \in \mathbb{Q}_c^{mp\times nq}(\alpha).$$

Definition 2.8. Let $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathbb{Q}_c^{m\times n}(\alpha)$ with $\mathbf{a}_i$ being the ith column of $\mathbf{A}$. Define the operator vec as

$$\mathrm{vec}(\mathbf{A}) = [\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_n^T]^T \in \mathbb{Q}_c^{mn}(\alpha).$$

Lemma 2.2. [22] *Let* $\mathbf{A} \in \mathbb{Q}_c^{m\times p}(\alpha)$, $\mathbf{B} \in \mathbb{Q}_c^{p\times q}(\alpha)$, $\mathbf{C} \in \mathbb{Q}_c^{q\times n}(\alpha)$. *Then,*

$$\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\mathrm{vec}(\mathbf{B}).$$

Next, we review adjoint matrix over the real number field $\mathbb{R}$.

Definition 2.9. [25] Let $A \in \mathbb{R}^{n\times n}$, the algebraic cofactor A_{ij} of each element a_{ij} in the determinant $\det(A)$, which forms the transpose of matrix (A_{ij}), and then, $(A_{ij})^T$ is defined as the adjoint matrix of the matrix A and denoted by A^*. The expression is as follows:

$$A^* = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

Finally, we give the definition of the zero divisor of the commutative quaternion ring.

Definition 2.10. Let $0 \neq \mathbf{p} \in \mathbb{Q}_c(\alpha)$; if $0 \neq \mathbf{q} \in \mathbb{Q}_c(\alpha)$ such that $\mathbf{pq} = 0$ or $\mathbf{qp} = 0$, $\mathbf{p}$ is defined to be the zero divisor of the commutative quaternion ring.

Obviously, the zero divisor does not have an inverse. For example, it is obvious that $\mathbf{p}$ does not have an inverse while $\mathbf{p} = \mathbf{i}$.

3 Cramer's rule for matrix equations (1.1)

In this section, we first transform the quaternion matrix equations into the corresponding real matrix equation using the real representation of commutative quaternion matrix. Then, we obtain the expression of Cramer's rule for the coupled Sylvester commutative quaternion matrix equations (1.1). Meanwhile, another method to solve the matrix equations (1.1) is used as a comparison.

Lemma 3.1. [25] (Cramer's rule) *For a system of linear equations of n variables over the real number field as follows*:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \qquad \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \tag{3.1}$$

If the determinant of the coefficients of a system of linear equations is not equal to zero, i.e.,

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

Then, the system of equations (3.1) *has a unique solution, and the solution can be expressed as*

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D},$$

where D_j $(j = 1, 2, \ldots, n)$ is the determinant of order n obtained by replacing the elements in the jth column of the coefficient determinant D with the constant term at the right end of the system of equations, i.e.,

$$D_j = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \quad (j = 1, 2, \ldots, n).$$

Lemma 3.2. [23] *The linear system of equations $Ax = b$ with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ has a unique solution $x \in \mathbb{R}^n$ if and only if $AA^\dagger b = b$. When the equations $Ax = b$ are compatible, the general solution can be represented as*

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. The linear system of equations $Ax = b$ has a unique solution if and only if $\operatorname{rank}(A) = n$. *In this case, the unique solution is $x = A^\dagger b$.*

Lemma 3.3. [25] *Let $A \in \mathbb{R}^{n\times n}$, matrix A is invertible if and only if its determinant* $\det(A) \neq 0$, *and when matrix A is invertible, we have*

$$A^{-1} = \frac{1}{\det(A)}A^*,$$

where A^* is the adjoint matrix of A.

Remark 3.1. Lemma 3.3 not only gives the condition of square matrix invertibility, but also gives the method of inverting matrix A^{-1} by adjoint matrix.

Theorem 3.1. *For coupled Sylvester commutative quaternion matrix equations* (1.1), *the coefficient matrix* $K \in \mathbb{R}^{8n^2\times 8n^2}$ *is invertible and* f_K^{-1} *is not a zero divisor. Then, for any* $c \in \mathbb{R}^{8n^2}$, *the system has a unique solution, and the solution is*

$$x_1 = \frac{K_1}{\det(K)}, \dots, x_{4n^2} = \frac{K_{4n^2}}{\det(K)}, \quad y_1 = \frac{K_{4n^2+1}}{\det(K)}, \dots, y_{4n^2} = \frac{K_{8n^2}}{\det(K)}, \tag{3.2}$$

where $K = \begin{bmatrix} \Upsilon_{\mathbf{B}_1^T\otimes\mathbf{A}_1} \Upsilon_{\mathbf{D}_1^T\otimes\mathbf{C}_1} \\ \Upsilon_{\mathbf{B}_2^T\otimes\mathbf{A}_2} \Upsilon_{\mathbf{D}_2^T\otimes\mathbf{C}_2} \end{bmatrix}$, $c = \begin{bmatrix} \Upsilon^c_{\mathrm{vec}(\mathbf{E}_1)} \\ \Upsilon^c_{\mathrm{vec}(\mathbf{E}_2)} \end{bmatrix}$, K_j $(j = 1, 2, \dots, 8n^2)$ *is the determinant of order* $8n^2$ *obtained by replacing the elements in the jth column of the coefficient determinant* $\det(K)$ *with the constant term at the right end of the system of equations.*

Here, the unique solutions $\mathbf{X} = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k}$, $\mathbf{Y} = Y_0 + Y_1\mathbf{i} + Y_2\mathbf{j} + Y_3\mathbf{k}$ *are corresponding to*

$$\Upsilon^c_{\mathrm{vec}(\mathbf{X})} = \begin{bmatrix} \mathrm{vec}(X_0) \\ \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_3) \end{bmatrix}, \quad \Upsilon^c_{\mathrm{vec}(\mathbf{Y})} = \begin{bmatrix} \mathrm{vec}(Y_0) \\ \mathrm{vec}(Y_1) \\ \mathrm{vec}(Y_2) \\ \mathrm{vec}(Y_3) \end{bmatrix}, \tag{3.3}$$

where $\mathrm{vec}(X_t) = [x_{tn^2+1}, \dots, x_{(t+1)n^2}]^T$, $\mathrm{vec}(Y_t) = [y_{tn^2+1}, \dots, y_{(t+1)n^2}]^T$, *for* $t = 0, 1, 2, 3$.

Proof. For $\mathbf{X} = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{Q}_c^{n\times n}(\alpha)$, $\mathbf{Y} = Y_0 + Y_1\mathbf{i} + Y_2\mathbf{j} + Y_3\mathbf{k} \in \mathbb{Q}_c^{n\times n}(\alpha)$, according to Definition 2.8, we have

$$\begin{cases} \mathrm{vec}(\mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{C}_1\mathbf{Y}\mathbf{D}_1) = \mathrm{vec}(\mathbf{E}_1), \\ \mathrm{vec}(\mathbf{A}_2\mathbf{X}\mathbf{B}_2 + \mathbf{C}_2\mathbf{Y}\mathbf{D}_2) = \mathrm{vec}(\mathbf{E}_2). \end{cases}$$

According to Lemma 2.2, we have

$$\begin{cases} (\mathbf{B}_1^T \otimes \mathbf{A}_1)\mathrm{vec}(\mathbf{X}) + (\mathbf{D}_1^T \otimes \mathbf{C}_1)\mathrm{vec}(\mathbf{Y}) = \mathrm{vec}(\mathbf{E}_1), \\ (\mathbf{B}_2^T \otimes \mathbf{A}_2)\mathrm{vec}(\mathbf{X}) + (\mathbf{D}_2^T \otimes \mathbf{C}_2)\mathrm{vec}(\mathbf{Y}) = \mathrm{vec}(\mathbf{E}_2). \end{cases}$$

According to Lemma 2.1, we have

$$\begin{cases} \Upsilon_{\mathbf{B}_1^T\otimes\mathbf{A}_1}\Upsilon^c_{\mathrm{vec}(\mathbf{X})} + \Upsilon_{\mathbf{D}_1^T\otimes\mathbf{C}_1}\Upsilon^c_{\mathrm{vec}(\mathbf{Y})} = \Upsilon^c_{\mathrm{vec}(\mathbf{E}_1)}, \\ \Upsilon_{\mathbf{B}_2^T\otimes\mathbf{A}_2}\Upsilon^c_{\mathrm{vec}(\mathbf{X})} + \Upsilon_{\mathbf{D}_2^T\otimes\mathbf{C}_2}\Upsilon^c_{\mathrm{vec}(\mathbf{Y})} = \Upsilon^c_{\mathrm{vec}(\mathbf{E}_2)}. \end{cases}$$

The system of equations can be equivalently expressed as follows:

$$Kz := \begin{bmatrix} \Upsilon_{\mathbf{B}_1^T\otimes\mathbf{A}_1} & \Upsilon_{\mathbf{D}_1^T\otimes\mathbf{C}_1} \\ \Upsilon_{\mathbf{B}_2^T\otimes\mathbf{A}_2} & \Upsilon_{\mathbf{D}_2^T\otimes\mathbf{C}_2} \end{bmatrix} \begin{bmatrix} \Upsilon^c_{\mathrm{vec}(\mathbf{X})} \\ \Upsilon^c_{\mathrm{vec}(\mathbf{Y})} \end{bmatrix} = \begin{bmatrix} \Upsilon^c_{\mathrm{vec}(\mathbf{E}_1)} \\ \Upsilon^c_{\mathrm{vec}(\mathbf{E}_2)} \end{bmatrix} := c, \tag{3.4}$$

where

$$K \in \mathbb{R}^{8n^2\times 8n^2}, \quad z \in \mathbb{R}^{8n^2}, \quad \text{and} \quad c \in \mathbb{R}^{8n^2}.$$

According to Definition 2.10 and Lemma 3.1, the determinant of the coefficients of a system of linear equations (3.4) is not equal to zero, i.e., $\det(K) \neq 0$ and f_K^{-1} is not a zero divisor. Then, the system of equations (3.4) has a unique solution, and the solution can be expressed as

$$x_1 = \frac{K_1}{\det(K)}, \dots, x_{4n^2} = \frac{K_{4n^2}}{\det(K)}, \quad y_1 = \frac{K_{4n^2+1}}{\det(K)}, \dots, y_{4n^2} = \frac{K_{8n^2}}{\det(K)},$$

where K_j ($j = 1, 2, \ldots, 8n^2$) is the determinant of order $8n^2$ obtained by replacing the element in the jth column of the coefficient determinant $\det(K)$ with the constant term at the right end of the system of equations. For the convenience of the following description, we denote $\hat{x} = \Upsilon^c_{\mathrm{vec}(\mathbf{X})}$ and $\hat{y} = \Upsilon^c_{\mathrm{vec}(\mathbf{Y})}$, and the unique solutions $[\mathbf{X}, \mathbf{Y}]$ are corresponding to

$$\hat{x} = \Upsilon^c_{\mathrm{vec}(\mathbf{X})} = \begin{bmatrix} \mathrm{vec}(X_0) \\ \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_3) \end{bmatrix}, \quad \hat{y} = \Upsilon^c_{\mathrm{vec}(\mathbf{Y})} = \begin{bmatrix} \mathrm{vec}(Y_0) \\ \mathrm{vec}(Y_1) \\ \mathrm{vec}(Y_2) \\ \mathrm{vec}(Y_3) \end{bmatrix},$$

where $\mathrm{vec}(X_t) = [x_{tn^2+1}, \ldots, x_{(t+1)n^2}]^T$, $\mathrm{vec}(Y_t) = [y_{tn^2+1}, \ldots, y_{(t+1)n^2}]^T$, for $t = 0, 1, 2, 3$. The proof is complete. □

Theorem 3.1 can prove the existence of the unique solution for the commutative quaternion matrix equations (1.1). The expression of the unique solution when the matrix equations (1.1) are consistent can also be obtained.

Corollary 3.1. *If the coupled commutative quaternion matrix equations* (1.1) *have a unique solution, if and only if the quaternion matrix K is a invertible matrix and f_K^{-1} is not a zero divisor. The solution is $z = K^{-1}c$, where z, K, and c are all defined as in* (3.4), *and the corresponding primitive equations*

$$\begin{cases} \mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{C}_1\mathbf{Y}\mathbf{D}_1 = \mathbf{O}, \\ \mathbf{A}_2\mathbf{X}\mathbf{B}_2 + \mathbf{C}_2\mathbf{Y}\mathbf{D}_2 = \mathbf{O}. \end{cases}$$

Only the zero solution $\mathbf{X} = \mathbf{Y} = \mathbf{O}$, where $\mathbf{O}$ is the zero matrix of order n.

Remark 3.2. Corollary 3.1 can be obtained by a proof method similar to Theorem 3.1.

Corollary 3.2. *For coupled Sylvester commutative quaternion matrix equations* (1.1), *the coefficient matrix $K \in \mathbb{R}^{8n^2 \times 8n^2}$ is invertible and f_K^{-1} is not a zero divisor. Then, for any $c \in \mathbb{R}^{8n^2}$, the solution is $z = K^{-1}c$, where z, K, and c are all defined as in* (3.4). *Therefore, the matrix equations* (1.1) *have a solution $z \in \mathbb{R}^{8n^2}$ if and only if*

$$(KK^\dagger - I_{8n^2})c = 0. \tag{3.5}$$

If (3.5) *holds, the solution set of the matrix equations* (1.1) *can be represented as*

$$\mathbb{S} = \left\{ [\mathbf{X}, \mathbf{Y}] \middle| \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = K^\dagger c + (I_{8n^2} - K^\dagger K)y \right\}, \tag{3.6}$$

where $y \in \mathbb{R}^{8n^2}$ is an arbitrary vector. Moreover, the matrix equations (1.1) *have unique solution $[\mathbf{X}', \mathbf{Y}']$ if and only if*

$$\mathrm{rank}(K) = 8n^2,$$

and the unique solution $[\mathbf{X}', \mathbf{Y}']$ satisfies

$$\begin{bmatrix} \widehat{x'} \\ \widehat{y'} \end{bmatrix} = K^\dagger c. \tag{3.7}$$

Proof. According to the proof of Theorem 3.1, Lemma 3.2, and the definition of Moore-Penrose generalized inverse, we have

$$\left\| K\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} - c \right\| = \left\| KK^\dagger K\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} - c \right\| = \|KK^\dagger c - c\| = \|(KK^\dagger - I_{8n^2})c\|.$$

Thus, the matrix equations (1.1) have the solution $[\mathbf{X}, \mathbf{Y}]$ if and only if $||\mathbf{AXB} + \mathbf{CYD} - \mathbf{E}|| = 0$, i.e.,

$$(KK^{\dagger} - I_{8n^2})c = 0.$$

On the other hand, if condition (3.5) is established, the solution $[\mathbf{X}, \mathbf{Y}]$ of the matrix equations (1.1) satisfies

$$K\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = c,$$

which has the least-squares solution

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = K^{\dagger}c + (I_{8n^2} - K^{\dagger}K)y, \quad \forall y \in \mathbb{R}^{8n^2}. \tag{3.8}$$

The proof is complete. □

Corollary 3.3. *For coupled Sylvester commutative quaternion matrix equations* (1.1), *the coefficient matrix* $K \in \mathbb{R}^{8n^2 \times 8n^2}$ *is invertible and* f_K^{-1} *is not a zero divisor. For any* $c \in \mathbb{R}^{8n^2}$, *the solution is* $z = K^{-1}c$, *where* z, K, *and* c *are all defined as in* (3.4). *Then, matrix equations* (1.1) *have a solution*

$$z = \frac{1}{\det(K)}K^{*}c.$$

Remark 3.3. According to Theorem 2.9 and Lemma 3.5, the proof of Corollary 3.5 is obviously easy to obtain. Corollary 3.5 gives another method to solve equations (1.1).

4 Algorithms and numerical experiments

In this section, based on the discussions in Section 3, we propose the corresponding algorithm.

Algorithm 4.1. Step 1. Input $\alpha \in \mathbb{R}$, $\mathbf{A}_i \in \mathbb{Q}_c^{n\times n}(\alpha)$, $\mathbf{B}_i \in \mathbb{Q}_c^{n\times n}(\alpha)$, $\mathbf{C}_i \in \mathbb{Q}_c^{n\times n}(\alpha)$, $\mathbf{D}_i \in \mathbb{Q}_c^{n\times n}(\alpha)$ and $\mathbf{E}_i \in \mathbb{Q}_c^{n\times n}(\alpha)$, output $\Upsilon_{\mathbf{B}_i^T \otimes \mathbf{A}_i}$, $\Upsilon_{\mathbf{D}_i^T \otimes \mathbf{C}_i}$ and $\Upsilon^c_{\mathrm{vec}(\mathbf{E}_i)}$ $(i = 1, 2)$ by the definition of the real representation of commutative quaternion matrix, Kronecker product, and the vec operator.

Step 2. Input K, output $\det(K)$ and $K_j (j = 1, 2, \ldots, 8n^2)$ by definition of determinant over the real number field and Cramer's rule.

Step 3. According to (3.2) and (3.3), calculate the unique solution $[\mathbf{X}, \mathbf{Y}]$ of matrix equations (1.1).

Now, we give some numerical examples to prove the feasibility of Algorithm 4.1.

Example 1. Let $n = 4$, and we take

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & -1 & 0 \\ 2\mathbf{i} & \mathbf{k} & 3 & \mathbf{j} \\ \mathbf{j} & 1 & 2 & 3 \\ 4 & \mathbf{k} & 1 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} -1 & 3 & 4\mathbf{i} & -\mathbf{j} \\ 2\mathbf{j} & \mathbf{k} & 1 & 0 \\ 4 & 4 & 5 & \mathbf{i} \\ \mathbf{k} & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 1 & 2\mathbf{i} & 3 & 4\mathbf{j} \\ 5 & \mathbf{k} & -1 & 0 \\ \mathbf{i} & \mathbf{j} & 0 & -1 \\ 4 & -1 & \mathbf{i} & 10 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 4 & 5\mathbf{i} & 3\mathbf{j} & -1 \\ 0 & -3 & 2\mathbf{j} & -4 \\ \mathbf{k} & 1 & 0 & -1 \\ 1 & 2 & \mathbf{i} & 4 \end{bmatrix},$$

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{i} & 7 & -1 & \mathbf{j} \\ 2\mathbf{i} & \mathbf{k} & 3 & \mathbf{j} \\ \mathbf{k} & 1 & \mathbf{j} & 3 \\ 4 & \mathbf{k} & 1 & 10 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 & \mathbf{j} & 5\mathbf{i} & -\mathbf{j} \\ 2\mathbf{j} & \mathbf{k} & 1 & 0 \\ 4 & 4\mathbf{j} & 5 & \mathbf{i} \\ \mathbf{k} & 3 & 5 & 1 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 1 & 3\mathbf{j} & 3 & 4\mathbf{j} \\ 5 & \mathbf{i} & -1 & \mathbf{j} \\ \mathbf{i} & \mathbf{j} & 0 & -1 \\ 4 & 0 & \mathbf{i} & 6 \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} 4 & 2\mathbf{i} & 3\mathbf{j} & -1 \\ 2 & \mathbf{j} & 2 & -1 \\ \mathbf{k} & 1 & \mathbf{j} & -3 \\ 1 & 2 & \mathbf{k} & 8 \end{bmatrix}.$$

Let

$$\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & 1 \\ \mathbf{j} & \mathbf{k} & 1 & \mathbf{k} \\ \mathbf{k} & 1 & \mathbf{k} & \mathbf{j} \\ 1 & \mathbf{k} & \mathbf{j} & \mathbf{i} \end{bmatrix} \in \mathbb{Q}_c^{4\times 4}(\alpha), \qquad \widehat{\mathbf{Y}} = \begin{bmatrix} 2 & 3\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\mathbf{i} & \mathbf{j} & \mathbf{k} & 4 \\ \mathbf{j} & \mathbf{k} & 4 & 5\mathbf{i} \\ \mathbf{k} & 4 & 5\mathbf{i} & \mathbf{j} \end{bmatrix} \in \mathbb{Q}_c^{4\times 4}(\alpha),$$

then we set

$$\begin{cases} \mathbf{E}_1 = \mathbf{A}_1\widehat{\mathbf{X}}\mathbf{B}_1 + \mathbf{C}_1\widehat{\mathbf{Y}}\mathbf{D}_1, \\ \mathbf{E}_2 = \mathbf{A}_2\widehat{\mathbf{X}}\mathbf{B}_2 + \mathbf{C}_2\widehat{\mathbf{Y}}\mathbf{D}_2. \end{cases}$$

Method 1. (Cramer's rule)

Using MATLAB and Algorithm 4.1, we obtain the unique solution $[\mathbf{X}_1, \mathbf{Y}_1]$ and the errors are as follows:

(1) when $\alpha = -1$, $\varepsilon = \|[\mathbf{X}_1, \mathbf{Y}_1] - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]\| = 3.2713 \times 10^{-13}$.

(2) when $\alpha = 0$, $\varepsilon = \|[\mathbf{X}_1, \mathbf{Y}_1] - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]\| = 8.0428 \times 10^{-13}$.

(3) when $\alpha = 1$, $\varepsilon = \|[\mathbf{X}_1, \mathbf{Y}_1] - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]\| = 1.5320 \times 10^{-12}$.

It is clear that no matter what the value of α is, the errors are very small. This shows the effectiveness of Algorithm 4.1.

Method 2. (Adjoint matrix method)

According to Lemma 3.3 and using MATLAB, we obtain the unique solution $[\mathbf{X}_2, \mathbf{Y}_2]$ and the errors are as follows:

(1) when $\alpha = -1$, $\varepsilon = \|[\mathbf{X}_2, \mathbf{Y}_2] - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]\| = 2.8116 \times 10^{-13}$.

(2) when $\alpha = 0$, $\varepsilon = \|[\mathbf{X}_2, \mathbf{Y}_2] - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]\| = 1.4843 \times 10^{-12}$.

(3) when $\alpha = 1$, $\varepsilon = \|[\mathbf{X}_2, \mathbf{Y}_2] - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]\| = 2.0554 \times 10^{-12}$.

Obviously, no matter what the value of α is, the errors also are very small, which show the effectiveness of adjoint matrix method. This gives another way to solve matrix equations (1.1) and also shows the feasibility of Algorithm 4.1.

Example 2. Consider the coupled Sylvester commutative quaternion matrix equations

$$\begin{cases} \mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{C}_1\mathbf{Y}\mathbf{D}_1 = \mathbf{E}_1, \\ \mathbf{A}_2\mathbf{X}\mathbf{B}_2 + \mathbf{C}_2\mathbf{Y}\mathbf{D}_2 = \mathbf{E}_2, \end{cases}$$

where $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$, and $\mathbf{E}_i \in \mathbb{Q}_c^{n\times n}(\alpha)$, $i = 1, 2$, and they have the following forms:

$$\begin{aligned} \mathbf{A}_i &= \mathrm{rand}(n, n) + \mathrm{rand}(n, n)\mathbf{i} + \mathrm{rand}(n, n)\mathbf{j} + \mathrm{rand}(n, n)\mathbf{k}, \\ \mathbf{B}_i &= \mathrm{rand}(n, n) + \mathrm{rand}(n, n)\mathbf{i} + \mathrm{rand}(n, n)\mathbf{j} + \mathrm{rand}(n, n)\mathbf{k}, \\ \mathbf{C}_i &= \mathrm{rand}(n, n) + \mathrm{rand}(n, n)\mathbf{i} + \mathrm{rand}(n, n)\mathbf{j} + \mathrm{rand}(n, n)\mathbf{k}, \\ \mathbf{D}_i &= \mathrm{rand}(n, n) + \mathrm{rand}(n, n)\mathbf{i} + \mathrm{rand}(n, n)\mathbf{j} + \mathrm{rand}(n, n)\mathbf{k}. \end{aligned}$$

Let

$$\begin{aligned} \widehat{\mathbf{X}} &= \widehat{X}_0 + \widehat{X}_1\mathbf{i} + \widehat{X}_2\mathbf{j} + \widehat{X}_3\mathbf{k} \in \mathbb{Q}_c^{n\times n}(\alpha), \\ \widehat{\mathbf{Y}} &= \widehat{Y}_0 + \widehat{Y}_1\mathbf{i} + \widehat{Y}_2\mathbf{j} + \widehat{Y}_3\mathbf{k} \in \mathbb{Q}_c^{n\times n}(\alpha), \end{aligned}$$

which are arbitrary both commutative quaternion matrices. Then, we set

$$\begin{cases} \mathbf{E}_1 = \mathbf{A}_1\widehat{\mathbf{X}}\mathbf{B}_1 + \mathbf{C}_1\widehat{\mathbf{Y}}\mathbf{D}_1, \\ \mathbf{E}_2 = \mathbf{A}_2\widehat{\mathbf{X}}\mathbf{B}_2 + \mathbf{C}_2\widehat{\mathbf{Y}}\mathbf{D}_2. \end{cases}$$

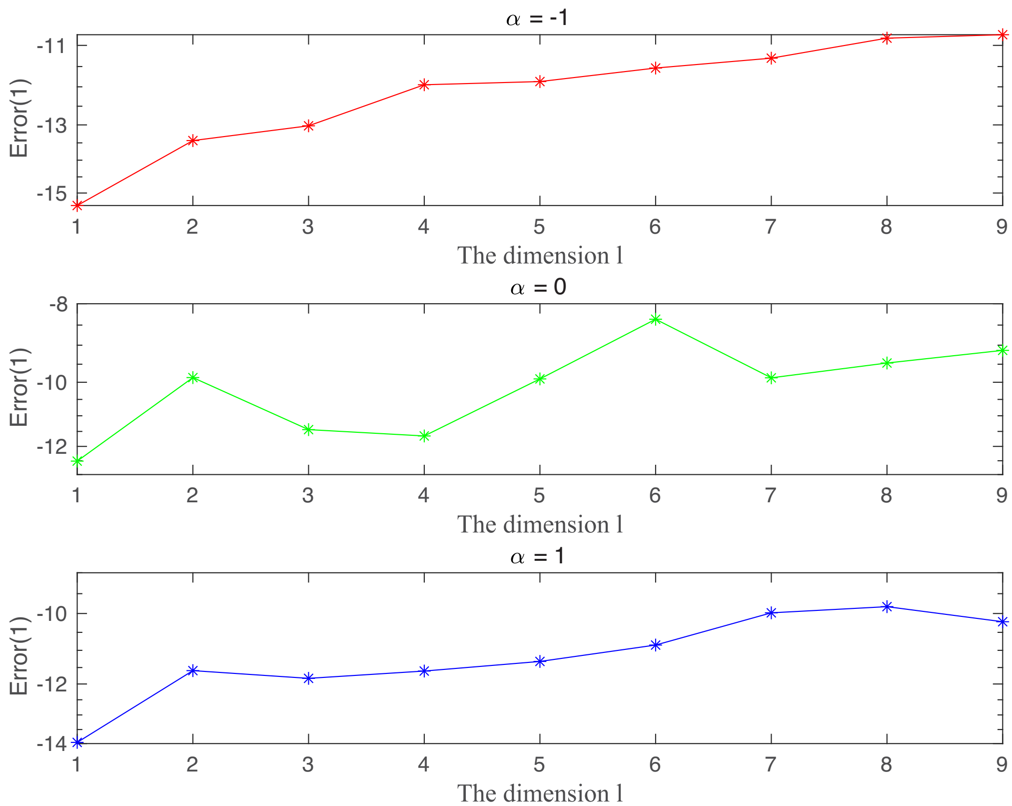

Figure 1: Error(l) with $\alpha = -1, 0, 1$.

In this case, the commutative quaternion matrix equations (1.1) have the unique solutions $[\mathbf{X}, \mathbf{Y}] \in \mathbb{Q}_c^{n\times n}(\alpha) \times \mathbb{Q}_c^{n\times n}(\alpha)$.

By Algorithm 4.1, we compute $[\mathbf{X}, \mathbf{Y}]$ and

$$\mathrm{Error}(l) = \log_{10}(\|[\mathbf{X}, \mathbf{Y}] - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]\|).$$

The relation between the dimension l and Error(l) is shown in Figure 1.

From Figure 1, it can see that Error(l) is very small. This confirms that the difference between the numerical solution and the exact solution is tiny. Therefore, our proposed algorithm is feasible.

5 Conclusions

Based on the real representation of commutative quaternion matrix, we transform the coupled Sylvester commutative quaternion matrix equations (1.1) into the corresponding real matrix equation. Then, we derive the expression of Cramer's rule for the coupled Sylvester commutative quaternion matrix equations (1.1). The expression we obtain is represented by real matrices only, and the algorithm only involves real operations, which greatly reduces the computational complexity. The last examples show that the proposed algorithm is feasible and convenient to solve the coupled Sylvester commutative quaternion matrix equations (1.1) using Cramer's rule.

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References

[1] S. D. Leo and G. Scolarici, *Right eigenvalue equation in quaternionic quantum mechanics*, J. Phys. A **33** (2000), 2971–2995, DOI: https://doi.org/10.1088/0305-4470/33/15/306.

[2] C. A. M. Melo and B. M. Pimentel, *Variational formulation for quaternionic quantum mechanics*, Adv. Appl. Clifford Algebr. **20** (2010), 745–763, DOI: https://doi.org/10.1007/s00006-010-0234-8.

[3] S. J. Sangwine and N. L. Bihan, *Quaternion singular value decomposition based on bidiagonalization to a real or complex matrix using quaternion Householder transformations*, Appl. Math. Comput. **182** (2006), 727–738, DOI: https://doi.org/10.1016/j.amc.2006.04.032.

[4] Z. G. Jia, M. K. Ng, and G. J. Song, *Lanczos method for large-scale quaternion singular value decomposition*, Numer. Algorithms **82** (2019), 699–717, DOI: https://doi.org/10.1007/s11075-018-0621-0.

[5] Z. G. Jia, M. K. Ng, and G. J. Song, *Robust quaternion matrix completion with applications to image in painting*, Numer. Linear Algebra Appl. **26** (2019), 2245–2280, DOI: https://doi.org/10.1002/nla.2245.

[6] Y. F. Ke, C. F. Ma, Z. G. Jia, Y. J. Xie, and R. W. Liao, *Quasi non-negative quaternion matrix factorization with application to color face recognition*, J. Sci. Comput. **95** (2023), 38, DOI: https://doi.org/10.1007/s10915-023-02157-x.

[7] G. J. Song and S. W. Yu, *Cramer's rule for the general solution to a restricted system of quaternion matrix equations*, Adv. Appl. Clifford Algebr. **29** (2019), 91–108, DOI: https://doi.org/10.1007/s00006-019-1000-1.

[8] I. Kyrchei, *Cramer's rules for Sylvester quaternion matrix equation and its special cases*, Adv. Appl. Clifford Algebr. **28** (2018), 1–26, DOI: https://doi.org/10.1007/s00006-018-0909-0.

[9] A. Rehman, I. I. Kyrchei, I. Ali, M. Akram, and A. Shakoor, *Constraint solution of a classical system of quaternion matrix equations and its Cramer's rule*, Iran. J. Sci. Technol. Trans. A Sci. **45** (2021), 1015–1024, DOI: https://doi.org/10.1007/s40995-021-01083-7.

[10] I. I. Kyrchei, *Crameras rule for quaternionic systems of linear equations*, J. Math. Sci. **155** (2008), 839–858, DOI: https://doi.org/10.1007/s10958-008-9245-6.

[11] G. J. Song, Q. W. Wang, and H. X. Chang, *Cramer rule for the unique solution of restricted matrix equations over the quaternion skew field*, Comput. Appl. Math. **61** (2011), 1576–1589, DOI: https://doi.org/10.1016/j.camwa.2011.01.026.

[12] C. Segre, *The real representations of complex elements and extension to bicomplex systems*, Math. Ann. **40** (1892), 413–467.

[13] H. H. Kösal and M. Tosun, *Commutative quaternion matrices*, Adv. Appl. Clifford Algebr. **24** (2014), 769–779, DOI: https://doi.org/10.1007/s00006-014-0449-1.

[14] H. H. Kösal, M. Akyiğit, and M. Tosun, *Consimilarity of commutative quaternion matrices*, Miskolc Math. Notes. **16** (2015), 965–977, DOI: https://doi.org/10.18514/MMN.2015.1421.

[15] F. Catoni, R. Cannata, V. Catoni, and P. Zampetti, *N-dimensional geometries generated by hypercomplex numbers*, Adv. Appl. Clifford Algebr. **15** (2005), 1–25, DOI: https://doi.org/10.1007/s00006-005-0001-4.

[16] F. Catoni, R. Cannata, and P. Zampetti, *An introduction to commutative quaternions*, Adv. Appl. Clifford Algebr. **16** (2006), 1–28, DOI: https://doi.org/10.1007/s00006-006-0002-y.

[17] B. Y. Ren, Q. W. Wang, and X. Y. Chen, *The η-Anti-Hermitian solution to a system of constrained matrix equations over the generalized segre quaternion algebra*, Symmetry **15** (2023), 592–607, DOI: https://doi.org/10.3390/sym15030592.

[18] X. Y. Chen and Q. W. Wang, *The η-(anti-) Hermitian solution to a constrained Sylvester-type generalized commutative quaternion matrix equation*, Banach J. Math. Anal. **17** (2023), 40–79, DOI: https://doi.org/10.1007/s43037-023-00262-5.

[19] H. H. Kösal, *Least-squares solutions of the reduced biquaternion matrix equation $AX = B$ and their applications in colour image restoration*, J. Mod. Opt. **66** (2019), 1802–1810, DOI: https://doi.org/10.1080/09500340.2019.1676474.

[20] H. H. Kösal and T. Bilgili, *Euler and De Moivreas Formulas for fundamental matrices of commutative quaternions*, Int. Electron. J. Geom. **13** (2020), 98–107, DOI: https://doi.org/10.36890/iejg.768821.
[21] E. Kemal and H. H. Kösal, *Numerical algorithm for solving general linear elliptic quaternionic matrix equations*, Fundam. J. Math. Appl. **4** (2021), 180–186, DOI: https://dx.doi.org/10.33401/fujma.888705.
[22] A. L. Wei, Y. Li, W. X. Ding, and J. L. Zhao, *Two algebraic methods for least squares L-structured and generalized L-structured problems of the commutative quaternion Stein matrix equation*, Comput. Appl. Math. **41** (2022), 1–23, DOI: https://doi.org/10.1007/s40314-022-01943-x.
[23] H. Dai, *Matrix Theory*, Science Press, Beijing, 2001.
[24] L. M. Xie and Q. W. Wang, *A system of matrix equations over the commutative quaternion ring*, Filomat **37** (2023), 97–106, DOI: https://doi.org/10.2298/FIL2301097X.
[25] E. F. Wang and S. M. Shi, *Advanced Algebra*, Shanghai Science and Technology Press, Shanghai, 1981.