Research Article

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Some Hardy's inequalities on conformable fractional calculus

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Abstract: In this article, we will demonstrate some Hardy's inequalities by utilizing Hölder inequality, integration by parts, and chain rule of the conformable fractional calculus. When $\alpha = 1$, we can obtain some of classical Copson's and Hardy's inequalities. In the end, we will obtain an application of Hardy's inequality utilizing the conformable fractional calculus.

Keywords: Hardy's inequality, chain rule, conformable fractional calculus, Hölder's inequality, Jensen's inequality

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1 Introduction

In 1920, Hardy [1] proved that

$$\sum_{\lambda=1}^{\infty}\left(\frac{1}{\lambda}\sum_{j=1}^{\lambda}w(j)\right)^{s} \leq \left(\frac{s}{s-1}\right)^{s}\sum_{\lambda=1}^{\infty}w^{s}(\lambda), \tag{1}$$

for a positive sequence $w(\lambda)$ for all $\lambda \geq 1$, $s > 1$, and $(s/(s-1))^{s}$ is the best constant we can obtain.

In 1925, Hardy [2] proved the integral inequality

$$\int_{0}^{\infty}\left(\frac{1}{\eta}\int_{0}^{\eta}q(\gamma)\mathrm{d}\gamma\right)^{s}\mathrm{d}\eta \leq \left(\frac{s}{s-1}\right)^{s}\int_{0}^{\infty}q^{s}(\eta)\mathrm{d}\eta, \quad s > 1, \tag{2}$$

where q is an integrable positive function over $(0, \eta)$ and q^{s} is convergent and integrable on $(0, \infty)$ such that $(s/(s-1))^{s}$ is the best constant we can obtain.

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In 1927, Copson [3] proved that if $\int_{\eta}^{\infty}(q(\gamma)/\gamma)\mathrm{d}\gamma$ converges for $\eta > 0$, then

$$\int_{0}^{\infty}\left(\int_{\eta}^{\infty}\frac{q(\gamma)}{\gamma}\mathrm{d}\gamma\right)^{s}\mathrm{d}\eta \le s^{s}\int_{0}^{\infty}q^{s}(\eta)\mathrm{d}\eta, \quad s > 1. \tag{3}$$

In 1928, Hardy [4] generalized inequality (2) and deduced that if $s > 1$, q is an integrable and positive function on $(0, \eta)$ such that q^{s} is integrable and convergent over $(0, \infty)$, then

$$\int_{0}^{\infty}\eta^{-\omega}\left(\int_{0}^{\eta}q(\gamma)\mathrm{d}\gamma\right)^{s}\mathrm{d}\eta \le \left(\frac{s}{\omega - 1}\right)^{s}\int_{0}^{\infty}\eta^{s-\omega}q^{s}(\eta)\mathrm{d}\eta, \quad \text{for } \omega > 1, \tag{4}$$

and

$$\int_{0}^{\infty}\eta^{-\omega}\left(\int_{\eta}^{\infty}q(\gamma)\mathrm{d}\gamma\right)^{s}\mathrm{d}\eta \le \left(\frac{s}{1 - \omega}\right)^{s}\int_{0}^{\infty}\eta^{s-\omega}q^{s}(\eta)\mathrm{d}\eta, \quad \text{for } 0 < \omega \le 1. \tag{5}$$

The constants $(s/(\omega - 1))^{s}$ and $(s/(1 - \omega))^{s}$ in (4) and (5) are the best constant we can obtain.

In 1964, Levinson [5] used Jensen's inequality that proved in [6] to obtain an extension of Hardy's inequality (2) as follows.

Let $s > 1$, $\hat{h}(\eta) > 0$, $\nu(\eta) > 0$ such that $\eta > 0$, $\phi(u)$ be a positive convex real-valued function for $u \in (0, \infty)$, $\Phi(\eta) = \int_{0}^{\eta}\nu(\gamma)\mathrm{d}\gamma$ and exists a constant $£ > 0$ such that

$$s - 1 + \frac{\nu'(\eta)\Phi(\eta)}{\nu^{2}(\eta)} \ge \frac{s}{£}, \quad \text{for } \eta > 0. \tag{6}$$

Then

$$\int_{0}^{\infty}\phi\left(\frac{1}{\Phi(\eta)}\int_{0}^{\eta}\nu(\gamma)\hat{h}(\gamma)\mathrm{d}\gamma\right)\mathrm{d}\eta \le £^{s}\int_{0}^{\infty}\phi(\hat{h}(\eta))\mathrm{d}\eta. \tag{7}$$

In 1976, Copson [7] proved that if

$$\Upsilon(\gamma) = \int_{0}^{\gamma}\nu(\eta)\mathrm{d}\eta \quad \text{and} \quad \Psi(\gamma) = \int_{0}^{\gamma}\nu(\eta)u(\eta)\mathrm{d}\eta,$$

then

$$\int_{0}^{\infty}\frac{\nu(\gamma)}{\Upsilon^{h}(\gamma)}\Psi^{k}(\gamma)\mathrm{d}\gamma \le \left(\frac{k}{h - 1}\right)^{k}\int_{0}^{\infty}\frac{\nu(\gamma)}{\Upsilon^{h-k}(\gamma)}u^{k}(\gamma)\mathrm{d}\gamma, \tag{8}$$

for $h > 1$, $k \ge 1$ and if $\bar{\Psi}(\gamma) = \int_{\gamma}^{\infty}\nu(\eta)u(\eta)\mathrm{d}\eta$, then

$$\int_{0}^{\infty}\frac{\nu(\gamma)}{\Upsilon^{h}(\gamma)}\bar{\Psi}^{k}(\gamma)\mathrm{d}\gamma \le \left(\frac{k}{1 - h}\right)^{k}\int_{0}^{\infty}\frac{\nu(\gamma)}{\Upsilon^{h-k}(\gamma)}u^{k}(\gamma)\mathrm{d}\gamma, \tag{9}$$

for $0 \le h < 1$ and $k \ge 1$.

In 1999, Yang and Hwang [8] extended Levinson inequality (7) and showed that, if

$$\Psi(\eta) = \int_{0}^{\eta}\nu(\gamma)\tau(\gamma)\hat{h}(\gamma)\mathrm{d}\gamma \quad \text{and} \quad \Phi(\eta) = \int_{0}^{\eta}\nu(\gamma)\tau(\gamma)\mathrm{d}\gamma,$$

where the functions $\nu(\eta)$, $\tau(\eta)$, and $\hat{h}(\eta)$ are nonnegative, and there is $£ > 0$ be a constant such that

$$s - 1 + \frac{\tau'(\eta)\Phi(\eta)}{\tau^{2}(\eta)\nu(\eta)} \ge \frac{s}{£}, \quad \text{for } \eta > 0 \quad \text{and} \quad s > 1,$$

then

$$\int_0^\infty v(\eta)\left(\frac{\Psi(\eta)}{\Phi(\eta)}\right)^s \mathrm{d}\eta \le \pounds^s \int_0^\infty v(\eta)\hat{h}^s(\eta)\mathrm{d}\eta. \tag{10}$$

Recently, many authors have made generalizations of fractional inequalities by using the conformable calculus like Chebyshev's inequality [9], Hermite-Hadamard's inequality [10–13], Opial's inequality [14–16], and Hardy's inequality [17–19].

In our article, we will deduce some fractional Hardy's inequalities by using integration by parts, Jensen's inequality, Hölder's inequality, and chain rule.

The article is divided into to three sections, the first section includes fundamentals of fractional calculus, the second section includes our main results and the third section includes an application of Hardy's inequality in conformable fractional calculus.

2 Preliminaries and basic lemmas

This section includes definitions and lemmas, which are fundamentals of conformable fractional calculus [20,21].

Definition 1. The conformable fractional derivative of order α for a function $\theta : [0, \infty) \to \mathbb{R}$ is defined by

$$D_\alpha\theta(v) = \lim_{\varepsilon\to 0}\frac{\theta(v+\varepsilon v^{1-\alpha}) - \theta(v)}{\varepsilon}, \quad v > 0 \quad \text{and} \quad \alpha \in (0,1]. \tag{11}$$

If $\lim_{v\to 0^+}\theta(v)$ exists and θ is α-differentiable on $(0, a)$ for $a > 0$, then we can define $D_\alpha\theta(0) = \lim_{v\to 0^+}D_\alpha\theta(v)$.

Definition 2. The conformable fractional integral of order α for a function $\theta : [0, \infty) \to \mathbb{R}$ is defined by

$$I_a^\alpha\theta(v) = \int_a^v \theta(\gamma)d_\alpha\gamma = \int_a^v \gamma^{\alpha-1}\theta(\gamma)\mathrm{d}\gamma, \quad \text{for } \alpha \in (0,1]. \tag{12}$$

The following gives the fundamentals of α-derivative of functions.

Theorem 1. *Assume that θ and Ω are α-differentiable, then for $\alpha \in (0,1]$, we obtain*

(1) *For all $a, b \in \mathbb{R}$, we have*

$$D_\alpha(a\theta + b\Omega)(v) = aD_\alpha\theta(v) + bD_\alpha\Omega(v). \tag{13}$$

(2) *The product rule*

$$D_\alpha(\theta\Omega)(v) = \theta D_\alpha\Omega(v) + \Omega D_\alpha\theta(v). \tag{14}$$

(3) *The quotient rule*

$$D_\alpha\left(\frac{\theta}{\Omega}\right)(v) = \frac{\Omega D_\alpha\theta(v) - \theta D_\alpha\Omega(v)}{\Omega^2}. \tag{15}$$

(4) *For a function θ, we have*

$$D_\alpha\theta(v) = v^{1-\alpha}\frac{\mathrm{d}\theta}{\mathrm{d}v}. \tag{16}$$

(5) *If $s \in \mathbb{R}$, then $D_\alpha(v^s) = sv^{s-\alpha}$.*

(6) *If $\theta(v) = \xi$ be a constant function, then $D_\alpha\theta(v) = 0$.*

Lemma 1. (Chain rule) *Let Ω be α-differentiable with respect to v and θ be differentiable with respect to Ω. Then*

$$D_\alpha\theta(\Omega(v)) = \Omega^{\alpha-1}(v)\cdot(D_\alpha\theta)(\Omega(v))\cdot D_\alpha\Omega(v). \tag{17}$$

Lemma 2. (Integration by parts) *Let θ and Ω be α-differentiable with respect to v on* $[0, \infty)$. *Then*

$$\int_0^\infty (D_\alpha\theta(v))\Omega(v)d_\alpha v = \theta(v)\Omega(v)|_0^\infty - \int_0^\infty \theta(v)(D_\alpha\Omega(v))d_\alpha v. \tag{18}$$

Lemma 3. (Hölder inequality) *Let* $0 < \alpha \le 1$ *and* $\theta, \Omega : [0, \infty] \to \mathbb{R}$. *Then*

$$\int_0^\infty |\theta(v)\Omega(v)|d_\alpha v \le \left(\int_0^\infty |\theta(v)|^\lambda d_\alpha v\right)^{\frac{1}{\lambda}} \left(\int_0^\infty |\Omega(v)|^\varepsilon d_\alpha v\right)^{\frac{1}{\varepsilon}}, \tag{19}$$

for $\frac{1}{\lambda} + \frac{1}{\varepsilon} = 1$ *and* $\lambda > 1$.

Lemma 4. (Jensen's inequality) *Let* $\Omega \in C([\omega, \varepsilon], \mathbb{R})$ *be convex,* $\omega, \varepsilon \in \mathbb{R}$, $a \in [0, \infty)$, $\hat{h} \in C([a, \infty), (\omega, \varepsilon))$, *and* $z \in C([a, \infty), \mathbb{R})$ *such that* $I_a^\alpha|k(\gamma)| = \int_a^\gamma |k(v)|d_\alpha v = \int_a^\gamma v^{\alpha-1}|k(v)|\mathrm{d}v > 0$. *Then*

$$\Omega\left(\frac{I_a^\alpha[|k(\gamma)|\hat{h}(\gamma)]}{I_a^\alpha|k(\gamma)|}\right) \le \frac{I_a^\alpha[|k(\gamma)|\Omega(\hat{h}(\gamma))]}{I_a^\alpha|k(\gamma)|}. \tag{20}$$

3 Main results

In this section, we will establish some formula of Yang and Hwang's and Pachpatte's inequalities, then we will prove some of Copson's and Hardy's inequalities from our results.

Theorem 2. *Let* $v, \tau, \hat{h}$ *be nonnegative functions,* τ *be an increasing function on* $[0, \infty)$ *and* $1 < \delta \le s$. *Define*

$$\Psi(\sigma) = \int_0^\sigma v(\gamma)\tau(\gamma)\hat{h}(\gamma)d_\alpha\gamma \quad \textit{and} \quad \Phi(\sigma) = \int_0^\sigma v(\gamma)\tau(\gamma)d_\alpha\gamma. \tag{21}$$

Let there is a positive constant K with

$$\delta - 1 + \frac{D_\alpha\tau(\sigma)\Phi(\sigma)}{v(\sigma)\tau^2(\sigma)} = \frac{s}{K}, \quad \textit{for} \;\; \sigma \in [0, \infty). \tag{22}$$

Then

$$\int_0^\infty \frac{v(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_\alpha\sigma \le K^s\int_0^\infty \Phi^{s-\delta}(\sigma)v(\sigma)\hat{h}^s(\sigma)d_\alpha\sigma. \tag{23}$$

Proof. By using integration by parts (18) on the term

$$\int_0^\infty \frac{v(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_\alpha\sigma,$$

with $\Omega(\sigma) = \frac{\Psi^s(\sigma)}{\tau(\sigma)}$ and $D_\alpha\theta(\sigma) = \frac{v(\sigma)\tau(\sigma)}{\Phi^\delta(\sigma)}$, then

$$\begin{aligned} \int_0^\infty \frac{v(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_\alpha\sigma &= \int_0^\infty \frac{v(\sigma)\tau(\sigma)}{\Phi^\delta(\sigma)}\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right)d_\alpha\sigma \\ &= \left[\theta(\sigma)\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right|_0^\infty + \int_0^\infty (-\theta(\sigma))D_\alpha\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right)d_\alpha\sigma. \end{aligned} \tag{24}$$

Since $\theta(\sigma) = -\int_\sigma^\infty \frac{\nu(\gamma)\tau(\gamma)}{\Phi^\delta(\gamma)} d_\alpha\gamma$, $\Psi(0) = 0$, and $\theta(\infty) = 0$, we obtain

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)} \Psi^s(\sigma) d_\alpha\sigma = \int_0^\infty (-\theta(\sigma)) D_\alpha\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right) d_\alpha\sigma. \tag{25}$$

Since $D_\alpha\Phi(\sigma) = \nu(\sigma)\tau(\sigma) > 0$, then by using chain rule (17), we obtain

$$\begin{aligned} D_\alpha\Phi^{1-\delta}(\sigma) &= \Phi^{\alpha-1}(\sigma)\cdot(D_\alpha\Phi^{1-\delta}(\sigma))(\Phi(\sigma))\cdot D_\alpha\Phi(\sigma) \\ &= \Phi^{\alpha-1}(\sigma)\cdot(1-\delta)\Phi^{1-\delta-\alpha}(\sigma)\cdot D_\alpha\Phi(\sigma) \\ &= -(\delta-1)\Phi^{-\delta}(\sigma) D_\alpha\Phi(\sigma) \\ &= -(\delta-1)\nu(\sigma)\tau(\sigma)\Phi^{-\delta}(\sigma), \end{aligned}$$

which can be written as follows:

$$\nu(\sigma)\tau(\sigma)\Phi^{-\delta}(\sigma) = \frac{-1}{\delta-1} D_\alpha\Phi^{1-\delta}(\sigma). \tag{26}$$

Therefore,

$$-\theta(\sigma) = \int_\sigma^\infty \frac{\nu(\gamma)\tau(\gamma)}{\Phi^\delta(\gamma)} d_\alpha\gamma = \frac{-1}{\delta-1}\int_\sigma^\infty D_\alpha\Phi^{1-\delta}(\gamma) d_\alpha\gamma = \frac{1}{\delta-1}\Phi^{1-\delta}(\sigma). \tag{27}$$

By combining (25) and (27), we have

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)} \Psi^s(\sigma) d_\alpha\sigma = \frac{1}{\delta-1}\int_0^\infty \Phi^{1-\delta}(\sigma) D_\alpha\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right) d_\alpha\sigma. \tag{28}$$

By using the quotient rule (15), we obtain

$$D_\alpha\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right) = \frac{\tau(\sigma) D_\alpha\Psi^s(\sigma) - \Psi^s(\sigma) D_\alpha\tau(\sigma)}{\tau^2(\sigma)}. \tag{29}$$

By using the chain rule (17) once more, we obtain

$$D_\alpha\Psi^s(\sigma) = s\Psi^{s-\alpha}(\sigma)\Psi^{\alpha-1}(\sigma) D_\alpha\Psi(\sigma) = s\Psi^{s-1}(\sigma) D_\alpha\Psi(\sigma). \tag{30}$$

As $D_\alpha\Psi(\sigma) = \nu(\sigma)\tau(\sigma)\hat{h}(\sigma) \geq 0$, then

$$D_\alpha\Psi^s(\sigma) = s\nu(\sigma)\tau(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma). \tag{31}$$

According to (29) and (31), we find

$$\begin{aligned} D_\alpha\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right) &= \frac{s\nu(\sigma)\tau^2(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma)}{\tau^2(\sigma)} - \frac{\Psi^s(\sigma) D_\alpha\tau(\sigma)}{\tau^2(\sigma)} \\ &= s\nu(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma) - \frac{\Psi^s(\sigma) D_\alpha\tau(\sigma)}{\tau^2(\sigma)}. \end{aligned} \tag{32}$$

From (28) and (32), we obtain

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)} \Psi^s(\sigma) d_\alpha\sigma = \frac{s}{\delta-1}\int_0^\infty \Phi^{1-\delta}(\sigma)\nu(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma) d_\alpha\sigma - \frac{1}{\delta-1}\int_0^\infty \frac{\Psi^s(\sigma) D_\alpha\tau(\sigma)}{\Phi^{\delta-1}(\sigma)\tau^2(\sigma)} d_\alpha\sigma.$$

Therefore,

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)} \Psi^s(\sigma)\left[\delta - 1 + \frac{\Phi(\sigma) D_\alpha\tau(\sigma)}{\nu(\sigma)\tau^2(\sigma)}\right] d_\alpha\sigma = s\int_0^\infty \Phi^{1-\delta}(\sigma)\nu(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma) d_\alpha\sigma. \tag{33}$$

By using (22) and Hölder's inequality (19) with indices $\lambda = s/(s-1)$ and $\varepsilon = s$, we see that

$$\begin{aligned}\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma &= K\int_0^\infty \nu(\sigma)\Phi^{1-\delta}(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma)d_a\sigma \\ &= K\int_0^\infty\left[\nu^{\frac{s-1}{s}}(\sigma)\Phi^{\frac{-\delta(s-1)}{s}}(\sigma)\Psi^{s-1}(\sigma)\right]\left[\Phi^{\frac{\delta(s-1)}{s}}(\sigma)\Phi^{1-\delta}(\sigma)\hat{h}(\sigma)\nu^{\frac{1}{s}}(\sigma)\right]d_a\sigma \\ &\le K\left[\int_0^\infty \nu(\sigma)\Phi^{-\delta}(\sigma)\Psi^s(\sigma)d_a\sigma\right]^{\frac{s-1}{s}}\left[\int_0^\infty \Phi^{s-\delta}(\sigma)\hat{h}^s(\sigma)\nu(\sigma)d_a\sigma\right]^{\frac{1}{s}}.\end{aligned}$$

Hence,

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma \le K^s\int_0^\infty \Phi^{s-\delta}(\sigma)\hat{h}^s(\sigma)\nu(\sigma)d_a\sigma.$$

Corollary 1. *In* Theorem 2, *if* $\tau(\sigma) = 1$, *then we obtain*

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma \le \left(\frac{s}{\delta-1}\right)^s\int_0^\infty \Phi^{s-\delta}(\sigma)\hat{h}^s(\sigma)\nu(\sigma)d_a\sigma,$$

which is Copson's inequality of Theorem 3 in [22]. *Also, if* $\alpha = 1$ *and* $\nu(\sigma) = 1$, *then we obtain*

$$\int_0^\infty \frac{1}{\sigma^\delta}\Psi^s(\sigma)\mathrm{d}\sigma \le \left(\frac{s}{\delta-1}\right)^s\int_0^\infty \sigma^{s-\delta}(\sigma)\hat{h}^s(\sigma)\mathrm{d}\sigma,$$

which is inequality of Hardy-Littlewood in Theorem 330 in [23] *such that* $\Phi(\infty) < \infty$ *and* $\delta > 1$.

Corollary 2. *In* Theorem 2, *if* $\alpha = 1$, *then we obtain*

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)\mathrm{d}\sigma \le K^s\int_0^\infty \Phi^{s-\delta}(\sigma)\hat{h}^s(\sigma)\nu(\sigma)\mathrm{d}\sigma, \tag{34}$$

which is inequality (7) *due to Levinson, where*

$$\delta - 1 + \frac{\tau'(\sigma)\Phi(\sigma)}{\nu(\sigma)\tau^2(\sigma)} = \frac{s}{K}, \quad \textit{for } \sigma > 0$$

and

$$\Psi(\sigma) = \int_0^\sigma \nu(\gamma)\tau(\gamma)\hat{h}(\gamma)\mathrm{d}\gamma \quad \textit{and} \quad \Phi(\sigma) = \int_0^\sigma \nu(\gamma)\tau(\gamma)\mathrm{d}\gamma.$$

Remark. In (34), if $\delta = s > 1$, then we have the inequality (10).

Remark. In (34), if $\tau(\sigma) = 1$, then we have inequality of Copson (8).

Remark. In (34), if $\nu(\sigma) = \tau(\sigma) = 1$, then we have inequality of Hardy (4).

Remark. In (34), if $\nu(\sigma) = \tau(\sigma) = 1$ and $\delta = s$, then we have the classical inequality of Hardy (2).

Theorem 3. *Let* $0 \le \delta < 1$, $s > 1$, $\tau(\sigma)$ *be an increasing function on* $[0, \infty)$, *and*

$$\Psi(\sigma) = \int_\sigma^\infty \nu(\gamma)\tau(\gamma)\hat{h}(\gamma)d_a\gamma \quad \textit{and} \quad \Phi(\sigma) = \int_0^\sigma \nu(\gamma)\tau(\gamma)d_a\gamma. \tag{35}$$

There is a positive constant K with

$$1-\delta-\frac{D_a\tau(\sigma)\Phi(\sigma)}{\nu(\sigma)\tau^2(\sigma)}=\frac{s}{K},\quad \textit{for }\sigma\in[0,\infty). \tag{36}$$

Then

$$\int_0^\infty\frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma\le K^s\int_0^\infty\Phi^{s-\delta}(\sigma)\nu(\sigma)\hat{h}^s(\sigma)d_a\sigma. \tag{37}$$

Proof. By using integration by parts (18) on the term

$$\int_0^\infty\frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma,$$

such that $\theta(\sigma)=\frac{\Psi^s(\sigma)}{\tau(\sigma)}$ and $D_a\Omega(\sigma)=\frac{\nu(\sigma)\tau(\sigma)}{\Phi^\delta(\sigma)}$, then

$$\begin{aligned}\int_0^\infty\frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma&=\int_0^\infty\frac{\nu(\sigma)\tau(\sigma)}{\Phi^\delta(\sigma)}\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right)d_a\sigma\\&=\left[\Omega(\sigma)\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right|_0^\infty+\int_0^\infty\Omega(\sigma)\left(-D_a\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right)\right)d_a\sigma.\end{aligned} \tag{38}$$

Since $\Omega(\sigma)=\int_0^\sigma\frac{\nu(s)\tau(s)}{\Phi^\delta(s)}d_as$, $\Psi(\infty)=0$ and $\Omega(0)=0$, we obtain

$$\int_0^\infty\frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma=\int_0^\infty\Omega(\sigma)\left(-D_a\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right)\right)d_a\sigma. \tag{39}$$

As $D_a\Phi(\sigma)=\nu(\sigma)\tau(\sigma)>0$, using the chain rule (17), we obtain

$$\begin{aligned}D_a\Phi^{1-\delta}(\sigma)&=\Phi^{a-1}(\sigma)\cdot(D_a\Phi^{1-\delta}(\sigma))(\Phi(\sigma))\cdot D_a\Phi(\sigma)\\&=\Phi^{a-1}(\sigma)\cdot(1-\delta)\Phi^{1-\delta-a}(\sigma)\cdot D_a\Phi(\sigma)\\&=(1-\delta)\nu(\sigma)\tau(\sigma)\Phi^{-\delta}(\sigma),\end{aligned}$$

which can be written as follows:

$$\nu(\sigma)\tau(\sigma)\Phi^{-\delta}(\sigma)=\frac{1}{1-\delta}D_a\Phi^{1-\delta}(\sigma). \tag{40}$$

Therefore,

$$\Omega(\sigma)=\int_0^\sigma\frac{\nu(s)\tau(s)}{\Phi^\delta(s)}d_as=\frac{1}{1-\delta}\int_0^\sigma D_a(\Phi^{1-\delta}(s))d_as=\frac{1}{1-\delta}\Phi^{1-\delta}(\sigma). \tag{41}$$

By combining (39) and (41), we have

$$\int_0^\infty\frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma=\frac{1}{1-\delta}\int_0^\infty\Phi^{1-\delta}(\sigma)\left(-D_a\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right)\right)d_a\sigma. \tag{42}$$

By using the quotient rule (15), we obtain

$$-D_a\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right)=\frac{-\tau(\sigma)D_a\Psi^s(\sigma)+\Psi^s(\sigma)D_a\tau(\sigma)}{\tau^2(\sigma)}. \tag{43}$$

By using the chain rule (17) once more, we obtain

$$-D_a\Psi^s(\sigma)=-s\Psi^{s-a}(\sigma)\Psi^{a-1}(\sigma)D_a\Psi(\sigma)=-s\Psi^{s-1}(\sigma)D_a\Psi(\sigma).$$

Since $D_a\Psi(\sigma) = -\nu(\sigma)\tau(\sigma)\hat{h}(\sigma) \le 0$, then

$$-D_a\Psi^s(\sigma) = s\nu(\sigma)\tau(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma). \tag{44}$$

By combining (43) and (44), we find

$$\begin{aligned} -D_a\left(\frac{\Psi^s(\sigma)}{\tau(\sigma)}\right) &= \frac{s\nu(\sigma)\tau^2(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma)}{\tau^2(\sigma)} + \frac{\Psi^s(\sigma)D_a\tau(\sigma)}{\tau^2(\sigma)} \\ &= s\nu(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma) + \frac{\Psi^s(\sigma)D_a\tau(\sigma)}{\tau^2(\sigma)}. \end{aligned} \tag{45}$$

From (42) and (45), we obtain

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma = \frac{s}{1-\delta}\int_0^\infty \Phi^{1-\delta}(\sigma)\nu(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma)d_a\sigma + \frac{1}{1-\delta}\int_0^\infty \frac{\Psi^s(\sigma)D_a\tau(\sigma)}{\Phi^{\delta-1}(\sigma)\tau^2(\sigma)}d_a\sigma. \tag{46}$$

Then

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)\left[1-\delta-\frac{\Phi(\sigma)D_a\tau(\sigma)}{\nu(\sigma)\tau^2(\sigma)}\right]d_a\sigma = s\int_0^\infty \Phi^{1-\delta}(\sigma)\nu(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma)d_a\sigma. \tag{47}$$

By employing (36) and Hölder's inequality (19) with indices $\lambda = s/(s-1)$ and $\varepsilon = s$, we see that

$$\begin{aligned} \int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma &= K\int_0^\infty \nu(\sigma)\Phi^{1-\delta}(\sigma)\hat{h}(\sigma)\Psi^{s-1}(\sigma)d_a\sigma \\ &= K\int_0^\infty\left[\nu^{\frac{s-1}{s}}(\sigma)\Phi^{\frac{-\delta(s-1)}{s}}(\sigma)\Psi^{s-1}(\sigma)\right]\left[\Phi^{\frac{\delta(s-1)}{s}}(\sigma)\Phi^{1-\delta}(\sigma)\hat{h}(\sigma)\nu^{\frac{1}{s}}(\sigma)\right]d_a\sigma \\ &\le K\left[\int_0^\infty \nu(\sigma)\Phi^{-\delta}(\sigma)\Psi^s(\sigma)d_a\sigma\right]^{\frac{s-1}{s}}\left[\int_0^\infty \Phi^{s-\delta}(\sigma)\hat{h}^s(\sigma)\nu(\sigma)d_a\sigma\right]^{\frac{1}{s}}. \end{aligned}$$

Hence,

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma \le K^s\int_0^\infty \nu(\sigma)\hat{h}^s(\sigma)\Phi^{s-\delta}(\sigma)d_a\sigma.$$

□

Corollary 3. *In* Theorem 3, *if* $\tau(\sigma) = 1$, *then we obtain*

$$\int_0^\infty \frac{\nu(\sigma)}{\Phi^\delta(\sigma)}\Psi^s(\sigma)d_a\sigma \le \left(\frac{s}{1-\delta}\right)^s\int_0^\infty \nu(\sigma)\hat{h}^s(\sigma)\Phi^{s-\delta}(\sigma)d_a\sigma, \tag{48}$$

which is Copson's inequality of Theorem 4 in [22].

Corollary 4. *In* Theorem 3, *if* $\alpha = 1$, *then we obtain inequality of Copson* (9), *where*

$$1-\delta-\frac{\tau'(\sigma)\Phi(\sigma)}{\nu(\sigma)\tau^2(\sigma)} = \frac{s}{K}, \quad \sigma\in[0,\infty),$$

and

$$\Psi(\sigma) = \int_\sigma^\infty \nu(\gamma)\tau(\gamma)\hat{h}(\gamma)\mathrm{d}\gamma \quad \textit{and} \quad \Phi(\sigma) = \int_0^\sigma \nu(\gamma)\tau(\gamma)\mathrm{d}\gamma.$$

Theorem 4. *Let $s > 1$, $f(w)$ be a positive nondecreasing convex function such that $w > 0$,*

$$\Phi(\sigma) = \int_0^\sigma \nu(\gamma) d_a \gamma, \quad \Psi(\sigma) = \int_0^\sigma \nu(\gamma)\hat{h}(\gamma) d_a \gamma,$$

$$\Lambda(\sigma) = \int_0^\sigma \nu(\gamma) f(\hat{h}(\gamma)) d_a \gamma \quad \text{and} \quad \Theta(\sigma) = \frac{\Lambda(\sigma)}{\Phi(\sigma)}. \tag{49}$$

Then

$$\int_0^\infty \nu(\sigma) f^s\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right) d_a\sigma \le \frac{s}{s-1} \int_0^\sigma \nu(\sigma) f(\hat{h}(\sigma)) \Theta^{s-1}(\sigma) d_a \sigma \le \left(\frac{s}{s-1}\right)^s \int_0^\infty \nu(\sigma) f^s(\hat{h}(\sigma)) d_a \sigma. \tag{50}$$

Proof. By using (49), we can write

$$\nu(\sigma) f(\hat{h}(\sigma)) = D_a(\Theta(\sigma)\Phi(\sigma)) = \Phi(\sigma) D_a \Theta(\sigma) + \Theta(\sigma) D_a \Phi(\sigma). \tag{51}$$

Then, we obtain

$$\begin{aligned}
&\nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\nu(\sigma) f(\hat{h}(\sigma))\Theta^{s-1}(\sigma) \\
&= \nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\Theta^{s-1}(\sigma)[\Phi(\sigma) D_a\Theta(\sigma) + \Theta(\sigma) D_a \Phi(\sigma)] \\
&= \nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\Theta^{s-1}(\sigma)\Phi(\sigma) D_a\Theta(\sigma) - \frac{s}{s-1}\Theta^s(\sigma) D_a\Phi(\sigma) \\
&= \nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\Theta^{s-1}(\sigma)\Phi(\sigma) D_a\Theta(\sigma) - \frac{s}{s-1}\Theta^s(\sigma)\nu(\sigma) \\
&= -\frac{1}{s-1}\Theta^s(\sigma)\nu(\sigma) - \frac{s}{s-1}\Theta^{s-1}(\sigma)\Phi(\sigma) D_a \Theta(\sigma).
\end{aligned} \tag{52}$$

By using the quotient rule (15), we obtain

$$\begin{aligned}
D_a\Theta(\sigma) = D_a\left(\frac{\Lambda(\sigma)}{\Phi(\sigma)}\right) &= \frac{\Phi(\sigma) D_a \Lambda(\sigma) - \Lambda(\sigma) D_a \Phi(\sigma)}{\Phi^2(\sigma)} \\
&= \frac{\Phi(\sigma)\nu(\sigma) f(\hat{h}(\sigma)) - \Lambda(\sigma)\nu(\sigma)}{\Phi^2(\sigma)} \\
&= \frac{\nu(\sigma)}{\Phi^2(\sigma)}\left[f(\hat{h}(\sigma))\int_0^\sigma \nu(\gamma) d_a\gamma - \int_0^\sigma \nu(\gamma) f(\hat{h}(\gamma)) d_a \gamma\right].
\end{aligned} \tag{53}$$

Since f is nondecreasing function, then

$$\int_0^\sigma \nu(\gamma) f(\hat{h}(\gamma)) d_a\gamma \le f(\hat{h}(\sigma)) \int_0^\sigma \nu(\gamma) d_a \gamma.$$

Therefore, $D_a\Theta(\sigma) > 0$. According to chain rule (17), we obtain

$$D_a \Theta^s(\sigma) = s\Theta^{s-\alpha}(\sigma)\Theta^{\alpha-1}(\sigma) D_a\Theta(\sigma) = s\Theta^{s-1}(\sigma) D_a \Theta(\sigma),$$

which can be written as follows:

$$\Theta^{s-1}(\sigma) D_a \Theta(\sigma) = \frac{1}{s} D_a \Theta^s(\sigma). \tag{54}$$

By combining (52) and (54), we obtain

$$\begin{aligned}
&\nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\nu(\sigma) f(\hat{h}(\sigma))\Theta^{s-1}(\sigma) \\
&= -\frac{1}{s-1}\Theta^s(\sigma)\nu(\sigma) - \frac{1}{s-1}\Phi(\sigma) D_a \Theta^s(\sigma) \\
&= -\frac{1}{s-1} D_a(\Phi(\sigma)\Theta^s(\sigma)).
\end{aligned} \tag{55}$$

By integrating both sides of (55) from 0 to z, we have

$$\int_0^z v(\sigma)\Theta^s(\sigma)d_a\sigma - \frac{s}{s-1}\int_0^z v(\sigma)f(\hat{h}(\sigma))\Theta^{s-1}(\sigma)d_a\sigma = -\frac{1}{s-1}\Phi(\sigma)\Theta^s(\sigma) \leq 0.$$

Therefore,

$$\begin{aligned}\int_0^z v(\sigma)\Theta^s(\sigma)d_a\sigma &\leq \frac{s}{s-1}\int_0^z v(\sigma)f(\hat{h}(\sigma))\Theta^{s-1}(\sigma)d_a\sigma \\ &= \frac{s}{s-1}\int_0^z v^{\frac{s-1}{s}}(\sigma)\Theta^{s-1}(\sigma)v^{\frac{1}{s}}(\sigma)f(\hat{h}(\sigma))d_a\sigma.\end{aligned}$$

By using (49) and inequality of Hölder (19) with indices $\lambda = s/(s-1)$ and $\varepsilon = s$, we obtain

$$\int_0^z v(\sigma)\Theta^s(\sigma)d_a\sigma \leq \frac{s}{s-1}\left(\int_0^z v(\sigma)\Theta^s(\sigma)d_a\sigma\right)^{\frac{s-1}{s}}\left(\int_0^z v(\sigma)f^s(\hat{h}(\sigma))d_a\sigma\right)^{\frac{1}{s}}.$$

Hence,

$$\int_0^z v(\sigma)\left(\frac{\Lambda(\sigma)}{\Phi(\sigma)}\right)^s d_a\sigma \leq \left(\frac{s}{s-1}\right)^s \int_0^z v(\sigma)f^s(\hat{h}(\sigma))d_a\sigma. \tag{56}$$

By using Jensen's inequality (20) with replace Ω by f and k by v, we obtain

$$f\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right) = f\left(\frac{\int_0^\sigma v(\gamma)\hat{h}(\gamma)d_a\gamma}{\int_0^\sigma v(\gamma)d_a\gamma}\right) \leq \frac{\int_0^\sigma v(\gamma)f(\hat{h}(\gamma))d_a\gamma}{\int_0^\sigma v(\gamma)d_a\gamma} = \frac{\Lambda(\sigma)}{\Phi(\sigma)}. \tag{57}$$

By substituting (57) into (56), we find

$$\int_0^z v(\sigma)f^s\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right)d_a\sigma \leq \int_0^z v(\sigma)\left(\frac{\Lambda(\sigma)}{\Phi(\sigma)}\right)^s d_a\sigma \leq \left(\frac{s}{s-1}\right)^s \int_0^z v(\sigma)f^s(\hat{h}(\sigma))d_a\sigma.$$

By assuming that $z \to \infty$, then we have

$$\int_0^\infty v(\sigma)f^s\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right)d_a\sigma \leq \left(\frac{s}{s-1}\right)^s \int_0^\infty v(\sigma)f^s(\hat{h}(\sigma))d_a\sigma.$$ □

Corollary 5. *In* Theorem 4, *if $f(w) = w$, then we obtain*

$$\int_0^\infty v(\sigma)\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right)^s d_a\sigma \leq \left(\frac{s}{s-1}\right)^s \int_0^\infty v(\sigma)\hat{h}^s(\sigma)d_a\sigma. \tag{58}$$

If $\alpha = 1$, then we obtain

$$\int_0^\infty v(\sigma)\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right)^s \mathrm{d}\sigma \leq \left(\frac{s}{s-1}\right)^s \int_0^\infty v(\sigma)\hat{h}^s(\sigma)\mathrm{d}\sigma, \tag{59}$$

where

$$\Phi(\sigma) = \int_0^\sigma v(\gamma)\mathrm{d}\gamma \quad \textit{and} \quad \Psi(\sigma) = \int_0^\sigma v(\gamma)\hat{h}(\gamma)\mathrm{d}\gamma,$$

which is an extension of Hardy's inequality.

Theorem 5. *Let $s > 1$, $f(w)$ be a positive convex nonincreasing function such that $w > 0$,*

$$\Phi(\sigma) = \int_{\sigma}^{\infty} \nu(\gamma) d_a\gamma, \quad \Psi(\sigma) = \int_{\sigma}^{\infty} \nu(\gamma)\hat{h}(\gamma) d_a\gamma,$$

$$\Lambda(\sigma) = \int_{\sigma}^{\infty} \nu(\gamma) f(\hat{h}(\gamma)) d_a\gamma \quad \text{and} \quad \Theta(\sigma) = \frac{\Lambda(\sigma)}{\Phi(\sigma)}. \tag{60}$$

Then

$$\int_0^{\infty} \nu(\sigma) f^s\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right) d_a\sigma \le \frac{s}{s-1} \int_0^{\infty} \nu(\sigma) f(\hat{h}(\sigma)) \Theta^{s-1}(\sigma) d_a\sigma \le \left(\frac{s}{s-1}\right)^s \int_0^{\infty} \nu(\sigma) f^s(\hat{h}(\sigma)) d_a\sigma. \tag{61}$$

Proof. From (60), we can write

$$\begin{aligned} \nu(\sigma) f(\hat{h}(\sigma)) &= -D_a(\Theta(\sigma)\Phi(\sigma)) \\ &= -(\Theta(\sigma) D_a\Phi(\sigma) + \Phi(\sigma) D_a\Theta(\sigma)) \\ &= -(-\nu(\sigma)\Theta(\sigma) + \Phi(\sigma) D_a\Theta(\sigma)) \\ &= \nu(\sigma)\Theta(\sigma) - \Phi(\sigma) D_a\Theta(\sigma). \end{aligned} \tag{62}$$

Then, we obtain

$$\begin{aligned} &\nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\nu(\sigma) f(\hat{h}(\sigma))\Theta^{s-1}(\sigma) \\ &= \nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\Theta^{s-1}(\sigma)[\nu(\sigma)\Theta(\sigma) - \Phi(\sigma) D_a\Theta(\sigma)] \\ &= \nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\Theta^{s-1}(\sigma)\nu(\sigma)\Theta(\sigma) + \frac{s}{s-1}\Theta^{s-1}(\sigma)\Phi(\sigma) D_a\Theta(\sigma) \\ &= \nu(\sigma)\Theta^s(\sigma) - \frac{s}{s-1}\Theta^{s}(\sigma)\nu(\sigma) + \frac{s}{s-1}\Theta^{s-1}(\sigma)\Phi(\sigma) D_a\Theta(\sigma) \\ &= -\frac{1}{s-1}\nu(\sigma)\Theta^s(\sigma) + \frac{s}{s-1}\Theta^{s-1}(\sigma)\Phi(\sigma) D_a\Theta(\sigma). \end{aligned} \tag{63}$$

According to the quotient rule (15), we obtain

$$\begin{aligned} D_a\Theta(\sigma) = D_a\left(\frac{\Lambda(\sigma)}{\Phi(\sigma)}\right) &= \frac{\Phi(\sigma) D_a\Lambda(\sigma) - \Lambda(\sigma) D_a\Phi(\sigma)}{\Phi^2(\sigma)} \\ &= \frac{\Phi(\sigma)(-\nu(\sigma) f(\hat{h}(\sigma))) - \Lambda(\sigma)(-\nu(\sigma))}{\Phi^2(\sigma)} \\ &= \frac{-\nu(\sigma)}{\Phi^2(\sigma)}\left[f(\hat{h}(\sigma)) \int_{\sigma}^{\infty} \nu(\gamma) d_a\gamma - \int_{\sigma}^{\infty} \nu(\gamma) f(\hat{h}(\gamma)) d_a\gamma \right]. \end{aligned} \tag{64}$$

Since f is nonincreasing function, then

$$\int_{\sigma}^{\infty} \nu(\gamma) f(\hat{h}(\gamma)) d_a\gamma \le f(\hat{h}(\sigma)) \int_{\sigma}^{\infty} \nu(\gamma) d_a\gamma.$$

Therefore, $D_a\Theta(\sigma) < 0$. Using chain rule (17), we obtain

$$D_a\Theta^s(\sigma) = s\Theta^{s-\alpha}(\sigma)\Theta^{\alpha-1}(\sigma) D_a\Theta(\sigma) = s\Theta^{s-1}(\sigma) D_a\Theta(\sigma),$$

which can be written as follows:

$$\Theta^{s-1}(\sigma) D_a\Theta(\sigma) = \frac{1}{s} D_a\Theta^s(\sigma). \tag{65}$$

By combining (63) and (65), we obtain

$$\begin{aligned}
&\nu(\sigma)\Theta^{s}(\sigma) - \frac{s}{s-1}\nu(\sigma)f(\hat{h}(\sigma))\Theta^{s-1}(\sigma) \\
&= -\frac{1}{s-1}\nu(\sigma)\Theta^{s}(\sigma) + \frac{1}{s-1}\Phi(\sigma)D_{a}\Theta^{s}(\sigma) \\
&= \frac{1}{s-1}(-\nu(\sigma)\Theta^{s}(\sigma) + \Phi(\sigma)D_{a}\Theta^{s}(\sigma)) \\
&= \frac{1}{s-1}(\Theta^{s}(\sigma)D_{a}\Phi(\sigma) + \Phi(\sigma)D_{a}\Theta^{s}(\sigma)) \\
&= -\frac{1}{s-1}D_{a}(\Phi(\sigma)\Theta^{s}(\sigma)).
\end{aligned} \tag{66}$$

By integrating both sides of (66) from 0 to z, we have

$$\begin{aligned}
&\int_{0}^{z}\nu(\sigma)\Theta^{s}(\sigma)d_{a}\sigma - \frac{s}{s-1}\int_{0}^{z}\nu(\sigma)f(\hat{h}(\sigma))\Theta^{s-1}(\sigma)d_{a}\sigma \\
&= \frac{1}{s-1}[\Phi(\sigma)\Theta^{s}(\sigma)|_{0}^{z} = \frac{1}{s-1}(\Phi(z)\Theta^{s}(z) - \Phi(0)\Theta^{s}(0)) \\
&\le \frac{\Phi(0)}{s-1}(\Theta^{s}(z) - \Theta^{s}(0)) \le 0,
\end{aligned}$$

where $D_{a}\Theta(\sigma) < 0$. Therefore,

$$\begin{aligned}
\int_{0}^{z}\nu(\sigma)\Theta^{s}(\sigma)d_{a}\sigma &\le \frac{s}{s-1}\int_{0}^{z}\nu(\sigma)f(\hat{h}(\sigma))\Theta^{s-1}(\sigma)d_{a}\sigma \\
&= \frac{s}{s-1}\int_{0}^{z}\nu^{\frac{s-1}{s}}(\sigma)\Theta^{s-1}(\sigma)\nu^{\frac{1}{s}}(\sigma)f(\hat{h}(\sigma))d_{a}\sigma.
\end{aligned}$$

Assume that $z \to \infty$, then we have

$$\begin{aligned}
\int_{0}^{\infty}\nu(\sigma)\Theta^{s}(\sigma)d_{a}\sigma &\le \frac{s}{s-1}\int_{0}^{\infty}\nu(\sigma)f(\hat{h}(\sigma))\Theta^{s-1}(\sigma)d_{a}\sigma \\
&= \frac{s}{s-1}\int_{0}^{\infty}\nu^{\frac{s-1}{s}}(\sigma)\Theta^{s-1}(\sigma)\nu^{\frac{1}{s}}(\sigma)f(\hat{h}(\sigma))d_{a}\sigma.
\end{aligned}$$

By using (60) and inequality of Hölder (19) with indices $\lambda = s/(s-1)$ and $\varepsilon = s$, we obtain

$$\int_{0}^{\infty}\nu(\sigma)\Theta^{s}(\sigma)d_{a}\sigma \le \frac{s}{s-1}\left(\int_{0}^{\infty}\nu(\sigma)\Theta^{s}(\sigma)d_{a}\sigma\right)^{\frac{s-1}{s}}\left(\int_{0}^{\infty}\nu(\sigma)f^{s}(\hat{h}(\sigma))d_{a}\sigma\right)^{\frac{1}{s}}.$$

Hence,

$$\int_{0}^{\infty}\nu(\sigma)\left(\frac{\Lambda(\sigma)}{\Phi(\sigma)}\right)^{s}d_{a}\sigma \le \left(\frac{s}{s-1}\right)^{s}\int_{0}^{\infty}\nu(\sigma)f^{s}(\hat{h}(\sigma))d_{a}\sigma. \tag{67}$$

By using Jensen's inequality (20) with replacing Ω by f and k by ν, we see that

$$f\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right) = f\left(\frac{\int_{\sigma}^{\infty}\nu(\gamma)\hat{h}(\gamma)d_{a}\gamma}{\int_{\sigma}^{\infty}\nu(\gamma)d_{a}\gamma}\right) \le \frac{\int_{\sigma}^{\infty}\nu(\gamma)f(\hat{h}(\gamma))d_{a}\gamma}{\int_{\sigma}^{\infty}\nu(\gamma)d_{a}\gamma} = \frac{\Lambda(\sigma)}{\Phi(\sigma)}. \tag{68}$$

By substituting (68) into (67), we obtain

$$\int_0^\infty \nu(\sigma) f^s\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right) d_\alpha\sigma \leq \int_0^\infty \nu(\sigma)\left(\frac{\Lambda(\sigma)}{\Phi(\sigma)}\right)^s d_\alpha\sigma \leq \left(\frac{s}{s-1}\right)^s \int_0^\infty \nu(\sigma) f^s(\hat{h}(\sigma)) d_\alpha\sigma. \qquad \square$$

Corollary 6. *In* Theorem 5, *if* $\alpha = 1$, *then we obtain the following Pachpatte's inequality*:

$$\int_0^\infty \nu(\sigma) f^s\left(\frac{\Psi(\sigma)}{\Phi(\sigma)}\right) \mathrm{d}\sigma \leq \left(\frac{s}{s-1}\right)^s \int_0^\infty \nu(\sigma) f^s(\hat{h}(\sigma)) \mathrm{d}\sigma,$$

where

$$\Phi(\sigma) = \int_\sigma^\infty \nu(\gamma)\mathrm{d}\gamma \quad \textit{and} \quad \Psi(\sigma) = \int_\sigma^\infty \nu(\gamma)\hat{h}(\gamma)\mathrm{d}\gamma.$$

4 Application

First, fractional integral versions of Hardy inequalities in [24], and their new generalizations will be obtained in this section, which is considered as basic items for our main results of higher integrability.

Theorem 6. *Let* $\omega \in \mathbb{R}$, $\Lambda : [0, \infty) \to \mathbb{R}$ *be a differentiable convex function and* $\Phi : [0, \omega] \to \mathbb{R}$ *be a nonnegative nonincreasing function. Then, we obtain the following inequality*:

$$\Lambda(0) + \int_0^\omega \Lambda'(\rho\Phi(\rho))\Phi(\rho) d_\alpha\rho \leq \Lambda\left(\int_0^\omega \Phi(\rho) d_\alpha\rho\right). \tag{69}$$

Proof. Let $\rho \in [0, \omega]$, where Φ is nonincreasing function. Then $\rho\Phi(\rho) \leq \int_0^\rho \Phi(\gamma) d_\alpha\gamma$. By using the definition $\Omega(\rho) = \int_0^\rho \Phi(\gamma) d_\alpha\gamma$, we have $D_\alpha\Omega(\rho) = \Phi(\rho) \geq 0$, which leads to Ω, which is an increasing function. Since Λ is convex function, then we obtain that

$$\Lambda'(\rho\Phi(\rho))\Phi(\rho) \leq \Lambda'\left(\int_0^\rho \Phi(\gamma) d_\alpha\gamma\right)\Phi(\rho) = \Lambda'(\Omega(\rho))\Phi(\rho). \tag{70}$$

By utilizing chain rule (17), we obtain

$$D_\alpha(\Lambda(\Omega(\rho))) = ((D_\alpha\Lambda(\Omega(\rho)))(\Omega(\rho)))\cdot \Omega^{\alpha-1}(\rho)\cdot D_\alpha\Omega(\rho) = \Lambda'(\Omega(\rho))D_\alpha\Omega(\rho),$$

then

$$D_\alpha(\Lambda(\Omega(\rho))) = \Lambda'(\Omega(\rho))D_\alpha\Omega(\rho) = \Lambda'(\Omega(\rho))\Phi(\rho), \tag{71}$$

where Ω be an increasing function. From (70) and (71), we find

$$\Lambda'(\rho\Phi(\rho))\Phi(\rho) \leq D_\alpha\Lambda(\Omega(\rho)). \tag{72}$$

By integrating the inequality (72) from 0 to ω, we obtain

$$\int_0^\omega \Lambda'(\rho\Phi(\rho))\Phi(\rho) d_\alpha\rho \leq \int_0^\omega D_\alpha\Lambda(\Omega(\rho)) d_\alpha\rho = \Lambda(\Omega(\omega)) - \Lambda(\Omega(0)) = \Lambda(\Omega(\omega)) - \Lambda(0). \qquad \square$$

Corollary 7. *In* Theorem 6, *if* $\Lambda(\nu) = \nu^s$ *with* $s \geq 1$, *then we obtain*

$$\int_0^\omega \rho^{s-1}\Phi^s(\rho) d_\alpha\rho \leq \frac{1}{s}\left(\int_0^\omega \Phi(\rho) d_\alpha\rho\right)^s. \tag{73}$$

Remark. By utilizing inequality of Hölder (19) on the right-hand side of (73) with indices $\lambda = (s-1)/s$ and $\varepsilon = 1/s$, we obtain that

$$\begin{aligned}\int_0^\omega \rho^{s-1}\Phi^s(\rho)d_\alpha\rho &\le \frac{1}{s}\left[\left(\int_0^\omega d_\alpha\rho\right)^{(s-1)/s}\left(\int_0^\omega \Phi^s(\rho)d_\alpha\rho\right)^{1/s}\right]^s\\ &= \frac{1}{s}\left(\int_0^\omega d_\alpha\rho\right)^{s-1}\int_0^\omega \Phi^s(\rho)d_\alpha\rho\\ &= \frac{1}{s}\left(\frac{\omega^\alpha}{\alpha}\right)^{s-1}\int_0^\omega \Phi^s(\rho)d_\alpha\rho.\end{aligned}$$

Remark. In (73), if we replace Φ by $\rho^{\xi-\alpha}\Phi$ such that $\xi < \alpha \le 1$, then we obtain

$$\int_0^\omega \rho^{(\xi-\alpha+1)s-1}\Phi^s(\rho)d_\alpha\rho \le \frac{1}{s}\left(\int_0^\omega \rho^{\xi-\alpha}\Phi(\rho)d_\alpha\rho\right)^s. \tag{74}$$

Remark. In (74), if we replace Φ by Φ^k and put $s = \ell/k \ge 1$, $\xi = 1/s \le 1$ such that $k \le \ell$, where $k, \ell \in \mathbb{R}^+$, then we obtain

$$\left(\int_0^\omega \rho^{\frac{k}{\ell}(1-\alpha)}\Phi^\ell(\rho)d_\alpha\rho\right)^{\frac{k}{\ell}} \le \frac{k}{\ell}\int_0^\omega \rho^{\frac{k}{\ell}-\alpha}\Phi^k(\rho)d_\alpha\rho. \tag{75}$$

In the following, we aim to obtain an important formula of Hardy's inequality, which is an extension of (2). First, we define a nonnegative function $y : [0, \check{T}] \to \mathbb{R}$ and G is defined as follows:

$$G(\rho) = \frac{1}{\rho}\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma, \quad \text{for } \gamma \in [0, \check{T}]. \tag{76}$$

Lemma 5. *Let $\rho \in [0, \check{T}]$ and $y : [0, \check{T}] \to \mathbb{R}$ be a nonnegative nonincreasing function. Then*

$$G(\rho) \le y(\rho).$$

Proof. Since y is nonincreasing, then we obtain

$$\begin{aligned}G(\rho) &= \frac{1}{\rho}\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma \le \frac{1}{\rho}\int_0^\rho \gamma^{1-\alpha}y(\rho)d_\alpha\gamma\\ &= \frac{1}{\rho}y(\rho)\int_0^\rho \gamma^{1-\alpha}d_\alpha\gamma = \frac{1}{\rho}y(\rho)\int_0^\rho \mathrm{d}\gamma = y(\rho).\end{aligned}$$ □

Now, we will prove that G is an nonincreasing function using the definition of y.

Lemma 6. *If $y : [0, \check{T}] \to \mathbb{R}$ is a nonnegative nonincreasing function, then G is also nonincreasing.*

Proof. According to the quotient rule (15), we obtain

$$D_\alpha G(\rho) = \frac{\rho D_\alpha\left(\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma\right) - \left(\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma\right)D_\alpha\rho}{\rho^2}$$

$$= \frac{\rho^{2-\alpha}y(\rho) - \rho^{1-\alpha}\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma}{\rho^2}$$
$$= \frac{\rho^{1-\alpha}y(\rho) - \rho^{-\alpha}\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma}{\rho}, \quad \rho \in [0, \check{T}].$$

By using Lemma 5, we obtain

$$\begin{aligned}\rho D_\alpha G(\rho) &= \rho^{1-\alpha}y(\rho) - \rho^{-\alpha}\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma \\ &= \rho^{1-\alpha}(y(\rho) - G(\rho)) \le 0, \quad \text{for } \rho \in [0, \check{T}].\end{aligned}$$

□

Now, we will obtain a new generalization for inequality of Hardy on conformable fractional calculus which is an extension of (2). To prove this, we will use the following algebraic inequality:

$$(\Omega + \theta)^s \ge \Omega^s + s\Omega^{s-1}\theta, \quad \text{where } s > 1 \quad \text{or} \quad s < 0, \tag{77}$$

which is considered as a variation of the Bernoulli inequality, and it is well defined for all $\Omega + \theta \ge 0$ and $\Omega \ge 0$, if $s > 1$, or for $\Omega + \theta > 0$ and $\Omega > 0$, if $s < 0$. The equality in (77) satisfied if $\theta = 0$ only.

Theorem 7. *Let $\omega \in \mathbb{R}$, $0 < \alpha \le 1$, $\beta > 1$, $\zeta < \alpha \le 1$ and y be a nonnegative nonincreasing function on $[0, \omega]$. Then we obtain*

$$\left(\frac{\beta}{\beta - \zeta}\right)^\beta \int_0^\omega \rho^{\zeta-\alpha}y^\beta(\rho)d_\alpha\rho \ge \int_0^\omega \rho^{\zeta-\alpha}G^\beta(\rho)d_\alpha\rho + \frac{\beta c^\zeta}{\beta - \zeta}G^\beta(\omega), \tag{78}$$

where G is defined as in (76).

Proof. Since G is nonincreasing on $[0, \omega]$, from Lemma 6, and according to the chain rule (17), we obtain

$$D_\alpha(G^\beta(\rho)) = \beta G^{\beta-\alpha}(\rho)G^{\alpha-1}(\rho)D_\alpha G(\rho) = \beta G^{\beta-1}(\rho)D_\alpha G(\rho).$$

Employing the product rule formula (14), and since $\rho G(\rho) = \int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma$, we obtain

$$\begin{aligned}D_\alpha(\rho G(\rho)) &= (D_\alpha\rho)G(\rho) + \rho D_\alpha G(\rho) \\ &= \rho^{1-\alpha}G(\rho) + \rho D_\alpha G(\rho) \\ &= D_\alpha\left(\int_0^\rho \gamma^{1-\alpha}y(\gamma)d_\alpha\gamma\right) = \rho^{1-\alpha}y(\rho).\end{aligned}$$

Therefore,

$$\rho^\zeta D_\alpha G(\rho) = \rho^{\zeta-1}[\rho^{1-\alpha}y(\rho) - \rho^{1-\alpha}G(\rho)] = \rho^{\zeta-\alpha}[y(\rho) - G(\rho)].$$

By using integration by parts (18) on the term

$$\int_0^\omega \rho^{\zeta-\alpha}G^\beta(\rho)d_\alpha\rho,$$

such that $\Omega(\rho) = \int_0^\rho \gamma^{\zeta-\alpha}d_\alpha\gamma$ and $\theta(\rho) = G^\beta(\rho)$, then

$$\begin{aligned}\int_0^\omega \rho^{\zeta-\alpha}G^\beta(\rho)d_\alpha\rho &= [\Omega(\rho)G^\beta(\rho)]_0^\omega - \int_0^\omega \Omega(\rho)D_\alpha(G^\beta(\rho))d_\alpha\rho \\ &= \Omega(\omega)G^\beta(\omega) - \lim_{\rho\to 0^+}\Omega(\rho)G^\beta(\rho) - \int_0^\omega \Omega(\rho)D_\alpha(G^\beta(\rho))d_\alpha\rho\end{aligned}$$

$$= \frac{\omega^\zeta}{\zeta} G^\beta(\omega) - \lim_{\rho\to 0^+} \Omega(\rho) G^\beta(\rho) - \frac{\beta}{\zeta}\int_0^\omega \rho^\zeta (D_\alpha G(\rho)) G^{\beta-1}(\rho) d_\alpha \rho$$

$$= \frac{\omega^\zeta}{\zeta} G^\beta(\omega) - \lim_{\rho\to 0^+} \Omega(\rho) G^\beta(\rho) - \frac{\beta}{\zeta}\int_0^\omega \rho^{\zeta-\alpha}[y(\rho) - G(\rho)] G^{\beta-1}(\rho) d_\alpha \rho$$

$$= \frac{\omega^\zeta}{\zeta} G^\beta(\omega) - \frac{\beta}{\zeta}\int_0^\omega \rho^{\zeta-\alpha} y(\rho) G^{\beta-1}(\rho) d_\alpha\rho + \frac{\beta}{\zeta}\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho - \lim_{\rho\to 0^+}\Omega(\rho) G^\beta(\rho),$$

where

$$\Omega(\rho) = \int_0^\rho \gamma^{\zeta-\alpha} d_\alpha\gamma = \int_0^\rho \gamma^{\zeta-1} \mathrm{d}\gamma = \left[\frac{\gamma^\zeta}{\zeta}\right]_0^\rho = \frac{\rho^\zeta}{\zeta}.$$

Also, since

$$\begin{aligned}\Omega(\rho)G^\beta(\rho) &= \int_0^\rho \gamma^{\zeta-\alpha} d_\alpha\gamma \left(\frac{1}{\rho}\int_0^\rho \gamma^{1-\alpha} y(\gamma) d_\alpha\gamma\right)^\beta \\ &= \left(\frac{1}{\rho}\right)^\beta \frac{\rho^\zeta}{\zeta}\left(\int_0^\rho \gamma^{1-\alpha} y(\gamma) d_\alpha\gamma\right)^\beta \\ &\le y^\beta(0)\left(\frac{1}{\rho}\right)^\beta \frac{\rho^\zeta}{\zeta}\left(\int_0^\rho \gamma^{1-\alpha} d_\alpha\gamma\right)^\beta \\ &= y^\beta(0)\left(\frac{1}{\rho}\right)^\beta \frac{\rho^\zeta}{\zeta}\rho^\beta = y^\beta(0)\frac{\rho^\zeta}{\zeta}.\end{aligned}$$

Since $\zeta < \alpha \le 1$, then $\lim_{\rho\to 0^+}\Omega(\rho)G^\beta(\rho) = 0$. Hence,

$$\frac{\beta-\zeta}{\zeta}\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho + \frac{\omega^\zeta}{\zeta} G^\beta(\omega) = \frac{\beta}{\zeta}\int_0^\omega \rho^{\zeta-\alpha} y(\rho) G^{\beta-1}(\rho) d_\alpha\rho. \tag{79}$$

By using inequality of Hölder (19) on right hand side of inequality (79) with indices $\lambda = \beta$ and $\varepsilon = \beta/(\beta-1)$, then we obtain

$$\frac{\beta-\zeta}{\zeta}\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho + \frac{\omega^\zeta}{\zeta} G^\beta(\omega) \le \frac{\beta}{\zeta}\left(\int_0^\omega \rho^{\zeta-\alpha} y^\beta(\rho) d_\alpha\rho\right)^{1/\beta}\left(\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho\right)^{(\beta-1)/\beta},$$

which can be rewritten in the following form:

$$\begin{aligned}\left(\frac{\beta}{\zeta}\right)^\beta \int_0^\omega \rho^{\zeta-\alpha} y^\beta(\rho) d_\alpha\rho &\ge \left[\frac{\frac{\beta-\zeta}{\zeta}\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho}{\left(\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho\right)^{(\beta-1)/\beta}} + \frac{\frac{\omega^\zeta}{\zeta} G^\beta(\omega)}{\left(\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho\right)^{(\beta-1)/\beta}}\right]^\beta \\ &= \left(\frac{\beta-\zeta}{\zeta}\right)^\beta \left[\left(\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho\right)^{1/\beta} + \frac{\frac{\omega^\zeta}{\beta-\zeta} G^\beta(\omega)}{\left(\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho\right)^{(\beta-1)/\beta}}\right]^\beta.\end{aligned}$$

Hence,

$$\left(\frac{\beta}{\beta-\zeta}\right)^\beta \int_0^\omega \rho^{\zeta-\alpha} y^\beta(\rho) d_\alpha\rho \ge \left[\left(\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho\right)^{1/\beta} + \frac{\frac{\omega^\zeta}{\beta-\zeta} G^\beta(\omega)}{\left(\int_0^\omega \rho^{\zeta-\alpha} G^\beta(\rho) d_\alpha\rho\right)^{(\beta-1)/\beta}}\right]^\beta.$$

By employing Bernoulli inequality (77), with

$$\Omega = \left(\int_0^{\omega} \rho^{\zeta-\alpha} G^{\beta}(\rho) d_{\alpha}\rho\right)^{1/\beta} \quad \text{and} \quad \theta = \frac{\frac{\omega^{\zeta}}{\beta-\zeta} G^{\beta}(\omega)}{\left(\int_0^{\omega} \rho^{\zeta-\alpha} G^{\beta}(\rho) d_{\alpha}\rho\right)^{(\beta-1)/\beta}},$$

then, we obtain that

$$\begin{aligned}
&\left(\frac{\beta}{\beta-\zeta}\right)^{\beta} \int_0^{\omega} \rho^{\zeta-\alpha} y^{\beta}(\rho) d_{\alpha}\rho \\
&\geq \int_0^{\omega} \rho^{\zeta-\alpha} G^{\beta}(\rho) d_{\alpha}\rho + \beta\left(\int_0^{\omega} \rho^{\zeta-\alpha} G^{\beta}(\rho) d_{\alpha}\rho\right)^{(\beta-1)/\beta} \frac{\frac{\omega^{\zeta}}{\beta-\zeta} G^{\beta}(\omega)}{\left(\int_0^{\omega} \rho^{\zeta-\alpha} G^{\beta}(\rho) d_{\alpha}\rho\right)^{(\beta-1)/\beta}} \\
&= \int_0^{\omega} \rho^{\zeta-\alpha} G^{\beta}(\rho) d_{\alpha}\rho + \frac{\beta\omega^{\zeta}}{\beta-\zeta} G^{\beta}(\omega).
\end{aligned}$$

□

Corollary 8. *In* Theorem 7, *if* $\zeta = z/s$ *and* $\beta = z$ *such that* $s \geq z > 1$, *then for* G *previously defined in* (76), *we obtain*

$$\int_0^{\omega} \rho^{\frac{z}{s}-\alpha} G^{z}(\rho) d_{\alpha}\rho + \frac{z\omega^{\frac{z}{s}}}{z-\frac{z}{s}} G^{z}(\omega) \leq \left(\frac{z}{z-\frac{z}{s}}\right)^{z} \left(\int_0^{\omega} \rho^{\frac{z}{s}-\alpha} y^{z}(\rho) d_{\alpha}\rho\right). \tag{80}$$

Remark. In Corollary 8, if $\alpha = 1$, $s = z$, and $G(\rho) = \frac{1}{\rho}\int_0^{\rho} y(\gamma)\mathrm{d}\gamma$, then we have an extension of the classical inequality of Hardy (2) to a class of decreasing functions.

Now, we will state some lemmas with their proofs, which are important for obtaining the higher integrability theorem.

Lemma 7. *Let* $a, \omega \in \mathbb{R}$, *and* τ, φ *be nonnegative functions defined on* $[a, \omega]$. *Then we have*

$$\int_a^{\omega} \tau(\gamma)\left(\int_{\gamma}^{\omega} \varphi(\rho) d_{\alpha}\rho\right) d_{\alpha}\gamma = \int_a^{\omega} \varphi(\gamma)\left(\int_a^{\gamma} \tau(\rho) d_{\alpha}\rho\right) d_{\alpha}\gamma.$$

Proof. Let $\Upsilon(\gamma) = \int_{\gamma}^{\omega} \varphi(\rho) d_{\alpha}\rho$ and utilizing integration by parts (18) with $\Omega(\gamma) = \Upsilon(\gamma)$ and $D_{\alpha}\theta(\gamma) = \tau(\gamma)$. Then we have

$$\int_a^{\omega} \tau(\gamma)\left(\int_{\gamma}^{\omega} \varphi(\rho) d_{\alpha}\rho\right) d_{\alpha}\gamma = \Upsilon(\gamma)\theta(\gamma)|_a^{\omega} - \int_a^{\omega} \theta(\gamma) D_{\alpha}\left(\int_{\gamma}^{\omega} \varphi(\rho) d_{\alpha}\rho\right) d_{\alpha}\gamma.$$

Since $\theta(\gamma) = \int_a^{\gamma} \tau(\rho) d_{\alpha}\rho$, $\theta(a) = 0$, and $\Upsilon(\omega) = 0$, then

$$\int_a^{\omega} \tau(\gamma)\left(\int_{\gamma}^{\omega} \varphi(\rho) d_{\alpha}\rho\right) d_{\alpha}\gamma = \int_a^{\omega} \theta(\gamma)\left(-D_{\alpha}\left(\int_{\gamma}^{\omega} \varphi(\rho) d_{\alpha}\rho\right)\right) d_{\alpha}\gamma = \int_a^{\omega} \varphi(\gamma)\left(\int_a^{\gamma} \tau(\rho) d_{\alpha}\rho\right) d_{\alpha}\gamma.$$

□

Lemma 8. *Let* $a, \omega \in \mathbb{R}$ *and* q *be a nonnegative increasing function on* $[a, \omega]$, *then for* $H(\rho) = \frac{1}{q(\rho)}\int_a^{\rho} \tau(\gamma) d_{\alpha}\gamma$ *and* $\mu < 0$, *we obtain*

$$\int_a^{\omega} q^{\mu}(\rho)(D_{\alpha}q(\rho))H(\rho) d_{\alpha}\rho = \frac{1}{\mu}\left[q^{\mu}(\omega)\int_a^{\omega} \tau(\rho) d_{\alpha}\rho - \int_a^{\omega} \tau(\rho) q^{\mu}(\rho) d_{\alpha}\rho\right]. \tag{81}$$

Proof. Since $H(\rho) = \frac{1}{\hbar(\rho)}\int_a^\rho \tau(\gamma)d_a\gamma$, we find

$$\begin{aligned}\int_a^\omega q^\mu(\rho)(D_a q(\rho))H(\rho)d_a\rho &= \int_a^\omega q^\mu(\rho)(D_a q(\rho))\left(\frac{1}{q(\rho)}\int_a^\rho \tau(\gamma)d_a\gamma\right)d_a\rho \\ &= \int_a^\omega q^{\mu-1}(\rho)(D_a q(\rho))\left(\int_a^\rho \tau(\gamma)d_a\gamma\right)d_a\rho \\ &= \int_a^\omega \tau(\rho)\left(\int_\rho^\omega q^{\mu-1}(\gamma)(D_a q(\gamma))d_a\gamma\right)d_a\rho.\end{aligned} \tag{82}$$

According to chain rule (17), we have

$$D_a(q^\mu(\gamma)) = \mu q^{\mu-a}(\gamma)\cdot q^{a-1}(\gamma)\cdot D_a q(\gamma) = \mu q^{\mu-1}(\gamma)D_a q(\gamma). \tag{83}$$

From (82) and (83), by using Lemma 7, we see that

$$\begin{aligned}\int_a^\omega q^\mu(\rho)(D_a q(\rho))H(\rho)d_a\rho &= \int_a^\omega \tau(\rho)\left(\frac{1}{\mu}\int_\rho^\omega D_a(q^\mu(\gamma))d_a\gamma\right)d_a\rho \\ &= \frac{1}{\mu}\int_a^\omega \tau(\rho)[q^\mu(\omega) - q^\mu(\rho)]d_a\rho \\ &= \frac{1}{\mu}\left[q^\mu(\omega)\int_a^\omega \tau(\rho)d_a\rho - \int_a^\omega \tau(\rho)q^\mu(\rho)d_a\rho\right].\end{aligned}$$

□

Lemma 9. *Let $s, z \in \mathbb{R} - \{0\}$ with $z > s > 0$ or $z < 0 < \omega$, $\aleph$ be a function, which is defined on $[0, 1]$ as follows*:

$$\aleph(s, z, \rho) = 1 - K^z(1-\rho)\left(\frac{z}{z - s\rho}\right)^{\frac{z}{s}}, \text{for } K > 1. \tag{84}$$

Then the equation

$$\aleph(s, z, \rho) = 0, \tag{85}$$

has just only one solution ρ_z and $\aleph(s, z, \rho) > 0$ if and only if $\rho \in (\rho_z, 1]$.

Proof. For the following auxiliary function:

$$q(\rho) = (1-\rho)\left(\frac{z}{z - s\rho}\right)^{\frac{z}{s}}.$$

which has the range $[0, 1]$. Now, since q is a nonincreasing in $[0, 1]$, then $\rho_z = q^{-1}(K^{-z})$ be a unique solution of equation $q(\rho) = K^{-z}$, which is equation of (85) such that $K^{-z} \in [0, 1]$. Also, as q is a nonincreasing function,

$$\aleph(s, z, \rho) > 0 \Leftrightarrow q(\rho) < K^{-z} \Leftrightarrow \rho > \rho_z.$$

□

Theorem 8. *Let $a, \omega \in \mathbb{R}$ such that $a < \omega$, Φ, and Υ be two positive functions defined on $[a, \omega]$ with $0 < b \le \Phi^s/\Upsilon^z \le B < \infty$. Then we have*

$$\left(\int_a^\omega \Phi(\rho)d_a\rho\right)^{\frac{1}{s}}\left(\int_a^\omega \Upsilon(\rho)d_a\rho\right)^{\frac{1}{z}} \le \left(\frac{B}{b}\right)^{\frac{1}{sz}}\left(\int_a^\omega \Phi^{\frac{1}{s}}(\rho)\Upsilon^{\frac{1}{z}}(\rho)d_a\rho\right), \tag{86}$$

for all $s > 1$ and $z > 1$ such that $\frac{1}{s} + \frac{1}{z} = 1$.

Proof. According to condition $\Phi^s/\Upsilon^z \le B$, we obtain $\Upsilon \ge B^{\frac{-1}{z}}\Phi^{\frac{s}{z}}$, then

$$\Phi\Upsilon \ge B^{\frac{-1}{z}}\Phi^{\frac{s}{z}+1} = B^{\frac{-1}{z}}\Phi^{\frac{s+z}{z}} = B^{\frac{-1}{z}}\Phi^{s}.$$

Hence, we obtain

$$\left(\int_a^\omega \Phi^s(\rho)d_\alpha\rho\right)^{\frac{1}{s}} \le B^{\frac{1}{sz}}\left(\int_a^\omega \Phi(\rho)\Upsilon(\rho)d_\alpha\rho\right)^{\frac{1}{s}}. \tag{87}$$

Similarly, according to condition $b \le \Phi^s/\Upsilon^z$, we obtain $\Phi \ge b^{\frac{1}{s}}\Upsilon^{\frac{z}{s}}$, then

$$\int_a^\omega \Phi(\rho)\Upsilon(\rho)d_\alpha\rho \ge b^{\frac{1}{s}}\int_a^\omega \Upsilon^{\frac{z}{s}+1}(\rho)d_\alpha\rho = b^{\frac{1}{s}}\int_a^\omega \Upsilon^{z}(\rho)d_\alpha\rho.$$

Hence,

$$\left(\int_a^\omega \Phi(\rho)\Upsilon(\rho)d_\alpha\rho\right)^{\frac{1}{z}} \ge b^{\frac{1}{sz}}\left(\int_a^\omega \Upsilon^z(\rho)d_\alpha\rho\right)^{\frac{1}{z}}. \tag{88}$$

From (87) and (88), we obtain

$$\left(\int_a^\omega \Phi^s(\rho)d_\alpha\rho\right)^{\frac{1}{s}}\left(\int_a^\omega \Upsilon^z(\rho)d_\alpha\rho\right)^{\frac{1}{z}} \le \left(\frac{B}{b}\right)^{\frac{1}{sz}}\left(\int_a^\omega \Phi(\rho)\Upsilon(\rho)d_\alpha\rho\right). \tag{89}$$

By replacing Φ^s and Υ^z by Φ and Υ, we obtain

$$\left(\int_a^\omega \Phi(\rho)d_\alpha\rho\right)^{\frac{1}{s}}\left(\int_a^\omega \Upsilon(\rho)d_\alpha\rho\right)^{\frac{1}{z}} \le \left(\frac{B}{b}\right)^{\frac{1}{sz}}\left(\int_a^\omega \Phi^{\frac{1}{s}}(\rho)\Upsilon^{\frac{1}{z}}(\rho)d_\alpha\rho\right).$$ □

Theorem 9. *Let $a, \omega \in \mathbb{R}$ such that $a < \omega$, $0 < \gamma < 1$ and two positive functions Φ and Υ be defined on $[a, \omega]$ with $0 < b \le \Phi^s \le B < \infty$. Then we have*

$$\left(\frac{1}{\rho}\int_a^\omega \Phi^s(\rho)d_\alpha\rho\right)^{\frac{1}{s}} \le \left(\frac{\omega^\alpha - a^\alpha}{\alpha}\right)^{\frac{-1}{\beta z}}\left(\frac{b}{B}\right)^{\frac{-1}{\beta sz}}\left(\frac{1}{\rho}\int_a^\omega \Phi^\gamma(\rho)d_\alpha\rho\right)^{\frac{1}{\gamma}}, \tag{90}$$

for $s > 1$, $z > 1$, $s > z$, and $\beta = \frac{1-\frac{1}{s}}{\frac{1}{\gamma}-\frac{1}{s}}$.

Proof. Assume $\Upsilon = 1$ in (89), then we have

$$\left(\int_a^\omega \Phi^s(\rho)d_\alpha\rho\right)^{\frac{1}{s}}\left(\frac{\omega^\alpha - a^\alpha}{\alpha}\right)^{\frac{1}{z}} \le \left(\frac{B}{b}\right)^{\frac{1}{sz}}\int_a^\omega \Phi(\rho)d_\alpha\rho = \left(\frac{b}{B}\right)^{\frac{-1}{sz}}\int_a^\omega \Phi(\rho)d_\alpha\rho,$$

which can be written as follows:

$$\left(\int_a^\omega \Phi^s(\rho)d_\alpha\rho\right)^{\frac{1}{s}} \le \left(\frac{\omega^\alpha - a^\alpha}{\alpha}\right)^{\frac{-1}{z}}\left(\frac{b}{B}\right)^{\frac{-1}{sz}}\int_a^\omega \Phi(\rho)d_\alpha\rho. \tag{91}$$

Since $\beta \in (0, 1)$ and $\frac{1-\beta}{s} + \frac{\beta}{\gamma} = 1$, then by using inequality of Hölder (19) with indices $\gamma = s/(1-\beta)$ and $\varepsilon = \gamma/\beta$, then we obtain

$$\left(\frac{1}{\omega}\int_a^\omega \Phi^s(\rho)d_\alpha\rho\right)^{\frac{1}{s}} \le \left(\frac{\omega^\alpha - a^\alpha}{\alpha}\right)^{\frac{-1}{z}}\left(\frac{b}{B}\right)^{\frac{-1}{sz}}\frac{1}{\omega}\int_a^\omega \Phi(\rho)d_\alpha\rho.$$

$$= \left(\frac{\omega^{\alpha} - a^{\alpha}}{\alpha}\right)^{\frac{-1}{z}} \left(\frac{b}{B}\right)^{\frac{-1}{sz}} \frac{1}{\omega} \int_{a}^{\omega} \Phi^{1-\beta}(\rho)\Phi^{\beta}(\rho) d_{\alpha}\rho$$

$$\leq \left(\frac{\omega^{\alpha} - a^{\alpha}}{\alpha}\right)^{\frac{-1}{z}} \left(\frac{b}{B}\right)^{\frac{-1}{sz}} \left(\frac{1}{\omega} \int_{a}^{\omega} \Phi^{s}(\rho) d_{\alpha}\rho\right)^{\frac{1-\beta}{s}} \left(\frac{1}{\omega} \int_{a}^{\omega} \Phi^{\gamma}(\rho) d_{\alpha}\rho\right)^{\frac{\beta}{\gamma}}.$$

Hence,

$$\left(\frac{1}{\omega} \int_{a}^{\omega} \Phi^{s}(\rho) d_{\alpha}\rho\right)^{\frac{1}{s}} \leq \left(\frac{\omega^{\alpha} - a^{\alpha}}{\alpha}\right)^{\frac{-1}{\beta z}} \left(\frac{b}{B}\right)^{\frac{-1}{\beta sz}} \left(\frac{1}{\omega} \int_{a}^{\omega} \Phi^{\gamma}(\rho) d_{\alpha}\rho\right)^{\frac{1}{\gamma}}. \qquad \square$$

Notice that the inequality (90) of Theorem 9 can be written as follows:

$$\left(\frac{1}{\omega} \int_{a}^{\omega} \Phi^{s}(\rho) d_{\alpha}\rho\right)^{\frac{1}{s}} \leq \tilde{K} \left(\frac{1}{\omega} \int_{a}^{\omega} \Phi^{\gamma}(\rho) d_{\alpha}\rho\right)^{\frac{1}{\gamma}}, \tag{92}$$

which is called the reverse Hölder's inequality.

For our main theorem, we will need to prove the next theorem.

Theorem 10. *Let $\omega \in \mathbb{R}$, $0 < \alpha \leq 1$, $0 < s < z$, $\tilde{K} > 1$, $\Phi : [0, \omega] \to \mathbb{R}$ be a nonnegative decreasing satisfying* (92) *and*

$$\aleph(s, z, \zeta, \tilde{K}) = 1 - (1 - \zeta)\tilde{K}^{z} \left(\frac{z}{z - s\zeta}\right)^{z/s}. \tag{93}$$

Then, for all $\zeta \in (\zeta_z, 1]$ such that ζ_z is the only root of the equation (84), *we obtain the inequality*

$$\int_{0}^{\omega} \rho^{\zeta-\alpha} \Phi^{z}(\rho) d_{\alpha}\rho \leq \frac{\omega^{\zeta-1}}{\aleph(s, z, \zeta, \tilde{K})} \int_{0}^{\omega} \rho^{1-\alpha} \Phi^{z}(\rho) d_{\alpha}\rho. \tag{94}$$

Proof. Since the inequality (94) is clearly satisfied when $\zeta = 1$, we can now assume $\zeta \in (\zeta_z, 1)$. In this case, we define a function $\Phi : [0, \omega] \to \mathbb{R}$, that satisfies the reverse Hölder's inequality (92). Therefore, we obtain the relation

$$G_z(\rho) \leq \tilde{K}^{z} (G_s(\rho))^{\frac{z}{s}}, \tag{95}$$

where

$$G_s(\rho) = \frac{1}{\rho} \int_{0}^{\rho} \gamma^{1-\alpha} \Phi^{s}(\gamma) d_{\alpha}\gamma \quad \text{and} \quad G_z(\rho) = \frac{1}{\rho} \int_{0}^{\rho} \gamma^{1-\alpha} \Phi^{z}(\gamma) d_{\alpha}\gamma.$$

Then by integrating (95) from 0 to ω, we have

$$\int_{0}^{\omega} \rho^{\zeta-\alpha} G_z(\rho) d_{\alpha}\rho \leq \tilde{K}^{z} \int_{0}^{\omega} \rho^{\zeta-\alpha} (G_s(\rho))^{z/s} d_{\alpha}\rho. \tag{96}$$

Applying Lemma 8 with $\mu = \zeta - 1$, $a = 0$, $\hat{h}(\rho) = \rho$, and $\tau(\rho) = \rho^{1-\alpha}\Phi^{z}(\rho)$ on left-hand side of (96), it follows that

$$\int_{0}^{\omega} \rho_{z}^{\zeta-\alpha} G(\rho) d_{\alpha}\rho = \frac{1}{\zeta - 1} \left[\omega^{\zeta-1} \int_{0}^{\omega} \rho^{1-\alpha} \Phi^{z}(\rho) d_{\alpha}\rho - \int_{0}^{\omega} \rho^{\zeta-\alpha} \Phi^{z}(\rho) d_{\alpha}\rho\right]. \tag{97}$$

By using Theorem 7 with $\beta = z/s$ and $y = \Phi^{s}$ to the right-hand side of inequality (96), we have

$$\int_{0}^{\omega} \rho^{\zeta-\alpha} (G_s(\rho))^{z/s} d_{\alpha}\rho \leq \left(\frac{z/s}{(z/s) - \zeta}\right)^{z/s} \left(\int_{0}^{\omega} \rho^{\zeta-\alpha} \Phi^{z}(\rho) d_{\alpha}\rho\right) - \frac{(z/s)\omega^{\zeta}}{(z/s) - \zeta} G^{z/s}(\omega) \tag{98}$$

$$\leq \left(\frac{z/s}{(z/s)-\zeta}\right)^{z/s}\left(\int_0^{\omega}\rho^{\zeta-\alpha}\Phi^z(\rho)d_\alpha\rho\right).$$

By substituting (98) and (97) into (96), we find

$$\frac{1}{\zeta-1}\left[\omega^{\zeta-1}\int_0^{\omega}\rho^{1-\alpha}\Phi^z(\rho)d_\alpha\rho-\int_0^{\omega}\rho^{\zeta-\alpha}\Phi^z(\rho)d_\alpha\rho\right]\leq \tilde{K}^z\left(\frac{z/s}{(z/s)-\zeta}\right)^{z/s}\int_0^{\omega}\rho^{\zeta-\alpha}\Phi^z(\rho)d_\alpha\rho.$$

Therefore,

$$\begin{aligned}\omega^{\zeta-1}\int_0^{\omega}\rho^{1-\alpha}\Phi^z(\rho)d_\alpha\rho &\geq (\zeta-1)\tilde{K}^z\left(\frac{z}{z-s\zeta}\right)^{z/s}\int_0^{\omega}\rho^{\zeta-\alpha}\Phi^z(\rho)d_\alpha\rho+\int_0^{\omega}\rho^{\zeta-\alpha}\Phi^z(\rho)d_\alpha\rho\\ &=\left[1-(1-\zeta)\tilde{K}^z\left(\frac{z}{z-s\zeta}\right)^{z/s}\right]\int_0^{\omega}\rho^{\zeta-\alpha}\Phi^z(\rho)d_\alpha\rho\\ &=\tilde{\aleph}(s,z,\zeta,\tilde{K})\int_0^{\omega}\rho^{\zeta-\alpha}\Phi^z(\rho)d_\alpha\rho.\end{aligned}$$

Finally, from Lemma 9, there exists a unique $\zeta_z \in (0,1)$ such that $\tilde{\aleph}(s,z,\zeta,\tilde{K}) > 0$ for $\zeta \in (\zeta_z, 1]$. This provides the inequality (94). □

Finally, we are able to state and prove the higher integrability theorem for decreasing functions on conformable calculus.

Theorem 11. *Assume that $\tilde{K} > 1$, $0 < s < z$, $\omega \in \mathbb{R}$, and $\Phi : [0,\omega] \to \mathbb{R}$ is a nonnegative decreasing function satisfying* (92) *such that $\Phi \in L^z[0,\omega]$. Then $\Phi \in L^\ell[0,\omega]$ with $z \leq \ell < z_0$ and z_0 is the only solution to equation*

$$\left(\frac{\rho}{\rho-z}\right)^{1/z}=\tilde{K}\left(\frac{\rho}{\rho-s}\right)^{1/s}, \tag{99}$$

and the following inequality holds

$$\left(\frac{1}{\omega}\int_0^{\omega}\rho^{\frac{z}{\ell}(1-\alpha)}\Phi^\ell(\rho)d_\alpha\rho\right)^{\frac{z}{\ell}}\leq \frac{z}{\ell\tilde{\aleph}(s,z,z/\ell,\tilde{K})}\left(\frac{1}{\omega}\int_0^{\omega}\rho^{1-\alpha}\Phi^z(\rho)d_\alpha\rho\right), \tag{100}$$

where

$$\tilde{K}=\left(\frac{\omega^\alpha-a^\alpha}{\alpha}\right)^{\frac{-1}{\beta z}}\left(\frac{b}{B}\right)^{\frac{-1}{\beta s z}}.$$

Proof. According to Theorem 10 with replacing ζ and ζ_z by z/ℓ and z/z_0, respectively, we obtain

$$\int_0^{\omega}\rho^{\frac{z}{\ell}-\alpha}\Phi^z(\rho)d_\alpha\rho\leq\frac{\omega^{\frac{z}{\ell}-1}}{\tilde{\aleph}(s,z,z/\ell,\tilde{K})}\int_0^{\omega}\rho^{1-\alpha}\Phi^z(\rho)d_\alpha\rho,\quad z\leq\ell<z_0. \tag{101}$$

By using the inequality (75) with $k = z$ to the left-hand side of inequality (101), we find

$$\left(\int_0^{\omega}\rho^{\frac{z}{\ell}(1-\alpha)}\Phi^\ell(\rho)d_\alpha\rho\right)^{\frac{z}{\ell}}\leq\frac{z\omega^{\frac{z}{\ell}-1}}{\ell\tilde{\aleph}(s,z,z/\ell,\tilde{K})}\int_0^{\omega}\rho^{1-\alpha}\Phi^z(\rho)d_\alpha\rho, z\leq\ell<z_0.$$

Therefore,

$$\left(\frac{1}{\omega}\int_0^\omega \rho^{\frac{z}{\ell}(1-\alpha)}\Phi^\ell(\rho)d_\alpha\rho\right)^{\frac{z}{\ell}} \le \frac{z}{\ell\tilde{\aleph}(s,z,z/\ell,\tilde{K})}\left(\frac{1}{\omega}\int_0^\omega \rho^{1-\alpha}\Phi^z(\rho)d_\alpha\rho\right),$$

that is the wanted inequality (100). Therefore, z_0 is the only solution to equation

$$\tilde{\aleph}(s,z,z/\ell,\tilde{K}) = 0,$$

which reduces to (99). □

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