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Research Article

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Solvability of infinite system of integral equations of Hammerstein type in three variables in tempering sequence spaces c_0^β and ℓ_1^β

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Abstract: The objective of this study is to determine the criteria under which the infinite system of integral equations in three variables has a solution in the Banach tempering sequence space c_0^β and ℓ_1^β, utilizing the Meir-Keeler condensing operators. Our research builds upon the findings of Malik and Jalal. Furthermore, we provide illustrative examples to demonstrate the implications of our established conditions.

Keywords: Hausdorff measure of non-compactness, Meir-Keeler condensing operators, system of integral equations, tempering sequence spaces

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1 Introduction and preliminaries

The study of functional integral equations is an important branch of non-linear analysis and has various applications in natural and applied sciences. Most of the problems, for instance, in mechanics and population dynamics can be formulated using integral equations. On the other hand, differential equations may be converted into integral equations and their solutions can be obtained.

The topic of solutions to functional equations has captured the attention of numerous researchers over an extensive period of time. Several approaches have been used to establish the existence of solutions of functional equations such as functional integral equations, differential equations, integro-differential equations, and fractional integro-differential equations [1–4].

Measures of non-compactness are important tool in fixed point theory, which is also crucial in the study of existence of solution of functional equations.

In 1930, Kuratowski [5] introduced the concept of measures of non-compactness, a function that determines the degree of non-compactness of a set. Later on, Banaś [6] generalized this concept into Banach space. In 1955, Darbo [7] used the Kuratowski measure of non-compactness to prove his famous fixed point theorem, which was a generalization of the classical Schauder's fixed point theorem and Banach's contraction principle. Darbo's fixed point theorem and it is several generalizations such as Meir-Keeler fixed point theorem are used

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by many researchers to study the existence of solutions of integral equations and differential equations in Banach spaces.

The study of sequence spaces has also attracted the interest of many researchers. Kreyszig [8] has shown some sequence spaces, which are also Banach spaces in his book titled introductory functional analysis with applications. Banaś and Mursaleen [9] discussed measures of non-compactness in some classical sequence spaces. Recently, several studies have been conducted on the investigation of the solvability of functional integral equations in some Banach sequence spaces. We can refer [10–13] for some works on existence of solutions of functional integral equations in Banach sequence spaces. This work extends and generalizes the results of Malik and Jalal [10] in tempered sequence spaces c_0^β and ℓ_1^β.

Sometimes it is not suitable to consider the classical sequence spaces for studying the solvability of functional equations such as initial value differential equations and integral equations, since the solution might be out of such sequences. Therefore, it is important to consider the enlarged sequence spaces.

These spaces are defined by fixing a non-increasing sequence $\beta = \beta_n$, where β_n is a real positive sequence for $n \in \mathbb{N}$. Such sequence β will be called the tempering sequence.

Let ω denote all sequences $y = (y_k)_{k=1}^{\infty}$ real or complex such that $\beta_n y_n \to 0$ as $n \to \infty$. We denote this sequence space by c_0^β. It is a Banach space under the norm

$$\|y\|_{c_0^\beta} = \|(y_n)\|_{c_0^\beta} = \sup\{\beta_n |y_n| : n \in \mathbb{N}\}.$$

In a similar manner, we examine the space ℓ_1^β, which encompasses real (or complex) sequences (y_n) with the property that $(\beta_n y_n)$ converges to a finite value. This particular space is a Banach space, characterized by the norm

$$\|y\|_{\ell_1^\beta} = \|y_n\|_{\ell_1^\beta} = \sum_{n=1}^{\infty} \beta_n |y_n|.$$

Banaś and Krajewska [2], proved that the pairs of Banach sequence spaces (c_0, c_0^β) and (ℓ_1, ℓ_1^β) are isometric. It is important to note that if we let $\beta_n = 1$, then we have $c_0^\beta = c_0$ and $\ell_1^\beta = \ell_1$. In order to obtain the enlarged sequence spaces c_0^β and ℓ_1^β, we assume that the sequence $\beta_n \to 0$ as $n \to \infty$.

In this research article, we will investigate the existence of solutions for an infinite system of integral equations with a Hammerstein-type structure in three variables of the form

$$x_n(\psi_1, \psi_2, \psi_3) = r_n(\psi_1, \psi_2, \psi_3) + \int_s^t \int_s^t \int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3) h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3)) \mathrm{d}\varsigma_1 \mathrm{d}\varsigma_2 \mathrm{d}\varsigma_3, \quad (1)$$

where $(\psi_1, \psi_2, \psi_3) \in [s, t] \times [s, t] \times [s, t]$.

Consider $\mathbb{R}_+ = [0, \infty)$, and suppose that $(M, \|.\|)$ is a real Banach space. If X is a non-empty subset of M, we use $\bar{X}$ and ConvX to represent the closure and convex closure of X, respectively. The family of all non-empty and bounded subsets of M is denoted as Q_M, and its subfamily consisting of all relatively compact sets is denoted as N_M. The space of continuously differentiable functions on $I^3 = [s, t] \times [s, t] \times [s, t]$ is denoted as $C^1(I^3, \mathbb{R})$.

In this article, the Hausdorff measure of non-compactness will be used to establish our results on the existence of solutions of a system of integral equations of Hammerstein type in three variables.

Definition 1.1. [9] Let $(\mathcal{X}, \rho)$ be a metric space and Q be a bounded subset of $\mathcal{X}$. Then, the Hausdorff measure of non-compactness also known as the χ-measure or ball measure of non-compactness of a set Q, denoted by $\chi(Q)$, is defined as the infimum of the set of all reals $\varepsilon > 0$ such that Q can be covered by a finite number of balls with radii $< \varepsilon$, i.e.,

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in \mathcal{X}, r_i < \varepsilon (i = 1, \ldots, n)\ n \in \mathbb{N}\right\}.$$

The following is the axiomatic definition of measures of non-compactness.

Definition 1.2. [9] A function $\lambda : Q_M \to \mathbb{R}_+$ is referred to as a measure of non-compactness when it fulfills the following conditions:

i. the family $\ker \lambda = \mathcal{Y} \in Q_M : \lambda(\mathcal{Y}) = 0$ is non-empty and $\ker \lambda \subset N_M$,
ii. $\mathcal{Y} \subset \mathcal{Z} \;\Rightarrow\; \lambda(\mathcal{Y}) \le \lambda(\mathcal{Z})$,
iii. $\lambda(\bar{\mathcal{Y}}) = \lambda(\mathcal{Y})$,
iv. $\lambda(\mathrm{Conv}\mathcal{Y}) = \lambda(\mathcal{Y})$,
v. $\lambda(k\mathcal{Y} + (1-k)\mathcal{Z}) \le k\lambda(\mathcal{Y}) + (1-k)\lambda(\mathcal{Z})$ for $k \in [0,1]$,
vi. If $(\mathcal{Y}_n)$ is a sequence of closed sets from Q_M such that $\mathcal{Y}_{n+1} \subset \mathcal{Y}_n$ for $n = 1, 2, 3, \dots$ and $\lim_{n\to\infty}\lambda(\mathcal{Y}_n) = 0$, then $\bigcap_{n=1}^{\infty}\mathcal{Y}_n$ is non-empty.

If a measure of non-compactness satisfies the following additional conditions, then it is called a regular measure:

i. $\lambda(\mathcal{Y}_1 \cup \mathcal{Y}_2) = \max\{\lambda(\mathcal{Y}_1), \lambda(\mathcal{Y}_2)\}$,
ii. $\lambda(\mathcal{Y}_1 + \mathcal{Y}_2) \le \lambda(\mathcal{Y}_1) + \lambda(\mathcal{Y}_2)$,
iii. $\lambda(k\mathcal{Y}) = |k|\lambda(\mathcal{Y})$,
iv. $\ker \lambda = N_M$.

The family $\ker \lambda$ is said to be the kernel of measure λ.

The following results obtained by Darbo [7] have been very useful for proving the existence of solutions of functional equations.

Definition 1.3. Consider two Banach spaces, M_1 and M_2, and let λ_1 and λ_2 be the arbitrary measures of non-compactness on M_1 and M_2, respectively. An operator f from M_1 to M_2 is referred to as a (λ_1, λ_2)-condensing operator if it satisfies the conditions of continuity and $\lambda_2(f(D)) < \lambda_1(D)$ for every set $D \in M_1$ with a compact closure.

Remark 1.1. If $M_1 = M_2$ and $\lambda_1 = \lambda_2 = \lambda$, then f is called a λ -condensing operator.

Theorem 1.1. [7] *If H is a non-empty, closed, bounded, and convex subset of a Banach space M, and a continuous mapping $f : H \to H$ satisfies the condition $\lambda_2(f(H)) < k\lambda_1(H)$ with a constant k in the interval* $[0, 1)$, *then there exists a fixed point of f in H.*

In 1969, Meir and Keeler [14] proved a very interesting fixed point, which is also a generalization of Banach's contraction principle and Schauder's fixed point theorem. Later on, Aghajan and Mursaleen [1] generalized the concept of Meir-Keeler contractions to Banach spaces and came up with the following results:

Definition 1.4. [1] Consider a non-empty subset C of a Banach space M, and let λ be an arbitrary measure of non-compactness defined on M. We say that an operator $T : C \to C$ is a Meir-Keeler condensing operator if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \le \lambda(X) < \varepsilon + \delta \;\Rightarrow\; \lambda(T(X)) < \varepsilon,$$

for any bounded subset X of C.

Theorem 1.2. [1] *Consider a non-empty subset C of a Banach space M, and let λ be an arbitrary measure of non-compactness defined on M. If we have a continuous and Meir-Keeler condensing operator $T : C \to C$, then T has at least one fixed point, and the set of all fixed points of T within C is a compact set.*

2 Hausdorff measure of non-compactness in Banach spaces c_0^β and ℓ_1^β

Banaś and Mursaleen [9] established the formula for the Hausdorff measure of non-compactness in the classical sequence spaces c_0 and ℓ_1 in one variable. We can consider publications [2,10,12,13] for the formulas of measures of non-compactness in some sequence spaces. In this section, the Hausdorff measure of non-compactness in sequence spaces c_0^β and ℓ_1^β in three variables is formulated.

Let H be a bounded subset of the Banach space $(c_0^\beta, \|.\|_{c_0^\beta})$, and the Hausdorff measure of non-compactness is given by

$$\chi(H) = \lim_{n\to\infty}\left\{\sup_{v(\psi_1,\psi_2,\psi_3)\in H}\left(\max_{k\geq n}\beta_k|v_k(\psi_1,\psi_2,\psi_3)|\right)\right\},$$

where $v(\psi_1,\psi_2,\psi_3) = (v_j(\psi_1,\psi_2,\psi_3))_{j=1}^{\infty} \in c_0^\beta$ for each $(\psi_1,\psi_2,\psi_3) \in I^3$ and $H \in Q_{c_0^\beta}$.

The Hausdorff measure of non-compactness χ in the Banach space ℓ_1^β is defined as follows:

$$\chi(H) = \lim_{n\to\infty}\left[\sup_{v(\psi_1,\psi_2,\psi_3)\in H}\left(\sum_{k=n}^{\infty}\beta_k|v_k(\psi_1,\psi_2,\psi_3)|\right)\right],$$

where $v(\psi_1,\psi_2,\psi_3) = (v_j(\psi_1,\psi_2,\psi_3))_{j=1}^{\infty} \in \ell_1^\beta$ for each $(\psi_1,\psi_2,\psi_3) \in I^3$ and $H \in Q_{\ell_1^\beta}$.

We will prove the existence of solutions for the infinite system of integral equation (1) in Banach spaces c_0^β and ℓ_1^β using the concept of Meir-Keeler condensing operators and the Hausdorff measure of non-compactness.

3 Existence of solution in the space c_0^β

Assume that the following conditions are satisfied:

i. The functions $(h_i)_{i=1}^{\infty}$ defined on the set $I^3 \times \mathbb{R}^\infty$ are continuous and have real values. The operator $T : I^3 \times c_0^\beta \to c_0^\beta$ is defined on the space $I^3 \times c_0^\beta$ as
$$(\psi_1,\psi_2,\psi_3,x) \to (Tx)(\psi_1,\psi_2,\psi_3) = (\beta_1 h_1(\psi_1,\psi_2,\psi_3), \beta_2 h_2(\psi_1,\psi_2,\psi_3), \ldots).$$
The collection of functions $\{(Tx)(\psi_1,\psi_2,\psi_3)\}_{(\psi_1,\psi_2,\psi_3)\in I^3}$ exhibits equicontinuity at all points in the space c_0^β. This means that for any given value of $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that
$$\|x-y\|_{c_0^\beta} \leq \delta \;\Rightarrow\; \|(Tx)(\psi_1,\psi_2,\psi_3) - (Ty)(\psi_1,\psi_2,\psi_3)\|_{c_0^\beta} \leq \varepsilon.$$

ii. For each fixed $(\psi_1,\psi_2,\psi_3) \in I^3$, $x(\psi_1,\psi_2,\psi_3) \in C^1(I^3, c_0^\beta)$, the following inequality holds:
$$|h_n(\psi_1,\psi_2,\psi_3), x(\psi_1,\psi_2,\psi_3)| \leq p_n(\psi_1,\psi_2,\psi_3) + q_n(\psi_1,\psi_2,\psi_3)\sup_{i\geq n}\{|\beta_i v_i|\},\ n \in \mathbb{N},$$
where $p_i(\psi_1,\psi_2,\psi_3)$ and $q_i(\psi_1,\psi_2,\psi_3)$ represent the continuous real-valued functions defined on I^3. Additionally, the sequence of functions $(q_i(\psi_1,\psi_2,\psi_3))_{i\in\mathbb{N}}$ is uniformly bounded on I^3, while the sequence of functions $(p_i(\psi_1,\psi_2,\psi_3))_{i\in\mathbb{N}}$ uniformly converges on I^3 to a function that becomes identically zero on I^3.

iii. The functions $K_n : I^6 \to \mathbb{R}$ are continuous throughout the entire domain I^6, where n belongs to the set of natural numbers $\mathbb{N}$. Additionally, these functions $K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)$ exhibit equicontinuity with respect to the variables (ψ_1,ψ_2,ψ_3), meaning that for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that
$$|K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3) - K_n(\bar\psi_1,\bar\psi_2,\bar\psi_3,\bar\varsigma_1,\bar\varsigma_2,\bar\varsigma_3)| < \varepsilon,$$
where $|\psi_1 - \bar\psi_1| < \delta$, $|\psi_2 - \bar\psi_2| < \delta$, and $|\psi_3 - \bar\psi_3| < \delta$, for all $(\varsigma_1,\varsigma_2,\varsigma_3) \in I^3$. In addition to that the function sequence $(K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3))$ is equibounded on the set I^3, and there exists a constant
$$K = \sup\{|K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)| : (\psi_1,\psi_2,\psi_3), (\varsigma_1,\varsigma_2,\varsigma_3) \in I^3, n \in \mathbb{N}\} < \infty.$$

iv. $r_n : I^3 \to \mathbb{R}$ is continuous; also, the sequence of functions (r_n) uniformly converges to zero on I^3. We define the finite constant $\mathcal{R}$ as

$$\mathcal{R} = \sup\{|r_n(\psi_1, \psi_2, \psi_3)| : (\psi_1, \psi_2, \psi_3) \in I^3, n \in \mathbb{N}\}.$$

Considering condition (ii), the following finite constants are defined:

$$\mathcal{Q} = \sup\{\beta_n q_n(\psi_1, \psi_2, \psi_3) : (\psi_1, \psi_2, \psi_3) \in I^3, n \in \mathbb{R}\},$$
$$\mathcal{P} = \sup\{\beta_n p_n(\psi_1, \psi_2, \psi_3) : (\psi_1, \psi_2, \psi_3) \in I^3, n \in \mathbb{N}\}.$$

Theorem 3.1. *Under assumptions* (i)–(iv), *the infinite system* (1) *has at least one solution* $x(\psi_1, \psi_2, \psi_3) = (x_i(\psi_1, \psi_2, \psi_3))_{i=1}^{\infty}$ *in* c_0^{β} *for fixed* $(\psi_1, \psi_2, \psi_3) \in I^3$, *whenever* $(t-s)^2 K\mathcal{Q} < 1$.

Proof. First, an operator Q on the space $C^1(I^3, c_0^{\beta})$ is defined by

$$\begin{aligned}(Qx)(\psi_1, \psi_2, \psi_3) &= ((Qx)_n(\psi_1, \psi_2, \psi_3))\\ &= r_n(\psi_1, \psi_2, \psi_3) + \int_s^t\int_s^t\int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)\beta_n h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\\ &= \Bigg(r_1(\psi_1, \psi_2, \psi_3) + \int_s^t\int_s^t\int_s^t K_1(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)\beta_1 h_1(\varsigma_1, \varsigma_2, \varsigma_3, x_1(\varsigma_1, \varsigma_2, \varsigma_3), x_2(\varsigma_1, \varsigma_2, \varsigma_3), \ldots)\\ &\quad \mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3, r_2(\psi_1, \psi_2, \psi_3) + \int_s^t\int_s^t\int_s^t K_2(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)\beta_2 h_2(\varsigma_1, \varsigma_2, \varsigma_3, x_1(\varsigma_1, \varsigma_2, \varsigma_3)\\ &\quad x_2(\varsigma_1, \varsigma_2, \varsigma_3), \ldots)\mathrm{d}\varsigma_1, \mathrm{d}\varsigma_2, \mathrm{d}\varsigma_3, \ldots\Bigg),\end{aligned} \tag{2}$$

for all $(\psi_1, \psi_2, \psi_3) \in I^3$ and $x = (x_i)_{i=1}^{\infty} \in c_0^{\beta}$. We show that Q maps $C^1(I^3, c_0^{\beta})$ into itself. Let $n \in \mathbb{N}$ and $(\psi_1, \psi_2, \psi_3) \in I^3$ and then, by assumption (i) and (iii), we have

$$\begin{aligned}|(Qx)_n(\psi_1, \psi_2, \psi_3)| &\le |r_n(\psi_1, \psi_2, \psi_3)| + \beta_n \left|\int_s^t\int_s^t\int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))\right|\\ &\le |r_n(\psi_1, \psi_2, \psi_3)| + K\int_s^t\int_s^t\int_s^t \beta_n|h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3.\end{aligned}$$

Thus, by assumption (iv) and the fact that $h_n(\psi_1, \psi_2, \psi_3, x(\psi_1, \psi_2, \psi_3))$ is in c_0^{β} space, we have

$$\lim_{n\to\infty}(|Q(x)_n(\psi_1, \psi_2, \psi_3)|) = 0.$$

We conclude that $(Qx)(\psi_1, \psi_2, \psi_3) \in c_0^{\beta}$ for any arbitrarily fixed $(\psi_1, \psi_2, \psi_3) \in I^3$.

Then, we have

$$\begin{aligned}\|x_n(\psi_1, \psi_2, \psi_3)\|_{c_0^{\beta}} &= \max_{n\ge1}\left\{\beta_n\left|r_n(\psi_1, \psi_2, \psi_3) + \int_s^t\int_s^t\int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\right\}\\ &\le \max_{n\ge1}\left\{\beta_n|r_n(\psi_1, \psi_2, \psi_3)| + \beta_n\left|\int_s^t\int_s^t\int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\right\}\\ &\le \mathcal{R} + \max_{n\ge1}\left\{\beta_n\int_s^t\int_s^t\int_s^t |K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)||h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right\}\\ &\le \mathcal{R} + K\int_s^t\int_s^t\int_s^t \max_{n\ge1}(p_n(\varsigma_1, \varsigma_2, \varsigma_3) + q_n(\varsigma_1, \varsigma_2, \varsigma_3)\sup_{i\ge n}\{\beta_n|x_i(\varsigma_1, \varsigma_2, \varsigma_3)|\})\\ &\le \mathcal{R} + K(t-s)^2[\mathcal{P} + \mathcal{Q}\ \|x_i(\varsigma_1, \varsigma_2, \varsigma_3)\|_{c_0^{\beta}}].\end{aligned}$$

So, we have

$$\begin{aligned}[1-(t-s)^2KQ]\|x(\psi_1,\psi_2,\psi_3)\|_{c_0^\beta} &\le \mathcal{R}+(t-s)^2K\mathcal{P}\\ \|x(\psi_1,\psi_2,\psi_3)\|_{c_0^\beta} &\le \frac{\mathcal{R}+(t-s)^2K\mathcal{P}}{1-(t-s)^2KQ}(=r).\end{aligned} \tag{3}$$

Therefore, using equation (2), it is evident that Q is a self-mapping on $C^1(I^3, c_0^\beta)$. Also, $\|(Qx)(\psi_1,\psi_2,\psi_3)-0\|_{c_0^\beta} \le r$, so the operator Q maps $B_r \in C^1(I^3, c_0^\beta)$ (ball of radius r centered at the origin) into itself.

Next, we demonstrate the continuity of the operator Q on B_r. To achieve this, we consider a fixed $\varepsilon > 0$ and an element $x \in B_r$. For any arbitrary, $y \in B_r$ satisfying $|y - x| \le \varepsilon$, as well as a fixed $(\psi_1,\psi_2,\psi_3) \in I^3$ and $n \in \mathbb{N}$. We deduce that

$$\begin{aligned}&\beta_n|(Qx)(\psi_1,\psi_2,\psi_3)-(Qy)(\psi_1,\psi_2,\psi_3)|\\ &= \beta_n\left|\int_s^t\int_s^t\int_s^t K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)[h_n(\varsigma_1,\varsigma_2,\varsigma_3,y(\varsigma_1,\varsigma_2,\varsigma_3))-h_n(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))]\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\\ &\le \int_s^t\int_s^t\int_s^t \beta_n|K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)||h_n(\varsigma_1,\varsigma_2,\varsigma_3,y(\varsigma_1,\varsigma_2,\varsigma_3))-h_n(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\end{aligned} \tag{4}$$

Using assumption (i), we establish the set $\delta(\varepsilon)$ as

$$\delta(\varepsilon) = \sup\{\beta_n|h_n(\psi_1,\psi_2,\psi_3,y)-h_n(\psi_1,\psi_2,\psi_3,x)| : x,y \in c_0^\beta, \|y-x\|_{c_0^\beta} < \varepsilon, n \in \mathbb{N}\}.$$

Then, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. So, by equation (4) and condition (iii), we have

$$|(Qy)-(Qx)| \le (t-s)^2K\delta(\varepsilon).$$

Thus, Q is a continuous operator on B_r.

Our final step in the proof is to demonstrate that Q qualifies as a Meir-Keeler condensing operator. This entails establishing the existence of positive values ε and δ such that

$$\varepsilon \le \chi(B_r) < \varepsilon + \delta \;\Rightarrow\; \chi(Q(B_r)) < \varepsilon.$$

Now, applying the Hausdorff measure of non-compactness in c_0^β and conditions (ii), (iii), and (iv), we have

$$\begin{aligned}\chi(Q(B_r)) &= \lim_{n\to\infty}\left[\sup_{x(\psi_1,\psi_2,\psi_3)\in B_r}\left\{\max_{k\ge n}\beta_n|x_k(\psi_1,\psi_2,\psi_3)|\right\}\right]\\ &\le \lim_{n\to\infty}\left[\sup_{x(\psi_1,\psi_2,\psi_3)\in B_r}\left\{\max_{k\ge n}\beta_n(|r_k(\psi_1,\psi_2,\psi_3)|\right.\right.\\ &\quad\left.\left. + \left|\int_s^t\int_s^t\int_s^t K_k(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)h_k(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\right)\right\}\Bigg]\\ &\le \lim_{n\to\infty}\left[\sup_{x(\psi_1,\psi_2,\psi_3)\in B_r}\left\{\max_{k\ge n}\beta_n\left(K\left|\int_s^t\int_s^t\int_s^t K_k(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)h_k(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\right)\right\}\right]\\ &\le \lim_{n\to\infty}\left[\sup_{x(\psi_1,\psi_2,\psi_3)\in B_r}\left\{\max_{k\ge n}\beta_n\left(K\int_s^t\int_s^t\int_s^t h_k(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right)\right\}\right]\\ &\le \lim_{n\to\infty}\left[\sup_{x(\psi_1,\psi_2,\psi_3)\in B_r}\left\{\max_{k\ge n}\beta_n\left(K\int_s^t\int_s^t\int_s^t (p_k(\psi_1,\psi_2,\psi_3)+q_k(\psi_1,\psi_2,\psi_3)\sup_{i\ge k}\{|x_i(\varsigma_1,\varsigma_2,\varsigma_3)|\})\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right)\right\}\right]\\ &\le (t-s)^2KQ\chi(B_r).\end{aligned}$$

Thus,

$$\chi(Q(B_r)) \leq (t-s)^2 KQ\chi(B_r) < \varepsilon \ \Rightarrow \ \chi(B_r) < \frac{\varepsilon}{(t-s)^2 KQ}.$$

Taking, $\delta = \frac{\varepsilon(1-(t-s)^2KQ)}{(t-s)^2KQ}$, we obtain $\varepsilon \leq \chi(B_r) < \delta + \varepsilon$.

It can be observed that Q satisfies the conditions of a Meir-Keeler condensing operator on $B_r \in c_0^\beta$. Consequently, all the requirements stated in Theorem 1.2 are met. Hence, the system (1) has a solution within the space c_0^β. □

Example 3.1. Consider the following infinite system of Hammerstein-type integral equations in three variables:

$$\begin{aligned} x_n(\psi_1, \psi_2, \psi_3) = \frac{1}{n}\arctan(\psi_1, \psi_2, \psi_3)^n + \int_1^2\int_1^2\int_1^2 \sin\left(\frac{\psi_1+\psi_2+\psi_3+\varsigma_1+\varsigma_2+\varsigma_3}{n}\right) \\ \times \ln\left(\frac{1+4n^4+(\varsigma_1+\varsigma_2+\varsigma_3)^2[4+\sup_{k\geq n}\{|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\}]}{4[(\varsigma_1+\varsigma_2+\varsigma_3)^2+n^4]}\right) \mathrm{d}\varsigma_1 \mathrm{d}\varsigma_2 \mathrm{d}\varsigma_3, \end{aligned} \tag{5}$$

for $(\psi_1, \psi_2, \psi_3) \in [1,2] \times [1,2] \times [1,2]$ and $n \in \mathbb{N}$.

We have

$$\begin{aligned} r_n(\psi_1,\psi_2,\psi_3) &= \frac{1}{n}\arctan(\psi_1,\psi_2,\psi_3)^n, \\ K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3) &= \sin\left(\frac{\psi_1+\psi_2+\psi_3+\varsigma_1+\varsigma_2+\varsigma_3}{n}\right), \\ h_n(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3)) &= \ln\left(\frac{1+4n^4+(\varsigma_1+\varsigma_2+\varsigma_3)^2[4+\sup_{k\geq n}\{|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\}]}{4[(\varsigma_1+\varsigma_2+\varsigma_3)^2+n^4]}\right) \\ &= \left(1+\frac{1+(\varsigma_1+\varsigma_2+\varsigma_3)\sup_{k\geq n}\{|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\}}{4[(\varsigma_1,\varsigma_2,\varsigma_3)^2+n^4]}\right). \end{aligned}$$

Consider $\beta = \beta_n = \frac{1}{n}$.

Let I denote the interval [1, 2], and we show that equation (5) satisfies all the conditions of Theorem 3.1. The operator Q_1 defined by $(Q_1x)(\psi_1,\psi_2,\psi_3) = (h_n(\psi_1,\psi_2,\psi_3,\beta_n x(\psi_1,\psi_2,\psi_3)))$ transforms the space $I^3 \times c_0^\beta$ into c_0^β.

To show that $(Q_1)(\psi_1,\psi_2,\psi_3)$ is equicontinuous at an arbitrary point $x \in c_0^\beta$, let $\varepsilon > 0$, $n \in \mathbb{N}$ and $(\psi_1,\psi_2,\psi_3) \in I^3$, and consider $y \in c_0^\beta$ such that $\|y-x\|_{c_0^\beta} < \varepsilon$. Then,

$$\begin{aligned} &|h_n(\psi_1,\psi_2,\psi_3,x) - h_n(\psi_1,\psi_2,\psi_3,y)| \\ &= \left|\ln\left(\frac{1+(\varsigma_1+\varsigma_2+\varsigma_3)^2\sup_{k\geq n}\left\{\frac{1}{n}|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\right\}}{4[(\varsigma_1+\varsigma_2+\varsigma_3)^2+n^4]}\right) - \ln\left(\frac{1+(\varsigma_1+\varsigma_2+\varsigma_3)^2\sup\limits_{k\geq n}\left\{\frac{1}{n}|y_k(\varsigma_1,\varsigma_2,\varsigma_3)|\right\}}{4[(\varsigma_1+\varsigma_2+\varsigma_3)^2+n^4]}\right)\right| \\ &\leq \left|\frac{(\varsigma_1+\varsigma_2+\varsigma_3)^2}{4[(\varsigma_1+\varsigma_2+\varsigma_3)^2+n^4]}\left[\sup_{k\geq n}\left\{\frac{1}{n}|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\right\} - \sup_{k\geq n}\left\{\frac{1}{n}|y_k(\varsigma_1,\varsigma_2,\varsigma_3)|\right\}\right]\right| \\ &\leq \frac{1}{16}\sup_{k\geq n}\left\{\frac{1}{n}|x_k - y_k|\right\}. \end{aligned}$$

Therefore, $\|h_n(\psi_1,\psi_2,\psi_3,x) - h_n(\psi_1,\psi_2,\psi_3,y)\| \leq \frac{1}{16}\|x-y\|_{c_0^\beta} \leq \frac{\varepsilon}{16}$, so the functions $\{(Q_1x)(\psi_1,\psi_2,\psi_3)\}$ are equicontinuous.

Fixing $(\psi_1, \psi_2, \psi_3) \in I^3$, $x \in c_0^\beta$, and $n \in \mathbb{N}$, we deduce

$$|h_n(\psi_1, \psi_2, \psi_3, x)| = \left| \ln\left(1 + \frac{1 + (\varsigma_1 + \varsigma_2 + \varsigma_3)^2 \sup_{k\geq n}\left\{\frac{1}{n}|x_k(\varsigma_1, \varsigma_2, \varsigma_3)|\right\}}{4[(\varsigma_1 + \varsigma_2 + \varsigma_3)^2 + n^4]}\right)\right|$$

$$\leq \frac{1 + (\varsigma_1 + \varsigma_2 + \varsigma_3)^2 \sup_{k\geq n}\left\{\frac{1}{n}|x_k(\varsigma_1, \varsigma_2, \varsigma_3)|\right\}}{4[(\varsigma_1 + \varsigma_2 + \varsigma_3)^2 + n^4]}$$

$$= \frac{1}{4[(\varsigma_1 + \varsigma_2 + \varsigma_3)^2 + n^4]} + \frac{\varsigma_1 + \varsigma_2 + \varsigma_3}{4[(\varsigma_1 + \varsigma_2 + \varsigma_3)^2 + n^4]} \sup_{k\geq n}\left\{\frac{1}{n}|x_k(\psi_1, \psi_2, \psi_3)|\right\}.$$

Taking $p_n(\psi_1, \psi_2, \psi_3) = \frac{1}{4[(\psi_1 + \psi_2 + \psi_3)^2 + n^4]}$ and $q_n(\psi_1, \psi_2, \psi_3) = \frac{\psi_1 + \psi_2 + \psi_3}{4[(\psi_1 + \psi_2 + \psi_3)^2 + n^4]}$, it is clear that both p_n and q_n are functions with real values, and $p_n(\psi_1, \psi_2, \psi_3)$ uniformly converges to zero.

Also, $|q_n(\psi_1, \psi_2, \psi_3)| \leq \frac{1}{4}$ $\forall n \in \mathbb{N}$. So $\mathcal{P} = \frac{1}{4}$ and

$$\sup_{(\psi_1,\psi_2,\psi_3)\in I^3} \{q_n(\psi_1, \psi_2, \psi_3)\} = \frac{1}{4}.$$

The functions $K_n(\psi_1, \psi_2, \psi_3, v_1, v_2, v_3)$ are continuous on I^6, and the function sequence $(K_n(\psi_1, \psi_2, \psi_3, v_1, v_2, v_3))$ is equibounded on I^6. And we have

$$K = \sup\{|K_n(\psi_1, \psi_2, \psi_3, v_1, v_2, v_3)| : (\psi_1, \psi_2, \psi_3), (v_1, v_2, v_3) \in I^3, n = 1, 2, \ldots\} = 1.$$

Now, fixing $\varepsilon > 0$, $(v_1, v_2, v_3) \in I^3$ and $n = 1, 2, \ldots$, then for arbitrary (ψ_1, ψ_2, ψ_3) and $(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3) \in I^3$ with

$$|\bar{\psi}_1 - \psi_1| < \frac{\varepsilon}{3}, |\bar{\psi}_2 - \psi_2| < \frac{\varepsilon}{3}, |\bar{\psi}_3 - \psi_3| < \frac{\varepsilon}{3},$$

we have

$$\begin{aligned}
&|K_n(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, v_1, v_2, v_3) - K_n(\psi_1, \psi_2, \psi_3, v_1, v_2, v_3)| \\
&\leq \left| \frac{\bar{\psi}_1 + \bar{\psi}_2 + \bar{\psi}_3 + v_1 + v_2 + v_3}{n} - \frac{\psi_1 + \psi_2 + \psi_3 + v_1 + v_2 + v_3}{n} \right| \\
&\leq \frac{1}{n}|(\bar{\psi}_1 - \psi_1) + (\bar{\psi}_2 - \psi_2) + (\bar{\psi}_3 - \psi_3)| \\
&\leq \frac{1}{n}(|\bar{\psi}_1 - \psi_1| + |\bar{\psi}_2 - \psi_2| + |\bar{\psi}_3 - \psi_3|) \\
&\leq \varepsilon.
\end{aligned}$$

Therefore, $(K_n(\psi_1, \psi_2, \psi_3, v_1, v_2, v_3))$ is equicontinuous.

Furthermore, the function $r_n(\psi_1, \psi_2, \psi_3)$ is continuous for all $(\psi_1, \psi_2, \psi_3) \in I^3$, and it converges uniformly to zero for all $n \in \mathbb{N}$.

The factor $(t - s)^2 KQ = \frac{1}{4} < 1$. All the conditions outlined in Theorem 3.1 are fulfilled. As a result, the infinite system (5) has a solution that belongs to the space c_0^β.

4 Existence of solution in the space ℓ_1^{β}

In this section, we establish the conditions for the infinite system (1) to have a solution in the tempering sequence space ℓ_1^{β}. We consider the following assumptions:

a. The functions $(h_i)_{i=1}^{\infty}$ are real-valued and continuous defined on the set $I^3 \times \mathbb{R}^{\infty}$. The operator $T : I^3 \times \ell_1^{\beta} \to \ell_1^{\beta}$ defined on the space $I^3 \times c_0^{\beta}$ as
$$(\psi_1, \psi_2, \psi_3, x) \to (Tx)(\psi_1, \psi_2, \psi_3) = (\beta_1 h_1(\psi_1, \psi_2, \psi_3), \beta_2 h_2(\psi_1, \psi_2, \psi_3), \dots).$$
The collection of functions $\{(Tx)(\psi_1, \psi_2, \psi_3)\}_{(\psi_1, \psi_2, \psi_3)\in I^3}$ exhibits equicontinuity at all points in the space c_0^{β}. This means that for any given value of $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that
$$\|x - y\|_{\ell_1^{\beta}} \le \delta \;\Rightarrow\; \|(Tx)(\psi_1, \psi_2, \psi_3) - (Ty)(\psi_1, \psi_2, \psi_3)\|_{\ell_1^{\beta}} \le \varepsilon.$$

b. For each fixed $(\psi_1, \psi_2, \psi_3) \in I^3$, $x(\psi_1, \psi_2, \psi_3) \in C^1(I^3, \ell_1^{\beta})$, the following inequality holds:
$$|h_n(\psi_1, \psi_2, \psi_3, x(\psi_1, \psi_2, \psi_3))| \le a_n(\psi_1, \psi_2, \psi_3) + b_n(\psi_1, \psi_2, \psi_3)|\beta_i v_i|, \; n \in \mathbb{N}.$$
The functions $a_i(\psi_1, \psi_2, \psi_3)$ and $b_i(\psi_1, \psi_2, \psi_3)$ are continuous real-valued functions defined on the interval I^3. The series of functions $\sum_{n=1}^{\infty} a_n(\psi_1, \psi_2, \psi_3)$ uniformly converges on the interval I^3, and the function sequence $(b_i(\psi_1, \psi_2, \psi_3))_{i\in\mathbb{N}}$ is equibounded on I^3. The function $a(\psi_1, \psi_2, \psi_3)$ given by $a(\psi_1, \psi_2, \psi_3) = \sum_{n=1}^{\infty} a_n(\psi_1, \psi_2, \psi_3)$ is continuous, and the finite constants A and B are defined as
$$A = \max\{a(\psi_1, \psi_2, \psi_3) : (\psi_1, \psi_2, \psi_3) \in I^3\},$$
$$B = \sup\{b_n(\psi_1, \psi_2, \psi_3) : (\psi_1, \psi_2, \psi_3) \in I^3\}.$$

c. The functions $K_n : I^6 \to \mathbb{R}$ are continuous throughout the interval I^6 for all $n \in \mathbb{N}$. Additionally, these functions $K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)$ exhibit equicontinuity with respect to (ψ_1, ψ_2, ψ_3). This means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
$$|K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3) - K_n(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, \varsigma_1, \varsigma_2, \varsigma_3)| < \varepsilon,$$
where $|\psi_1 - \bar{\psi}_1| < \delta$, $|\psi_2 - \bar{\psi}_2| < \delta$, and $|\psi_3 - \bar{\psi}_3| < \delta$, for all $(\varsigma_1, \varsigma_2, \varsigma_3) \in I^3$. In addition to that the function sequence $(K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3))$ is equibounded on the set I^3, and there exists a constant
$$K = \sup\{\beta_n |K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)| : (\psi_1, \psi_2, \psi_3), (\varsigma_1, \varsigma_2, \varsigma_3) \in I^3, n \in \mathbb{N}\} < \infty.$$

d. The functions $r_n : I^3 \to \mathbb{R}$ are continuous, and the sequence of functions (r_n) belongs to the space $C^1(I^3, \ell_1^{\beta})$.

Remark 4.1. Due to the fact that $I^3 = [s, t] \times [s, t] \times [s, t]$ is a compact subset of $\mathbb{R}^3$, the continuity condition in (d) implies that $r_n : I^3 \to \mathbb{R}$ is uniformly continuous. This, in turn, implies that the sequence of functions $(r_n(\psi_1, \psi_2, \psi_3))$ is equicontinuous on I^3. For any given $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $(\psi_1, \psi_2, \psi_3), (v_1, v_2, v_3) \in I^3$,
$$\|(r_n(\psi_1, \psi_2, \psi_3)) - r_n(v_1, v_2, v_3)\|_{\ell_1^{\beta}} \le \sum_{n=1}^{\infty} |r_n(\psi_1, \psi_2, \psi_3) - r_n(v_1, v_2, v_3)| < \varepsilon, \tag{6}$$
whenever $|(\psi_1, \psi_2, \psi_3) - (v_1, v_2, v_3)| < \delta$. In addition, from inequality (6), it is evident that the function series $\sum_{n=1}^{\infty} r_n(\psi_1, \psi_2, \psi_3)$ is convergent on I^3 and the function
$$r(\psi_1, \psi_2, \psi_3) = \sum_{n=1}^{\infty} r_n(\psi_1, \psi_2, \psi_3)$$
is equicontinuous on I^3. Therefore, the constant given by
$$\mathcal{R} = \max\{r(\psi_1, \psi_2, \psi_3) : (\psi_1, \psi_2, \psi_3) \in I^3\}$$
is finite.

Theorem 4.1. *Under assumptions* (a)–(d), *the infinite system* (1) *possesses at least one solution denoted as* $x(\psi_1, \psi_2, \psi_3) = (x_i(\psi_1, \psi_2, \psi_3))_{i=1}^{\infty}$, *belonging to* ℓ_1^{β}, *for a fixed* $(\psi_1, \psi_2, \psi_3) \in I^3$. *This is guaranteed whenever the condition* $(t - s)^2 KB < 1$ *holds.*

Proof. An operator T is defined on the space $\Lambda = C^1(I^3, \ell_1^{\beta})$ by

$$
\begin{aligned}
(Tx)(\psi_1, \psi_2, \psi_3) &= ((Tx)_n(\psi_1, \psi_2, \psi_3))\\
&= r_n(\psi_1, \psi_2, \psi_3) + \int_s^t\int_s^t\int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)\beta_n h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\\
&= (r_1(\psi_1, \psi_2, \psi_3)) + \int_s^t\int_s^t\int_s^t K_1(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)\beta_1 h_1(\varsigma_1, \varsigma_2, \varsigma_3, x_1(\varsigma_1, \varsigma_2, \varsigma_3), \ldots)\\
&\quad \times \mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3, (r_2(\psi_1, \psi_2, \psi_3)) + \int_s^t\int_s^t\int_s^t K_1(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)\beta_2 h_1\\
&\quad \times (\varsigma_1, \varsigma_2, \varsigma_3, x_2(\varsigma_1, \varsigma_2, \varsigma_3), \ldots)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3,
\end{aligned}
\tag{7}
$$

for all $(\psi_1, \psi_2, \psi_3) \in I^3$ and $x = (x_i)_{i=1}^{\infty} \in \ell_1^{\beta}$.

First, we will show that T is a self-map of Λ into itself. Let $n \in \mathbb{N}$ and $(\psi_1, \psi_2, \psi_3) \in I^3$ using assumption (a) and (c), we have

$$
\begin{aligned}
\|(Tx)_n(\psi_1, \psi_2, \psi_3)\|_{\ell_1^{\beta}} &= \sum_{n=0}^{\infty} \beta_n |(Tx)_n(\psi_1, \psi_2, \psi_3)|\\
&\le \sum_{n=1}^{\infty} \beta_n |r_n(\psi_1, \psi_2, \psi_3)| + \sum_{n=1}^{\infty} \beta_n \left|\int_s^t\int_s^t\int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3) h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\\
&\le R + \sum_{n=1}^{\infty}\int_s^t\int_s^t\int_s^t \beta_n |K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)||h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\\
&\le R + K\sum_{n=1}^{\infty}\int_s^t\int_s^t\int_s^t [a_n(\varsigma_1, \varsigma_2, \varsigma_3) + b_n(\varsigma_1, \varsigma_2, \varsigma_3)\beta_n |x_n(\varsigma_1, \varsigma_2, \varsigma_3)|]\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\\
&\le R + K\sum_a^b\left(\int_s^t\int_s^t\int_s^t a_n(\varsigma_1, \varsigma_2, \varsigma_3)\right)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}s_3 + KB\sum_{n=1}^{\infty}\left(\int_s^t\int_s^t\int_s^t \beta_n |x_n(\varsigma_1, \varsigma_2, \varsigma_3)|\right)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3.
\end{aligned}
$$

Applying the Lebesgue monotone convergence theorem [15], we deduce

$$
\sum_{n=1}^{\infty} |(Tx)_n(\psi_1, \psi_2, \psi_3)| \le R + K\sum_a^b\left(\int_s^t\int_s^t\int_s^t a_n(\varsigma_1, \varsigma_2, \varsigma_3)\right)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}s_3 + KB\sum_{n=1}^{\infty}\left(\int_s^t\int_s^t\int_s^t \beta_n |x_n(\varsigma_1, \varsigma_2, \varsigma_3)|\right)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3 \tag{8}
$$

$$
\begin{aligned}
&\le R + KA(t - s)^2 + KB(t - s)^2 \sup\{\|x(\psi_1, \psi_2, \psi_3)\|_{\ell_1^{\beta}} : (\psi_1, \psi_2, \psi_3) \in I^3\}\\
&\le R + KA(t - s)^2 + KB(t - s)^2\|x\|_{\ell_1^{\beta}}\\
&< \infty.
\end{aligned}
\tag{9}
$$

Thus, the operator $(Tx)(\psi_1, \psi_2, \psi_3)$ belongs to the space ℓ_1^{β} for any fixed $(\psi_1, \psi_2, \psi_3) \in I^3$. Furthermore,

$$
\begin{aligned}
\|x(\psi_1, \psi_2, \psi_3)\|_{\ell_1^{\beta}} &= \sum_{n=1}^{\infty} \beta_n |r_n(\psi_1, \psi_2, \psi_3)| + \sum_{n=1}^{\infty} \beta_n \left|\int_s^t\int_s^t\int_s^t K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)\right.\\
&\quad \left. \times\, h_n(\varsigma_1, \varsigma_2, \varsigma_3, x(\varsigma_1, \varsigma_2, \varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\\
&\le R + KA(t - s)^2 + KB(t - s)^2\|x\|_{\ell_1^{\beta}}.
\end{aligned}
\tag{10}
$$

Using (10), since $KB(t-s)^2 < 1$, we have

$$[1-(t-s)^2KB]\|x(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta} \le R + (t-s)^2KA, \tag{11}$$

$$\|x(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta} \le \frac{R+(t-s)^2KA}{1-(t-s)^2KB} = (r_1).$$

By employing equation (8), we can deduce that T is a self-map on Λ.

Additionally, due to the fact that $|(Tx)(\psi_1,\psi_2,\psi_3)| < r_1$, it follows that the operator T maps the ball (B_{r_1}), which is centered at the origin and has a radius of r_1, into itself.

Now, we show that T is continuous on B_{r_1}. Let $\varepsilon > 0$ and $x \in B_{r_1}$. Then, for arbitrary $z \in B_{r_1}$ with $\|z-x\|_{\ell_1^\beta} \le \varepsilon$, arbitrary fixed $(\psi_1,\psi_2,\psi_3) \in I^3$, and $n \in \mathbb{N}$.

$$\begin{aligned}
&\|(Tz)(\psi_1,\psi_2,\psi_3)-(Tx)(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta}\\
&= \sum_{n=1}^{\infty}\beta_n\left|\int_s^t\int_s^t\int_s^t K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)[h_n(\varsigma_1,\varsigma_2,\varsigma_3,z(\varsigma_1,\varsigma_2,\varsigma_3))-h_n(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))]\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\\
&\le \sum_{n=1}^{\infty}\beta_n\int_s^t\int_s^t\int_s^t |K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)||h_n(\varsigma_1,\varsigma_2,\varsigma_3,z(\varsigma_1,\varsigma_2,\varsigma_3))-h_n(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\\
&\le K\sum_{n=1}^{\infty}\beta_n\int_s^t\int_s^t\int_s^t |h_n(\varsigma_1,\varsigma_2,\varsigma_3,z(\varsigma_1,\varsigma_2,\varsigma_3))-h_n(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3.
\end{aligned} \tag{12}$$

Using condition (a), we define the set $\delta(\varepsilon)$ as follows:

$$\delta(\varepsilon) = \sup\{\beta_n|h_n(\psi_1,\psi_2,\psi_3,z)-h_n(\psi_1,\psi_2,\psi_3,x)| : x,z \in B_{r_1}, \|z-x\|_{\ell_1^\beta} < \varepsilon, (\psi_1,\psi_2,\psi_3) \in I^3,\ n \in \mathbb{N}\}.$$

Then, we have that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Using assumption (c) and applying the Lebesgue monotone convergence theorem [15], we can deduce the following result based on inequality (11):

$$\begin{aligned}
&\|(Tz(\psi_1,\psi_2,\psi_3))-(Tx)(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta}\\
&\le K\sum_{n=1}^{\infty}\beta_n\int_s^t\int_s^t\int_s^t |h_n(\varsigma_1,\varsigma_2,\varsigma_3,z(\varsigma_1,\varsigma_2,\varsigma_3))-h_n(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\\
&\le K\int_s^t\int_s^t\int_s^t \|h_n(\varsigma_1,\varsigma_2,\varsigma_3,z)-h_n(\varsigma_1,\varsigma_2,\varsigma_3,x)\|_{\ell_1^\beta}\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\\
&\le (t-s)^2K\delta(\varepsilon).
\end{aligned} \tag{13}$$

Since (13) holds for arbitrary fixed $(\psi_1,\psi_2,\psi_3) \in I^3$,

$$\|Tz-Tx\|_{\ell_1^\beta} \le \sup_{(\psi_1,\psi_2,\psi_3)\in I^3}\{\|(Tz)(\psi_1,\psi_2,\psi_3)-(Tz)(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta}\} \le (t-s)^2K\delta(\varepsilon).$$

Thus, T is a continuous operator on B_{r_1}.

To demonstrate that T is a Meir-Keeler condensing operator, we need to establish the existence of $\varepsilon > 0$ and $\delta > 0$ such that

$$\varepsilon \le \chi(B_{r_1}) < \varepsilon + \delta \ \Rightarrow\ \chi(T(B_{r_1})) < \varepsilon.$$

Making use of the Hausdorff measure of non-compactness in ℓ_1^β and assumptions (b)–(d), we deduce

$$\begin{aligned}\chi(T(B_{r_1})) &= \lim_{n\to\infty}\left(\sup_{x(\psi_1,\psi_2,\psi_3)\in B_{r_1}}\left\{\sum_{k\geq n}\beta_n(x_k(\psi_1,\psi_2,\psi_3))\right\}\right)\\
&\leq \lim_{n\to\infty}\left[\sup_{x(\psi_1,\psi_2,\psi_3)\in B_{r_1}}\left\{\sum_{k\geq n}(\beta_n|r_k(\psi_1,\psi_2,\psi_3)|\right.\right.\\
&\quad+\left.\left.\left|\int_s^t\int_s^t\int_s^t\beta_n K_k(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)h_k(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right|\right)\right\}\right]\\
&\leq \lim_{n\to\infty}\left[\sup_{x(\psi_1,\psi_2,\psi_3)\in B_{r_1}}\left\{\sum_{k\geq n}\left(\beta_n|r_k(\psi_1,\psi_2,\psi_3)|+K\int_s^t\int_s^t\int_s^t\beta_n|h_k(\varsigma_1,\varsigma_2,\varsigma_3,x(\varsigma_1,\varsigma_2,\varsigma_3))\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3|\right)\right\}\right]\\
&\leq \lim_{n\to\infty}\left[\sum_{k\geq n}\beta_k|r_n(\psi_1,\psi_2,\psi_3)|+K\sup_{x(\psi_1,\psi_2,\psi_3)\in B_{r_1}}\left\{\sum_{k\geq n}\int_s^t\int_s^t\int_s^t a_k(\varsigma_1,\varsigma_2,\varsigma_3)\right.\right.\\
&\quad\left.\left.+\,b_k(\varsigma_1,\varsigma_2,\varsigma_3)\beta_k|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right\}\right].\end{aligned}$$

Applying the Lebesgue-dominated convergence theorem, we obtain

$$\begin{aligned}\chi(T(B_{r_1})) &\leq \lim_{n\to\infty}\left[\sum_{k\geq n}\beta_k|r_k(\psi_1,\psi_2,\psi_3)|+K\left\{\int_s^t\int_s^t\int_s^t\sum_{k\geq n}a_k(\varsigma_1,\varsigma_2,\varsigma_3)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right.\right.\\
&\quad\left.\left.+\,B\int_s^t\int_s^t\int_s^t\sup_{x(\psi_1,\psi_2,\psi_3)\in B_{r_1}}\left(\sum_{k\geq n}\beta_k|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\right)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right\}\right]\\
&\leq KB\int_s^t\int_s^t\int_s^t\lim_{n\to\infty}\left\{\sup_{x(\psi_1,\psi_2,\psi_3)\in B_{r_1}}\left(\sum_{k\geq n}\beta_k|x_k(\varsigma_1,\varsigma_2,\varsigma_3)|\right)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3\right\}\\
&\leq (t-s)^2KB\chi(B_{r_1}).\end{aligned}$$

Therefore,

$$\chi(T(B_{r_1}))\leq(t-s)^2KB\chi(B_{r_1})<\varepsilon\ \Rightarrow\ \chi(B_{r_1})<\frac{\varepsilon}{(t-s)^2KB}.$$

Taking $\delta=\frac{\varepsilon(1-(t-s)^2KB)}{(t-s)^2KB}$, we obtain that $\varepsilon\leq\chi(B_{r_1})<\varepsilon+\delta$.

Hence, T qualifies as a Meir-Keeler condensing operator on $B_{r_1}\in\ell_1^\beta$. Consequently, according to Theorem 3.1, T possesses a fixed point within B_{r_1}, implying the existence of a solution for the system (1) in ℓ_1^β. □

Example 4.1. Let us consider an infinite set of Hammerstein-type integral equations in three variables:

$$\begin{aligned}x_n(\psi_1,\psi_2,\psi_3) &= \frac{1}{n^2}\ln[(\psi_1+\psi_2+\psi_3)+n]+\int_1^2\int_1^2\int_1^2\arctan(\psi_1+\psi_2+\psi_3+\varsigma_1+\varsigma_2+\varsigma_3+n)\\
&\quad\times\left((\varsigma_1+\varsigma_2+\varsigma_3)^2e^{-n(\varsigma_1+\varsigma_2+\varsigma_3)}+\frac{\sin n(\varsigma_1+\varsigma_2+\varsigma_3)}{(\varsigma_1+\varsigma_2+\varsigma_3)^2+n}\,\frac{x_n^2(\varsigma_1,\varsigma_2,\varsigma_3)}{1+x_1^2(\varsigma_1,\varsigma_2,\varsigma_3)+\ldots+x_n^2(\varsigma_1,\varsigma_2,\varsigma_3)}\right)\mathrm{d}\varsigma_1\mathrm{d}\varsigma_2\mathrm{d}\varsigma_3,\end{aligned}\tag{14}$$

for $(\psi_1,\psi_2,\psi_3),(\varsigma_1,\varsigma_2,\varsigma_3)\in[1,2]\times[1,2]\times[1,2]$.

From equation (14), we have

$r_n(\psi_1,\psi_2,\psi_3)=\frac{1}{n^2}\ln[(\psi_1+\psi_2+\psi_3)+n]$,

$K_n(\psi_1,\psi_2,\psi_3,\varsigma_1,\varsigma_2,\varsigma_3)=\arctan(\psi_1+\psi_2+\psi_3+\varsigma_1+\varsigma_2+\varsigma_3+n)$,

$$h_n(\psi_1, \psi_2, \psi_3, x_1, x_2, \ldots) = (\varsigma_1 + \varsigma_2 + \varsigma_3)^2 e^{-n(\varsigma_1+\varsigma_2+\varsigma_3)} + \frac{\sin n(\varsigma_1 + \varsigma_2 + \varsigma_3)}{(\varsigma_1 + \varsigma_2 + \varsigma_3)^2 + n} \times \frac{x_n^2(\psi_1, \psi_2, \psi_3)}{1 + x_1^2(\psi_1, \psi_2, \psi_3) + \ldots + x_n^2(\psi_1, \psi_2, \psi_3)}.$$

It is clear that $r_n(\psi_1, \psi_2, \psi_3)$ is continuous on $I^3 = [1, 2] \times [1, 2] \times [1, 2]$ for $(\psi_1, \psi_2, \psi_3), (\varsigma_1, \varsigma_2, \varsigma_3) \in I^3$ and $n \in \mathbb{N}$.

Taking the tempering sequence $\beta = \beta_n = \frac{1}{n^3}$, for fixed $(\psi_1, \psi_2, \psi_3), (u_1, u_2, u_3) \in I^3$, we observe that

$$\begin{aligned} &\|(r_n)(\psi_1, \psi_2, \psi_3) - (r_n)(u_1, u_2, u_3)\| \\ &= \sum_{n=1}^{\infty} \beta_n |r_n(\psi_1, \psi_2, \psi_3) - r_n(u_1, u_2, u_3)| \\ &= \sum_{n=1}^{\infty} \frac{1}{n^5} |\ln[(\psi_1 + \psi_2 + \psi_3) + n] - \ln[(u_1 + u_2 + u_3) + n]| \\ &= \sum_{n=1}^{\infty} \frac{1}{n^5} \left| \ln\left(1 + \frac{\psi_1 + \psi_2 + \psi_3 - u_1 - u_2 - u_3}{u_1 + u_2 + u_3 + n}\right)\right| \\ &\le \sum_{n=1}^{\infty} \frac{1}{n^6} |\psi_1 + \psi_2 + \psi_3 - u_1 - u_2 - u_3| \\ &\le [|\psi_1, -u_1| + |\psi_2, -u_2| + |\psi_3 - u_3|]\zeta(6), \end{aligned}$$

where the symbol $\zeta(s)$ represents the Riemann zeta function.

If we choose $\delta = \frac{\varepsilon}{\zeta(6)}$, so that

$$|\psi_1, -u_1| < \frac{\delta}{3}, \ |\psi_1, -u_1| < \frac{\delta}{3}, \ |\psi_3 - u_3| < \frac{\delta}{3},$$

we obtain

$$\|(r_n)(\psi_1 + \psi_2 + \psi_3) - (r_n)(u_1 + u_2 + u_3)\| < \varepsilon.$$

Moreover, for every $(\psi_1, \psi_2, \psi_3) \in I^3$, we deduce

$$r_n(\psi_1, \psi_2, \psi_3) \le \frac{1}{n^5} \ln(4 + n) \le \frac{1}{n^5}\sqrt{4 + n} \le \left(\frac{2}{n^5} + \frac{1}{n^{6/5}}\right).$$

Thus,

$$\begin{aligned} R &= \max\left\{\sum_{n=1}^{\infty} r_n(\psi_1, \psi_2, \psi_3) : (\psi_1, \psi_2, \psi_3) \in I^3\right\} \\ &= \sum_{n=1}^{\infty}\left(\frac{2}{n^5} + \frac{1}{n^{6/5}}\right) \\ &= 2\zeta(5) + \zeta(1.2) \\ &< \infty. \end{aligned} \tag{15}$$

Thus, assumption (d) and Remark (4.1) are satisfied. Also, the function K_n is continuous on I^6 and

$$K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3) = |\arctan(\psi_1 + \psi_2 + \psi_3 + \varsigma_1 + \varsigma_2 + \varsigma_3 + n)| \le \frac{\pi}{2}.$$

Therefore, the sequence of functions (K_n) is uniformly bounded on the interval I^6.

Also, for fixed $(\psi_1, \psi_2, \psi_3), (u_1, u_2, u_3) \in I^3$, we have

$$\begin{aligned} &|K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3) - K_n(u_1, u_2, u_3, \varsigma_1, \varsigma_2, \varsigma_3)| \\ &= |\arctan(\psi_1 + \psi_2 + \psi_3 + \varsigma_1 + \varsigma_2 + \varsigma_3 + n) - \arctan(u_1 + u_2 + u_3 + \varsigma_1 + \varsigma_2 + \varsigma_3 + n)| \\ &\le |\psi_1, -u_1| + |\psi_2, -u_2| + |\psi_3 - u_3|. \end{aligned}$$

It is evident that the function sequence $K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3)$ is equicontinuous with respect to $(\psi_1, \psi_2, \psi_3) \in I^3$ uniformly with respect to $(\varsigma_1, \varsigma_2, \varsigma_3) \in I^3$, and the value of the constant K is given as

$$K = \sup\{K_n(\psi_1, \psi_2, \psi_3, \varsigma_1, \varsigma_2, \varsigma_3) : (\psi_1, \psi_2, \psi_3), (\varsigma_1, \varsigma_2, \varsigma_3) \in I^3, n \in \mathbb{N}\} = \frac{\pi}{2}. \tag{16}$$

Thus, all conditions of assumption (c) are satisfied.

Again,

$$\begin{aligned} |h_n(\psi_1, \psi_2, \psi_3, x)| &\le (\psi_1 + \psi_2 + \psi_3)^2 e^{-n(\psi_1+\psi_2+\psi_3)} + \frac{1}{n^3}\left|\frac{\sin n(\psi_1 + \psi_2 + \psi_3)}{(\psi_1 + \psi_2 + \psi_3)^2 + n^3}\frac{x_n^2}{1 + x_1^2 + x_2^2 + \ldots + x_n^2}\right| \\ &\le (\psi_1 + \psi_2 + \psi_3)^2 e^{-n(\psi_1+\psi_2+\psi_3)} + \frac{1}{(\psi_1 + \psi_2 + \psi_3)^2 + n^3}\frac{1}{n^3}\left|\frac{x_n^2}{1 + x_1^2 + x_2^2 + \ldots + x_n^2}\right| \\ &\le (\psi_1 + \psi_2 + \psi_3)^2 e^{-n(\psi_1+\psi_2+\psi_3)} + \frac{1}{(\psi_1 + \psi_2 + \psi_3)^2 + n^3}\frac{1}{n^3}\frac{|x_n|}{1 + x_n^2}(|x_n|) \\ &\le (\psi_1 + \psi_2 + \psi_3)^2 e^{-n(\psi_1+\psi_2+\psi_3)} + \frac{1}{2[(\psi_1 + \psi_2 + \psi_3)^2 + n^3]}\frac{1}{n^3}|x_n|. \end{aligned}$$

Taking $a_n(\psi_1, \psi_2, \psi_3) = (\psi_1 + \psi_2 + \psi_3)^2 e^{-n(\psi_1+\psi_2+\psi_3)}$ and $b_n(\psi_1, \psi_2, \psi_3) = \frac{1}{2[(\psi_1 + \psi_2 + \psi_3)^2 + n^3]}$, we have

$$|h_n(\psi_1, \psi_2, \psi_3, x)| \le a_n(\psi_1, \psi_2, \psi_3) + b_n(\psi_1, \psi_2, \psi_3)\beta_n|x_n|.$$

It is obvious that the functions $a_n(\psi_1, \psi_2, \psi_3)$ are continuous on I^3, for any $(\psi_1, \psi_2, \psi_3) \in I^3$ we have $|a_n(\psi_1, \psi_2, \psi_3)| \le \frac{9}{n^3}e^{-3}$ and the function series

$$a(\psi_1, \psi_2, \psi_3) = \sum_{n=1}^{\infty} a_n(\psi_1, \psi_2, \psi_3) = \frac{(\psi_1 + \psi_2 + \psi_3)^2}{e^{\psi_1+\psi_2+\psi_3} - 1}$$

is uniformly convergent on I^3.

Also,

$$b_n(\psi_1, \psi_2, \psi_3) = \frac{1}{2[(\psi_1 + \psi_2 + \psi_3)^2 + n^3]} \le \frac{1}{2n^3} \le \frac{1}{2},$$

for all $n \in \mathbb{N}$. Thus, the sequence of functions $(b_n(\psi_1, \psi_2, \psi_3))$ is uniformly bounded on the interval I^3. The values of the constants A and B are as follows:

$$A = \max\{a(\psi_1, \psi_2, \psi_3) : (\psi_1, \psi_2, \psi_3) \in I^3\} = \frac{36}{e^3 - 1} \quad \text{and} \quad B = \frac{1}{2}, \tag{17}$$

and $(t - s)^2 KB = \frac{\pi}{8}$.

Using, (11), (15), (16), and (17), we obtain

$$r_1 = \frac{2\zeta(5) + \zeta(1.2) + (2 - 1)^2 \times \frac{1}{2} \times \frac{36}{e^3 - 1}}{1 - \frac{\pi}{8}}.$$

Finally, we check if assumption (a) is satisfied. Fixing $x = (x_n) \in B_{r_1} \subset \ell_1^\beta$ and $\varepsilon > 0$, then for fixed $(\psi_1, \psi_2, \psi_3) \in I^3$, we have

$$\begin{aligned}
&\|(Tz)(\psi_1,\psi_2,\psi_3)-(Tx)(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta}\\
&=\sum_{n=1}^{\infty}\beta_n|h_n(\psi_1,\psi_2,\psi_3,z)-h_n(\psi_1,\psi_2,\psi_3,x)|\\
&\le\sum_{n=1}^{\infty}\frac{1}{n^3}\left|\frac{\sin n(\psi_1+\psi_2+\psi_3)}{(\psi_1+\psi_2+\psi_3)^2+n^3}\right|\left|\frac{z_n^2}{1+z_1^2+\ldots+z_n^2}-\frac{x_n^2}{1+x_1^2+\ldots+x_n^2}\right|\\
&\le\sum_{n=1}^{\infty}\frac{1}{n^6}|z_n^2(1+x_1^2+\ldots+x_n^2)-x_n^2(1+z_1^2+\ldots+z_n^2)|\\
&\le\sum_{n=1}^{\infty}\frac{1}{n^6}[|z_n^2-x_n^2|+|z_n^2(x_1^2+\ldots+x_n^2)-z_n^2(z_1^2+\ldots+z_n^2)||z_n^2(z_1^2+\ldots+z_n^2)-x_n^2(z_1^2+\ldots+z_n^2)|]\\
&\le\sum_{n=1}^{\infty}\frac{1}{n^6}[|z-n^2-x_n^2|+z_n^2(|x_1^2-z_1^2|+\ldots+|x_n^2-z_n^2|)|z_n^2-x_n^2|(z_1^2+\ldots+z_n^2)].
\end{aligned}$$

Since $x_n, z_n \in B_{r_1}, n \in \mathbb{N}, |x_n| \le r_1, |z_n < r_1$, so

$$\begin{aligned}
&\|(Tz)(\psi_1,\psi_2,\psi_3)-(Tx)(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta}\\
&\le\sum_{n=1}^{\infty}\frac{1}{n^6}(|z_n-x_n|(|z_n|+|x_n|))(1+z_1^2+\ldots+z_n^2)+z_n^2(|x_1-z_1|(|x_1|+|z_1|)+\ldots+|x_n-z_n|(|x_n|+|z_n|))\\
&<2r_1\sum_{n=1}^{\infty}\left[\frac{1}{n^6}|z_n-x_n|(1+nr_1^2)+r_1^2\left(\sum_{i=1}^{\infty}|x_i-z_i|\right)\right]\\
&=2r_1\|z-x\|_{\ell_1^\beta}\sum_{n=1}^{\infty}\frac{1}{n^6}[(1+nr_1^2)+r_1^2]\\
&=2r_1\|z-x\|_{\ell_1^\beta}([1+r_1^2]\zeta(6)+r_1^2\zeta(5)).
\end{aligned}$$

Thus, choose $\delta = \frac{\varepsilon}{2r_1(1-r_1^2)(\zeta(6)+r_1^2\zeta(5))}$, and then for $\|z-x\|_{\ell_1^\beta} < \delta$ we have

$$\|T(z)(\psi_1,\psi_2,\psi_3)-T(x)(\psi_1,\psi_2,\psi_3)\|_{\ell_1^\beta} < \varepsilon.$$

Consequently, assumption (a) is fulfilled as well. As a result, according to Theorem 3.1, we can deduce that the system (14) has a solution within the subset B_{r_1} of ℓ_1^β.

5 Conclusion

In this work, we have used the Hausdorff measure of non-compactness to prove the existence of solutions for a system of integral equations of Hammerstein type in the space of tempering sequences c_0^β and ℓ_1^β. Our results improve the results of Malik and Jalal [10]. Our results are more generalized in larger sequence spaces.

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