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Research Article

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Derivation of Hermite-Hadamard-Jensen-Mercer conticrete inequalities for Atangana-Baleanu fractional integrals by means of majorization

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Abstract: This article is mainly concerned to link the Hermite-Hadamard and the Jensen-Mercer inequalities by using majorization theory and fractional calculus. We derive the Hermite-Hadamard-Jensen-Mercer-type inequalities in conticrete form, which serve as both discrete and continuous inequalities at the same time, for majorized tuples in the framework of the famous Atangana-Baleanu fractional operators. Also, the main inequalities encompass the previously established inequalities as special cases. Using generalized Mercer's inequality, we also investigate the weighted forms of our major inequalities for certain monotonic tuples. Furthermore, the derivation of new integral identities enables us to construct bounds for the discrepancy of the terms concerning the main results. These bounds are constructed by incorporating the convexity of $|f'|$ and $|f'|^q$ $(q > 1)$ and making use of power mean and Hölder inequalities along with the established identities.

Keywords: convex function, majorization, Atangana-Baleanu fractional operators

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1 Introduction

Mathematical inequalities are immensely useful when dealing with the quantities that fall under conditions such as greater than, greater or equal to, less than, and less than or equal to. They are also needed when we are required to approximate a solution instead of giving an exact solution to a mathematical problem [1–3]. Applications of this field can be observed in different scientific fields such as Information Theory [4], Economics [5], Mathematical Statistics [6], and Engineering [7]. The theory of convex functions further strengthens the literature of the field of inequalities [8–11]. Consequently, researchers used the concept of convexity and obtained a large number of inequalities. Among these inequalities, some well-known inequalities are the Jensen [4], the Jensen-Mercer [12], Fejér [13], the Hermite-Hadamard [14,15], and the Ostrowski [16,17] inequalities. This article commences with a description of the Hermite-Hadamard and Jensen-Mercer inequalities.

The celebrated Hermite-Hadamard inequality, which applies to convex function f with $\theta_1 < \theta_2$, is written as

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$$f\left(\frac{\theta_1+\theta_2}{2}\right) \le \frac{1}{\theta_2-\theta_1}\int\limits_{\theta_1}^{\theta_2} f(u)\mathrm{d}u \le \frac{f(\theta_1)+f(\theta_2)}{2}. \tag{1}$$

The classical Jensen-Mercer inequality that applies to a convex function f defined on $[\theta_1, \theta_2]$ and $\beta_\zeta \in [\theta_1, \theta_2]$, $\sigma_\zeta \ge 0$ for all $\zeta = 1, 2, \ldots, \omega$, with $\sum_{\zeta=1}^{\omega}\sigma_\zeta = 1$, is given as

$$f\left(\theta_1+\theta_2-\sum_{\zeta=1}^{\omega}\sigma_\zeta\beta_\zeta\right) \le f(\theta_1)+f(\theta_2)-\sum_{\zeta=1}^{\omega}\sigma_\zeta f(\beta_\zeta). \tag{2}$$

A handful amount of research has been devoted to the Hermite-Hadamard inequality in recent years [18–20]. To obtain its various shapes, researchers have either applied convexity of the function in various forms such as strong convexity [21], s-convexity [22], coordinated convexity [23], and η-convexity [24], or they have mixed convex theory with some other well-known concepts such as majorization and fractional calculus [25–30].

The area of fractional calculus has been the subject of research for many researchers due to its extensive variety of applications in numerous scientific fields such as medicine [31], biology [32], geophysics [33], and mathematical analysis [34]. Earlier, this journey began with a query about the possibility of a solution of the differential equation if the integer order of the involved derivative is replaced by fractional order. However, it has now been evolved with a variety of fractional derivatives and integral operators. These operators increase the connection between mathematics and other disciplines regarding their application areas by putting out approaches that are extremely appropriate for real-world problems. Riemann-Liouville fractional integrals played a significant role in this regard, and their discovery gave rise to numerous extended and generalized forms of fractional operators [35]. Some most commonly used operators include Caputo [36], Caputo-Fabrizio [37], k-Caputo [27,38], Hadamard [39], and Katugampola [40]. In the following, the definitions of some well-known fractional operators are discussed.

Definition 1. [35] **Riemann-Liouville fractional integral operators:** Let a function $f : [\theta_1, \theta_2] \to \mathbb{R}$, gamma function $\Gamma(.)$, and $\eta > 0$. Then, the below-mentioned integrals are termed as Riemann-Liouville fractional integral operators:

$$J_{\theta_1^+}^{\eta} f(z) = \frac{1}{\Gamma(\eta)}\int\limits_{\theta_1}^{z}(z-u)^{\eta-1}f(u)\mathrm{d}u, \quad z > \theta_1, \qquad \text{(left-sided integral)}$$

and

$$J_{\theta_2^-}^{\eta} f(z) = \frac{1}{\Gamma(\eta)}\int\limits_{z}^{\theta_2}(u-z)^{\eta-1}f(u)\mathrm{d}u, \quad z < \theta_2, \qquad \text{(right-sided integral)}.$$

Note that $f(z) = J_{\theta_2^-}^{0} f(z) = J_{\theta_1^+}^{0} f(z)$.

Definition 2. [35,36] **Caputo fractional derivative operators:** Let a function $f : [\theta_1, \theta_2] \to \mathbb{R}$ be such that $f \in C^n[\theta_1, \theta_2]$, gamma function $\Gamma(.)$ and $\eta > 0$. Then, the below-mentioned derivatives are termed as Caputo fractional derivative operators:

$${}^{c}D_{\theta_1^+}^{\eta} f(z) = \frac{1}{\Gamma(n-\eta)}\int\limits_{\theta_1}^{z}\frac{f^{(n)}(u)}{(z-u)^{\eta-n+1}}\mathrm{d}u, \quad z > \theta_1, \qquad \text{(left-sided derivative operator)}$$

and

$${}^{c}D_{\theta_2^-}^{\eta} f(z) = \frac{(-1)^n}{\Gamma(n-\eta)}\int\limits_{z}^{\theta_2}\frac{f^{(n)}(u)}{(u-z)^{\eta-n+1}}\mathrm{d}u, \quad z < \theta_2, \qquad \text{(right-sided derivative operator)}.$$

Note that for $\eta = 0$ and $n = 1$, we have

$$f(z) = ({}^{c}D_{\theta_2^-}^{0} f)(z) = ({}^{c}D_{\theta_1^+}^{0} f)(z).$$

Definition 3. [37]. **Caputo-Fabrizio fractional derivative operator:** Let a function $f \in H(0, \theta_2)$ ("H" denotes Hilbert space), $\theta_2 > \theta_1$, $\eta \in [0, 1]$, and normalization function $B(\eta)$. Then, the below-mentioned derivative is termed as Caputo-Fabrizio fractional derivative operator:

$$ {}^{CF}D^{\eta}f(z) = \frac{B(\eta)}{1-\eta}\int_{\theta_1}^{z} f'(u)\exp\left[-\frac{\eta}{1-\eta}(z-u)\right]\mathrm{d}u. \tag{3}$$

The associated fractional integrals to the aforementioned operator are given in [41].

The Riemann-Liouville and Caputo operators contain power law as part of the kernel. In their applications, these operators have a number of benefits. However, it is important to note that power law behavior is not always present in nature. To address this issue, a new derivative and fractional operator known as Atangana-Baleanu (AB) fractional operator was introduced [42,43]. This operator contains the Mittag-Leffler function as part of its kernel, which can best model several real-world processes. One of the attractive advantages of this operator is that this operator reduces to the original function when the value of the parameter is selected zero. This characteristic distinguishes AB fractional operator over the Caputo-Fabrizio operator, which does not reduce to the original function for specific value of the parameter. Another disguising feature of this operator is that it has not been immensely used in connection to the inequalities and specially to the Hermite-Hadamard-type inequalities. The definition of this operator is expressed as follows.

Definition 4. [42] **AB fractional integral operators:** Let a function $f \in H(\theta_1, \theta_2)$ with $\theta_2 > \theta_1$ and $\eta \in [0, 1]$. Then, the below-mentioned integrals are termed as AB fractional integral operators:

$$ {}^{AB}_{\theta_1}I^{\eta}f(z) = \frac{1-\eta}{B(\eta)}f(z) + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_{\theta_1}^{z}(z-u)^{\eta-1}f(u)\mathrm{d}u. \tag{4}$$

The right-sided AB fractional integral operator is defined in [43] as follows:

$$ {}^{AB}I^{\eta}_{\theta_2}f(z) = \frac{1-\eta}{B(\eta)}f(z) + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_{z}^{\theta_2}(u-z)^{\eta-1}f(u)\mathrm{d}u. \tag{5}$$

Another concept that is related to our results is the majorization concept. Majorization symbolizes a partial order relation between the two tuples. It describes how far off one tuple is from the other tuple or how one tuple's entries are almost equal to those of the other tuple. It is defined as follows:

Definition 5. [44] For any two tuples $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_\omega)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_\omega)$ whose components are arranged in decreasing order as $\vartheta_{[\omega]} \le \vartheta_{[\omega-1]} \le \cdots \le \vartheta_{[1]}$, $\theta_{[\omega]} \le \theta_{[\omega-1]} \le \cdots \le \theta_{[1]}$, then we say that the tuple $\boldsymbol{\vartheta}$ majorizes the tuple $\boldsymbol{\theta}$ or $\boldsymbol{\theta}$ is majorized by $\boldsymbol{\vartheta}$ (in symbols $\boldsymbol{\theta} \prec \boldsymbol{\vartheta}$), if:

$$\sum_{\zeta=1}^{k}\theta_{[\zeta]} \le \sum_{\zeta=1}^{k}\vartheta_{[\zeta]} \quad \text{for } k = 1, 2, \ldots, \omega - 1,$$

and

$$\sum_{\zeta=1}^{\omega}\vartheta_{\zeta} = \sum_{\zeta=1}^{\omega}\theta_{\zeta}.$$

Majorization turns complex optimization-related problems into straightforward ones that are then simple to address [45,46]. Moreover, majorization has also played a crucial role in the theory of matrices and led to the derivation of a number of matrix inequalities and norm inequalities [47]. Nowadays, researchers are using the concept of majorization to construct inequalities connecting various fields. Deng et al. [4] refined the Jensen inequality using the idea of majorization and convexity of the function. In addition to the bounds for quasi-arithmetic and power means, they also provided applications for various divergences. Saeed et al. [48]

presented a refinement of the integral version of the Jensen inequality by incorporating the idea of majorization and convexity. With the aid of the major obtained results, they also established an improvement and refinement of the Hermite-Hadamard and Hölder inequalities, respectively. Furthermore, applications of the major results in the field of information theory have also been discussed.

Research has been carried out continuously to construct inequalities in either discrete or continuous sense [28–30,49]. However, the derivation of compact inequalities containing both continuous and discrete inequalities simultaneously was the need of the hour. To address this issue, Faisal et al. [50–52] used the concept of majorization together with convex theory and various fractional operators and constructed such remarkable inequalities that not only serve as combined form of discrete and continuous inequalities but also produce the previously existing inequalities in the literature, as particular cases. The authors also obtained integral identities that have been further used to construct bounds for the established inequalities.

To obtain our main findings, let us first remind the below-extended form of the Jensen-Mercer inequality established with the aid of majorization.

Theorem 1. [53] *Assume a convex function* $f: I \to \mathbb{R}$, *a tuple* $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_\omega)$, *and* $n \times \omega$ *real matrix* $(x_{i\zeta})$ *such that* $\rho_\zeta, x_{i\zeta} \in I$ *for all* $i = 1, 2, \ldots, n$, $\zeta = 1, 2, \ldots, \omega$, *and* $\sigma_i \geq 0$ *for* $i = 1, 2, \ldots, n$ *with* $\sum_{i=1}^{n}\sigma_i = 1$. *If each row of the matrix* $(x_{i\omega})$ *is majorized by the tuple* $\boldsymbol{\rho}$, *then*

$$f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\sum_{i=1}^{n}\sigma_i x_{i\zeta}\right) \leq \sum_{\zeta=1}^{\omega} f(\rho_\zeta) - \sum_{\zeta=1}^{\omega-1}\sum_{i=1}^{n}\sigma_i f(x_{i\zeta}).$$

This article is categorized into four sections. Section 1 consists of the derivation of new inequalities in terms of the majorization concept and fractional calculus. These results are accompanied by the supporting remarks which show that the specific values of the parameter transform the freshly constructed inequalities to other classical inequalities and unified inequalities for majorized tuples that contain Riemann integrals. Section 2 provides weighted forms of the results obtained in Section 1 by incorporating Lemmas 1 and 2. In order to make the established inequalities more powerful and attractive, we derive integral identities in Section 3. These identities work as a tool that bound the major results when applied along with the Hölder and power mean inequalities and assumptions of convexity of the function. Section 5 contains a detailed conclusion.

2 Main results

Our first major finding in the context of AB fractional integral operators is demonstrated as follows:

Theorem 2. *Assume a convex function* $f: [\nu_\omega, \mu_\omega] \to \mathbb{R}$ *and three tuples* $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_\omega)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_\omega)$, $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_\omega)$ *such that* $\rho_\zeta, \mu_\zeta, \nu_\zeta \in [\nu_\omega, \mu_\omega]$ *for all* $\zeta = 1, 2, \ldots, \omega$ *and* $\eta > 0$. *Also, if* $\boldsymbol{\mu} \prec \boldsymbol{\rho}$ *and* $\boldsymbol{\nu} \prec \boldsymbol{\rho}$, *then*

$$\begin{aligned}
&f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta + \nu_\zeta}{2}\right)\right)\\
&\leq \frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)\right)^\eta}\left[[{}^{AB}_{\nu_\omega}I^\eta f(\mu_\omega) + {}^{AB}I^\eta_{\mu_\omega} f(\nu_\omega)] - \frac{1-\eta}{B(\eta)}[f(\mu_\omega) + f(\nu_\omega)]\right]\\
&\leq \frac{1}{2}[f(\mu_\omega) + f(\nu_\omega)] \leq \sum_{\zeta=1}^{\omega} f(\rho_\zeta) - \frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1} f(\mu_\zeta) + \sum_{\zeta=1}^{\omega-1} f(\nu_\zeta)\right].
\end{aligned} \tag{6}$$

Proof. By the fact that $\boldsymbol{\mu} \prec \boldsymbol{\rho}$ and $\boldsymbol{\nu} \prec \boldsymbol{\rho}$, we have

$$\sum_{\zeta=1}^{\omega} \rho_\zeta = \sum_{\zeta=1}^{\omega} \mu_\zeta \quad \text{and} \quad \sum_{\zeta=1}^{\omega} \rho_\zeta = \sum_{\zeta=1}^{\omega} \nu_\zeta,$$

which implies

$$\sum_{\zeta=1}^{\omega} \rho_\zeta - \sum_{\zeta=1}^{\omega-1} \mu_\zeta = \mu_\omega \quad \text{and} \quad \sum_{\zeta=1}^{\omega} \rho_\zeta - \sum_{\zeta=1}^{\omega-1} \nu_\zeta = \nu_\omega.$$

Now, we start by considering that

$$f\left(\sum_{\zeta=1}^{\omega} \rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta + \nu_\zeta}{2}\right)\right) = f\left(\frac{\mu_\omega + \nu_\omega}{2}\right) = f\left[\frac{\xi\mu_\omega + (1-\xi)\nu_\omega + \xi\nu_\omega + (1-\xi)\mu_\omega}{2}\right]. \tag{7}$$

Using convexity of f in (7), we have

$$f\left(\sum_{\zeta=1}^{\omega} \rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta + \nu_\zeta}{2}\right)\right) \le \frac{1}{2}[f(\xi\mu_\omega + (1-\xi)\nu_\omega) + f(\xi\nu_\omega + (1-\xi)\mu_\omega)]. \tag{8}$$

Multiplying (8) by $\frac{\eta}{B(\eta)\Gamma(\eta)}\xi^{\eta-1}$ and then integrating, we obtain

$$\begin{aligned} &\frac{1}{B(\eta)\Gamma(\eta)} f\left(\sum_{\zeta=1}^{\omega} \rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta + \nu_\zeta}{2}\right)\right) \\ &\le \frac{1}{2}\left[\frac{\eta}{B(\eta)\Gamma(\eta)}\int_0^1 \xi^{\eta-1} f(\xi\mu_\omega + (1-\xi)\nu_\omega)\mathrm{d}\xi + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_0^1 \xi^{\eta-1} f(\xi\nu_\omega + (1-\xi)\mu_\omega)\mathrm{d}\xi\right]. \end{aligned} \tag{9}$$

Using AB fractional integrals in (9), we obtain

$$f\left(\sum_{\zeta=1}^{\omega} \rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta + \nu_\zeta}{2}\right)\right) \le \frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)\right)^\eta}\left[\left[{}^{AB}_{\nu_\omega}I^\eta f(\mu_\omega) + {}^{AB}I^\eta_{\mu_\omega} f(\nu_\omega)\right] - \frac{1-\eta}{B(\eta)}[f(\mu_\omega) + f(\nu_\omega)]\right]. \tag{10}$$

The proof for the first part of inequality (6) is now finished. In order to establish the second inequality, we make use of the convexity of f as shown in the following:

$$f(\xi\mu_\omega + (1-\xi)\nu_\omega) \le \xi f(\mu_\omega) + (1-\xi)f(\nu_\omega), \tag{11}$$

$$f(\xi\nu_\omega + (1-\xi)\mu_\omega) \le \xi f(\nu_\omega) + (1-\xi)f(\mu_\omega). \tag{12}$$

Adding (11) and (12), and using Theorem 1 with $n = 1$ and $\sigma_1 = 1$, we obtain

$$f(\xi\mu_\omega + (1-\xi)\nu_\omega) + f(\xi\nu_\omega + (1-\xi)\mu_\omega) \le f(\mu_\omega) + f(\nu_\omega) \le 2\sum_{\zeta=1}^{\omega} f(\rho_\zeta) - \left[\sum_{\zeta=1}^{\omega-1} f(\mu_\zeta) + \sum_{\zeta=1}^{\omega-1} f(\nu_\zeta)\right]. \tag{13}$$

By integrating the inequality that results from multiplying (13) with $\frac{\eta}{B(\eta)\Gamma(\eta)}\xi^{\eta-1}$, we derive the remaining inequalities in (6). □

Remark 1.

(1) By considering the hypotheses of Theorem 2 and taking $\eta = 1$, we obtain the following inequality given in [50]:

$$\begin{aligned} f\left(\sum_{\zeta=1}^{\omega} \rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta + \nu_\zeta}{2}\right)\right) &\le \frac{1}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)}\int_{\sum_{\zeta=1}^{\omega-1}\mu_\zeta}^{\sum_{\zeta=1}^{\omega-1}\nu_\zeta} f\left(\sum_{\zeta=1}^{\omega} \rho_\zeta - u\right)\mathrm{d}u \\ &\le \sum_{\zeta=1}^{\omega} f(\rho_\zeta) - \frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1} f(\mu_\zeta) + \sum_{\zeta=1}^{\omega-1} f(\nu_\zeta)\right]. \end{aligned} \tag{14}$$

(2) By choosing $\omega = 2$, inequality (6) produces the following inequality given in [54]:

$$f\left(\rho_1+\rho_2-\frac{\mu_1+\nu_1}{2}\right) \le \frac{B(\eta)\Gamma(\eta)}{2(\nu_1-\mu_1)^\eta}\Bigg[[{}_{(\rho_1+\rho_2-\nu_1)}^{AB}I^\eta f(\rho_1+\rho_2-\mu_1) + {}^{AB}I^\eta_{(\rho_1+\rho_2-\mu_1)} f(\rho_1+\rho_2-\nu_1)] - \frac{1-\eta}{B(\eta)}[f(\rho_1+\rho_2-\mu_1)+f(\rho_1+\rho_2-\nu_1)]\Bigg] \le f(\rho_1)+f(\rho_2)-\frac{1}{2}[f(\mu_1)+f(\nu_1)]. \tag{15}$$

(3) If we choose $\omega = 2$ and $\eta = 1$ in (6), then we obtain the succeeding result obtained by Kian and Moslehian [55]:

$$f\left(\rho_1+\rho_2-\frac{\mu_1+\nu_1}{2}\right) \le \frac{1}{\nu_1-\mu_1}\int_{\mu_1}^{\nu_1} f(\rho_1+\rho_2-u)\mathrm{d}u \le f(\rho_1)+f(\rho_2)-\frac{1}{2}[f(\mu_1)+f(\nu_1)]. \tag{16}$$

As previously mentioned, we construct our next finding as follows:

Theorem 3. *Assuming that all the conditions specified in Theorem 2 are satisfied, then*

$$f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right) \le \frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^\eta}\Bigg[\Bigg[{}^{AB}I^\eta_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)} f(\nu_\omega) + {}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}^{AB}I^\eta f(\mu_\omega)\Bigg] - \frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\Bigg] \le \sum_{\zeta=1}^{\omega} f(\rho_\zeta)-\frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1} f(\mu_\zeta)+\sum_{\zeta=1}^{\omega-1} f(\nu_\zeta)\right]. \tag{17}$$

Proof. For $\xi \in [0,1]$, it can be expressed that

$$f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right) = f\left(\frac{\mu_\omega+\nu_\omega}{2}\right) = f\left[\frac{1}{2}\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right) + \left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right]. \tag{18}$$

Using the convexity of f in (18), we establish that

$$f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right) \le \frac{1}{2}\left[f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right) + f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right]. \tag{19}$$

Multiplying (19) by $\frac{\eta}{B(\eta)\Gamma(\eta)}\xi^{\eta-1}$ and then integrating yield

$$\frac{1}{B(\eta)\Gamma(\eta)}f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right) \le \frac{1}{2}\left[\frac{\eta}{B(\eta)\Gamma(\eta)}\int_0^1 \xi^{\eta-1} f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\mathrm{d}\xi + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_0^1 \xi^{\eta-1} f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\mathrm{d}\xi\right]. \tag{20}$$

Using AB fractional integrals in (20), we arrive at

$$\begin{aligned} & f\left(\sum_{\zeta=1}^{\omega}\rho_{\zeta}-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_{\zeta}+\nu_{\zeta}}{2}\right)\right) \\ & \leq \frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_{\zeta}-\mu_{\zeta})\right)^{\eta}}\left[{}^{AB}I^{\eta}_{\left(\sum_{\zeta=1}^{\omega}\rho_{\zeta}-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_{\zeta}+\nu_{\zeta}}{2}\right)\right)}f(\nu_{\omega})+{}^{AB}_{\left(\sum_{\zeta=1}^{\omega}\rho_{\zeta}-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_{\zeta}+\nu_{\zeta}}{2}\right)\right)}I^{\eta}f(\mu_{\omega})\right] \\ & -\frac{(1-\eta)\Gamma(\eta)}{2}[f(\mu_{\omega})+f(\nu_{\omega})]. \end{aligned} \tag{21}$$

Thus, the first part of Inequality (17) is completed. In order to demonstrate the second inequality of (17), we employ Theorem 1 with $\sigma_1=\frac{\xi}{2}$, $\sigma_2=\frac{2-\xi}{2}$ and $n=2$ in the following manner:

$$f\left(\sum_{\zeta=1}^{\omega}\rho_{\zeta}-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_{\zeta}+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_{\zeta}\right)\right)\leq\sum_{\zeta=1}^{\omega}f(\rho_{\zeta})-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}f(\mu_{\zeta})+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}f(\nu_{\zeta})\right) \tag{22}$$

and

$$f\left(\sum_{\zeta=1}^{\omega}\rho_{\zeta}-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_{\zeta}+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_{\zeta}\right)\right)\leq\sum_{\zeta=1}^{\omega}f(\rho_{\zeta})-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}f(\nu_{\zeta})+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}f(\mu_{\zeta})\right). \tag{23}$$

Adding (22) and (23), we obtain

$$\begin{aligned} & f\left(\sum_{\zeta=1}^{\omega}\rho_{\zeta}-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_{\zeta}+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_{\zeta}\right)\right)+f\left(\sum_{\zeta=1}^{\omega}\rho_{\zeta}-\left(\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_{\zeta}+\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_{\zeta}\right)\right) \\ & \leq 2\sum_{\zeta=1}^{\omega}f(\rho_{\zeta})-\left(\sum_{\zeta=1}^{\omega-1}f(\mu_{\zeta})+\sum_{\zeta=1}^{\omega-1}f(\nu_{\zeta})\right). \end{aligned} \tag{24}$$

By multiplying (24) by $\frac{\eta}{B(\eta)\Gamma(\eta)}\xi^{\eta-1}$ and then integrating the resulting inequality, we obtain the second inequality of (17). □

Remark 2. Inequality (17) produces the below result obtained by Kian and Moslehian given in [55] by choosing $\omega=2$ and $\eta=1$:

$$f\left(\rho_1+\rho_2-\frac{\mu_1+\nu_1}{2}\right)\leq\frac{1}{\nu_1-\mu_1}\int_{\mu_1}^{\nu_1}f(\rho_1+\rho_2-u)\mathrm{d}u\leq f(\rho_1)+f(\rho_2)-\frac{1}{2}[f(\mu_1)+f(\nu_1)]. \tag{25}$$

3 Weighted forms of the main results

This section of the manuscript is devoted to the derivation of weighted forms of our major results. We obtain these weighted forms by taking into account certain monotonic tuples.

To obtain our new results in weighted forms, we employ the following lemmas [50].

Lemma 1. *Assume a convex function $f: I\to\mathbb{R}$, two tuples $\boldsymbol{\rho}=(\rho_1,\ldots,\rho_{\omega})$, $\mathbf{r}=(r_1,\ldots,r_{\omega})$, and $n\times\omega$ real matrix $(x_{i\zeta})$ such that $\rho_{\zeta}, x_{i\zeta}\in I$, $r_{\zeta}\geq 0$ with $r_{\omega}\neq 0$, $\varepsilon=\frac{1}{r_{\omega}}$ for all $\zeta=1,\ldots,\omega$, $i=1,2,\ldots,n$ and $\sigma_i\geq 0$ for $i=1,2,\ldots,n$ with $\sum_{i=1}^{n}\sigma_i=1$. If for every $i=1,2,\ldots,n$, the tuple $(x_{i1},\ldots,x_{i\omega})$ is decreasing and*

$$\sum_{\zeta=1}^{k}r_{\zeta}x_{i\zeta}\leq\sum_{\zeta=1}^{k}r_{\zeta}\rho_{\zeta}\quad\text{for}\quad k=1,2,\ldots,\omega-1,\qquad\sum_{\zeta=1}^{\omega}r_{\zeta}\rho_{\zeta}=\sum_{\zeta=1}^{\omega}r_{\zeta}x_{i\zeta},$$

then

$$f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\sum_{i=1}^{n}\varepsilon\sigma_i r_\zeta x_{i\zeta}\right)\leq\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta f(\rho_\zeta)-\sum_{\zeta=1}^{\omega-1}\sum_{i=1}^{n}\varepsilon\sigma_i r_\zeta f(x_{i\zeta}).$$

Lemma 2. *Assume a convex function $f:I\to\mathbb{R}$, two tuples $\boldsymbol{\rho}=(\rho_1,\ldots,\rho_\omega)$, $\mathbf{r}=(r_1,\ldots,r_\omega)$, and $n\times\omega$ real matrix $(x_{i\zeta})$ such that $\rho_\zeta, x_{i\zeta}\in I$, $r_\zeta\geq 0$ with $r_\omega\neq 0$, $\varepsilon=\frac{1}{r_\omega}$ for all $\zeta=1,\ldots,\omega$, $i=1,2,\ldots,n$ and $\sigma_i\geq 0$ for $i=1,2,\ldots,n$ with $\sum_{i=1}^{n}\sigma_i=1$. If $x_{i\zeta}$ and $(\rho_\zeta-x_{i\zeta})$ are showing the same monotonic behavior for each $i=1,\ldots,n$, and*

$$\sum_{\zeta=1}^{\omega}r_\zeta\rho_\zeta=\sum_{\zeta=1}^{\omega}r_\zeta x_{i\zeta},$$

then

$$f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\sum_{i=1}^{n}\varepsilon\sigma_i r_\zeta x_{i\zeta}\right)\leq\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta f(\rho_\zeta)-\sum_{\zeta=1}^{\omega-1}\sum_{i=1}^{n}\varepsilon\sigma_i r_\zeta f(x_{i\zeta}).$$

The following result is based on Lemma 1.

Theorem 4. *Assume a convex function $f:[\nu_\omega,\mu_\omega]\to\mathbb{R}$ and four tuples $\boldsymbol{\rho}=(\rho_1,\rho_2,\ldots,\rho_\omega)$, $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_\omega)$, $\boldsymbol{\nu}=(\nu_1,\nu_2,\ldots,\nu_\omega)$, and $\mathbf{r}=(r_1,r_2,\ldots,r_\omega)$ such that $\rho_\zeta,\mu_\zeta,\nu_\zeta\in[\nu_\omega,\mu_\omega]$, $r_\zeta\geq 0$ with $r_\omega\neq 0$ for all $\zeta=1,2,\ldots,\omega$ and $\varepsilon=\frac{1}{r_\omega}$, $\eta>0$. Also, if the tuples $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are decreasing and*

$$\sum_{\zeta=1}^{k}r_\zeta\mu_\zeta\leq\sum_{\zeta=1}^{k}r_\zeta\rho_\zeta,\ \sum_{\zeta=1}^{k}r_\zeta\nu_\zeta\leq\sum_{\zeta=1}^{k}r_\zeta\rho_\zeta,\quad \text{for } k=1,\ldots,\omega-1,$$

$$\sum_{\zeta=1}^{\omega}r_\zeta\rho_\zeta=\sum_{\zeta=1}^{\omega}r_\zeta\mu_\zeta,\ \sum_{\zeta=1}^{\omega}r_\zeta\rho_\zeta=\sum_{\zeta=1}^{\omega}r_\zeta\nu_\zeta,$$

then

$$\begin{aligned}&f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\varepsilon\sum_{\zeta=1}^{\omega-1}\left(\frac{r_\zeta\mu_\zeta+r_\zeta\nu_\zeta}{2}\right)\right)\\&\leq\frac{B(\eta)\Gamma(\eta)}{2(\mu_\omega-\nu_\omega)^\eta}\left[[{}^{AB}_{\nu_\omega}I^{\eta}f(\mu_\omega)+{}^{AB}I^{\eta}_{\mu_\omega}f(\nu_\omega)]-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]\\&\leq\frac{1}{2}[f(\mu_\omega)+f(\nu_\omega)]\\&\leq\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta f(\rho_\zeta)-\frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\mu_\zeta)+\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\nu_\zeta)\right].\end{aligned}\tag{26}$$

Proof. By taking

$$\sum_{\zeta=1}^{\omega}r_\zeta\rho_\zeta=\sum_{\zeta=1}^{\omega}r_\zeta\mu_\zeta\quad\text{and}\quad\sum_{\zeta=1}^{\omega}r_\zeta\rho_\zeta=\sum_{\zeta=1}^{\omega}r_\zeta\nu_\zeta,$$

we arrive at

$$\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta\mu_\zeta=\mu_\omega\quad\text{and}\quad\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta\nu_\zeta=\nu_\omega.$$

Now, for $\xi\in[0,1]$, we may write that

$$f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\varepsilon\sum_{\zeta=1}^{\omega-1}r_\zeta\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)=f\left(\frac{\mu_\omega+\nu_\omega}{2}\right)=f\left[\frac{\xi\mu_\omega+(1-\xi)\nu_\omega+\xi\nu_\omega+(1-\xi)\mu_\omega}{2}\right]. \tag{27}$$

Using the convexity of f in (27), we obtain

$$f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\varepsilon\sum_{\zeta=1}^{\omega-1}r_\zeta\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\leq\frac{1}{2}[f(\xi\mu_\omega+(1-\xi)\nu_\omega)+f(\xi\nu_\omega+(1-\xi)\mu_\omega)]. \tag{28}$$

Multiplying (28) by $\frac{\eta}{B(\eta)\Gamma(\eta)}\xi^{\eta-1}$ and then integrating yield

$$\begin{aligned}&\frac{1}{B(\eta)\Gamma(\eta)}f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\varepsilon\sum_{\zeta=1}^{\omega-1}r_\zeta\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\\&\leq\frac{1}{2}\left[\frac{\eta}{B(\eta)\Gamma(\eta)}\int_0^1\xi^{\eta-1}f(\xi\mu_\omega+(1-\xi)\nu_\omega)\mathrm{d}\xi+\frac{\eta}{B(\eta)\Gamma(\eta)}\int_0^1\xi^{\eta-1}f(\xi\nu_\omega+(1-\xi)\mu_\omega)\mathrm{d}\xi\right].\end{aligned} \tag{29}$$

Using AB fractional integrals in (29), we obtain

$$f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\varepsilon\sum_{\zeta=1}^{\omega-1}r_\zeta\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\leq\frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^\eta}\left[\left[{}^{AB}_{\nu_\omega}I^{\eta}f(\mu_\omega)+{}^{AB}I^{\eta}_{\mu_\omega}f(\nu_\omega)\right]-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]. \tag{30}$$

Hence, we have successfully accomplished the first part of (26). To obtain the second part, we use the convexity of f to express that

$$f(\xi\mu_\omega+(1-\xi)\nu_\omega)\leq\xi f(\mu_\omega)+(1-\xi)f(\nu_\omega), \tag{31}$$

$$f(\xi\nu_\omega+(1-\xi)\mu_\omega)\leq\xi f(\nu_\omega)+(1-\xi)f(\mu_\omega). \tag{32}$$

Adding (31) and (32) and then using Lemma 1 for $n=2$, $\sigma_1=\xi$, and $\sigma_2=1-\xi$, we obtain

$$\begin{aligned}f(\xi\mu_\omega+(1-\xi)\nu_\omega)+f(\xi\nu_\omega+(1-\xi)\mu_\omega)&\leq f(\mu_\omega)+f(\nu_\omega)\\&\leq 2\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta f(\rho_\zeta)-\left[\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\mu_\zeta)+\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\nu_\zeta)\right].\end{aligned} \tag{33}$$

By integrating the inequality that results from multiplying (33) with $\frac{\eta}{B(\eta)\Gamma(\eta)}\xi^{\eta-1}$, we obtain the remaining inequalities in (26). □

Remark 3. By setting $\eta=1$, Inequality (26) transforms into the underlying inequality given in [50]:

$$\begin{aligned}f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\varepsilon\sum_{\zeta=1}^{\omega-1}\left(\frac{r_\zeta\mu_\zeta+r_\zeta\nu_\zeta}{2}\right)\right)&\leq\frac{1}{\sum_{\zeta=1}^{\omega-1}(\varepsilon r_\zeta\nu_\zeta-\varepsilon r_\zeta\mu_\zeta)}\int_{\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta\mu_\zeta}^{\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta\nu_\zeta}f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-u\right)\mathrm{d}u\\&\leq\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta f(\rho_\zeta)-\frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\mu_\zeta)+\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\nu_\zeta)\right].\end{aligned} \tag{34}$$

Lemma 2 is used to establish the following result.

Theorem 5. *Assume a convex function* $f:[\nu_\omega,\mu_\omega]\to\mathbb{R}$ *and four tuples* $\boldsymbol{\rho}=(\rho_1,\rho_2,\ldots,\rho_\omega)$, $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_\omega)$, $\boldsymbol{\nu}=(\nu_1,\nu_2,\ldots,\nu_\omega)$, *and* $\mathbf{r}=(r_1,r_2,\ldots,r_\omega)$ *such that* $\rho_\zeta,\mu_\zeta,\nu_\zeta\in[\nu_\omega,\mu_\omega]$, $r_\zeta\geq0$ *with* $r_\omega\neq0$ *for all* $\zeta=1,2,\ldots,\omega$, *and* $\varepsilon=\frac{1}{r_\omega}$, $\eta>0$. *If* $\boldsymbol{\rho}-\boldsymbol{\mu}$, $\boldsymbol{\mu}$, $\boldsymbol{\rho}-\boldsymbol{\nu}$, *and* $\boldsymbol{\nu}$ *are monotonically showing the similar behavior and*

$$\sum_{\zeta=1}^{\omega}r_\zeta\rho_\zeta=\sum_{\zeta=1}^{\omega}r_\zeta\mu_\zeta,\ \sum_{\zeta=1}^{\omega}r_\zeta\rho_\zeta=\sum_{\zeta=1}^{\omega}r_\zeta\nu_\zeta,$$

then

$$\begin{aligned}
&f\left(\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta\rho_\zeta-\varepsilon\sum_{\zeta=1}^{\omega-1}\left(\frac{r_\zeta\mu_\zeta+r_\zeta\nu_\zeta}{2}\right)\right)\\
&\le\frac{B(\eta)\Gamma(\eta)}{2(\mu_\omega-\nu_\omega)^\eta}\left[\left[{}^{AB}_{\nu_\omega}I^{\eta}f(\mu_\omega)+{}^{AB}I^{\eta}_{\mu_\omega}f(\nu_\omega)\right]-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]\\
&\le\frac{1}{2}[f(\mu_\omega)+f(\nu_\omega)]\\
&\le\sum_{\zeta=1}^{\omega}\varepsilon r_\zeta f(\rho_\zeta)-\frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\mu_\zeta)+\sum_{\zeta=1}^{\omega-1}\varepsilon r_\zeta f(\nu_\zeta)\right].
\end{aligned}\tag{35}$$

Proof. We prove (35) by using Lemma 2 and similar steps taken in the proof of Theorem 4. Thus, the proof is complete. □

Remark 4. Theorems 4 and 5 serve as weighted versions of Theorem 2, and corresponding weighted forms for Theorem 3 can be obtained in the similar manner.

4 Derivation of new integral identities and bounds for the main results

Integral identities are the tools that lead us to more powerful inequalities when combined with some assumptions. In this section, first, we derive the identities for a differentiable function and then using these identities, we construct bounds for the major inequalities.

Lemma 3. *Assume a differentiable function* f *on* $[\nu_\omega,\mu_\omega]$ *and three tuples* $\boldsymbol{\rho}=(\rho_1,\rho_2,\ldots,\rho_\omega)$ *and* $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_\omega)$, $\boldsymbol{\nu}=(\nu_1,\nu_2,\ldots,\nu_\omega)$ *such that* $\rho_\zeta,\mu_\zeta,\nu_\zeta\in[\nu_\omega,\mu_\omega]$ *for all* $\zeta=1,2,\ldots,\omega$, $\eta>0$ *and* $\xi\in[0,1]$. *If* $f'\in L(I)$, *then*

$$\begin{aligned}
&\frac{1}{2}[f(\mu_\omega)+f(\nu_\omega)]-\frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^\eta}\left[\left[{}^{AB}_{\nu_\omega}I^{\eta}f(\mu_\omega)+{}^{AB}I^{\eta}_{\mu_\omega}f(\nu_\omega)\right]-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]\\
&=\frac{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)}{2}\int_0^1(\xi^\eta-(1-\xi)^\eta)f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta+(1-\xi)\nu_\zeta)\right)\mathrm{d}\xi.
\end{aligned}\tag{36}$$

Proof. In order to demonstrate our desired result, we take into consideration that

$$\begin{aligned}
I&=\int_0^1(\xi^\eta-(1-\xi)^\eta)f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta+(1-\xi)\nu_\zeta)\right)\mathrm{d}\xi\\
&=\int_0^1\xi^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta+(1-\xi)\nu_\zeta)\right)\mathrm{d}\xi-\int_0^1(1-\xi)^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta+(1-\xi)\nu_\zeta)\right)\mathrm{d}\xi\\
&=I_1-I_2.
\end{aligned}\tag{37}$$

Here,

$$\begin{aligned}I_1 &= \int_0^1 \xi^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\mathrm{d}\xi \\ &= \left.\frac{\xi^\eta f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)}\right|_0^1 - \frac{\eta}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)} \\ &\quad\times \int_0^1 \xi^{\eta-1} f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\mathrm{d}\xi \\ &= \frac{f(\mu_\omega)}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)} - \frac{\eta}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)}\int_0^1 \xi^{\eta-1} f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\mathrm{d}\xi.\end{aligned} \tag{38}$$

Likewise,

$$\begin{aligned}I_2 &= \int_0^1 (1-\xi)^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\mathrm{d}\xi \\ &= \left.\frac{(1-\xi)^\eta f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)}\right|_0^1 + \frac{\eta}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)} \\ &\quad\times \int_0^1 (1-\xi)^{\eta-1} f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\mathrm{d}\xi \\ &= -\frac{f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)} + \frac{\eta}{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)}\int_0^1 (1-\xi)^{\eta-1} f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\mathrm{d}\xi.\end{aligned} \tag{39}$$

By inserting (38) and (39) into (37) and then using the definition of AB fractional integrals and multiplying both sides by $\frac{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)}{2}$, we obtain (36). □

Remark 5.

(1) For $\omega = 2$, Equality (36) transforms into the following equality, which has been proved in [54]:

$$\begin{aligned}&\frac{1}{2}[f(\mu_1) + f(\nu_1)] - \frac{B(\eta)\Gamma(\eta)}{2(\nu_1 - \mu_1)^\eta}\Big[\left[{}_{(\rho_1+\rho_2-\nu_1)}^{AB}I^\eta f(\rho_1 + \rho_2 - \mu_1) + {}^{AB}I^\eta_{(\rho_1+\rho_2-\mu_1)} f(\rho_1 + \rho_2 - \nu_1)\right] \\ &\quad - \frac{1-\eta}{B(\eta)}[f(\rho_1 + \rho_2 - \mu_1) + f(\rho_1 + \rho_2 - \nu_1)]\Big] \\ &= \frac{(\nu_1 - \mu_1)}{2}\int_0^1 (\xi^\eta - (1-\xi)^\eta) f'(\rho_1 + \rho_2 - (\xi\mu_1 + (1-\xi)\nu_1))\mathrm{d}\xi.\end{aligned} \tag{40}$$

(2) By taking $\eta = 1$, $\omega = 2$, $\mu_1 = \rho_1$, and $\nu_1 = \rho_2$ in (36), we obtain the following inequality given in [14]:

$$\frac{f(\rho_1) + f(\rho_2)}{2} - \frac{1}{\rho_2 - \rho_1}\int_{\rho_1}^{\rho_2} f(u)\mathrm{d}u = \frac{\rho_2 - \rho_1}{2}\int_0^1 (2\xi - 1) f'(\xi\rho_2 + (1-\xi)\rho_1)\mathrm{d}\xi. \tag{41}$$

Lemma 4. *Assuming that all the conditions specified in Lemma* 3*are satisfied, then*

$$\frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^{\eta}}\left[\left[{}^{AB}I^{\eta}_{\left[\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right]}f(\nu_\omega)+{}_{\left[\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right]}{}^{AB}I^{\eta}f(\mu_\omega)\right]-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]$$
$$-f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right) \tag{42}$$
$$=\frac{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)}{4}\int_0^1\xi^{\eta}\left[f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)-f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right]\mathrm{d}\xi.$$

Proof. It is very simple to derive Equality (42) by applying similar steps as presented in the proof of Lemma 3. □

Now, we present the following result based on Lemma 3.

Theorem 6. *Assume a differentiable function* f *on* $[\nu_\omega,\mu_\omega]$ *and three tuples* $\boldsymbol{\rho}=(\rho_1,\rho_2,\ldots,\rho_\omega)$, $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_\omega)$, *and* $\boldsymbol{\nu}=(\nu_1,\nu_2,\ldots,\nu_\omega)$ *such that* $\rho_\zeta,\mu_\zeta,\nu_\zeta\in[\nu_\omega,\mu_\omega]$ *for all* $\zeta=1,2,\ldots,\omega$ and $\eta>0$. If $|f'|$ *is convex,* $\boldsymbol{\mu}\prec\boldsymbol{\rho}$ *and* $\boldsymbol{\nu}\prec\boldsymbol{\rho}$, *then*

$$\left|\frac{1}{2}[f(\mu_\omega)+f(\nu_\omega)]-\frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^{\eta}}\left[[{}_{\nu_\omega}^{AB}I^{\eta}f(\mu_\omega)+{}^{AB}I^{\eta}_{\mu_\omega}f(\nu_\omega)]-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]\right|$$
$$\le\frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|}{\eta+1}\left(1-\frac{1}{2^{\eta}}\right)\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|+\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right]\right]. \tag{43}$$

Proof. Using Lemma 3, we can write

$$\left|\frac{1}{2}[f(\mu_\omega)+f(\nu_\omega)]-\frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^{\eta}}\left[[{}_{\nu_\omega}^{AB}I^{\eta}f(\mu_\omega)+{}^{AB}I^{\eta}_{\mu_\omega}f(\nu_\omega)]+\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]\right|$$
$$=\left|\frac{1}{2}\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\int_0^1(\xi^{\eta}-(1-\xi)^{\eta})f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta+(1-\xi)\nu_\zeta)\right)\mathrm{d}\xi\right| \tag{44}$$
$$\le\frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\int_0^1|(\xi^{\eta}-(1-\xi)^{\eta})|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta+(1-\xi)\nu_\zeta)\right)\right|\mathrm{d}\xi.$$

The convexity of $|f'|$ allows us to obtain the following by applying Theorem 1 with $\sigma_1=\xi$, $n=2$, and $\sigma_2=1-\xi$ in (44):

$$\le\frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\int_0^1|(\xi^{\eta}-(1-\xi)^{\eta})|\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|+(1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right)\right]\mathrm{d}\xi,$$
$$=\frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\int_0^{\frac{1}{2}}((1-\xi)^{\eta}-\xi^{\eta})\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|+(1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right)\right]\mathrm{d}\xi\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}(\xi^{\eta}-(1-\xi)^{\eta})\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|+(1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right)\right]\mathrm{d}\xi\right], \tag{45}$$
$$=\frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|(B_1+B_2).$$

Note that we have

$$\begin{aligned} B_1 &= \int_0^{\frac{1}{2}}((1-\xi)^\eta - \xi^\eta)\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)| - \left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)| + (1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right)\right]\mathrm{d}\xi \\ &= \sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|\left(\frac{1}{\eta+1} - \frac{\frac{1}{2^\eta}}{\eta+1}\right) - \left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|\left(\frac{1}{(\eta+1)(\eta+2)} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right) + \sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\left(\frac{1}{\eta+2} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right)\right] \end{aligned} \tag{46}$$

and

$$\begin{aligned} B_2 &= \int_{\frac{1}{2}}^{1}(\xi^\eta - (1-\xi)^\eta)\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)| - \left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)| + (1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right)\right]\mathrm{d}\xi \\ &= \sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|\left(\frac{1}{\eta+1} - \frac{\frac{1}{2^\eta}}{\eta+1}\right) - \left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|\left(\frac{1}{\eta+2} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right) + \sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\left(\frac{1}{(\eta+1)(\eta+2)} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right)\right]. \end{aligned} \tag{47}$$

Substituting (46) and (47) into (45), we obtain (43). □

Another result is obtained as follows:

Theorem 7. *Assume a differentiable function f on $[\nu_\omega, \mu_\omega]$ and three tuples $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_\omega)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_\omega)$, and $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_\omega)$ such that $\rho_\zeta, \mu_\zeta, \nu_\zeta \in [\nu_\omega, \mu_\omega]$ for all $\zeta = 1, 2, \ldots, \omega$ and $\eta > 0$. If $|f'|^q$ $(q > 1)$ is convex, $\boldsymbol{\mu} \prec \boldsymbol{\rho}$ and $\boldsymbol{\nu} \prec \boldsymbol{\rho}$, then*

$$\begin{aligned} &\left|\frac{1}{2}[f(\mu_\omega) + f(\nu_\omega)] - \frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)\right)^\eta}\left[[{}^{AB}_{\nu_\omega}I^\eta f(\mu_\omega) + {}^{AB}I^\eta_{\mu_\omega}f(\nu_\omega)] - \frac{(1-\eta)}{B(\eta)}[f(\mu_\omega) + f(\nu_\omega)]\right]\right| \\ &\quad \le \frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|}{\eta+1}\left(1 - \frac{1}{2^\eta}\right)\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + \sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right]\right]^{\frac{1}{q}}. \end{aligned} \tag{48}$$

Proof. Using Lemma 3, we have

$$\begin{aligned} &\left|\frac{1}{2}[f(\mu_\omega) + f(\nu_\omega)] - \frac{B(\eta)\Gamma(\eta)}{2\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)\right)^\eta}\left[[{}^{AB}_{\nu_\omega}I^\eta f(\mu_\omega) + {}^{AB}I^\eta_{\mu_\omega}f(\nu_\omega)] - \frac{(1-\eta)}{B(\eta)}[f(\mu_\omega) + f(\nu_\omega)]\right]\right| \\ &= \left|\frac{1}{2}\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)\int_0^1(\xi^\eta - (1-\xi)^\eta)f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\mathrm{d}\xi\right| \\ &\le \frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|\int_0^1|\xi^\eta - (1-\xi)^\eta|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\right|\mathrm{d}\xi. \end{aligned}$$

The utilization of the power mean inequality to the aforementioned integral yields

$$\begin{aligned} &\le \frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|\left(\int_0^1|\xi^\eta - (1-\xi)^\eta|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\left(\int_0^1|\xi^\eta - (1-\xi)^\eta|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}} \\ &= \frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|\left(\int_0^{\frac{1}{2}}((1-\xi)^\eta - \xi^\eta)\mathrm{d}\xi + \int_{\frac{1}{2}}^1(\xi^\eta - (1-\xi)^\eta)\mathrm{d}\xi\right)^{1-\frac{1}{q}}\left(\int_0^1|\xi^\eta - (1-\xi)^\eta|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}(\xi\mu_\zeta + (1-\xi)\nu_\zeta)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}}. \end{aligned} \tag{49}$$

The convexity of $|f'|^q$ allows us to obtain the following by applying Theorem 1 with $\sigma_1 = \xi$, $n = 2$, and $\sigma_2 = 1 - \xi$ in (49):

$$\begin{aligned}
&= \frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|\left(\int_0^{\frac{1}{2}}((1-\xi)^\eta - \xi^\eta)\mathrm{d}\xi + \int_{\frac{1}{2}}^1(\xi^\eta - (1-\xi)^\eta)\mathrm{d}\xi\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1|\xi^\eta - (1-\xi)^\eta|\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + (1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)\mathrm{d}\xi\right)^{\frac{1}{q}}\\
&= \frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|\left(\int_0^{\frac{1}{2}}((1-\xi)^\eta - \xi^\eta)\mathrm{d}\xi + \int_{\frac{1}{2}}^1(\xi^\eta - (1-\xi)^\eta)\mathrm{d}\xi\right)^{1-\frac{1}{q}}\\
&\quad\times\left[\int_0^{\frac{1}{2}}((1-\xi)^\eta - \xi^\eta)\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + (1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)\mathrm{d}\xi\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1(\xi^\eta - (1-\xi)^\eta)\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + (1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)\mathrm{d}\xi\right]^{\frac{1}{q}}\\
&= \frac{1}{2}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|\left(\frac{2\left(1-\frac{1}{2^\eta}\right)}{\eta+1}\right)^{1-\frac{1}{q}}(C_1 + C_2)^{\frac{1}{q}}.
\end{aligned}\tag{50}$$

Note that we have

$$\begin{aligned}
C_1 &= \int_0^{\frac{1}{2}}((1-\xi)^\eta - \xi^\eta)\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + (1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)\mathrm{d}\xi\\
&= \sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q\left(\frac{1}{\eta+1} - \frac{\frac{1}{2^\eta}}{\eta+1}\right) - \left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\left(\frac{1}{(\eta+1)(\eta+2)} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right)\right.\\
&\quad\left.+\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\left(\frac{1}{\eta+2} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right)\right]
\end{aligned}\tag{51}$$

and

$$\begin{aligned}
C_2 &= \int_{\frac{1}{2}}^1(\xi^\eta - (1-\xi)^\eta)\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \left(\xi\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + (1-\xi)\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)\mathrm{d}\xi\\
&= \sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q\left(\frac{1}{\eta+1} - \frac{\frac{1}{2^\eta}}{\eta+1}\right) - \left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\left(\frac{1}{\eta+2} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right)\right.\\
&\quad\left.+\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\left(\frac{1}{(\eta+1)(\eta+2)} - \frac{\frac{1}{2^{\eta+1}}}{\eta+1}\right)\right].
\end{aligned}\tag{52}$$

By inserting (51) and (52) into (50), we obtain (48). □

By considering Lemma 4, we establish the following results.

Theorem 8. *Assuming that all the conditions specified in Theorem* 6 *are satisfied, then*

$$\left|\frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^{\eta}}\left[\left[{}^{AB}I^{\eta}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}f(\nu_\omega)+{}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}{}^{AB}I^{\eta}f(\mu_\omega)\right]\right.\right.$$
$$\left.\left.-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]-f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\right| \tag{53}$$
$$\leq\frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|}{2(\eta+1)}\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|+\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right]\right].$$

Proof. Using Lemma 4, we have

$$\left|\frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^{\eta}}\left[\left[{}^{AB}I^{\eta}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}f(\nu_\omega)+{}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}{}^{AB}I^{\eta}f(\mu_\omega)\right]\right.\right.$$
$$\left.\left.-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]-f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\right|$$
$$=\left|\frac{1}{4}\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\left[\int_0^1\xi^{\eta}f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\mathrm{d}\xi-\int_0^1\xi^{\eta}f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\mathrm{d}\xi\right]\right|$$
$$\leq\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\int_0^1|\xi^{\eta}|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|\mathrm{d}\xi\right. \tag{54}$$
$$\left.+\int_0^1|\xi^{\eta}|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|\mathrm{d}\xi\right]$$
$$=\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\int_0^1\xi^{\eta}\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|\mathrm{d}\xi+\int_0^1\xi^{\eta}\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|\mathrm{d}\xi\right]$$

.

Using Theorem 1 for $n=2$, $\sigma_1=\frac{2-\xi}{2}$, and $\sigma_2=\frac{\xi}{2}$ in (54), we obtain

$$\leq\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\int_0^1\xi^{\eta}\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right)\right)\mathrm{d}\xi\right.$$
$$\left.+\int_0^1\xi^{\eta}\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|\right)\right)\mathrm{d}\xi\right]$$
$$=\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\frac{\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|}{\eta+1}-\frac{\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|}{\eta+1}+\frac{\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|}{2(\eta+2)}-\frac{\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|}{2(\eta+2)}\right.$$
$$\left.+\frac{\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|}{\eta+1}-\frac{\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|}{\eta+1}+\frac{\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|}{2(\eta+2)}-\frac{\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|}{2(\eta+2)}\right]$$
$$=\frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|}{2(\eta+1)}\left[\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|-\frac{1}{2}\left[\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|+\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|\right]\right].$$

Henceforth, the proof is complete. □

Theorem 9. *Assume a differentiable function* f *on* $[\nu_\omega, \mu_\omega]$ *and three tuples* $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_\omega)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_\omega)$, *and* $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_\omega)$ *such that* $\rho_\zeta, \mu_\zeta, \nu_\zeta \in [\nu_\omega, \mu_\omega]$ *for all* $\zeta = 1, 2, \ldots, \omega$ *and* $\eta > 0$. *If* $|f'|^q$ *is convex,* $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\boldsymbol{\mu} \prec \boldsymbol{\rho}$, *and* $\boldsymbol{\nu} \prec \boldsymbol{\rho}$, *then*

$$\begin{aligned}
&\left|\frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)\right)^\eta}\left[\left[{}^{AB}I^\eta_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)} f(\nu_\omega) + {}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}{}^{AB}I^\eta f(\mu_\omega)\right]\right.\right.\\
&\left.\left.\quad - \frac{(1-\eta)}{B(\eta)}[f(\mu_\omega) + f(\nu_\omega)]\right] - f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\right|\\
&\le \frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|}{4(\eta p + 1)^{\frac{1}{p}}}\left[\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \frac{1}{4}\left(3\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + \sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right.\\
&\left.\quad + \left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \frac{1}{4}\left(3\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q + \sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{55}$$

Proof. Using Lemma 4, we can write

$$\begin{aligned}
&\left|\frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)\right)^\eta}\left[\left[{}^{AB}I^\eta_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)} f(\nu_\omega) + {}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}{}^{AB}I^\eta f(\mu_\omega)\right]\right.\right.\\
&\left.\left.\quad - \frac{(1-\eta)}{B(\eta)}[f(\mu_\omega) + f(\nu_\omega)]\right] - f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\right|\\
&= \left|\frac{\sum_{\zeta=1}^{\omega-1}(\nu_\zeta - \mu_\zeta)}{4}\left[\int_0^1 \xi^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta + \frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\mathrm{d}\xi\right.\right.\\
&\left.\left.\quad - \int_0^1 \xi^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta + \frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\mathrm{d}\xi\right]\right|\\
&\le \frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|}{4}\left[\int_0^1\left|\xi^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta + \frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|\mathrm{d}\xi\right.\\
&\left.\quad + \int_0^1\left|\xi^\eta f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta + \frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|\mathrm{d}\xi\right]\\
&\le \frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|}{4}\left[\int_0^1|\xi^\eta|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta + \frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|\mathrm{d}\xi\right.\\
&\left.\quad + \int_0^1|\xi^\eta|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta - \left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta + \frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|\mathrm{d}\xi\right].
\end{aligned}$$

Using Hölder's inequality to the aforementioned integral, yields

$$\begin{aligned}
&\leq \frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\left(\int_0^1|\xi^\eta|^p\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1|\xi^\eta|^p\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}}\right]\\
&=\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\left(\int_0^1\xi^{\eta p}\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\xi^{\eta p}\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}}\right]\\
&=\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left(\int_0^1\xi^{\eta p}\mathrm{d}\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|^q\mathrm{d}\xi\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{56}$$

The convexity of $|f'|^q$ allows us to obtain the following by applying Theorem 1 with $\sigma_1=\frac{2-\xi}{2}$, $n=2$, and $\sigma_2=\frac{\xi}{2}$ in (56):

$$\begin{aligned}
&=\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left(\frac{1}{\eta p+1}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)\mathrm{d}\xi\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\right)\right)\mathrm{d}\xi\right)^{\frac{1}{q}}\right]\\
&=\frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|}{4(\eta p+1)^{\frac{1}{p}}}\left[\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\frac{1}{4}\left(3\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q+\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\frac{1}{4}\left(3\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q+\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Hence, the proof is complete. □

Theorem 10. *Assuming that all the conditions specified in Theorem 7are satisfied, then*

$$\begin{aligned}
&\left|\frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^\eta}\left[\left[{}^{AB}I^{\eta}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}f(\nu_\omega)+{}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}^{AB}I^{\eta}f(\mu_\omega)\right]\right.\right.\\
&\quad\left.\left.-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]-f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\right|\\
&\leq\frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|}{4(\eta+1)^{1-\frac{1}{q}}}\left[\left(\frac{1}{\eta+1}\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\left(\frac{\eta+3}{2(\eta+1)(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q+\frac{1}{2(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{1}{\eta+1}\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\left(\frac{\eta+3}{2(\eta+1)(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q+\frac{1}{2(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{57}$$

Proof. Using Lemma 4, we have

$$\left|\frac{2^{\eta-1}B(\eta)\Gamma(\eta)}{\left(\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\right)^{\eta}}\left[\left[{}^{AB}I^{\eta}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}f(\nu_\omega)+{}_{\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)}^{AB}I^{\eta}f(\mu_\omega)\right]\right.\right.$$
$$\left.\left.-\frac{(1-\eta)}{B(\eta)}[f(\mu_\omega)+f(\nu_\omega)]\right]-f\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\sum_{\zeta=1}^{\omega-1}\left(\frac{\mu_\zeta+\nu_\zeta}{2}\right)\right)\right|$$
$$=\left|\frac{1}{4}\sum_{\zeta=1}^{\omega-1}(\nu_\zeta-\mu_\zeta)\left[\int_0^1\xi^{\eta}f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\mathrm{d}\xi\right.\right.$$
$$\left.\left.-\int_0^1\xi^{\eta}f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\mathrm{d}\xi\right]\right|$$
$$\leq\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\int_0^1|\xi^{\eta}|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|\mathrm{d}\xi\right.$$
$$\left.+\int_0^1|\xi^{\eta}|\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|\mathrm{d}\xi\right].$$

The utilization of power mean inequality to the aforementioned integral, yields

$$\leq\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left[\left(\int_0^1\xi^{\eta}\mathrm{d}\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\xi^{\eta}\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|^{q}\mathrm{d}\xi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\xi^{\eta}\mathrm{d}\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\xi^{\eta}\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|^{q}\mathrm{d}\xi\right)^{\frac{1}{q}}\right]$$
$$=\frac{1}{4}\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|\left(\frac{1}{\eta+1}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\xi^{\eta}\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta\right)\right)\right|^{q}\mathrm{d}\xi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\xi^{\eta}\left|f'\left(\sum_{\zeta=1}^{\omega}\rho_\zeta-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}\nu_\zeta+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}\mu_\zeta\right)\right)\right|^{q}\mathrm{d}\xi\right)^{\frac{1}{q}}\right].\tag{58}$$

The convexity of $|f'|^q$ allows us to achieve the following by using Theorem 1 with $\sigma_1=\frac{2-\xi}{2}$, $n=2$, and $\sigma_2=\frac{\xi}{2}$ in (58):

$$=\frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta-\mu_\zeta|}{4(\eta+1)^{1-\frac{1}{q}}}\left[\left(\int_0^1\xi^{\eta}\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)\mathrm{d}\xi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\xi^{\eta}\left(\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q-\left(\frac{2-\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q+\frac{\xi}{2}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\right)\right)\mathrm{d}\xi\right)^{\frac{1}{q}}\right]$$

$$= \frac{\sum_{\zeta=1}^{\omega-1}|\nu_\zeta - \mu_\zeta|}{4(\eta+1)^{1-\frac{1}{q}}}\left[\left(\frac{1}{\eta+1}\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \left(\frac{\eta+3}{2(\eta+1)(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q + \frac{1}{2(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{\eta+1}\sum_{\zeta=1}^{\omega}|f'(\rho_\zeta)|^q - \left(\frac{\eta+3}{2(\eta+1)(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\nu_\zeta)|^q + \frac{1}{2(\eta+2)}\sum_{\zeta=1}^{\omega-1}|f'(\mu_\zeta)|^q\right)\right)^{\frac{1}{q}}\right]. \tag{59}$$

□

5 Conclusion

Numerous mathematicians are presently conducting extensive study on the popular Hermite-Hadamard inequality. This inequality is used as a tool for error estimation in numerical integration. It guarantees that the convex function is integrable and offers approximations for the integral mean of the convex function. Additionally, it has been established for a variety of convex functions, with particular emphasis on s-convex, η-convex, coordinate convex, and strongly convex functions. The construction of several integral identities for this inequality has also led to the creation of numerous other inequalities. In this particular study, we have used the idea of majorization to obtain combined inequalities of the Hermite-Hadamard and the Jensen-Mercer within the framework of fractional calculus. The idea of majorization was used as a connecting mechanism between these two inequalities, and the result is a single inequality that contains both discrete and continuous inequalities. These inequalities have been constructed with the help of AB fractional operators. The given remarks demonstrated that these inequalities cover the previously known inequalities. Additionally, weighted versions of the main inequalities have been found using different kinds of majorized tuples. We have also developed bounds for the discrepancy of the terms pertaining to the major inequalities with the aid of integral identities. The convexity of $|f'|$ and $|f'|^q$, $(q > 1)$ as well as power mean and Hölder inequalities are used to build these bounds along with the identities that have already been established. The findings of this research would be a valuable addition to the theory of mathematical inequalities.

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