

Research Article

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Spectral properties for a system of Dirac equations with nonlinear dependence on the spectral parameter

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Abstract: We consider the boundary value problem generated by a system of Dirac equations with polynomials of spectral parameter in the boundary condition. We investigate the continuity of the scattering function and provide Levinson-type formula, which shows that the increment of the scattering function's logarithm is related to the number of eigenvalues of the boundary value problem.

Keywords: Levinson-type formula, scattering data, spectral parameter dependent boundary condition

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1 Introduction

In 1949, Levinson established a theorem for the Schrödinger equation in nonrelativistic quantum mechanics [1]. The theorem relates the number of bound states in a given angular momentum channel to the phase shift at threshold, and it is also responsible for the inverse scattering problem which consists in the recovery of potential from spectral characteristics of the boundary value problem.

Klaus [2] presented the relativistic analog of Levinson's theorem. The author considered the Dirac equation for a particle moving in a central electrostatic potential $V(r)$. With the help of separation of variables, the following systems of equations were studied:

$$H_{\kappa}(c)\psi = c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi' + \begin{pmatrix} mc^2 + V(r) & \kappa/r \\ \kappa/r & -mc^2 + V(r) \end{pmatrix} \psi = E\psi, \quad \psi = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix}$$

on $0 < r < \infty$. Here, m is the mass of the particle, c is the velocity of light, E is the energy (in units where $\hbar = 1$), and κ is a nonzero integer. It is assumed that

$$\int_0^{\infty} (1+r)|V(r)|dr < \infty, \quad (1)$$

which guarantees that the differential operator H is limit point at zero (it is always limit point at infinity) so that H_{κ} can be considered as a self-adjoint operator in a Hilbert space of vector-valued functions. The spectrum of H_{κ} is absolutely continuous on $(-\infty, -mc^2] \cup [mc^2, \infty)$ and consists of at most finitely many (simple) eigenvalues in the gap $[-mc^2, mc^2]$.

In his work, Klaus [2] showed that there exists a deep connection between the continuous part and the discrete part of the spectrum, which is the content of Levinson's theorem in the Schrödinger case. When the

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potential satisfied the condition (1), a relation between the number of bound states in the spectral gap $[-m, m]$ and the variation of an appropriate phase $\delta_\kappa(-m) - \delta_\kappa(m)$ along the continuous part of the spectrum $(-\infty, -m] \cup [m, \infty)$ was established for the Dirac equation.

Relativistic versions of Levinson's theorem for the Dirac operator were presented in the monographs [3–9]. Also, Levinson's theorem was studied by many authors in [10–18] and references therein. For the Sturm Liouville operator, Levinson formula was obtained in [19]. In the case of the boundary condition containing spectral parameter, Levinson-type formulas were provided in [20–23]. There were also studies on scattering problems in [24–30] and other papers.

In the present work, we consider the boundary value problem generated by the canonical system of Dirac differential equations

$$By' + mTy + \Omega(x)y = \lambda y, \quad (0 \leq x < \infty) \quad (2)$$

with the boundary condition

$$(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y_1(0) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)y_2(0) = 0, \quad (3)$$

where λ is a spectral parameter,

$$y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$m > 0$ is the mass and the potential matrix $\Omega(x)$ has the canonical form

$$\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$$

with $p(x)$ and $q(x)$ being real valued and the inequalities

$$|p(x)| \leq \frac{c}{(1+x)^{2+\varepsilon}}, \quad |q(x)| \leq \frac{c}{(1+x)^{1+\varepsilon}} \quad (4)$$

being satisfied for positive numbers c and ε .

Assume that the relations

$$\alpha_0\beta_1 - \alpha_1\beta_0 < 0, \quad \alpha_0\beta_2 - \alpha_2\beta_0 = 0, \quad \alpha_1\beta_2 - \alpha_2\beta_1 < 0 \quad (5)$$

hold for $\alpha_i, \beta_j \in \mathbb{R}$ ($i, j = 0, 1, 2$).

The purpose of this article is to establish Levinson-type formula for a system of Dirac equations with spectral parameter in the boundary condition. In the process, we examine the continuity of the scattering function for the boundary value problem (2)–(4) and present the relation between the number of eigenvalues of the boundary value problem (2)–(4) in the interval $(-m, m)$ and the variation of the argument of $S(\lambda)$ over $(-\infty, -m] \cup [m, \infty)$.

The rest of this article is organized as follows. In Section 2, the required theorems and lemmas for the boundary value problem (2)–(4) are provided. In Section 3, the continuity of the scattering function is investigated. Finally, Levinson-type formula is presented in Section 4.

2 Preliminaries

In this section, we mention basic tools needed and results from the work [25] that allow us to achieve this research.

It is easily seen from Theorem 1.2.1 in [24] that when $\Omega(x)$ satisfies condition (4), then equation (2) has a unique matrix solution $f(x, \lambda)$ of order (2×1) which tends to $f_0(x, \lambda)$ as $x \rightarrow \infty$, $\text{Im } \lambda \geq 0$, and there exists a matrix function $A(x, t)$ of order 2 such that

$$f(x, \lambda) = f_0(x, \lambda) + \int_x^\infty A(x, t)f_0(t, \lambda)dt. \quad (6)$$

The matrix function

$$f_0(x, \lambda) = \begin{pmatrix} \frac{\lambda + m}{k} \\ -i \end{pmatrix} e^{ikx}$$

is a solution of (2) when $\Omega(x) = 0$, where $k = \lambda \sqrt{1 - \frac{m^2}{\lambda^2}}$, $|\lambda| > m$. Furthermore, the components of the matrix kernel $A(x, t)$ satisfy the inequalities

$$|A_{ij}| \leq \frac{c_1}{(1+x)(1+t)^{1+\varepsilon}}, \quad i \neq j, \quad (7)$$

$$|A_{ii}| \leq \frac{c_2}{(1+t)^{1+\varepsilon}}, \quad i = 1, 2. \quad (8)$$

Let us denote by

$$\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}$$

the matrix solution of order 2×1 of equation (2) with the initial data

$$\varphi_1(0, \lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2, \quad \varphi_2(0, \lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2.$$

It is evident that the solution $\varphi(x, \lambda)$ satisfies the boundary condition (3).

Lemma 1. [25] *For all λ in the intervals $(-\infty, -m)$ and (m, ∞) , the following identity is valid:*

$$2i \frac{\lambda + m}{k} \frac{\varphi(x, \lambda)}{E(\lambda)} = \overline{f(x, \bar{\lambda})} - S(\lambda) f(x, \lambda), \quad (9)$$

where

$$E(\lambda) = (\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2) f_1(0, \lambda) - (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2) f_2(0, \lambda), \quad (10)$$

$$S(\lambda) = \frac{\overline{E(\bar{\lambda})}}{E(\lambda)}, \quad (11)$$

and

$$S^{-1}(\lambda) = \overline{S(\bar{\lambda})}, \quad |S(\lambda)| = 1.$$

The function $S(\lambda)$ defined by (11) is called the scattering function of the boundary value problem (2)–(4).

Lemma 2. [25] *The function $E(\lambda)$ is analytic in the upper half plane $\text{Im } \lambda > 0$, continuous along the real axis except at $\lambda = m$ and has only a finite number of zeros in the interval $(-m, m)$. All the zeros in $(-m, m)$ are simple.*

The zeros λ_k , $k = 1, \dots, n$, of the function $E(\lambda)$ are called the singular values of the boundary value problem (2)–(4).

The numbers m_k , $k = 1, \dots, n$, are defined by using

$$m_k^{-2} \equiv \int_0^\infty f^*(x, \lambda_k) f(x, \lambda_k) dx + \frac{[\alpha_1 \beta_0 - \alpha_0 \beta_1 + (\alpha_2 \beta_1 - \alpha_1 \beta_2) |\lambda_k|^2]}{|\alpha_0 + \alpha_1 \lambda_k + \alpha_2 \lambda_k^2|^2} |f_2(0, \lambda_k)|^2$$

and called the norming numbers of boundary value problem (2)–(4), where f^* denotes the transposed vector of $\bar{f}$.

Definition 1. [25] The collection

$$\{S(\lambda), \{\lambda_k\}_{k=1}^n, \{m_k\}_{k=1}^n\}$$

is called scattering data of the boundary value problem (2)–(4).

Theorem 1. [25] For every fixed $x \geq 0$, the kernel $A(x, t)$ of solution (6) satisfies the integral equation

$$A(x, y) + F(x + y) + \int_x^\infty A(x, t)F(t + y)dt = 0, \quad y > x, \quad (12)$$

where

$$F(x + y) = \operatorname{Re} \frac{1}{2\pi} \int_{|\lambda| > m} \left(\frac{\alpha_2 - i\beta_2}{\alpha_2 + i\beta_2} - S(\lambda) \right) \begin{pmatrix} \frac{\lambda + m}{k} & -i \\ -i & -\frac{k}{\lambda + m} \end{pmatrix} e^{ik(x+y)} d\lambda - \sum_{j=1}^n m_j^2 f_0(x, \lambda_j) \widetilde{f_0}(y, \lambda_j),$$

where $\widetilde{f_0}$ denotes the transposed vector of f_0 .

The integral equation (12) is called the main equation of the inverse scattering problem for the boundary value problem (2)–(4).

3 Scattering function

In this section, we present the continuity of the scattering function $S(\lambda)$ defined by (11).

Theorem 2. $S(\lambda)$ is the quotient of two functions so that the numerator and the denominator are analytic in the lower and upper half plane, respectively. Furthermore, the limits

$$\lim_{\lambda \rightarrow \pm m} S(\lambda)$$

exist, and as $\lambda \rightarrow \pm\infty$

$$S(\lambda) = S_0(\lambda) + O(1),$$

where $S_0(\lambda) = \frac{\alpha_2 - i\beta_2}{\alpha_2 + i\beta_2}$. Moreover, $S(\lambda)$ is continuous in $(-\infty, -m]$ and $[m, \infty)$.

Proof. From Lemma 1, it is evident that $S(\lambda)$ is the quotient of two functions. Also, Lemma 2 shows that the numerator and the denominator of $S(\lambda)$ are analytic in the lower and upper half plane, respectively (see the proof in [25]). $E(\lambda) \neq 0$ for all λ in the intervals $(-\infty, -m)$ and (m, ∞) . Hence, $S(\lambda)$ is defined on $(-\infty, -m)$ and (m, ∞) and continuous in these intervals. We need to show the continuity of $S(\lambda)$ at $\lambda = \pm m$. For this, we rewrite the function $E(\lambda)$ in the form

$$\begin{aligned} E(\lambda) &= (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)f_1(0, \lambda) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)f_2(0, \lambda) \\ &= (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \left\{ \frac{\lambda + m}{k} + \int_0^\infty \left[A_{11}(0, t) \frac{\lambda + m}{k} - iA_{12}(0, t) \right] e^{ikt} dt \right\} \\ &\quad - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \left\{ -i + \int_0^\infty \left[A_{21}(0, t) \frac{\lambda + m}{k} - iA_{22}(0, t) \right] e^{ikt} dt \right\}. \end{aligned}$$

Let us define the function

$$\Psi(\lambda) = k \cdot E(\lambda)$$

and let l_1 and l_2 be limits of $\lim_{\lambda \rightarrow m} \frac{k}{\lambda + m} E(\lambda)$ and $\lim_{\lambda \rightarrow -m} E(\lambda)$, respectively.

Moreover,

$$\begin{aligned} l_1 &= \lim_{\lambda \rightarrow m} \frac{k}{\lambda + m} E(\lambda) \\ &= \lim_{\lambda \rightarrow m} \left[(\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2) \left\{ \left(1 + \int_0^\infty A_{11}(0, t) e^{ikt} dt \right) - i \frac{k}{\lambda + m} \int_0^\infty A_{12}(0, t) e^{ikt} dt \right\} \right. \\ &\quad \left. - (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2) \left\{ \int_0^\infty A_{21}(0, t) e^{ikt} dt - i \frac{k}{\lambda + m} \left(1 + \int_0^\infty A_{22}(0, t) e^{ikt} dt \right) \right\} \right] \\ &= (\alpha_0 + \alpha_1 m + \alpha_2 m^2) \left(1 + \int_0^\infty A_{11}(0, t) dt \right) - (\beta_0 + \beta_1 m + \beta_2 m^2) \int_0^\infty A_{21}(0, t) dt \end{aligned}$$

and

$$\begin{aligned} \Psi(\lambda) &= (\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2) \left\{ (\lambda + m) \left(1 + \int_0^\infty A_{11}(0, t) e^{ikt} dt \right) - ik \int_0^\infty A_{12}(0, t) e^{ikt} dt \right\} \\ &\quad - (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2) \left\{ (\lambda + m) \int_0^\infty A_{21}(0, t) e^{ikt} dt - ik \left(1 + \int_0^\infty A_{22}(0, t) e^{ikt} dt \right) \right\}. \end{aligned}$$

Hence, for the case $l_1 \neq 0$, we have

$$\Psi(\lambda) = 2ml_1 + O(1), \quad \lambda \rightarrow m.$$

Now, we shall investigate $\Psi(\lambda)$ for the case $l_1 = 0$. Therefore, we substitute $x = 0$ into the main equation (12) and obtain the following components of the first column

$$A_{11}(0, y) + F_{11}(y) + \int_0^\infty \{A_{11}(0, t)F_{11}(t + y) + A_{12}(0, t)F_{21}(t + y)\} dt = 0 \quad (13)$$

and

$$A_{21}(0, y) + F_{21}(y) + \int_0^\infty \{A_{21}(0, t)F_{11}(t + y) + A_{22}(0, t)F_{21}(t + y)\} dt = 0. \quad (14)$$

Integrating (13) with respect to y from z to ∞ and letting $t + y = \xi$, we have

$$\int_z^\infty A_{11}(0, y) dy + \int_z^\infty F_{11}(y) dy + \int_0^\infty A_{11}(0, t) \int_{t+z}^\infty F_{11}(\xi) d\xi dt + \int_0^\infty A_{12}(0, t) \int_{t+z}^\infty F_{21}(\xi) d\xi dt = 0. \quad (15)$$

Applying partial integration to the third term on the left-hand side of equality (15), we find

$$\int_0^\infty A_{11}(0, t) \int_{t+z}^\infty F_{11}(\xi) d\xi dt = \left(\int_z^\infty F_{11}(\xi) d\xi \right) \left(\int_0^\infty A_{11}(0, y) dy \right) - \int_0^\infty F_{11}(t + z) \int_t^\infty A_{11}(0, y) dy dt.$$

Calculating the fourth term with similar process and substituting both of these integrals into equality (15), we obtain

$$\begin{aligned} &\int_z^\infty A_{11}(0, y) dy + \left(1 + \int_0^\infty A_{11}(0, y) dy \right) \left(\int_z^\infty F_{11}(\xi) d\xi \right) - \int_0^\infty F_{11}(t + z) \int_t^\infty A_{11}(0, y) dy dt \\ &+ \left(\int_0^\infty A_{12}(0, y) dy \right) \left(\int_z^\infty F_{21}(\xi) d\xi \right) - \int_0^\infty F_{21}(t + z) \int_t^\infty A_{12}(0, y) dy dt = 0. \end{aligned} \quad (16)$$

We now apply the same procedure to equation (14) to obtain

$$\begin{aligned} & \int_z^\infty A_{21}(0, y) dy + \left(1 + \int_0^\infty A_{22}(0, y) dy \right) \left(\int_z^\infty F_{21}(\xi) d\xi \right) - \int_0^\infty F_{21}(t+z) \int_t^\infty A_{22}(0, y) dy dt \\ & + \left(\int_0^\infty A_{21}(0, y) dy \right) \left(\int_z^\infty F_{11}(\xi) d\xi \right) - \int_0^\infty F_{11}(t+z) \int_t^\infty A_{21}(0, y) dy dt = 0. \end{aligned} \quad (17)$$

Multiplying equation (16) by $(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)$ and equation (17) by $(\beta_0 + \beta_1\lambda + \beta_2\lambda^2)$, then subtracting the latter from the former, we have

$$\begin{aligned} & \left[(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \left(1 + \int_0^\infty A_{11}(0, y) dy \right) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \int_0^\infty A_{21}(0, y) dy \right] \int_z^\infty F_{11}(y) dy \\ & + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \int_z^\infty A_{11}(0, y) dy - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \int_z^\infty A_{21}(0, y) dy - \int_0^\infty F_{11}(t+z) \\ & \times \left[(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \int_t^\infty A_{11}(0, y) dy - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \int_t^\infty A_{21}(0, y) dy \right] dt \\ & - \int_0^\infty F_{21}(t+z) \left[(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \int_t^\infty A_{12}(0, y) dy - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \int_t^\infty A_{22}(0, y) dy \right] dt \\ & + \left[(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \int_0^\infty A_{12}(0, y) dy - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \left(1 + \int_0^\infty A_{22}(0, y) dy \right) \right] \int_z^\infty F_{21}(y) dy = 0. \end{aligned}$$

Letting $\lambda \rightarrow m$ and considering $l_1 = 0$, we obtain

$$\begin{aligned} & (\alpha_0 + \alpha_1m + \alpha_2m^2) \int_z^\infty A_{11}(0, y) dy - (\beta_0 + \beta_1m + \beta_2m^2) \int_z^\infty A_{21}(0, y) dy \\ & - \int_0^\infty F_{11}(t+z) \left[(\alpha_0 + \alpha_1m + \alpha_2m^2) \int_t^\infty A_{11}(0, y) dy - (\beta_0 + \beta_1m + \beta_2m^2) \int_t^\infty A_{21}(0, y) dy \right] dt \\ & - \int_0^\infty F_{21}(t+z) \left[(\alpha_0 + \alpha_1m + \alpha_2m^2) \int_t^\infty A_{12}(0, y) dy - (\beta_0 + \beta_1m + \beta_2m^2) \int_t^\infty A_{22}(0, y) dy \right] dt \\ & + \left[(\alpha_0 + \alpha_1m + \alpha_2m^2) \int_0^\infty A_{12}(0, y) dy - (\beta_0 + \beta_1m + \beta_2m^2) \left(1 + \int_0^\infty A_{22}(0, y) dy \right) \right] \int_z^\infty F_{21}(y) dy = 0. \end{aligned}$$

Here, we define the functions $H(z)$ and $G(z)$ as follows:

$$\begin{aligned} H(z) &= (\alpha_0 + \alpha_1m + \alpha_2m^2) \int_z^\infty A_{11}(0, y) dy - (\beta_0 + \beta_1m + \beta_2m^2) \int_z^\infty A_{21}(0, y) dy, \\ G(z) &= \int_0^\infty F_{21}(t+z) \left[(\alpha_0 + \alpha_1m + \alpha_2m^2) \int_t^\infty A_{12}(0, y) dy - (\beta_0 + \beta_1m + \beta_2m^2) \int_t^\infty A_{22}(0, y) dy \right] dt \\ & - \left[(\alpha_0 + \alpha_1m + \alpha_2m^2) \int_0^\infty A_{12}(0, y) dy - (\beta_0 + \beta_1m + \beta_2m^2) \left(1 + \int_0^\infty A_{22}(0, y) dy \right) \right] \int_z^\infty F_{21}(y) dy. \end{aligned}$$

Hence, we find an integral equation

$$H(z) - \int_0^{\infty} F_{11}(t+z)H(t)dt = G(z).$$

$H(z)$ is a bounded solution of the equation

$$H(z) - \int_0^{\infty} F_{11}(t+z)H(t)dt = 0, \quad 0 \leq z < \infty,$$

and every bounded solution of this equation is summable on the half line $[0, \infty)$. It means that $H(z) \in L_1(0, \infty)$ (see [19] p. 211).

Returning to the function $\Psi(\lambda)$ and by partial integration, we obtain

$$\begin{aligned} \Psi(\lambda) = & (\lambda + m) \left\{ (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \left(1 + \int_0^{\infty} A_{11}(0, t)dt \right) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \int_0^{\infty} A_{21}(0, t)dt \right\} \\ & + ik(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \left\{ (\lambda + m) \int_0^{\infty} \int_t^{\infty} A_{11}(0, y)dy e^{ikt} dt - \int_0^{\infty} A_{12}(0, t) e^{ikt} dt \right\} \\ & - ik(\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \left\{ (\lambda + m) \int_0^{\infty} \int_t^{\infty} A_{21}(0, y)dy e^{ikt} dt - \left(1 + \int_0^{\infty} A_{22}(0, t) e^{ikt} dt \right) \right\}. \end{aligned}$$

Because of the fact that $l_1 = 0$, the factor of the first term of $\Psi(\lambda)$ equals:

$$\begin{aligned} & (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \left(1 + \int_0^{\infty} A_{11}(0, t)dt \right) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \int_0^{\infty} A_{21}(0, t)dt \\ & = (\alpha_1(\lambda - m) + \alpha_2(\lambda^2 - m^2)) \left(1 + \int_0^{\infty} A_{11}(0, t)dt \right) - (\beta_1(\lambda - m) + \beta_2(\lambda^2 - m^2)) \int_0^{\infty} A_{21}(0, t)dt. \end{aligned}$$

After substituting the obtained value into the function $\Psi(\lambda)$, we conclude that

$$\Psi(\lambda) = \sqrt{\lambda - m} \widehat{K(\lambda)}, \quad (18)$$

where

$$\begin{aligned} \widehat{K(\lambda)} = & (\lambda + m) \left\{ (\alpha_1\sqrt{\lambda - m} + \alpha_2\sqrt{\lambda - m}(\lambda + m)) \left(1 + \int_0^{\infty} A_{11}(0, t)dt \right) \right. \\ & \left. - (\beta_1\sqrt{\lambda - m} + \beta_2\sqrt{\lambda - m}(\lambda + m)) \int_0^{\infty} A_{21}(0, t)dt \right\} \\ & + i\sqrt{\lambda + m}(\lambda + m) \int_0^{\infty} \int_t^{\infty} [(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)A_{11}(0, y) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)A_{21}(0, y)]dy e^{ikt} dt \\ & - i\sqrt{\lambda + m} \left\{ (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \int_0^{\infty} A_{12}(0, t) e^{ikt} dt - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) \left(1 + \int_0^{\infty} A_{22}(0, t) e^{ikt} dt \right) \right\}. \end{aligned}$$

Hence, in the present case,

$$\begin{aligned} \widehat{K(m)} = & i(2m)^{3/2} \int_0^{\infty} H(t)dt - i(2m)^{1/2} \left\{ (\alpha_0 + \alpha_1m + \alpha_2m^2) \int_0^{\infty} A_{12}(0, t)dt - (\beta_0 + \beta_1m + \beta_2m^2) \right. \\ & \left. \times \left(1 + \int_0^{\infty} A_{22}(0, t)dt \right) \right\} \end{aligned}$$

and

$$S(\lambda) = \frac{\overline{\widehat{K(\lambda)}}}{\widehat{K(\lambda)}}.$$

Taking into account the first components of (6) and (9), we obtain

$$\begin{aligned} 2i(\lambda + m)\varphi_1(x, \lambda) &= \widehat{K(\lambda)}\{\sqrt{\lambda - m} \overline{f_1(x, \lambda)} - S(\lambda)\sqrt{\lambda - m}f_1(x, \lambda)\} \\ &= \widehat{K(\lambda)}\left\{\sqrt{\lambda + m}e^{-ikx} + \int_x^\infty [A_{11}(x, t)\sqrt{\lambda + m} + iA_{12}(x, t)]e^{-ikt}dt \right. \\ &\quad \left. - S(\lambda)\left[\sqrt{\lambda + m}e^{ikx} + \int_x^\infty [A_{11}(x, t)\sqrt{\lambda + m} - iA_{12}(x, t)]e^{ikt}dt\right]\right\}, \end{aligned}$$

from which it follows that $\widehat{K(m)} \neq 0$, otherwise it would be $\varphi_1(x, m) = 0$ and it contradicts $\varphi_1(x, \lambda) \neq 0$. This shows that $S(\lambda)$ is continuous at $\lambda = m$ in the case $l_1 = 0$.

Consequently, as $\lambda \rightarrow m$, we have

$$\Psi(\lambda) = \begin{cases} 2ml_1 + O(1), & l_1 \neq 0 \\ \sqrt{\lambda - m}\widehat{K(m)}, & l_1 = 0. \end{cases} \quad (19)$$

If the condition $l_1 \neq 0$ holds, it is obvious that $S(\lambda)$ is continuous at $\lambda = m$.

In the case that $l_2 = \lim_{\lambda \rightarrow -m} E(\lambda)$, using a similar process, we obtain the result

$$\Psi(\lambda) = \begin{cases} \sqrt{\lambda + m}(c_1 + O(1)) & l_2 \neq 0 \\ (\lambda + m)(c_2 + O(1)) & l_2 = 0, \end{cases} \quad (20)$$

which shows the continuity of $S(\lambda)$ at $\lambda = -m$.

To complete the proof, using the definition of $S(\lambda)$, we take $\lambda \rightarrow \pm\infty$ and the following result is obtained

$$S(\lambda) = \frac{\alpha_2 - i\beta_2}{\alpha_2 + i\beta_2} + O(1).$$

The theorem is proved. □

4 Levinson-type formula

Now, after stating the above preparations, we can describe Levinson-type formula for the boundary value problem (2)–(4).

Theorem 3. *The increment of the logarithm of the scattering function $S(\lambda)$ is related to the number of eigenvalues of problems (2)–(4) through the equality*

$$n - 2 = \frac{\ln S(m + 0) - \ln S(\infty)}{2\pi i} + \frac{\ln S(-\infty) - S(-m - 0)}{2\pi i} - \frac{S(-m - 0) - \ln S(m + 0)}{4}. \quad (21)$$

Proof. To achieve formula (21), we apply the argument principle to the function $\Psi(\lambda)$. This function is analytic in the upper half plane and continuous along the real axis. Also, for $|\lambda| < m$, $\text{Im } \lambda = 0$, we have $\text{Im } \Psi(\lambda) = 0$. Hence, the increment of its argument $\arg \Psi(\lambda)$ equals zero as λ runs over the curve

$$\Gamma_{\delta, R} = \{\lambda | m + \delta \leq \text{Re } \lambda \leq R, \text{ Im } \lambda = 0; |\lambda \pm m| = \delta, |\lambda - \lambda_j| = \delta, j = \overline{1, n}, |\lambda| = R, \text{ Im } \lambda \geq 0\},$$

i.e.,

$$\arg \Psi(\lambda)_{\Gamma_{\delta,R}} = 0.$$

Let the argument of the function $\Psi(\lambda)$ be

$$\arg \Psi(\lambda)_{\Gamma_{\delta,R}} = \mu(\lambda).$$

As $R \rightarrow \infty$, we obtain

$$\begin{aligned} & \{\mu(-m - \delta) - \mu(-\infty)\} + \{\mu(-m + \delta) - \mu(-m - \delta)\} + \{\mu(m + \delta) - \mu(m - \delta)\} + \{\mu(\infty) - \mu(m + \delta)\} \\ & + \sum_{j=1}^n \{\mu(\lambda_j + \delta) - \mu(\lambda_j - \delta)\} + 3\pi = 0 \end{aligned}$$

because of that

$$\Psi(\lambda) = \lambda^3(\alpha_2 - \beta_2) + O(1), \quad |\lambda| \rightarrow \infty$$

for $\operatorname{Im} \lambda \geq 0$. By using (19) and (20), as $\delta \rightarrow 0$, we find that

$$\{\mu(-m - 0) - \mu(-\infty)\} + \{\mu(\infty) - \mu(m + 0)\} - n\pi + 3\pi = \begin{cases} \frac{\pi}{2}, & l_1 \neq 0, \quad l_2 \neq 0, \\ \pi, & l_1 = 0, \quad l_2 \neq 0, \\ \pi, & l_1 \neq 0, \quad l_2 = 0, \\ \frac{3\pi}{2}, & l_1 = 0, \quad l_2 = 0. \end{cases} \quad (22)$$

On the other hand, from (19) and (20), we deduce that

$$S(-m - 0) = \begin{cases} -1, & l_2 \neq 0, \\ 1, & l_2 = 0, \end{cases}$$

and

$$S(m + 0) = \begin{cases} 1, & l_1 \neq 0, \\ -1, & l_1 = 0. \end{cases}$$

Since $\ln S(\lambda) = -2i\mu(\lambda)$, we substitute the required values into (22) and obtain the desired result

$$\begin{aligned} \frac{\ln S(m + 0) - \ln S(\infty)}{2\pi i} + \frac{\ln S(-\infty) - \ln S(-m - 0)}{2\pi i} &= \frac{\mu(\infty) - \mu(m + 0)}{\pi} + \frac{\mu(-m - 0) - \mu(-\infty)}{\pi} \\ &= n - 2 + \frac{S(-m - 0) - S(m + 0)}{4}. \end{aligned} \quad \square$$

Definition 2. The formula defined by (21) is called Levinson-type formula for the boundary value problem (2)–(4).

5 Conclusion

In quantum scattering theory, Levinson's theorem deals with the relation between the number of bound states in a spectral gap and the variation of an appropriate phase along the continuous part of the spectrum. This article presents Levinson-type formula for a system of Dirac equations with polynomials of spectral parameter in the boundary condition. In this view, the continuity of the scattering function has been investigated and the variation of its argument has been formulated. The formula demonstrates that the increment of the logarithm of $S(\lambda)$ over $(-\infty, -m] \cup [m, \infty)$ is related to the number of eigenvalues of the boundary value problem (2)–(4) in the interval $(-m, m)$ through equality (21).

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