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Research Article

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Global smooth solution to the n-dimensional liquid crystal equations with fractional dissipation

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Abstract: In this article, we focus on the global regularity of n-dimensional liquid crystal equations with fractional dissipation terms $(-\Delta)^{\alpha}u$ and $(-\Delta)^{\beta}d$. We show that the equations have a unique global smooth solution if $\alpha \geq \frac{1}{2} + \frac{n}{4}$ and $\beta \geq \frac{1}{2} + \frac{n}{4}$.

Keywords: liquid crystal equations, fractional dissipation, global regularity

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1 Introduction

The n-dimensional incompressible fractional liquid crystal equations are

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mu(-\Delta)^{\alpha} u + \nabla p = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \partial_t d + u \cdot \nabla d + \gamma(-\Delta)^{\beta} d = -f(d), \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), d(x,0) = d_0(x), \end{cases} \tag{1.1}$$

where $u \in \mathbb{R}^n$ denotes the velocity field, $d \in \mathbb{R}^n$ denotes the macroscopic average of molecular orientation field, and p stands the scalar pressure. The symbol $\nabla d \odot \nabla d$ denotes one $n \times n$ matrix whose (i,j)th entry is $\sum_{k=1}^{3} \partial_i d_k \partial_j d_k (i, j \leq n)$, and $f(d) = \frac{1}{|\eta|^2}(|d|^2 - 1)d$. And here, μ, λ, γ, and η are all positive constants. For simplicity, we shall assume $\mu = \lambda = \gamma = \eta = 1$ throughout this article. The fractional Laplacian operator $(-\Delta)^{\alpha}$ is defined by the Fourier transform

$$\widehat{(-\Delta)^{\alpha} f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

The original liquid crystal equations were established by Ericksen [1] and Leslie [2], which is too much complicated to study in mathematics. Thus, Lin [3] simplified the original liquid crystal equations to this forms that in the above liquid crystal equations (1.1) that but still retains most of the essential features of the original liquid crystal equations .

When $\alpha = \beta = 1$, System (1.1) reduces to the standard liquid crystal equations. Lin and Liu [4,5] proved that the liquid crystal equations have a unique smooth solution globally in two dimensions and locally in three dimensions. They also proved the global existence of weak solutions. But the regularity of solutions is still a difficult problem. Inspired by the classical regularity criteria of weak solutions of the Navier-Stokes equation,

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researchers obtained different types of regularity criteria for the liquid crystal equations, for instance, the well-known Prodi-Serrin-type regularity criterion and the type of logarithmical regularity criterion, and for details the readers may refer to [6–14] and references therein.

Here, we consider the generalized values of the dissipation indexes α and β. When the orientation field d equals to a constant, the nematic liquid crystal flows reduces to the generalized incompressible Navier-Stokes equations. As known, for the generalized incompressible Navier-Stokes equations, for any smooth initial datum $u_0 \in H^s(\mathbb{R}^n)$ with $s > 1 + \frac{n}{2}$, there exists a unique global-in-time solution with $\alpha \geq \frac{1}{2} + \frac{n}{4}$. Wu [15] studied the generalized magnetohydrodynamic (MHD) equations and obtained smooth solutions with $\alpha, \beta \geq \frac{1}{2} + \frac{n}{4}$. Wu [16] improved the result by $\alpha \geq \frac{1}{2} + \frac{n}{4}$, $\beta > 0$, and $\alpha + \beta \geq 1 + \frac{n}{2}$, which was also sharpened by a logarithmic term in the meantime. Based on these results, much attention has been paid to reducing the dissipation in the liquid crystal equations. For the 2D liquid crystal equations, when $\alpha = \beta = 1$, System (1.1) has a unique global smooth solution. Wang and Zhou [17] established the global existence of smooth solutions with $\alpha = 0$ and $\beta > 1$. Subsequently, Jin et al. [18] obtained the global regularity for the two cases $\alpha > 0$, $\beta = 1$, and $\alpha + \beta = 2$, $0 < \beta < 1$. If $\gamma = 0$, Yuan and Wei [19] studied the case $\alpha \geq 1 + \frac{n}{2}$ and $\beta = 0$ with a logarithm operation on the dissipation term for general n-dimension and proved the global existence of a smooth solution.

In this article, we will study the n-dimensional generalized liquid crystal equations for general α and β. The condition $\alpha \geq \frac{1}{2} + \frac{n}{4}$ is clearly needed due to the first equation in (1.1). Because System (1.1) has higher derivative term $\nabla\cdot(\nabla d \odot \nabla d) = \frac{1}{2}\nabla(|\nabla d|^2) + \Delta d_i \cdot \nabla d_i$ than the MHD equations, as a result, $\beta \geq \frac{1}{2} + \frac{n}{4}$ is also needed. Next, we state our result.

Theorem 1.1. *Let the initial data* $(u_0, d_0) \in H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)$ *with* $s > 1 + \frac{n}{2}$ *and* $\nabla\cdot u_0 = 0$. *If dissipation indexes* α *and* β *satisfy*

$$\alpha \geq \frac{1}{2} + \frac{n}{4}, \quad \text{and} \quad \beta \geq \frac{1}{2} + \frac{n}{4},$$

then System (1.1) *has a unique global smooth solution satisfying*

$$u, \nabla d \in L^\infty(0, T; H^s(\mathbb{R}^n)),$$

for any given $T > 0$.

2 Proof of Theorem 1.1

In this section, we concentrate on the proof of Theorem 1.1. The proof of existence of the local solution to (1.1) is a standard process, readers may refer to [20] for details. Here, we only give the *a priori* estimate of the local solution to ensure $\|(u, \nabla d)\|_{H^s}$ bounded at all time. By adopting the basic energy method and taking use of commutator estimate, Hölder's inequality, interpolation inequality, and so on, we will improve the regularity of the local solution step by step and obtain the higher-order derivative estimate at last.

In this article, we will utilize some simplified notations. The letter C means a generic constant, which may change in different places. In addition, the notation $\|\cdot\|_{L^p}$ represents the norm of a Lebesgue space and $\|\cdot\|_{H^s}$ represents the norm of a Sobolev space, which is defined as follows:

$$\|\cdot\|_{H^s} = \left(\int (1 + |\xi|^2)^s |\hat{\cdot}|^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

At first, multiplying the second equation in (1.1) with d and then integrating it on $\mathbb{R}^n$ yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|d\|_{L^2}^2 + \|\Lambda^\beta d\|_{L^2}^2 + \|d\|_{L^4}^4 \leq \|d\|_{L^2}^2,$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$, and the following equality has been used:

$$\int_{\mathbb{R}^n} (u\cdot\nabla d)\cdot d\,\mathrm{d}x = 0.$$

Using Gronwall's inequality, we have

$$\|d\|_{L^2}^2 + 2\int_0^T (\|d\|_{L^4}^4 + \|\Lambda^\beta d\|_{L^2}^2)\,\mathrm{d}t \le C\|d_0\|_{L^2}^2.$$

Next, multiplying u and $-\Delta d$ to the first and second equations in System (1.1), respectively, and then integrating that two equations on $\mathbb{R}^n$ and adding together, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^{\beta+1} d\|_{L^2}^2 + \|d|\nabla d|\|_{L^2}^2 + \frac{1}{2}\|\nabla |d|^2\|_{L^2}^2 \le \|\nabla d\|_{L^2}^2,$$

where we have used the following equalities:

$$\int_{\mathbb{R}^n} (u\cdot\nabla u)\cdot u\,\mathrm{d}x = 0, \quad \int_{\mathbb{R}^n} u\cdot\nabla\left(\frac{|\nabla d|^2}{2}\right)\mathrm{d}x = 0, \quad \text{and} \quad \int_{\mathbb{R}^n} \nabla p\cdot u\,\mathrm{d}x = 0.$$

Hence, by Gronwall's inequality, we deduce that

$$(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + 2\int_0^T (\|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^{\beta+1} d\|_{L^2}^2 + \|d|\nabla d|\|_{L^2}^2 + \frac{1}{2}\|\nabla |d|^2\|_{L^2}^2)\,\mathrm{d}t \le C(T)(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2), \tag{2.1}$$

where $C(T)$ is a constant depending on T. In addition, the upper bound of $\|d\|_{L^\infty}$ can be evaluated, which will be used in the following part. Multiplying the second equation in (1.1) by $p\,|d|^{p-2}d$ for $p > 2$ and integrating resulting equation on $\mathbb{R}^n$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d\|_{L^p}^p + \int_{\mathbb{R}^n} \Lambda^{2\beta} d\cdot p\,|d|^{p-2} d\,\mathrm{d}x + p\|d\|_{L^{p+2}}^{p+2} = p\|d\|_{L^p}^p.$$

According to [21] and [22], we have

$$\int_{\mathbb{R}^n} \Lambda^{2\beta} d\cdot p\,|d|^{p-2} d\,\mathrm{d}x \ge 0.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d\|_{L^p} \le \|d\|_{L^p}.$$

According to Gronwall's inequality and letting $p \to \infty$, it can be seen that

$$\|d\|_{L^\infty} \le \mathrm{e}^T\|d_0\|_{L^\infty} < \infty.$$

Next, we will show the bounds for the norms $\|u\|_{H^1}$ and $\|\nabla d\|_{H^1}$. And now, we first recall the following commutator estimates, for details referring pages 1–2 in [23].

Lemma 2.1. *Let $s > 0$, $1 < p < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$ with $p_2, p_3 \in (1, +\infty)$, and $p_1, p_4 \in [1, +\infty]$. Then,*

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|g\|_{L^{p_1}}\|\Lambda^s f\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}),$$
$$\|[\Lambda^s, f\cdot\nabla]g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|\nabla g\|_{L^{p_4}}),$$

where $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$.

Applying the first equation in (1.1) with Δ, multiplying u to the resulting equation, and then integrating it, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \|\Lambda^{\alpha+1}u\|_{L^2}^2 = \int_{\mathbb{R}^n}(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x + \int_{\mathbb{R}^n}\nabla\cdot(\nabla d\odot\nabla d)\cdot\Delta u\,\mathrm{d}x. \tag{2.2}$$

Applying Δ to the second equation in (1.1), multiplying Δd to that equation, and then integrating it, it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta d\|_{L^2}^2 + \|\Lambda^{\beta+2}d\|_{L^2}^2 = -\int_{\mathbb{R}^n}\Delta(u\cdot\nabla d)\cdot\Delta d\,\mathrm{d}x - \int_{\mathbb{R}^n}\Delta f(d)\cdot\Delta d\,\mathrm{d}x. \tag{2.3}$$

Combining (2.2) and (2.3), it will be clear that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \|\Lambda^{\alpha+1}u\|_{L^2}^2 + \|\Lambda^{\beta+2}d\|_{L^2}^2\\
&= \int_{\mathbb{R}^n}(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x + \int_{\mathbb{R}^n}\nabla\cdot(\nabla d\odot\nabla d)\cdot\Delta u\,\mathrm{d}x - \int_{\mathbb{R}^n}\Delta(u\cdot\nabla d)\cdot\Delta d\,\mathrm{d}x - \int_{\mathbb{R}^n}\Delta f(d)\cdot\Delta d\,\mathrm{d}x\\
&\coloneqq I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.4}$$

For the first term I_1 mentioned earlier, employing Lemma 2.1, Hölder's inequality, interpolation inequality, and Young's inequality, we obtain

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n}[\nabla, u\cdot\nabla]u\cdot\nabla u\,\mathrm{d}x\\
&\le \|[\nabla, u\cdot\nabla]u\|_{L^2}\|\nabla u\|_{L^2}\\
&\le C\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\nabla u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^2}\\
&\le C\|\nabla u\|_{H^{\alpha-1}}\|u\|_{L^2}^{\theta}\|u\|_{H^{\alpha+1}}^{1-\theta}\|\nabla u\|_{L^2}\\
&\le \frac{1}{2}\|u\|_{H^{\alpha+1}}^{2(1-\theta)} + C\|u\|_{H^\alpha}^2\|\nabla u\|_{L^2}^2\\
&\le \frac{1}{2}\|\Lambda^{\alpha+1}u\|_{L^2}^2 + C(1+\|u\|_{L^2}^2) + C\|u\|_{H^\alpha}^2\|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.5}$$

Here, $0\le\theta = \frac{4n\alpha-n-6}{4n(\alpha+1)} < 1$, and we have used the Sobolev embedding

$$H^{\alpha-1}\hookrightarrow L^{\frac{4n}{n+2}},\quad \|\nabla u\|_{L^{\frac{4n}{n+2}}}\le C\|\nabla u\|_{H^{\alpha-1}},$$

for $\alpha\ge\frac{1}{2}+\frac{n}{4}$. There shows that the index α cannot be smaller, and note that $\|\nabla u\|_{L^{\frac{4n}{n-2}}}\le C\|u\|_{L^2}^{\theta}\|u\|_{H^{\alpha+1}}^{1-\theta}$ is not valid for $n=2$, $\alpha=1$. However, we can use $\|[\nabla, u\cdot\nabla]u\|_{L^2}\le C\|\nabla u\|_{L^4}^2\le C\|u\|_{H^\alpha}\|u\|_{H^{\alpha+1}}$ in this case, and the final result is similar to (2.5).

Observing that there are some terms in I_2 and I_3 having nice canceling properties through integrating by parts, it can be shown that

$$\begin{aligned}
I_2 + I_3 &= \int_{\mathbb{R}^n}\left[\left(\nabla\frac{|\nabla d|^2}{2} + \Delta d\cdot\nabla d\right)\cdot\Delta u - \Delta(u\cdot\nabla d)\cdot\Delta d\right]\mathrm{d}x\\
&= \int_{\mathbb{R}^n}[\Delta d\cdot\nabla d\cdot\Delta u - \nabla\cdot(\nabla u\cdot\nabla d + u\cdot\nabla\nabla d)\cdot\Delta d]\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n}[\Delta d\cdot\nabla d\cdot\Delta u - (\Delta u\cdot\nabla d + 2\nabla u\cdot\nabla\nabla d + u\cdot\nabla\Delta d)\cdot\Delta d]\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n} - 2\nabla u\cdot\nabla\nabla d\cdot\Delta d\,\mathrm{d}x.
\end{aligned}$$

Therefore,

$$\begin{aligned} I_2+I_3 &\le C\int_{\mathbb{R}^n}|\nabla u||\nabla\nabla d||\Delta d|\,\mathrm{d}x \le C\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\Delta d\|^2_{L^{\frac{8n}{3n-2}}} \\ &\le C\|u\|_{H^\alpha}\|\Delta d\|_{L^2}^{\frac{8\beta-n-2}{4\beta}}\|\Lambda^{\beta+2}d\|_{L^2}^{\frac{n+2}{4\beta}} \\ &\le C\|u\|_{H^\alpha}^{\frac{8\beta}{8\beta-n-2}}\|\Delta d\|_{L^2}^2+\frac{1}{4}\|\Lambda^{\beta+2}d\|_{L^2}^2, \end{aligned} \tag{2.6}$$

where $\beta \ge \frac{1}{2}+\frac{n}{4}$ makes $\frac{8\beta}{8\beta-n-2}\le 2$ and the Sobolev embedding inequality $\|\nabla u\|_{L^{\frac{4n}{n+2}}}\le C\|u\|_{H^\alpha}$ is used. For I_4, utilizing the norm $\|d\|_{L^\infty}$ that has been estimated, it can be achieved that

$$\begin{aligned} I_4 &\le \int_{\mathbb{R}^n}|\Delta d|^2+\Delta(|d|^2 d)\cdot\Delta d\,\mathrm{d}x \\ &\le \|\Delta d\|_{L^2}^2+C(\|\Delta|d|^2\|_{L^2}\|d\|_{L^\infty}+\|\Delta d\|_{L^2}\||d|^2\|_{L^\infty})\|\Delta d\|_{L^2} \\ &\le C\|\Delta d\|_{L^2}^2. \end{aligned} \tag{2.7}$$

Here, we have used the following inequality that comes from Lemma 2.1

$$\|\Delta\,|d|^2\|_{L^2}\le C(\|\Delta d\|_{L^2}\|d\|_{L^\infty}+\|\Delta d\|_{L^2}\|d\|_{L^\infty}).$$

Now, inserting the estimates (2.5)–(2.7) into (2.4) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2)+\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\beta+2}d\|_{L^2}^2 \\ &\quad\le\ C(1+\|u\|_{H^\alpha}^2+\|\Delta d\|_{L^2}^2)(1+\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2). \end{aligned}$$

Thanks to Inequality (2.1), applying the Gronwall's inequality yields

$$\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\int_0^T(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\beta+2}d\|_{L^2}^2)\,\mathrm{d}t\le C(\|\nabla u_0\|_{L^2}^2+\|\Delta d_0\|_{L^2}^2). \tag{2.8}$$

Finally, with the estimates mentioned earlier, we are ready to show the global H^s bounds of u and ∇d.

Taking Λ^s and Λ^{s+1} with $s>1+\frac{n}{2}$ on the first equation and second equation in System (1.1), respectively, and multiplying $\Lambda^s u$ and $\Lambda^{s+1}d$ to the resulting equations, then the following equation can be arrived by integration

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^{s+1}d\|_{L^2}^2)+\|\Lambda^{s+\alpha}u\|_{L^2}^2+\|\Lambda^{s+\beta+1}d\|_{L^2}^2 \\ &\quad= -\int_{\mathbb{R}^n}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,\mathrm{d}x-\int_{\mathbb{R}^n}\Lambda^s(\nabla d\cdot\Delta d)\cdot\Lambda^s u\,\mathrm{d}x \\ &\qquad-\int_{\mathbb{R}^n}\Lambda^{s+1}(u\cdot\nabla d)\cdot\Lambda^{s+1}d\,\mathrm{d}x-\int_{\mathbb{R}^n}\Lambda^{s+1}f(d)\cdot\Lambda^{s+1}d\,\mathrm{d}x \\ &\quad\coloneqq J_1+J_2+J_3+J_4. \end{aligned} \tag{2.9}$$

To estimate J_1, by the condition $\nabla\cdot u=0$ and Lemma 2.1, it follows that

$$\begin{aligned} J_1 &= -\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u\,\mathrm{d}x \le \|[\Lambda^s,u\cdot\nabla]u\|_{L^{\frac{4}{3}}}\|\Lambda^s u\|_{L^4} \\ &\le C(\|\nabla u\|_{L^2}\|\Lambda^{s-1}\nabla u\|_{L^4}+\|\Lambda^s u\|_{L^4}\|\nabla u\|_{L^2})\|\Lambda^s u\|_{L^4} \\ &\le C\|\nabla u\|_{L^2}\|\Lambda^s u\|_{L^2}^{\frac{4\alpha-n}{2\alpha}}\|\Lambda^{s+\alpha}u\|_{L^2}^{\frac{n}{2\alpha}} \\ &\le C\|\Lambda^s u\|_{L^2}^2+\frac{1}{4}\|\Lambda^{s+\alpha}u\|_{L^2}^2. \end{aligned} \tag{2.10}$$

Then, using the Sobolev embedding inequality and commutator estimate, we have the following for J_2:

$$\begin{aligned}
J_2 &= -\int_{\mathbb{R}^n} \Lambda^s(\nabla d \cdot \Delta d)\Lambda^s u \,\mathrm{d}x \\
&\le \|\Lambda^{s-\alpha}(\nabla d_j \cdot \Delta d_j)\|_{L^2}\|\Lambda^{s+\alpha}u\|_{L^2} \\
&\le (\|\nabla d\|_{L^{\frac{4n}{n-2}}}\|\Lambda^{s-\alpha}\Delta d\|_{L^{\frac{4n}{n+2}}} + \|\Delta d\|_{L^4}\|\Lambda^{s-\alpha}\nabla d\|_{L^4})\|\Lambda^{s+\alpha}u\|_{L^2} \\
&\le C(\|d\|_{H^{\beta+1}}\|d\|_{H^{s+1}} + \|d\|_{H^{\beta+2}}\|d\|_{H^{s+1}})\|\Lambda^{s+\alpha}u\|_{L^2} \\
&\le C\|d\|_{H^{\beta+2}}\|d\|_{H^{s+1}}\|\Lambda^{s+\alpha}u\|_{L^2} \\
&\le C\|d\|^2_{H^{2+\beta}}\|\Lambda^{s+1}d\|^2_{L^2} + \frac{1}{4}\|\Lambda^{s+\alpha}u\|^2_{L^2} + C\|d\|^2_{H^{\beta+2}}.
\end{aligned} \tag{2.11}$$

Here, we make use of the following Sobolev embedding inequalities:

$$\|\nabla d\|_{L^{\frac{4n}{n-2}}} \le C\|d\|_{H^{\beta+1}} \le C\|d\|_{H^{\beta+2}}, \quad \|\Lambda^{s-\alpha}\Delta d\|_{L^{\frac{4n}{n+2}}} \le C\|d\|_{H^{s+1}},$$

$$\|\Delta d\|_{L^4} \le C\|d\|_{H^{\beta+2}}, \quad \|\Lambda^{s-\alpha}\nabla d\|_{L^4} \le C\|d\|_{H^{s+1}}.$$

Note that these inequalities can also be arrived by the interpolation inequality. For J_3, evaluating similarly as we did to J_2, one has

$$\begin{aligned}
J_3 &= \int_{\mathbb{R}^n} \Lambda^s(u \cdot \nabla d)\Lambda^{s+2}d \,\mathrm{d}x \\
&\le \|\Lambda^s(u \cdot \nabla d)\|_{L^{\frac{4n}{3n-2}}}\|\Lambda^{s+2}d\|_{L^{\frac{4n}{n+2}}} \\
&\le C(\|\Lambda^s u\|_{L^2}\|\nabla d\|_{L^{\frac{4n}{n-2}}} + \|\Lambda^{s+1}d\|_{L^2}\|u\|_{L^{\frac{4n}{n-2}}})\|\Lambda^{s+2}d\|_{L^{\frac{4n}{n+2}}} \\
&\le C(\|\Lambda^s u\|_{L^2}\|d\|_{H^{\beta+1}} + \|\Lambda^{s+1}d\|_{L^2}\|u\|_{H^{\alpha+1}})\|d\|_{H^{s+\beta+1}} \\
&\le \frac{1}{2}\|d\|^2_{H^{s+\beta+1}} + C\|d\|^2_{H^{\beta+2}}\|\Lambda^s u\|^2_{L^2} + C\|u\|^2_{H^{\alpha+1}}\|\Lambda^{s+1}d\|^2_{L^2} \\
&\le \frac{1}{2}\|\Lambda^{s+\beta+1}d\|^2_{L^2} + C(\|d\|^2_{H^{\beta+2}} + \|u\|^2_{H^{\alpha+1}})(\|\Lambda^s u\|^2_{L^2} + \|\Lambda^{s+1}d\|^2_{L^2}) + C\|d\|^2_{L^2}.
\end{aligned} \tag{2.12}$$

Similarly as I_4, Hölder's inequality and Young's inequality imply

$$\begin{aligned}
J_4 &= \int_{\mathbb{R}^n} \Lambda^{s+1}(d - |d|^2 d)\Lambda^{s+1}d \,\mathrm{d}x \\
&\le \|\Lambda^{s+1}d\|^2_{L^2} + \|\Lambda^{s+1}(|d|^2 d)\|_{L^2}\|\Lambda^{s+1}d\|_{L^2} \\
&\le \|\Lambda^{s+1}d\|^2_{L^2} + C(\|d\|_{L^\infty}\|\Lambda^{s+1}\,|d|^2\|_{L^2} + \|\Lambda^{s+1}d\|_{L^2}\||d|^2\|_{L^\infty})\|\Lambda^{s+1}d\|_{L^2} \\
&\le \|\Lambda^{s+1}d\|^2_{L^2} + C\|\Lambda^{s+1}d\|^2_{L^2} \\
&\le C\|\Lambda^{s+1}d\|^2_{L^2}.
\end{aligned} \tag{2.13}$$

Combining up the formulas (2.9)–(2.13), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|^2_{L^2} + \|\Lambda^{s+1}d\|^2_{L^2}) + \|\Lambda^{s+\alpha}u\|^2_{L^2} + \|\Lambda^{s+\beta+1}d\|^2_{L^2} \\
&\quad\le C(1 + \|d\|^2_{H^{\beta+2}} + \|u\|^2_{H^{\alpha+1}})(\|\Lambda^s u\|^2_{L^2} + \|\Lambda^{s+1}d\|^2_{L^2}) + C(\|d\|^2_{L^2} + \|d\|^2_{H^{\beta+2}}).
\end{aligned} \tag{2.14}$$

Here, we can deduce from Inequality (2.8) that

$$\int_0^T (\|u\|^2_{H^{\alpha+1}} + \|d\|^2_{H^{\beta+2}})\,\mathrm{d}t \le C(\|\nabla u_0\|^2_2 + \|\Delta d_0\|^2_2).$$

Thus, it is not difficult to obtain that by employing Gronwall's inequality to (2.14),

$$\|\Lambda^s u\|^2_{L^2} + \|\Lambda^{s+1}d\|^2_{L^2} + \int_0^T (\|\Lambda^{s+\alpha}u\|^2_{L^2} + \|\Lambda^{s+\beta+1}d\|^2_{L^2})\,\mathrm{d}t \le C(\|\Lambda^s u_0\|^2_{L^2} + \|\Lambda^{s+1}d_0\|^2_{L^2}).$$

The proof of Theorem 1.1 is complete.

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