

Research Article

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Composition of some positive linear integral operators

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Abstract: This article is devoted to constructing sequences of integral operators with the same Voronovskaja formula as the generalized Baskakov operators, but having different behavior in other respects. For them, we investigate the eigenstructure, the inverses, and the corresponding Voronovskaja type formulas. A general result of Voronovskaja type for composition of operators is given and applied to the new operators. The asymptotic behavior of differences between the operators is investigated, and as an application, we obtain a formula involving Euler's gamma function.

Keywords: integral operators, composition, eigenstructure, Voronovskaja type formulas, differences of operators, Euler's gamma function

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1 Introduction

Let $c \in \mathbb{R}$, $n \in \mathbb{R}$, $n > c$ for $c \geq 0$ and $-n/c \in \mathbb{N}$ for $c < 0$. Furthermore, let $I_c = [0, \infty)$ for $c \geq 0$ and $I_c = [0, -1/c]$ for $c < 0$. Take $f: I_c \rightarrow \mathbb{R}$ given in such a way that the corresponding integrals and series are convergent.

The Baskakov-type operators are defined by [1]

$$B_n^{[c]}(f; x) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) f\left(\frac{k}{n}\right), \quad (1.1)$$

with the corresponding basis functions

$$p_{n,k}^{[c]}(x) = \begin{cases} \frac{n^k}{k!} x^k e^{-nx}, & c = 0, \\ \frac{n^{c,\bar{k}}}{k!} x^k (1 + cx)^{-\left(\frac{n}{c} + k\right)}, & c \neq 0, \end{cases} \quad (1.2)$$

and $a^{c,\bar{k}} := \prod_{l=0}^{k-1} (a + cl)$, $a^{c,\bar{0}} := 1$.

In case $c < 0$, the sum in (1.1) is finite. For $c = -1$, we recover the Bernstein operators, $c < 0$ leads to the Szász-Mirakjan operators, and for $c = 1$, we obtain the classical Baskakov operators. Therefore, the family $(B_n^{[c]})$ generalizes the important sequences of the classical Bernstein, Szász-Mirakjan, and Baskakov operators.

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Studying the properties of the operators $B_n^{[c]}$ provides an unified vision of the properties of the three classical mentioned operators and suggests new properties of the individual classical operators. In this sense, see [2–5].

The Baskakov-type operators are discrete operators and satisfy the following Voronovskaja type formula (for details and extensions, see [3]).

Theorem 1.1. *Let $c \geq 0$ and $f \in C[0, \infty)$ with $\sup_{t \geq 0} \frac{|f(t)|}{1+t^2} < \infty$. Suppose that $x \in (0, \infty)$, and there exists $f''(x) \in \mathbb{R}$. Then,*

$$\lim_{n \rightarrow \infty} n(B_n^{[c]}(f; x) - f(x)) = \frac{x(1 + cx)}{2} f''(x).$$

In this article, we construct sequences of integral operators with the same Voronovskaja formula, but having different behavior in other respects. For these integral operators, it is easy to obtain explicit expressions of the moments and central moments. Moreover, the eigenstructure can be described in detail, which enables us to obtain properties of the inverse operators on the polynomials. This is not the case with the Baskakov type operators, for which in turn it is easy to give explicit expressions of the images of the exponential functions. With these images, it is possible to investigate the characteristic functions of the random variables associated to the operators and furthermore to obtain convergence results in the spirit of [2,6,7]. The operators that will be introduced and investigated are compositions of the following three operators, the Rathore operator

$$W_n f(x) = \frac{1}{\Gamma(nx)} \int_0^\infty e^{-t} t^{nx-1} f\left(\frac{t}{n}\right) dt,$$

the Gamma operator

$$G_n f(x) = \frac{1}{n!} \int_0^\infty e^{-s} s^n f\left(\frac{nx}{s}\right) ds,$$

and the Post-Widder operator

$$P_n f(x) = \frac{1}{(n-1)!} \left(\frac{n}{x}\right)^n \int_0^\infty e^{-nu/x} u^{n-1} f(u) du.$$

One example is

$$\mathcal{A}_n := W_n P_n.$$

For $c > 0$, we consider the operators

$$\mathcal{A}_{n,c}(f(t); x) := \mathcal{A}_{n/c}\left(f\left(\frac{t}{c}\right); cx\right).$$

In [8], the following operator was introduced

$$\mathcal{F}_n := W_n G_n.$$

We consider also the operators

$$\mathcal{F}_{n,c}(f(t); x) := \mathcal{F}_{n/c}\left(f\left(\frac{t}{c}\right); cx\right), \quad c > 0.$$

Section 2 is devoted to the eigenstructure of the operator $\mathcal{A}_{n,c}$. The results are connected to those presented in [9]. In Section 3, we present Voronovskaja type formulas for the operators $\mathcal{A}_{n,c}$ and $\mathcal{A}_{n,c}^{-1}$. Section 4 is concerned with a general Voronovskaja type result for operators obtained as composition of two positive linear operators. It can be applied to the compositions previously presented. The asymptotic behavior of the differences between the operators $\mathcal{F}_{n,c}$, $\mathcal{A}_{n,c}$, and $B_n^{[c]}$ is investigated in Section 5. As a byproduct we get a new

proof of a relation concerning Euler's gamma function. Moreover, in this section, we give examples of functions for which one of these operators provides a better approximation than another one. Conclusions and further work are discussed in Section 6.

2 Eigenstructure of $\mathcal{A}_{n,c}$

Denote $e_k(t) = t^k$, $t \geq 0$, $k = 0, 1, \dots$. By an elementary computation, we find that

$$\mathcal{A}_{n,c}e_k(x) = \frac{n^{c,\bar{k}}(nx)^{1,\bar{k}}}{n^{2k}}. \quad (2.1)$$

Let Π_n be the space of polynomial functions of degree at most n . From (2.1), it follows that the eigenvalues of the operator $\mathcal{A}_{n,k} : \Pi_n \rightarrow \Pi_n$ are the numbers

$$\alpha_{k,c}^{(n)} = \frac{n^{c,\bar{k}}}{n^k}, \quad 0 \leq k \leq n. \quad (2.2)$$

Let

$$v_{k,c}^{(n)}(x) = \sum_{j=0}^k a(n, k, j, c)x^j, \quad (2.3)$$

be the associated monic eigenpolynomials. This means that

$$\mathcal{A}_{n,c}v_{k,c}^{(n)} = \alpha_{k,c}^{(n)}v_{k,c}^{(n)}, \quad k = 0, \dots, n. \quad (2.4)$$

In particular, $v_{0,c}^{(n)} = e_0$, $v_{1,c}^{(n)} = e_1$, and so

$$a(n, 0, 0, c) = 1, \quad a(n, 1, 0, c) = 0, \quad a(n, 1, 1, c) = 1. \quad (2.5)$$

From (2.1) and (2.3), it follows

$$\mathcal{A}_{n,c}v_{k,c}^{(n)}(x) = \sum_{j=0}^k a(n, k, j, c) \frac{n^{c,\bar{j}}(nx)^{1,\bar{j}}}{n^{2j}} = \frac{n^{c,\bar{k}}}{n^k} \sum_{j=0}^k a(n, k, j, c)x^j. \quad (2.6)$$

From the definition of the Stirling numbers of the first kind $s(j, i)$, we obtain

$$(nx)^{1,\bar{j}} = \sum_{i=0}^j s(j, i)(-1)^{j-i}n^i x^i, \quad (2.7)$$

Combining (2.6) and (2.7), we obtain

$$\sum_{i=0}^k \sum_{j=i}^k a(n, k, j, c) \frac{n^{c,\bar{j}}s(j, i)(-1)^{j-i}n^i}{n^{2j}} x^i = \sum_{i=0}^k \frac{n^{c,\bar{k}}}{n^k} a(n, k, i, c)x^i,$$

and therefore,

$$\sum_{j=i}^k a(n, k, j, c) \frac{n^{c,\bar{j}}s(j, i)(-1)^{j-i}}{n^{2j}} = \frac{n^{c,\bar{k}}}{n^{k+i}} a(n, k, i, c), \quad i = 0, \dots, k.$$

This can be written as

$$a(n, k, i, c) \frac{n^{c,\bar{i}}}{n^{2i}} + \sum_{j=i+1}^k a(n, k, j, c) \frac{n^{c,\bar{j}}s(j, i)(-1)^{j-i}}{n^{2j}} = \frac{n^{c,\bar{k}}}{n^{k+i}} a(n, k, i, c), \quad i = 0, \dots, k.$$

Solving for $a(n, k, i, c)$, we obtain

$$a(n, k, i, c) = \frac{n^{2i+k}}{n^{c,\bar{k}}n^i - n^{c,\bar{i}}n^k} \sum_{j=i+1}^k a(n, k, j, c) \frac{n^{c,\bar{j}}s(j, i)(-1)^{j-i}}{n^{2j}}, \quad i = k-1, k-2, \dots, 0. \quad (2.8)$$

Since $v_{k,c}^{(n)}$ is a monic polynomial, we have

$$a(n, k, k, c) = 1. \quad (2.9)$$

From (2.8) and (2.9), we can determine recurrently the coefficients $a(n, k, i, c)$, $i = k-1, k-2, \dots, 0$.

Denote

$$a^*(k, j, c) := \lim_{n \rightarrow \infty} a(n, k, j, c), \quad j = 0, \dots, k.$$

From (2.5), we obtain

$$a^*(0, 0, c) = 1, \quad a^*(1, 0, c) = 0, \quad a^*(1, 1, c) = 1.$$

Moreover, (2.9) shows that

$$a^*(k, k, c) = 1. \quad (2.10)$$

Lemma 2.1. For $k \geq 2$ and $j = 0, \dots, k$, one has

$$a^*(k, j, c) = \left(-\frac{1}{c}\right)^{k-j} \prod_{l=1}^{k-j} \frac{(k-l+1)(k-l)}{l(l-2k+1)}. \quad (2.11)$$

Proof. We prove (2.11) by induction. It is true for $j = k$ according to (2.10). Suppose that it is true for all $j \geq i+1$, and let us prove it for $j = i$.

Formula (2.8) can be written as follows:

$$a(n, k, i, c) = \frac{n^{2i+k}}{n^{c,\bar{k}}n^i - n^{c,\bar{i}}n^k} \left\{ \sum_{j=i+2}^k a(n, k, j, c) \frac{n^{c,\bar{j}}s(j, i)(-1)^{j-i}}{n^{2j}} - a(n, k, i+1, c) \frac{n^{c,\bar{i+1}}s(i+1, i)}{n^{2i+2}} \right\}.$$

Passing to the limit when $n \rightarrow \infty$, we obtain after some calculations

$$a^*(k, i, c) = \frac{i(i+1)}{c(k-i)(k+i-1)} a^*(k, i+1, c).$$

Consequently, using (2.11) for $j = i+1$, it follows that

$$\begin{aligned} a^*(k, i, c) &= \frac{i(i+1)}{c(k-i)(k+i-1)} \cdot \left(-\frac{1}{c}\right)^{k-i-1} \prod_{l=1}^{k-i-1} \frac{(k-l+1)(k-l)}{l(l-2k+1)} \\ &= \left(-\frac{1}{c}\right)^{k-i} \prod_{l=1}^{k-i} \frac{(k-l+1)(k-l)}{l(l-2k+1)}, \end{aligned} \quad (2.12)$$

and so the proof by induction is complete. $\square$

Remark 2.1. Using (2.3), we obtain

$$v_k^*(x) := \lim_{n \rightarrow \infty} v_{k,c}^{(n)}(x) = \sum_{i=0}^k a^*(k, i, c) x^i = \sum_{i=0}^k \left(-\frac{1}{c}\right)^{k-i} c^*(i, k) x^i, \quad (2.13)$$

where $c^*(i, k)$ are the coefficients from [9, Theorem 4.1].

More precisely,

$$\sum_{i=0}^k c^*(i, k) x^i = p_k^*(x), \quad k \geq 0,$$

where $p_0^*(x) = 1$, $p_1^*(x) = x - \frac{1}{2}$, and for $k \geq 2$,

$$p_k^*(x) = \frac{k!(k-2)!}{(2k-2)!} x(x-1) P_{k-2}^{(1,1)}(2x-1).$$

Here, $P_m^{(1,1)}$ are the Jacobi polynomials, orthogonal with respect to the weight $(1-t)(1+t)$ on the interval $[-1, 1]$.

3 Voronovskaja type results

By using a general result from [3], we prove a Voronovskaja type formula for the operators $\mathcal{A}_{n,c}$. The eigenstructure of $\mathcal{A}_{n,c}$ restricted to polynomials is used to obtain a Voronovskaja type formula for the sequence $(\mathcal{A}_{n,c}^{-1})_{n \geq 1}$.

Let $\varphi \in C^2[0, \infty)$, $\varphi(0) = 0$, $\varphi'(t) > 0$, $t \in (0, \infty)$, $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. Denote

$$E_\varphi := \left\{ f \in C[0, \infty) \mid \sup_{t \geq 0} \frac{|f(t)|}{1 + \varphi^2(t)} < \infty \right\}$$

and $\|f\|_\varphi = \sup_{t \geq 0} \frac{|f(t)|}{1 + \varphi^2(t)}$, $f \in E_\varphi$.

Theorem 3.1. [3] Let $x > 0$ be given, and let $\Psi_x(t) := \varphi(t) - \varphi(x)$, $t \geq 0$. Denote by E_φ^x a linear subspace of $C[0, \infty)$ such that $E_\varphi \subset E_\varphi^x$ and $\Psi_x^4 \in E_\varphi^x$. Let $L_n : E_\varphi^x \rightarrow C[0, \infty)$ be a sequence of positive linear operators such that

- (i) $\lim_{n \rightarrow \infty} n(L_n e_0(x) - 1) = 0$,
- (ii) $\lim_{n \rightarrow \infty} n L_n \Psi_x(x) = b(x)$,
- (iii) $\lim_{n \rightarrow \infty} n L_n \Psi_x^2(x) = 2a(x)$,
- (iv) $\lim_{n \rightarrow \infty} n L_n \Psi_x^4(x) = 0$.

If $f \in E_\varphi$ and there exists $f''(x) \in \mathbb{R}$, then

$$\lim_{n \rightarrow \infty} n(L_n f(x) - f(x)) = \frac{a(x)}{\varphi'(x)^2} f''(x) + \frac{b(x) \varphi'(x)^2 - a(x) \varphi''(x)}{\varphi'(x)^3} f'(x). \quad (3.1)$$

We apply Theorem 3.1 for the sequence $(\mathcal{A}_{n,c})_{n \geq 1}$ with $\varphi(t) = t$. To this end, we need the following identities

$$\begin{aligned} \mathcal{A}_{n,c} e_0(x) &= 1, \quad \mathcal{A}_n \Psi_x(x) = 0, \\ \mathcal{A}_{n,c} \Psi_x^2(x) &= \frac{x(cnx + c + n)}{n^2}, \\ \mathcal{A}_{n,c} \Psi_x^4(x) &= \frac{3x}{n^6} \{x(cx+1)^2 n^4 + 2(cx+1)(c^2 x^2 + 6cx+1)n^3 + c(12c^2 x^2 + 35cx+12)n^2 + 22c^2(cx+1)n + 12c^3\}. \end{aligned}$$

It follows immediately that conditions (i)–(iv) are fulfilled with $b(x) = 0$ and $a(x) = \frac{x(cx+1)}{2}$. So we have proved the following Voronovskaja type result.

Theorem 3.2. Let $\varphi(t) = t$, $t \geq 0$, $f \in E_\varphi$ and suppose that there exists $f''(x) \in \mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} n(\mathcal{A}_{n,c} f(x) - f(x)) = \frac{x(cx+1)}{2} f''(x).$$

In the sequel we consider the restriction $\mathcal{A}_{n,c} : \Pi_n \rightarrow \Pi_n$, which is bijective because all the eigenvalues $\alpha_{k,c}^{(n)}$, $k = 0, \dots, n$, are different from 0. We will establish a Voronovskaja type result for $\mathcal{A}_{n,c}^{-1} : \Pi_n \rightarrow \Pi_n$. Theorem 3.1 is not applicable because the operators $\mathcal{A}_{n,c}^{-1}$ are not positive. Our main tool will be the previously described eigenstructure.

Theorem 3.3. Let $m \geq 1$, $p \in \Pi_m$, and $x \geq 0$ be given. Then

$$\lim_{n \rightarrow \infty} n(\mathcal{A}_{n,c}^{-1} p(x) - p(x)) = -\frac{x(cx+1)}{2} p''(x). \quad (3.2)$$

Proof. Let $n \geq m$. Then $\{v_{0,c}^{(n)}(x), v_{1,c}^{(n)}(x), \dots, v_{m,c}^{(n)}(x)\}$ and $\{v_{0,c}^*(x), v_{1,c}^*(x), \dots, v_{m,c}^*(x)\}$ are bases of Π_m . Therefore, $p \in \Pi_m$ can be represented as follows:

$$p(x) = \sum_{k=0}^m a_{nk} v_{k,c}^{(n)}(x),$$

respectively

$$p(x) = \sum_{k=0}^m a_k v_{k,c}^*(x),$$

with $a_{n,k} \in \mathbb{R}$ and $a_k \in \mathbb{R}$.

We know from Remark 2.1 that $\lim_{n \rightarrow \infty} v_{k,c}^{(n)} = v_{k,c}^*$, and consequently, $\lim_{n \rightarrow \infty} a_{nk} = a_k$, $k = 0, \dots, m$.

Using (2.4), we can write

$$\mathcal{A}_{n,c}^{-1} p(x) = \sum_{k=0}^m a_{nk} \frac{1}{\alpha_{k,c}^{(n)}} v_{k,c}^{(n)}(x),$$

and consequently from (2.2), we obtain

$$\lim_{n \rightarrow \infty} n(\mathcal{A}_{n,c}^{-1} p(x) - p(x)) = \lim_{n \rightarrow \infty} n \sum_{k=0}^m a_{nk} v_{k,c}^{(n)}(x) \left(\frac{1}{\alpha_{k,c}^{(n)}} - 1 \right) = - \sum_{k=0}^m a_k v_{k,c}^*(x) \frac{c(k-1)k}{2}. \quad (3.3)$$

To prove (3.2), it remains to verify that

$$c \sum_{k=0}^m a_k v_{k,c}^*(x) (k-1)k = x(1+cx)p''(x),$$

namely,

$$x(1+cx)(v_{k,c}^*(x))'' = c(k-1)k v_{k,c}^*(x). \quad (3.4)$$

From (2.12) and (2.13), we obtain

$$v_{k,c}^*(x) = \sum_{j=0}^k \left(-\frac{1}{c} \right)^{k-j} \prod_{l=1}^{k-j} \frac{(k-l+1)(k-l)}{l(l-2k+1)} x^j.$$

Let us compute

$$\begin{aligned} x(1+cx)(v_{k,c}^*(x))'' &= \sum_{j=2}^k \left(-\frac{1}{c} \right)^{k-j} \prod_{l=1}^{k-j} \frac{(k-l+1)(k-l)}{l(l-2k+1)} j(j-1) x^{j-1} \\ &\quad + c \sum_{j=2}^k \left(-\frac{1}{c} \right)^{k-j} \prod_{l=1}^{k-j} \frac{(k-l+1)(k-l)}{l(l-2k+1)} j(j-1) x^j \\ &= \sum_{j=0}^{k-1} \left(-\frac{1}{c} \right)^{k-j-1} \prod_{l=1}^{k-j-1} \frac{(k-l+1)(k-l)}{l(l-2k+1)} j(j+1) x^j \\ &\quad + c \sum_{j=0}^k \left(-\frac{1}{c} \right)^{k-j} \prod_{l=1}^{k-j} \frac{(k-l+1)(k-l)}{l(l-2k+1)} j(j-1) x^j \\ &= ck(k-1) \sum_{j=0}^k \left(-\frac{1}{c} \right)^{k-j} \prod_{l=1}^{k-j} \frac{(k-l+1)(k-l)}{l(l-2k+1)} x^j \\ &= c(k-1)k v_{k,c}^*(x). \end{aligned}$$

Thus, (3.4) is verified and the proof is complete. $\square$

Remark 3.1. Voronovskaja type results for the inverses of Bernstein, Durrmeyer, Kantorovich, genuine-Bernstein-Durrmeyer, $\overline{B}_n$, and Bernstein-Schnabl operators acting on polynomials were obtained in [10].

In all these cases, the differential operators from the right-hand side of the Voronovskaja formulas for the operators, and their inverses have the sum equal to zero. For a general result, in this sense, see [10].

4 A Voronovskaja type result for composition of operators

We provide a general result concerning the Voronovskaja type formula for composition of operators. Consider two sequences of positive linear operators R_n, Q_n satisfying the hypotheses of Theorem 3.1 with $\varphi(t) = t$, $t \in [0, \infty)$. In particular, suppose that

$$R_n e_0 = Q_n e_0 = e_0, \quad n \geq 1, \quad (4.1)$$

$$R_n(\Pi_m) \subset \Pi_m, \quad Q_n(\Pi_m) \subset \Pi_m, \quad n, m \geq 1, \quad (4.2)$$

$$\lim_{n \rightarrow \infty} n R_n(e_1 - x e_0)(x) = b_1(x), \quad (4.3)$$

$$\lim_{n \rightarrow \infty} n Q_n(e_1 - x e_0)(x) = b_2(x), \quad (4.4)$$

$$\lim_{n \rightarrow \infty} n R_n(e_1 - x e_0)^2(x) = 2a_1(x), \quad (4.5)$$

$$\lim_{n \rightarrow \infty} n Q_n(e_1 - x e_0)^2(x) = 2a_2(x), \quad (4.6)$$

$$\lim_{n \rightarrow \infty} n R_n(e_1 - x e_0)^4(x) = \lim_{n \rightarrow \infty} n Q_n(e_1 - x e_0)^4(x) = 0. \quad (4.7)$$

Theorem 4.1. *Under the aforementioned assumptions, it holds that*

$$\lim_{n \rightarrow \infty} n(Q_n R_n f(x) - f(x)) = \lim_{n \rightarrow \infty} n(R_n Q_n f(x) - f(x)) = (a_1(x) + a_2(x))f''(x) + (b_1(x) + b_2(x))f'(x), \quad x > 0. \quad (4.8)$$

Proof. Formulas (4.1)–(4.7) show that the hypotheses of Theorem 3.1 are satisfied with $\varphi(t) = t$, $t \in [0, \infty)$. Consequently,

$$\begin{aligned} \lim_{n \rightarrow \infty} n(R_n f(x) - f(x)) &= a_1(x)f''(x) + b_1(x)f'(x), \\ \lim_{n \rightarrow \infty} n(Q_n f(x) - f(x)) &= a_2(x)f''(x) + b_2(x)f'(x). \end{aligned}$$

Denote

$$\begin{aligned} U_{n,j}(x) &:= \frac{1}{j!} R_n(e_1 - x e_0)^j(x), \quad V_{n,i}(x) := \frac{1}{i!} Q_n(e_1 - x e_0)^i(x), \\ M_{n,m}(x) &:= \frac{1}{m!} (R_n Q_n)(e_1 - x e_0)^m(x) \quad m, i, j \geq 0, x \in [0, \infty). \end{aligned}$$

According to [11],

$$M_{n,m} = \sum_{\substack{i,k \geq 0 \\ i+k=m}} \sum_{j=k}^m \binom{j}{k} U_{n,j} V_{n,i}^{(j-k)}.$$

In particular,

$$M_{n,1} = U_{n,1} V_{n,0} + U_{n,0} V_{n,1} + U_{n,1} V_{n,1}^{(1)}, \quad (4.9)$$

$$M_{n,2} = U_{n,2} V_{n,0} + U_{n,1} V_{n,1} + 2U_{n,2} V_{n,1}^{(1)} + U_{n,0} V_{n,2} + U_{n,1} V_{n,2}^{(1)} + U_{n,2} V_{n,2}^{(2)} \quad (4.10)$$

and

$$\begin{aligned} M_{n,4} = & U_{n,4}V_{n,0} + U_{n,3}V_{n,1} + 4U_{n,4}V_{n,1}^{(1)} + U_{n,2}V_{n,2} + 3U_{n,3}V_{n,2}^{(1)} \\ & + 6U_{n,4}V_{n,2}^{(2)} + U_{n,1}V_{n,3} + 2U_{n,2}V_{n,3}^{(1)} + 3U_{n,3}V_{n,3}^{(2)} + 4U_{n,4}V_{n,3}^{(3)} \\ & + U_{n,0}V_{n,4} + U_{n,1}V_{n,4}^{(1)} + U_{n,2}V_{n,4}^{(2)} + U_{n,3}V_{n,4}^{(3)} + U_{n,4}V_{n,4}^{(4)}. \end{aligned} \quad (4.11)$$

We have $R_n Q_n e_0 = e_0$, $n \geq 1$. From (4.9), (4.3), and (4.4), we obtain after some calculation

$$\lim_{n \rightarrow \infty} n R_n Q_n (e_1 - x e_0)(x) = b_1(x) + b_2(x). \quad (4.12)$$

Similarly, from (4.10), (4.5), and (4.6), we obtain

$$\lim_{n \rightarrow \infty} n R_n Q_n (e_1 - x e_0)^2(x) = 2a_1(x) + 2a_2(x), \quad (4.13)$$

while (4.11) and (4.7) lead to

$$\lim_{n \rightarrow \infty} n R_n Q_n (e_1 - x e_0)^4(x) = 0. \quad (4.14)$$

Now Theorem 3.1 shows that

$$\lim_{n \rightarrow \infty} n(R_n Q_n f(x) - f(x)) = (a_1(x) + a_2(x))f''(x) + (b_1(x) + b_2(x))f'(x). \quad (4.15)$$

Clearly, the same arguments produce

$$\lim_{n \rightarrow \infty} n(Q_n R_n f(x) - f(x)) = (a_1(x) + a_2(x))f''(x) + (b_1(x) + b_2(x))f'(x). \quad (4.16)$$

□

Remark 4.1. By direct calculation, it can be proved that under the hypotheses (4.1)–(4.2), conditions (4.3)–(4.7) are equivalent to

$$\lim_{n \rightarrow \infty} n(R_n(f; x) - f(x)) = a_1(x)f''(x) + b_1(x)f'(x), \quad (4.17)$$

$$\lim_{n \rightarrow \infty} n(Q_n(f; x) - f(x)) = a_2(x)f''(x) + b_2(x)f'(x). \quad (4.18)$$

In other words, (4.1), (4.2), (4.17), and (4.18) imply (4.15) and (4.16).

Remark 4.2. It is not difficult to establish the Voronovskaja type results for Post-Widder and for Rathore operators [12]. By using them in conjunction with Theorem 4.1, we obtain another proof of Theorem 3.2 expressing the Voronovskaja formula for the operators $\mathcal{A}_{n,c}$. Starting with the Voronovskaja formulas for Rathore and Gamma operators and using again Theorem 4.1, we obtain the Voronovskaja formula for the sequence $(\mathcal{F}_{n,c})$. It was obtained in [13] with a different approach. The Voronovskaja formula for the sequence $(\mathcal{F}_n)$ was established in [14].

5 Differences of operators and their asymptotic behavior

Differences of operators were investigated from various points of view in [15–18] and the references therein.

One method is based on Voronovskaja type formulas. Let us describe it briefly.

Let $(H_n)_{n \geq 1}$ and $(K_n)_{n \geq 1}$ be two sequences of positive linear operators for which we know Voronovskaja formulas of the following form:

$$\lim_{n \rightarrow \infty} n(H_n f(x) - f(x)) = V_1 f(x), \quad (5.1)$$

$$\lim_{n \rightarrow \infty} n(K_n f(x) - f(x)) = V_2 f(x), \quad (5.2)$$

$$\lim_{n \rightarrow \infty} n[n(H_n f(x) - f(x)) - V_1 f(x)] = V_3 f(x), \quad (5.3)$$

$$\lim_{n \rightarrow \infty} n[n(K_n f(x) - f(x)) - V_2 f(x)] = V_4 f(x), \quad (5.4)$$

$$\lim_{n \rightarrow \infty} n\{n[H_n f(x) - f(x)] - V_1 f(x)\} = V_5 f(x), \quad (5.5)$$

$$\lim_{n \rightarrow \infty} n\{n[K_n f(x) - f(x)] - V_2 f(x)\} = V_6 f(x). \quad (5.6)$$

From (5.1) and (5.2), we obtain

$$\lim_{n \rightarrow \infty} n(H_n f(x) - K_n f(x)) = V_1 f(x) - V_2 f(x),$$

and this gives us an information about how close are the operators H_n and K_n .

If $V_1 = V_2$, we obtain

$$\lim_{n \rightarrow \infty} n^2(H_n f(x) - K_n f(x)) = V_3 f(x) - V_4 f(x), \quad (5.7)$$

which shows that H_n and K_n are closer than in the previous case. Pairs (H_n, K_n) with $V_1 = V_2$ can be constructed directly, as in the previous sections. In Subsection 5.1, we consider the pairs $(\mathcal{A}_{n,c}, \mathcal{F}_{n,c}), (\mathcal{A}_{n,c}, B_n^{[c]}), (\mathcal{F}_{n,c}, B_n^{[c]})$ and illustrate (5.7).

If in addition $V_3 = V_4$, then

$$\lim_{n \rightarrow \infty} n^3(H_n f(x) - K_n f(x)) = V_5 f(x) - V_6 f(x). \quad (5.8)$$

In Subsection 5.2, we present another approach, illustrating (5.8). Namely, we consider operators $H_n = P_n \circ Q_n$ and $K_n = Q_n \circ P_n$ for suitable P_n, Q_n . In this context, Theorem 4.1 guarantees that $V_1 = V_2$.

5.1 Comparison based on asymptotic behavior

In this subsection, we investigate the asymptotic behavior of the differences $\mathcal{F}_{n,c} - \mathcal{A}_{n,c}$, $\mathcal{F}_{n,c} - B_n^{[c]}$, and $\mathcal{A}_{n,c} - B_n^{[c]}$. This enables us to construct functions for which the approximation provided by one of the operators $\mathcal{F}_{n,c}, \mathcal{A}_{n,c}, B_n^{[c]}$ is better than the approximation furnished by another one.

Theorem 5.1. *Let f be in the domains of $\mathcal{F}_{n,c}$ and $\mathcal{A}_{n,c}$ and $x \in [0, \infty)$ such that $f^{(4)}(x)$ exists and is finite. Then*

$$\lim_{n \rightarrow \infty} n^2(\mathcal{F}_{n,c} f(x) - \mathcal{A}_{n,c} f(x)) = \frac{1}{2} c^2 x^2 f''(x) + \frac{1}{3} c^2 x^3 f'''(x). \quad (5.9)$$

Proof. Let $\mu_{n,j}(x) := \mathcal{F}_{n,c}(e_1 - x e_0)^j(x)$, $\nu_{n,j}(x) := \mathcal{A}_{n,c}(e_1 - x e_0)^j(x)$. It is not difficult to prove that

$$\mu_{n,0}(x) = \nu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \nu_{n,1}(x) = 0, \quad (5.10)$$

$$\lim_{n \rightarrow \infty} n^2(\mu_{n,2}(x) - \nu_{n,2}(x)) = c^2 x^2, \quad (5.11)$$

$$\lim_{n \rightarrow \infty} n^2(\mu_{n,3}(x) - \nu_{n,3}(x)) = 2c^2 x^3, \quad (5.12)$$

$$\lim_{n \rightarrow \infty} n^2(\mu_{n,4}(x) - \nu_{n,4}(x)) = 0, \quad (5.13)$$

$$\lim_{n \rightarrow \infty} n^2 \mu_{n,5}(x) = \lim_{n \rightarrow \infty} n^2 \mu_{n,6}(x) = 0, \quad (5.14)$$

$$\lim_{n \rightarrow \infty} n^2 \nu_{n,5}(x) = \lim_{n \rightarrow \infty} n^2 \nu_{n,6}(x) = 0. \quad (5.15)$$

Now according to a classical result of Sikkema [19], one has

$$\lim_{n \rightarrow \infty} n^2 \left\{ \mathcal{F}_{n,c} f(x) - f(x) - \mu_{n,1}(x) f'(x) - \frac{1}{2!} \mu_{n,2}(x) f''(x) - \frac{1}{3!} \mu_{n,3}(x) f'''(x) - \frac{1}{4!} \mu_{n,4}(x) f^{(4)}(x) \right\} = 0, \quad (5.16)$$

$$\lim_{n \rightarrow \infty} n^2 \left\{ \mathcal{A}_{n,c} f(x) - f(x) - v_{n,1}(x) f'(x) - \frac{1}{2!} v_{n,2}(x) f''(x) - \frac{1}{3!} v_{n,3}(x) f'''(x) - \frac{1}{4!} v_{n,4}(x) f^{(4)}(x) \right\} = 0. \quad (5.17)$$

By using (5.10)–(5.17), we obtain (5.9), and this concludes the proof. $\square$

Remark 5.1. In a similar way, one can prove that

$$\lim_{n \rightarrow \infty} n^2 (\mathcal{F}_{n,c} f(x) - B_n^{[c]} f(x)) = \frac{x(1+cx)}{6} (3cf''(x) + (1+2cx)f'''(x)), \quad (5.18)$$

$$\lim_{n \rightarrow \infty} n^2 (\mathcal{A}_{n,c} f(x) - B_n^{[c]} f(x)) = \frac{x}{6} (3cf''(x) + (1+3cx)f'''(x)). \quad (5.19)$$

Remark 5.2. Let $x > 0$ be fixed such that $f''(x) > 0$ and $f'''(x) > 0$. Then (5.18) shows that for sufficiently large n we have

$$\mathcal{F}_{n,c} f(x) \geq B_n^{[c]} f(x). \quad (5.20)$$

Moreover, if f is convex, then (5.20) can be completed as follows:

$$\mathcal{F}_{n,c} f(x) \geq B_n^{[c]} f(x) \geq f(x). \quad (5.21)$$

Remark 5.3. Let $Uf(x) := 3cf''(x) + (1+2cx)f'''(x)$, $x \geq 0$, $c > 0$. For $\alpha \in \mathbb{R}$ let $f_\alpha(x) := (1+2cx)^\alpha$, $x \geq 0$. Then $Uf_\alpha(x) = 4c^3\alpha(\alpha-1)(2\alpha-1)(1+2cx)^{\alpha-2}$. Taking into account the sign of $Uf_\alpha(x)$, (5.18) shows that for sufficiently large n ,

$$\begin{aligned} f_\alpha &\leq \mathcal{F}_{n,c} f_\alpha \leq B_n^{[c]} f_\alpha, & \text{for } \alpha \in (-\infty, 0), \\ f_\alpha &\geq \mathcal{F}_{n,c} f_\alpha \geq B_n^{[c]} f_\alpha, & \text{for } \alpha \in \left[0, \frac{1}{2}\right], \\ \mathcal{F}_{n,c} f_\alpha &\geq B_n^{[c]} f_\alpha \geq f_\alpha, & \text{for } \alpha \in (1, \infty). \end{aligned}$$

This shows that for $\alpha \in (-\infty, 0) \cup \left[0, \frac{1}{2}\right]$, the approximation of f_α furnished by $\mathcal{F}_{n,c}$ is better than the approximation furnished by $B_n^{[c]}$.

Remark 5.4. Similar inequalities as in Remark 5.2 and Remark 5.3 involving $\mathcal{A}_{n,c}$ and $\mathcal{F}_{n,c}$, respectively, $\mathcal{A}_{n,c}$ and $B_n^{[c]}$, can be obtained by using (5.9), respectively (5.19).

Table 1: Approximation by the operators $\mathcal{F}_{n,1}$ and $B_n^{[1]}$

x	1	1.1	1.2	1.3	1.4
f	5.0000	5.929	6.912	7.943	9.016
$\mathcal{F}_{n,1}f$	5.117346939	6.049999999	7.033673469	8.061989797	9.128571427
$B_n^{[1]}f$	5.117600000	6.050783200	7.035129600	8.064274400	9.131852800
x	1.5	1.6	1.7	1.8	1.9
f	10.125	11.264	12.427	13.608	14.801
$\mathcal{F}_{n,1}f$	10.22704081	11.35102040	12.49413265	13.65000000	14.81224490
$B_n^{[1]}f$	10.23150000	11.35685120	12.50154160	13.65920640	14.82348080

Example 5.1. Let $f(x) = 6x^2 - x^3$. Then, we have

$$\mathcal{F}_{n,1}f(x) = -\frac{x(nx+1)(nx-6n+14)}{(n-2)(n-1)},$$

$$B_n^{[1]}f(x) = -\frac{x(n^2x^2 - 6n^2x + 3nx^2 - 3nx + 2x^2 - 6n + 3x + 1)}{n^2}.$$

Table 1 presents numerical values for $n = 50$. We can remark that for this specific example the approximation provided by the operator $\mathcal{F}_{n,1}$ is better than that provided by $B_n^{[1]}$.

Remark 5.5. As an application of Theorem 5.1, we give an alternative proof of the following known result concerning Euler's Gamma function,

$$\lim_{n \rightarrow \infty} n^2 \left(\frac{\Gamma(n+1-\lambda)}{n^{1-\lambda}\Gamma(n)} - \frac{\Gamma(n+\lambda)}{n^\lambda\Gamma(n)} \right) = \frac{\lambda(\lambda-1)(2\lambda-1)}{6}, \quad \lambda \in \mathbb{R}. \quad (5.22)$$

To prove (5.22), let $e_\lambda(t) = t^\lambda$, $t > 0$. For $n \in \mathbb{N}$ such that $n + \lambda > 0$ and $n + 1 - \lambda > 0$, we have

$$\mathcal{F}_{n,1}e_\lambda(1) = \frac{1}{n!} \frac{\Gamma(n+\lambda)}{\Gamma(n)} \Gamma(n+1-\lambda), \quad (5.23)$$

$$\mathcal{A}_{n,1}e_\lambda(1) = \frac{1}{n!} \frac{\Gamma(n+\lambda)}{\Gamma(n)} \frac{\Gamma(n+\lambda)}{n^{2\lambda-1}}. \quad (5.24)$$

According to Theorem 5.1,

$$\lim_{n \rightarrow \infty} n^2 (\mathcal{F}_{n,1}e_\lambda(1) - \mathcal{A}_{n,1}e_\lambda(1)) = \frac{\lambda(\lambda-1)(2\lambda-1)}{6}. \quad (5.25)$$

From (5.23), (5.24), and (5.25), we deduce

$$\lim_{n \rightarrow \infty} n^2 (\mathcal{F}_{n,1}e_\lambda(1) - \mathcal{A}_{n,1}e_\lambda(1)) = \lim_{n \rightarrow \infty} n^2 \frac{\Gamma(n+\lambda)}{n^\lambda\Gamma(n)} \left(\frac{\Gamma(n+1-\lambda)}{n^{1-\lambda}\Gamma(n)} - \frac{\Gamma(n+\lambda)}{n^\lambda\Gamma(n)} \right). \quad (5.26)$$

Since $\lim_{n \rightarrow \infty} \frac{\Gamma(n+\lambda)}{n^\lambda\Gamma(n)} = 1$ from (5.25) and (5.26), we obtain (5.22).

5.2 Lupas operators and Phillips operators

In this subsection, we consider the Lupas operators [20]

$$U_n^1 f(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^{1,\bar{k}}}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0,$$

and the Phillips operators [21]

$$\tilde{S}_n f(x) = n \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k-1}(t) f(t) dt + e^{-nx} f(0), \quad s_{n,k}(x) = \frac{(nx)^k}{k!} e^{-nx}, \quad x \geq 0.$$

Recall the Szasz-Mirakjan operators

$$S_n f(x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \geq 0.$$

Then $U_n^1 = W_n \circ S_n$ and $\tilde{S}_n = S_n \circ W_n$. This means that the sequences (U_n^1) and $(\tilde{S}_n)$ have the same Voronovskaja operator of order one (Theorem 4.1). In the language of equations (5.1) and (5.2), we have an example with $V_1 = V_2$. We have also $V_3 = V_4$. For related results, see [10] and [22].

Theorem 5.2. Let f be in the domains of U_n^1 and $\tilde{S}_n$ and $x \in [0, \infty)$ such that $f^{(6)}(x)$ exists and is finite. Then

$$\lim_{n \rightarrow \infty} n^3 (U_n^1 f(x) - \tilde{S}_n f(x)) = \frac{x}{12} f^{(4)}(x). \quad (5.27)$$

Proof. Let $\omega_{n,j}(x) = U_n^1(e_1 - xe_0)^j(x)$, $\theta_{n,j}(x) = \tilde{S}_n(e_1 - xe_0)^j(x)$. It is not difficult to prove that

$$\omega_{n,0}(x) = \theta_{n,0}(x) = 1, \quad \omega_{n,1}(x) = \theta_{n,1}(x) = 0, \quad (5.28)$$

$$\lim_{n \rightarrow \infty} n^3 (\omega_{n,j}(x) - \theta_{n,j}(x)) = 0, \quad j \in \{2, 3, 5, 6\} \quad (5.29)$$

$$\lim_{n \rightarrow \infty} n^3 (\omega_{n,4}(x) - \theta_{n,4}(x)) = 2x, \quad (5.30)$$

$$\lim_{n \rightarrow \infty} n^3 \omega_{n,7}(x) = \lim_{n \rightarrow \infty} n^3 \omega_{n,8}(x) = 0, \quad (5.31)$$

$$\lim_{n \rightarrow \infty} n^3 \theta_{n,7}(x) = \lim_{n \rightarrow \infty} n^3 \theta_{n,8}(x) = 0. \quad (5.32)$$

By Sikkema's theorem [19],

$$\begin{aligned} \lim_{n \rightarrow \infty} n^3 \left\{ U_n^1 f(x) - f(x) - \omega_{n,1}(x) f'(x) - \frac{1}{2!} \omega_{n,2}(x) f''(x) - \frac{1}{3!} \omega_{n,3}(x) f'''(x) - \frac{1}{4!} \omega_{n,4}(x) f^{(4)}(x) \right. \\ \left. - \frac{1}{5!} \omega_{n,5}(x) f^{(5)}(x) - \frac{1}{6!} \omega_{n,6}(x) f^{(6)}(x) \right\} = 0, \end{aligned} \quad (5.33)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n^3 \left\{ \tilde{S}_n f(x) - f(x) - \theta_{n,1}(x) f'(x) - \frac{1}{2!} \theta_{n,2}(x) f''(x) - \frac{1}{3!} \theta_{n,3}(x) f'''(x) - \frac{1}{4!} \theta_{n,4}(x) f^{(4)}(x) \right. \\ \left. - \frac{1}{5!} \theta_{n,5}(x) f^{(5)}(x) - \frac{1}{6!} \theta_{n,6}(x) f^{(6)}(x) \right\} = 0. \end{aligned} \quad (5.34)$$

By using (5.28)–(5.34), we obtain (5.27) and this concludes the proof. $\square$

6 Conclusions and further work

It is well known that with the generalized Baskakov operators, it is possible to give explicit expressions of the images of exponential functions. This fact is very useful for getting certain results concerning convergence of sequences of positive linear operators. But in this context, it is not easy to compute the moments of the operators. In this article, we construct sequences of integral operators having the same Voronovskaja formula as the generalized Baskakov operators and for which the moments and the central moments can be explicitly computed. Consequently, it is possible to investigate the eigenstructure and the inverse of such an integral operator. Indeed, the eigenstructure is completely described. The Voronovskaja type results for the newly introduced operators and for their inverses are provided. Since our operators are compositions of other operators, we obtain a general Voronovskaja type result for such compositions. Differences between the new operators are investigated from the point of view of the asymptotic behavior, and the results are applied to obtain an alternative proof of a formula involving the Euler's gamma function.

There is a rich literature concerning the convergence of some special sequences of positive linear operators towards suitable positive linear operators ([2,7,23,24] and the references therein). In this context, the following relations are involved:

$$G_m \left(f \left(\frac{t}{v} \right); vx \right) = G_m(f(t); x), \quad m \geq 1, \quad v > 0, \quad t \geq 0, \quad (6.1)$$

and

$$P_m \left(f \left(\frac{t}{v} \right); vx \right) = P_m(f(t); x), \quad m \geq 1, \quad v > 0, \quad t \geq 0. \quad (6.2)$$

See also [23, Example 3].

By using (6.1) and (6.2), we obtain for $m \geq 1$, $v > 0$, $t \geq 0$,

$$\mathcal{F}_{m,c}\left(f\left(\frac{t}{v}\right); vx\right) = \mathcal{F}_{m,c}(f(t); x)$$

and

$$\mathcal{A}_{m,c}\left(f\left(\frac{t}{v}\right); vx\right) = \mathcal{A}_{m,c}(f(t); x).$$

On the other hand,

$$W_{mv}\left(f(vt); \frac{x}{v}\right) = W_m(f(t); x). \quad (6.3)$$

The relations (6.1), (6.2), and (6.3) can be used to establish properties of type (C_1) and (C_2) from [23, Definition 1]. This will be the subject of a forthcoming paper.

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References

- [1] M. Heilmann, *Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren*, Habilitationsschrift, Universität Dortmund, 1992.
- [2] A. M. Acu, M. Heilmann, I. Rasa, and A. Seserman, *Poisson approximation to the binomial distribution: extensions to the convergence of positive operators*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A: Mat. **117** (2023), 162, DOI: <https://doi.org/10.1007/s13398-023-01497-5>.
- [3] A. M. Acu, M. Heilmann, I. Rasa, and A. E. Şteopoaie, *Voronovskaja type results for the Aldaz-Kounchev-Render version of generalized Baskakov operators* (submitted). Appl Anal Discret Math. 2024, DOI: <https://doi.org/10.2298/AADM230725016A> (in press).
- [4] E. Berdysheva, *Studying Baskakov-Durrmeyer operators and quasi-interpolants via special functions*, J. Approx. Theory **149** (2007), no. 2, 131–150, DOI: <https://doi.org/10.1016/j.jat.2007.04.009>.
- [5] M. Wagner, *Quasi-Interpolanten zu Genuinen Baskakov-Durrmeyer-Typ Operatoren*, Shaker Verlag, Aachen, 2013.
- [6] J. A. Adell and J. De la Cal, *Bernstein-Durrmeyer operators*, Comput. Math. Appl. **30** (1995), no. 3–6, 1–14, DOI: [https://doi.org/10.1016/0898-1221\(95\)00081-X](https://doi.org/10.1016/0898-1221(95)00081-X).
- [7] J. de la Cal and F. Luquin, *A note on limiting properties of some Bernstein-type operators*, J. Approx. Theory **68** (1992), no. 3, 322–329, DOI: [https://doi.org/10.1016/0021-9045\(92\)90108-Z](https://doi.org/10.1016/0021-9045(92)90108-Z).
- [8] F. Sofonea, *On a sequence of linear and positive operators*, Gen. Math. **16** (2008), no. 4, 155–165.
- [9] S. Cooper and S. Waldron, *The eigenstructure of the Bernstein operator*, J. Approx. Theory **105** (2000), 133–165, DOI: <https://doi.org/10.1006/jath.2000.3464>.

- [10] F. Nasaireh and I. Raşa, *Another look at Voronovskaja type formulas*, J. Math. Inequal. **12** (2018), no. 1, 95–105, DOI: <https://doi.org/10.7153/jmi-2018-12-07>.
- [11] H. Gonska and I. Raşa, *On the composition and decomposition of positive linear operators (II)*, Studia Sci. Math. Hungar. **47** (2010), 448–461, DOI: <https://doi.org/10.1556/sscmath.2009.1144>.
- [12] I. C. Buşcu, V. Paşca, and A. Seserman, *On Rathore type operators*, Gen. Math. **30** (2022), no. 2, 35–39, DOI: <https://doi.org/10.2478/gm-2022-0013>.
- [13] G. Motronea and A. E. Steopoaie, *Eigenstructure and Voronovskaja type formula for a sequence of integral operators*, Gen. Math. **30** (2022), no. 2, 29–34.
- [14] A. M. Acu, I. Raşa, and A. Seserman, *Composition and decomposition of positive linear operators (VIII)*, Axioms **12** (2023), 228, DOI: <https://doi.org/10.2478/gm-2022-0012>.
- [15] A. Aral and H. Erbay, *A note on the difference of positive operators and numerical aspects*, Mediterr. J. Math. **17** (2020), 45, DOI: <https://doi.org/10.1007/s00009-020-1489-5>.
- [16] V. Gupta, *Differences of operators of Lupaş type*, Constr. Math. Anal. **1** (2018), no. 1, 9–14.
- [17] V. Gupta and G. Tachev, *A note on the differences of two positive linear operators*, Constr. Math. Anal. **2** (2019), 1–7, DOI: <https://doi.org/10.33205/cma.452962>.
- [18] V. Gupta, T. M. Rassias, P. N. Agrawal, and A. M. Acu, *Recent Advances in Constructive Approximation Theory*, Springer, Cham, 2018.
- [19] P. C. Sikkema, *On some linear positive operators*, Indag. Math. **32** (1970), 327–337, DOI: [https://doi.org/10.1016/S1385-7258\(70\)80037-3](https://doi.org/10.1016/S1385-7258(70)80037-3).
- [20] U. Abel and M. Ivan, *On a generalization of an approximation operator defined by A. Lupaş*, Gen. Math. **15** (2007), no. 1, 21–34.
- [21] M. Heilmann and G. Tachev, *Commutativity, direct and strong converse results for Phillips operators*, East J. Approx. **17** (2011), no. 3, 299–317.
- [22] M. Heilmann, F. Nasaireh, and I. Raşa, *Complements to Voronovskaya's Formula*, In: D. Ghosh, D. Giri, R. Mohapatra, K. Sakurai, E. Savas, T. Som (eds), Mathematics and Computing. ICMC 2018. Springer Proceedings in Mathematics & Statistics, vol. 253, Springer, Singapore, 2018.
- [23] A. M. Acu, V. Gupta, I. Raşa, and F. Sofonea, *Convergence of special sequences of semi-exponential operators*, Mathematics **10** (2022), 2978, DOI: <https://doi.org/10.3390/math10162978>.
- [24] V. Gupta, *A form of Gamma operator due to Rathore*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A: Mat. **117** (2023), 81, DOI: <https://doi.org/10.1007/s13398-023-01413-x>.