Research Article

Kuljeet Singh, Sandeep Sharma*, and Arun Kumar Bhardwaj

# Some aspects of normal curve on smooth surface under isometry

https://doi.org/10.1515/dema-2024-0017
received September 17, 2023; accepted May 10, 2024

**Abstract:** The normal curve is a space curve that plays an important role in the field of differential geometry. This research focuses on analyzing the properties of normal curves on smooth immersed surfaces, considering their invariance under isometric transformations. The primary contribution of this article is to explore the requirements for the image of a normal curve that preserves its invariance under isometric transformations. In this article, we investigate the invariant condition for the component of the position vector of the normal curves under isometry and compute the expression for the normal and geodesic curvature of such curves. Moreover, it has been investigated that the geodesic curvature and Christoffel symbols remain unchanged under the isometry of surfaces in $\mathbb{R}^3$.

**Keywords:** normal curves, isometry, geodesic, normal curvature, orthonormal frame

**MSC 2020:** 53A04, 53A05, 53A15, 51M05

## 1 Introduction and preliminaries

The area of mathematics that analyzes the geometry of curved and flat surfaces is known as differential geometry. A differential geometer is primarily interested in the characteristics of curves and surfaces that hold for a "generic" point and tends to overlook what occurs at "particular" places. However, an analyst is interested in what occurs at "specific" sites and is especially interested in providing examples that refute the general assertion the geometer is attempting to make. Generally speaking, a geometer is not concerned with the exact class of a curve being discussed as long as it is high enough to allow him to discuss any pertinent curve-related issues. However, a differential geometer cannot afford to overlook the problems of identifying the lowest class of the curve or pertinent functions that are involved in a theorem's hypothesis or analysis.

The smoothness of a curve is determined by the number of continuous derivatives it possesses. Moreover, a curve is considered regular if it can be differentiated and has no zero derivative. These concepts of regularity and smoothness are fundamental in the field of differential geometry. For further insights into regular and smooth curves, readers are encouraged to refer to [1–3]. In this article, the terms "motion," "transformation," and "map" are used interchangeably to denote the same concept.

Various methods exist to classify transformations, but our focus will be on those that preserve specific characteristics. Based on the invariant properties of Gaussian curvatures (K) and mean curvatures (H), transformations can be classified into three equivalence classes, namely, conformal, isometric, and non-

*** Corresponding author: Sandeep Sharma**, School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, Jammu and Kashmir, India, e-mail: sandeep.greater123@gmail.com
**Kuljeet Singh:** School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, Jammu and Kashmir, India, e-mail: kulljeet83@gmail.com
**Arun Kumar Bhardwaj:** Department of Mathematics and Astronomy, University of Lucknow, Lucknow-226025, Uttar Pradesh, India, e-mail: arunb8593@gmail.com

conformal [4]. An isometric transformation preserves both the lengths and angles between curves on surfaces. However, the mean curvature (H) is altered while the Gaussian curvature (K) remains invariant under isometric transformations. While conformal motion preserves the angles between two directed curves, it may not always retain their lengths.

Conformal transformation plays a crucial role in cartography, and one of the most well-known examples of this type of transformation is the stereographic projection, which maps a sphere onto a plane. In 1569, Gerardus Mercator pioneered the use of conformal maps to create the first angle-preserving map, known as "Mercator's globe map." In 2018, Gunn and Bobenko published an animated film based on the Springer videoMATH, which focuses on a conformal map [5]. However, it is important to note that for general motion, any pair of intersecting curves on the surface does not preserve their lengths or angles.

The concepts of normal, rectifying, and osculating curves are widely studied in books on the differential geometry of curves and surfaces due to their significance. For more information, we refer the reader to see [1,3]. Chen [2] introduced the notion of a rectifying curve as a space curve whose position vector always lies in its rectifying plane. Later, Chen and Dillen [6] conducted a study on the motion of rectifying curves and investigated various properties associated with such curves. Deshmukh et al. [7] investigated rectifying curves in Euclidean 3-space and derived a necessary and sufficient condition for the centrode of a unit speed curve in Euclidean 3-space to be a rectifying curve. Recently, Shaikh et al. [8–10] initiated the study of surface curves in a different way, especially, rectifying, osculating, and normal curves on a surface by considering isometry and conformal map between two surfaces, and investigated their invariancy under such maps. Moreover, Shaikh and Ghosh [11,12] explored the sufficient conditions under which osculating and rectifying curves on smooth surfaces remain invariant under the isometry of surfaces.

Lone [4] investigated the invariant properties of normal curves under conformal transformations in the three-dimensional Euclidean space $\mathbb{E}^3$, specifically focusing on the normal and tangential components of these curves. The author drew inspiration from the work of Shaikh and Ghosh [11,12], who explored the geometric invariant characteristics of rectifying curves on smooth surfaces under the isometry of the surfaces. He et al. [13] studied the variation of Gaussian curvature and its application using the concept of conformal mapping. Ilarslan and Nešović [14] investigated normal curves in Minkowski three-dimensional space and characterized them as timelike and null normal curves. Camci et al. [15] examined a space curve by confining its position vector to three mutually orthogonal planes on the surface and established the existence of such a curve.

The motivation for the study and its findings are quite interesting as the study integrates some isometric behavior of normal curves on smooth surfaces. Inspired by prior literature, our aim is to obtain the invariance nature of normal curvature, geodesic curvatures, and Christoffel symbols for normal curves under isometric transformations. Additionally, we analyze the deviations in the position vector of the normal curve along the binormal component to the surface, deriving conditions for the binormal component's invariance under isometric transformations between smooth surfaces. This article is organized as follows: Section 1 provides the introduction and preliminary aspects of the study. Section 2 presents the isometric behavior of normal curves and the main findings. Section 3 presents the conclusion and future scope of this study.

## 2 Isometric image of a normal curve

Let $P$ and $\tilde{P}$ be two smooth immersed surfaces in the three-dimensional Euclidean space $\mathbb{R}^3$ and $G : P \to \tilde{P}$ being a smooth map. The dilation function, denoted by $\zeta(x, y)$, is proportional to the area element of both $P$ and $\tilde{P}$. For more information on the dilation function, one can refer to [7]. The conformal transformation belongs to a generalized class of specific transformations [16] and is defined as follows:

If the dilation function $\zeta(x, y) = c$, where $c$ is a constant and $c \neq 0, 1$, then $G$ is classified as a homothetic transformation. If the function $\zeta(x, y) = 1$, then $G$ becomes an isometry. Let $E$, $F$, and $G$ represent the coefficients of the first fundamental form of the surface $P$. Similarly, $\tilde{E}$, $\tilde{F}$, and $\tilde{G}$ are the coefficients of the first fundamental form of the surface $\tilde{P}$. The invariance of these coefficients is a necessary and sufficient condition for the surfaces $P$ and $\tilde{P}$ to be isometric [4], i.e., $\tilde{E} = E, \tilde{F} = F$, and $\tilde{G} = G$.

Let $\delta : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth parameterized curve having unit speed, at least a fourth-order continuous derivative, and an arc length $(r)$. Then, the tangent, normal, and binormal of the curve $\delta$ is denoted by $\vec{t}$, $\vec{n}$, and $\vec{b}$, respectively. At each point on the curve $\delta(r)$, the vectors $\vec{t}$, $\vec{n}$, and $\vec{b}$ are mutually perpendicular to each other and so the triplet $\{\vec{t}, \vec{n}, \vec{b}\}$ forms an orthonormal frame.

Consider $\vec{t}'(r) \neq 0$, if the unit normal vector $\vec{n}$ along the tangent at a point on the curve $\delta$, then we can write $\vec{t}'(r) = \kappa(r)\vec{n}(r)$, where $\vec{t}'(r)$ is the derivative of $\vec{t}$ with respect to arc length parameter "$r$" and $\kappa(r)$ is the curvature of $\delta(r)$. Also, binormal vector $\vec{b} = \vec{t} \times \vec{n}$, and we can write $\vec{b}'(r) = \tau(r)\vec{n}(r)$, where $\tau(r)$ is another curvature function known as torsion of the curve $\delta(r)$. Then, the Serret-Frenet equations are given as follows:

$$\vec{t}'(r) = \kappa(r)\vec{n}(r),$$
$$\vec{n}'(r) = -\kappa(r)\vec{t}(r) + \tau(r)\vec{b}(r),$$
$$\vec{b}'(r) = -\tau(r)\vec{n}(r),$$

where the functions $\kappa$ and $\tau$ are, respectively, called the curvature and torsion of the curve $\delta$, satisfying the following conditions:

$$\vec{t}(r) = \delta'(r), \quad \vec{n}(r) = \frac{\vec{t}'(r)}{\kappa(r)} \quad \text{and} \quad \vec{b}(r) = \vec{t}(r) \times \vec{n}(r).$$

From an arbitrary point $\delta(r)$ on the curve $\delta$, the plane spanned by $\{\vec{t}, \vec{n}\}$ is known as the osculating plane, while the plane spanned by $\{\vec{t}, \vec{b}\}$ is referred to as the rectifying plane. Similarly, the plane spanned by $\{\vec{n}, \vec{b}\}$ is called the normal plane. Previous studies [17–21] discuss the position vectors of space curves and characterize different classes of curves based on the locations of their position vectors. They define these curves as follows: a curve can be classified as a normal curve if its position vector is located in the normal plane. Similarly, a curve is categorized as a rectifying curve if its position vector lies in the rectifying plane. Furthermore, if a curve's position vector lies in the osculating plane, it is termed an osculating curve.

If the position vector of a curve is located in the normal plane, then the curve is said to be a normal curve, i.e.,

$$\delta(r) = \mu_1(r)\vec{n}(r) + \mu_2(r)\vec{b}(r), \tag{1}$$

where $\mu_1$ and $\mu_2$ are two smooth functions.

Let $\sigma : U \to P$ be the coordinate chart map on the smooth surface $P$ and the smooth parameterized unit speed curve $\delta(r) : I \to P$, where $I = (a, b) \subset \mathbb{R}$ and $U \subset \mathbb{R}^2$. As a result, the curve $\delta(r)$ is given by

$$\delta(r) = \sigma(x(r), y(r)). \tag{2}$$

Using the chain rule to differentiate (2), with respect to $r$, we obtain

$$\delta'(r) = \sigma_x x' + \sigma_y y'. \tag{3}$$

Now, $\vec{t}(r) = \delta'(r)$. Then, from equation (3), we find that

$$\vec{t}(r) = \sigma_x x' + \sigma_y y'. \tag{4}$$

When we differentiate equation (4) again in terms of $r$, we obtain

$$\vec{t}'(r) = x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}.$$

If $N$ is the normal to the surface $P$ and $\kappa(r)$ is the curvature of the curve $\delta(r)$, then the normal vector $\vec{n}(r)$ can be written as

$$\vec{n}(r) = \frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}). \tag{5}$$

Now, the binormal vector $\vec{b}(r)$ can be written as

$$\vec{b}(r) = \vec{t}(r) \times \vec{n}(r).$$

By substituting the value of $\vec{t}(r)$ and $\vec{n}(r)$ from equations (4) and (5), we obtain

$$\begin{aligned}\vec{b}(r) &= \frac{1}{\kappa(r)}[(\sigma_x x' + \sigma_y y') \times (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})],\\ &= \frac{1}{\kappa(r)}[(y''x' - y'x'')N + x'^3\sigma_x \times \sigma_{xx} + 2x'^2y'\sigma_x \times \sigma_{xy} + x'y'^2\sigma_x \times \sigma_{yy} + x'^2y'\sigma_y \times \sigma_{xx}\\ &\quad + 2x'y'^2\sigma_y \times \sigma_{xy} + y'^3\sigma_y \times \sigma_{yy}].\end{aligned} \tag{6}$$

Substituting equations (5) and (6) into (1), we obtain

$$\begin{aligned}\delta(r) = {}&\frac{\mu_1(r)}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}) + \frac{\mu_2(r)}{\kappa(r)}\{(x'y'' - x''y')N\\ &+ x'^3\sigma_x \times \sigma_{xx} + 2x'^2y'\sigma_x \times \sigma_{xy} + x'y'^2\sigma_x \times \sigma_{yy} + x'^2y'\sigma_y \times \sigma_{xx} + 2x'y'^2\sigma_y \times \sigma_{xy} + y'^3\sigma_y \times \sigma_{yy}\}.\end{aligned} \tag{7}$$

**Definition 1.** [18] Let $P$ and $\tilde{P}$ be two smooth immersed surfaces in $\mathbb{R}^3$. Then, a diffeomorphism $G : P \to \tilde{P}$ is an isometry if $G$ maps the curve of the same length from $P$ to $\tilde{P}$.

**Definition 2.** [2] Let $P$ and $\tilde{P}$ be two regular surfaces in the Euclidean surface $\mathbb{R}^3$ and $\delta(r)$ be an arc length parameterized curve lying on the surface $P$. Then, $\delta'(r)$ is perpendicular to normal surface $N$, and also, $\delta'(r)$ and $\delta''(r)$ are perpendicular. Thus, $\delta''$ can be represented as the linear combination of $N$ and $N \times \delta'$, i.e.,

$$\delta'' = \kappa_n N + \kappa_g N \times \delta',$$

where the parameters $\kappa_n$ and $\kappa_g$, which are commonly known as the normal and geodesic curvatures of the curve $\delta$, are given by

$$\kappa_n = \delta'' \cdot N,$$
$$\kappa_g = \delta'' \cdot (N \times \delta').$$

**Definition 3.** If there exists a neighborhood $V$ of $G(q) \in \tilde{P}$ such that $G : U \to V$ is an isometry, then the diffeomorphism $G : U \subset P \to \tilde{P}$ of the neighborhood $U$ of a point "$q$" in $P$ is referred to as a local isometry at "$q$." The surfaces $P$ and $\tilde{P}$ are said to be locally isometric when the local isometry exists at every point of $P$. In general, $G$ is termed a global isometry if it acts as a local isometry at every point of the surface $P$.

Next, we will take into account the equation $G_*(\delta(r))$, which results from the product of the $3 \times 3$ orthogonal matrix $G_*$ and the $3 \times 1$ matrix $\delta(r)$ [14].

**Theorem 1.** *Let $P$ and $\tilde{P}$ be two smooth surfaces in the Euclidean surface $\mathbb{R}^3$ and $G : P \to \tilde{P}$ be an isometry. If $\delta(r)$ and $\tilde{\delta}(r)$ are the normal curves on the surface $P$ and $\tilde{P}$, respectively, then*
*(i)* $\kappa_n = x'^2X + 2x'y'Y + y'^2Z$.
*(ii) The geodesic curvature $\kappa_g$ of the curve $\delta(r)$ is invariant under the isometry $G$, i.e., $\tilde{\kappa}_g = \kappa_g$.*

**Proof.**

- (i) Given that $G$ is an isometry between the surfaces $P$ and $\tilde{P}$. Let $\sigma(x, y)$ and $\tilde{\sigma}(x, y)$ be the coordinate chart maps of the surfaces $P$ and $\tilde{P}$, respectively, such that $\tilde{\sigma}(x, y) = G \circ \sigma(x, y)$. We know that $E, F, G$ and $\tilde{E}, \tilde{F}, \tilde{G}$ are the magnitudes of the first fundamental forms of $\sigma$ and $\tilde{\sigma}$, respectively; therefore, we can write

$$\tilde{E} = E, \ \tilde{F} = F, \ \tilde{G} = G. \tag{8}$$

Now, we can write $E = (\sigma_x \cdot \sigma_x)$, $F = (\sigma_x \cdot \sigma_y)$, and $G = (\sigma_y \cdot \sigma_y)$. As $E$, $F$, and $G$ are the functions of both $x$ and $y$, then on differentiating with respect to $x$ and $y$, we find that

$$E_x = (\sigma_x \cdot \sigma_x)_x = 2\sigma_{xx} \cdot \sigma_x, \\ \Rightarrow \sigma_{xx} \cdot \sigma_x = \frac{E_x}{2}. \tag{9}$$

Similarly, we can find

$$\left.\begin{aligned} &\sigma_{xx} \cdot \sigma_y = F_x - \frac{E_y}{2},\ \ \sigma_{xy} \cdot \sigma_x = \frac{E_y}{2},\ \ \sigma_{xy} \cdot \sigma_y = \frac{G_x}{2}, \\ &\sigma_{yy} \cdot \sigma_x = F_y - \frac{G_x}{2},\ \text{ and } \sigma_{yy} \cdot \sigma_y = \frac{G_y}{2}. \end{aligned}\right\} \tag{10}$$

Since $\delta(r)$ is a normal curve, we can write

$$\delta(r) = \mu_1(r)\vec{n}(r) + \mu_2(r)\vec{b}(r), \tag{11}$$

for some smooth functions $\mu_1(r)$ and $\mu_2(r)$. On differentiating (11), with respect to "$r$" we have

$$\delta'(r) = \mu_1'(r)\vec{n}(r) + \mu_1(r)\vec{n}' + \mu_2'(r)\vec{b}(r) + \mu_2(r)\vec{b}'(r). \tag{12}$$

Chen [2] derived a condition for $\mu_1(r)$ and $\mu_2(r)$ that satisfy the equation and by applying the Serret-Frenet equation, we have

$$\delta''(r) = \kappa(r)\vec{n}(r).$$

Now, normal curvature,

$$\begin{aligned} \kappa_n &= \delta''(r) \cdot N, \\ &= \kappa(r)\vec{n}(r) \cdot N, \\ &= \{(x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}) + (x''\sigma_x + y''\sigma_y)\} \cdot N, \\ &= x'^2X + 2x'y'Y + y'^2Z, \end{aligned}$$

where $X$, $Y$, and $Z$ are the coefficients of the second fundamental form of $\sigma$. This proves (i).

- (ii) Now, for the geodesic curvature $\kappa_g$, we have

$$\begin{aligned} \kappa_g &= \delta'' \cdot (N \times \delta'), \\ &= \delta'' \cdot \{(\sigma_x \times \sigma_y) \times (x'\sigma_x + y'\sigma_y)\}, \\ &= \{(x''\sigma_x + y''\sigma_y) + (x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\} \cdot \{x'(E\sigma_y - F\sigma_x) + y'(F\sigma_y - G\sigma_x)\}, \\ &= x'x''(EF - FE) + x'y''(EG - F^2) + y'y''(FG - GF) + x'^3(E\sigma_{xx} \cdot \sigma_y - F\sigma_{xx} \cdot \sigma_x) \\ &\quad + x'^2y'(F\sigma_{xx} \cdot \sigma_y - G\sigma_{xx} \cdot \sigma_x) + 2x'^2y'(E\sigma_{xy} \cdot \sigma_y - F\sigma_{xy} \cdot \sigma_x) + 2x'y'^2(F\sigma_{xy} \cdot \sigma_y - G\sigma_{xy} \cdot \sigma_x) \\ &\quad + x'y'^2(E\sigma_{yy} \cdot \sigma_y - F\sigma_{yy} \cdot \sigma_x) + y'^3(F\sigma_{yy} \cdot \sigma_y - G\sigma_{yy} \cdot \sigma_x). \end{aligned} \tag{13}$$

On substituting equation (10) into equation (13), we find that

$$\begin{aligned} \kappa_g &= y'x''(F^2 - GE) + x'y''(EG - F^2) + \frac{1}{2}x'^3(2EF_x - EE_y - FE_x) \\ &\quad + \frac{1}{2}x'^2y'(2FF_x - FE_y - GE_x) + x'^2y'(EG_x - FE_y) \\ &\quad + x'y'^2(FG_x - GE_y) + \frac{1}{2}x'y'^2(EG_y - 2FF_y - FG_x) + \frac{1}{2}y'^3(FG_y - 2GF_y - GG_x). \end{aligned} \tag{14}$$

Since $\tilde{\delta}(r)$ is a normal curve in $\tilde{P}$, the geodesic curvature $\tilde{\kappa}_g$ is given by

$$\begin{aligned}\tilde{\kappa}_g &= y'x''(\tilde{F}^2 - \tilde{G}\tilde{E}) + x'y''(\tilde{E}\tilde{G} - \tilde{F}^2) + \frac{1}{2}x'^3(2\tilde{E}\tilde{F}_x - \tilde{E}\tilde{E}_y - \tilde{F}\tilde{E}_x)\\ &+ \frac{1}{2}x'^2y'(2\tilde{F}\tilde{F}_x - \tilde{F}\tilde{E}_y - \tilde{G}\tilde{E}_x) + x'^2y'(\tilde{E}\tilde{G}_x - \tilde{F}\tilde{E}_y) + x'y'^2(\tilde{F}\tilde{G}_x - \tilde{G}\tilde{E}_y)\\ &+ \frac{1}{2}x'y'^2(\tilde{E}\tilde{G}_y - 2\tilde{F}\tilde{F}_y - \tilde{F}\tilde{G}_x) + \frac{1}{2}y'^3(\tilde{F}\tilde{G}_y - 2\tilde{G}\tilde{F}_y - \tilde{G}\tilde{G}_x).\end{aligned} \tag{15}$$

By subtracting equation (14) from equation (15), we obtained

$$\tilde{\kappa}_g - \kappa_g = 0, \Rightarrow \tilde{\kappa}_g = \kappa_g.$$

Thus, the geodesic curvature for a normal curve is invariant under isometry. □

**Theorem 2.** *Let $G : P \to \tilde{P}$ represent an isometry between two smooth surfaces $P$ and $\tilde{P}$, and $\delta(r)$ represent a normal curve on $P$. Then, under $G$, the Christoffel symbols of the second kind are isometrically invariant.*

**Proof.** It is noted that $G : P \to \tilde{P}$ is isometry and $\delta(r)$ represents a normal curve on the surface $P$. The surface area element of $P$ and $\tilde{P}$ are proportional to a differential function, which is commonly called a dilation function, and are denoted by $\zeta(x, y)$, as we know about the conformal transformation, which is a generalized class of function. For the case of isometry, we take $\zeta(x, y) = 1$.

Since $E, F$, and $G$ are the magnitudes of the first fundamental form, $E = (\sigma_x \cdot \sigma_x)$, $F = (\sigma_x \cdot \sigma_y)$, $G = (\sigma_y \cdot \sigma_y)$. As $E, F$, and $G$ are the functions of both $x$ and $y$, then by differentiating with respect to both $x$ and $y$, we find that

$$E_x = (\sigma_x \cdot \sigma_x)_x = 2\sigma_{xx} \cdot \sigma_x, \Rightarrow \sigma_{xx} \cdot \sigma_x = \frac{E_x}{2}. \tag{16}$$

Similarly, we can find

$$\left.\begin{aligned}&\sigma_{xx} \cdot \sigma_y = F_x - \frac{E_y}{2}, \ \sigma_{xy} \cdot \sigma_x = \frac{E_y}{2}, \ \sigma_{xy} \cdot \sigma_y = \frac{G_x}{2},\\ &\sigma_{yy} \cdot \sigma_x = F_y - \frac{G_x}{2}, \ \sigma_{yy} \cdot \sigma_y = \frac{G_y}{2}.\end{aligned}\right\} \tag{17}$$

Let $\Gamma^n_{lm}$, where $\{l, m, n = 1, 2\}$, be the Christoffel symbols of second kind. Now, for the conformal transformation, these symbols take the form

$$\left.\begin{aligned}&\tilde{\Gamma}^1_{11} = \Gamma^1_{11} + \Theta^1_{11}, \ \tilde{\Gamma}^2_{11} = \Gamma^2_{11} + \Theta^2_{11}, \ \tilde{\Gamma}^1_{12} = \Gamma^1_{12} + \Theta^1_{12},\\ &\tilde{\Gamma}^2_{12} = \Gamma^2_{12} + \Theta^2_{12}, \ \tilde{\Gamma}^1_{22} = \Gamma^1_{22} + \Theta^1_{22}, \ \tilde{\Gamma}^2_{22} = \Gamma^2_{22} + \Theta^2_{22},\end{aligned}\right\} \tag{18}$$

where

$$\left.\begin{aligned}\Theta^1_{11} &= \frac{EG\zeta_x - 2F^2\zeta_x + FE\zeta_y}{\zeta\mathbb{H}^2}\\ \Theta^2_{11} &= \frac{EF\zeta_x - E^2\zeta_y}{\zeta\mathbb{H}^2},\\ \Theta^1_{12} &= \frac{EG\zeta_y - FG\zeta_x}{\zeta\mathbb{H}^2},\\ \Theta^2_{12} &= \frac{EG\zeta_x - FE\zeta_y}{\zeta\mathbb{H}^2},\\ \Theta^1_{22} &= \frac{GF\zeta_y - G^2\zeta_x}{\zeta\mathbb{H}^2},\\ \Theta^2_{22} &= \frac{EG\zeta_y - 2F^2\zeta_v + FG\zeta_u}{\zeta\mathbb{H}^2}.\end{aligned}\right\} \tag{19}$$

Now, for the isometry between the surfaces, we have $\zeta = 1$, then on substituting $\zeta = 1$ into equation (19), we find that all $\Theta_{lm}^{n} = 0$, for $l, m, n = \{1, 2\}$. On substituting $\Theta_{l,m}^{n} = 0$ for $l, m, n = \{1, 2\}$ into equation (18), we obtain

$$\tilde{\Gamma}_{lm}^{n} = \Gamma_{lm}^{n}.$$

This proves that for the normal curve, the second kind of Christoffel symbol is isometrically invariant under $G$. □

**Theorem 3.** *Let $G : P \to \tilde{P}$ be an isometry between two smooth immersed surfaces $P$ and $\tilde{P}$, and* let *$\delta(r)$ be a normal curve on P. Then, $\tilde{\delta}(r)$ is a normal curve on the surface $\tilde{P}$ if the following condition holds*:

$$\begin{aligned}\tilde{\delta}(r) - G_*(\delta) = {} & \frac{\mu_1(r)}{\kappa(r)}\left\{x'^2\frac{\partial G_*}{\partial x}\sigma_x + y'^2\frac{\partial G_*}{\partial y}\sigma_y + 2x'y'\frac{\partial G_*}{\partial x}\sigma_y\right\} \\ & + \frac{\mu_2(r)}{\kappa(r)}\left\{x'^3\left(G_*\sigma_x \times \frac{\partial G_*}{\partial x}\sigma_x\right) + 2x'^2y'\left(G_*\sigma_x \times \frac{\partial G_*}{\partial x}\sigma_y\right)\right. \\ & + x'y'^2\left(G_*\sigma_x \times \frac{\partial G_*}{\partial y}\sigma_y\right) + x'^2y'\left(G_*\sigma_y \times \frac{\partial G_*}{\partial x}\sigma_x\right) \\ & \left. + 2x'y'^2\left(G_*\sigma_y \times \frac{\partial G_*}{\partial x}\sigma_y\right) + y'^3\left(G_*\sigma_y \times \frac{\partial G_*}{\partial y}\sigma_y\right)\right\}.\end{aligned} \tag{20}$$

**Proof.** Given that $G$ is an isometry from $P$ and $\tilde{P}$. Let $\sigma$ and $\tilde{\sigma}$ be the coordinate maps for the surfaces $P$ and $\tilde{P}$, respectively. Then, we can write $\tilde{\sigma} = G \circ \sigma$ as the differential map $G_*$ of $G$ takes linearly independent vectors $\sigma_x$ and $\sigma_y$ of the tangent plane $T_p(P)$ to a dilated tangent vectors $\tilde{\sigma}_x$ and $\tilde{\sigma}_y$ of the tangent plane $T_{G(p)(P)}$. Also, the normals to the surfaces $P$ and $\tilde{P}$ are $\vec{N}$ and $\vec{\tilde{N}}$, respectively. Now,

$$\left.\begin{aligned}\tilde{\sigma}_x(x, y) = G_*(\sigma(x, y))\sigma_x = G_*\sigma_x, \\ \tilde{\sigma}_y(x, y) = G_*(\sigma(x, y))\sigma_y = G_*\sigma_y.\end{aligned}\right\} \tag{21}$$

On differentiating equation (21), partially with respect to both "$x$" and "$y$," we obtain

$$\left.\begin{aligned}\tilde{\sigma}_{xx} &= \frac{\partial G_*}{\partial x}\sigma_x + G_*\sigma_{xx}, \\ \tilde{\sigma}_{yy} &= \frac{\partial G_*}{\partial y}\sigma_y + G_*\sigma_{yy}, \\ \tilde{\sigma}_{xy} &= \frac{\partial G_*}{\partial x}\sigma_y + G_*\sigma_{xy}, \\ \tilde{\sigma}_{yx} &= \frac{\partial G_*}{\partial y}\sigma_x + G_*\sigma_{yx}.\end{aligned}\right\} \tag{22}$$

Now,

$$G_*\sigma_x \times \frac{\partial G_*}{\partial x}\sigma_x = G_*\sigma_x \times \left(\frac{\partial G_*}{\partial x}\sigma_x + G_*\sigma_{xx}\right) - G_*(\sigma_x \times \sigma_{xx}), = \tilde{\sigma}_x \times \tilde{\sigma}_{xx} - G_*(\sigma_x \times \sigma_{xx}). \tag{23}$$

On a similar pattern, we find that

$$\left.\begin{aligned}G_*\sigma_x \times \frac{\partial G_*}{\partial x}\sigma_y &= \tilde{\sigma}_x \times \tilde{\sigma}_{xy} - G_*(\sigma_x \times \sigma_{xy}), \\ G_*\sigma_x \times \frac{\partial G_*}{\partial y}\sigma_y &= \tilde{\sigma}_x \times \tilde{\sigma}_{yy} - G_*(\sigma_x \times \sigma_{yy}), \\ G_*\sigma_y \times \frac{\partial G_*}{\partial x}\sigma_x &= \tilde{\sigma}_y \times \tilde{\sigma}_{xx} - G_*(\sigma_y \times \sigma_{xx}), \\ G_*\sigma_y \times \frac{\partial G_*}{\partial x}\sigma_y &= \tilde{\sigma}_y \times \tilde{\sigma}_{xy} - G_*(\sigma_y \times \sigma_{xy}), \\ G_*\sigma_y \times \frac{\partial G_*}{\partial y}\sigma_y &= \tilde{\sigma}_y \times \tilde{\sigma}_{yy} - G_*(\sigma_y \times \sigma_{yy}).\end{aligned}\right\} \tag{24}$$

From equations (20), (23), and (24), we obtain

$$\begin{aligned}\tilde{\delta}(r) = G_*\Bigg\{&\frac{\mu_1(r)}{\kappa(r)}\{(x''\sigma_x + y''\sigma_y) + (x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\}\\ &+ \frac{\mu_2(r)}{\kappa(r)}\{(x'y'' - x''y')N + x'^3\sigma_x \times \sigma_{xx} + 2x'^2y'\sigma_x \times \sigma_{xy}\\ &+ x'y'^2\sigma_x \times \sigma_{yy} + x'^2y'\sigma_y \times \sigma_{xx} + 2x'y'^2\sigma_y \times \sigma_{xy} + y'^3\sigma_y \times \sigma_{yy}\}\Bigg\}\\ &+ \frac{\mu_1(r)}{\kappa(r)}\left\{x'^2\frac{\partial G_*}{\partial x}\sigma_x + y'^2\frac{\partial G_*}{\partial y}\sigma_y + 2x'y'\frac{\partial G_*}{\partial x}\sigma_y\right\}\\ &+ \frac{\mu_2(r)}{\kappa(r)}\Bigg\{x'^3\left(G_*\sigma_x \times \frac{\partial G_*}{\partial x}\sigma_x\right) + 2x'^2y'\left(G_*\sigma_x \times \frac{\partial G_*}{\partial x}\sigma_y\right)\\ &+ x'y'^2\left(G_*\sigma_x \times \frac{\partial G_*}{\partial y}\sigma_y\right) + x'^2y'\left(G_*\sigma_y \times \frac{\partial G_*}{\partial x}\sigma_x\right)\\ &+ 2x'y'^2\left(G_*\sigma_y \times \frac{\partial G_*}{\partial x}\sigma_y\right) + y'^3\left(G_*\sigma_y \times \frac{\partial G_*}{\partial y}\sigma_y\right)\Bigg\}.\end{aligned}$$

The aforementioned expression can be written as

$$\begin{aligned}\tilde{\delta}(r) = &\frac{\tilde{\mu}_1(r)}{\tilde{\kappa}(r)}\{(x''\tilde{\sigma}_x + y''\tilde{\sigma}_y) + (x'^2\tilde{\sigma}_{xx} + 2x'y'\tilde{\sigma}_{xy} + y'^2\tilde{\sigma}_{yy})\}\\ &+ \frac{\tilde{\mu}_2(r)}{\tilde{\kappa}(r)}\{(x'y'' - x''y')\tilde{N} + x'^3\tilde{\sigma}_x \times \tilde{\sigma}_{xx} + 2x'^2y'\tilde{\sigma}_x \times \tilde{\sigma}_{xy}\\ &+ x'y'^2\tilde{\sigma}_x \times \tilde{\sigma}_{yy} + x'^2y'\tilde{\sigma}_y \times \tilde{\sigma}_{xx} + 2x'y'^2\tilde{\sigma}_y \times \tilde{\sigma}_{xy} + y'^3\tilde{\sigma}_y \times \tilde{\sigma}_{yy}\},\\ \Rightarrow \tilde{\delta}(r) = &\frac{\tilde{\mu}_1(r)}{\tilde{\kappa}(r)}\vec{\tilde{n}}(r) + \frac{\tilde{\mu}_2(r)}{\tilde{\kappa}(r)}\vec{\tilde{b}}(r),\end{aligned}$$

for some smooth function $\tilde{\mu}_1$ and $\tilde{\mu}_2$; without loss of generality, we assume that $\frac{\tilde{\mu}_1}{\tilde{\kappa}} = \frac{\mu_1}{\kappa}$ and $\frac{\tilde{\mu}_2}{\tilde{\kappa}} = \frac{\mu_2}{\kappa}$. This proves that $\tilde{\delta}(r)$ is a normal curve on the surface $\tilde{P}$. □

**Theorem 4.** *Consider two smooth surfaces $P$ and $\tilde{P}$ with $G : P \to \tilde{P}$ being an isometry. Let $\delta(r)$ be a normal curve on surface $P$. Then, along the surface binormal $B$, the component of the curve $\delta(r)$ satisfies the following:*

$$\tilde{\delta} \cdot \tilde{B} - \delta \cdot B = \frac{\mu_2(r)}{\kappa(r)}\{a(Fy' - Ex') + b(Gy' - Fx')\}(\tilde{\kappa}_n - \kappa_n),$$

*for some real numbers $a$ and $b$.*

**Proof.** It is noted that $\delta(r)$ is a normal curve on the surface $P$. Then, we can write

$$\delta(r) = \mu_1(r)\vec{n}(r) + \mu_2(r)\vec{b}(r).$$

Furthermore, we know that $B = T \times N$, where $T = a\sigma_x + b\sigma_y$ with real numbers $a$ and $b$. Also, the vectors $T$ and $N$ are perpendicular at $\delta(r)$, and so the triplet $\{T, N, B\}$ forms an orthonormal system at each point of the curve $\delta(r)$. Now, from equation (6), we find that

$$\begin{aligned}\delta \cdot \sigma_x = &\frac{\mu_1(r)}{\kappa(r)}\{x''E + y''F + x'^2\sigma_{xx} \cdot \sigma_x + 2x'y'\sigma_{xy} \cdot \sigma_x + y'^2\sigma_{yy} \cdot \sigma_x\}\\ &+ \frac{\mu_2(r)}{\kappa(r)}\{x'^2y'X + 2x'y'^2Y + y'^3Z\}.\end{aligned} \tag{25}$$

Now, for the isometric images of $\delta(r)$ and $\sigma_x$, we have

$$\begin{aligned}\tilde{\delta}\cdot\tilde{\sigma}_x = {} & \frac{\mu_1(r)}{\kappa(r)}\{x''\tilde{E} + y''\tilde{F} + x'^2\tilde{\sigma}_{xx}\cdot\tilde{\sigma}_x + 2x'y'\tilde{\sigma}_{xy}\cdot\tilde{\sigma}_x + y'^2\tilde{\sigma}_{yy}\cdot\tilde{\sigma}_x\} \\ & + \frac{\mu_2(r)}{\kappa(r)}\{x'^2y'\tilde{X} + 2x'y'^2\tilde{Y} + y'^3\tilde{Z}\}.\end{aligned} \tag{26}$$

Since we have

$$\tilde{E} = E,\ \ \tilde{F} = F,\ \ \tilde{G} = G,$$

also

$$E = \tilde{E} = (G_*\sigma_x)\cdot(G_*\sigma_x) = (\sigma_x\cdot\sigma_x).$$

After differentiating with respect to $x$, we obtain

$$\tilde{\sigma}_{xx}\cdot\tilde{\sigma}_x = \sigma_{xx}\cdot\sigma_x.$$

Similarly, we can prove that

$$\tilde{\sigma}_{xy}\cdot\tilde{\sigma}_x = \sigma_{xy}\cdot\sigma_x \quad \text{and} \quad \tilde{\sigma}_{yy}\cdot\tilde{\sigma}_x = \sigma_{yy}\cdot\sigma_x.$$

On substituting the aforementioned values into equation (26), we obtain

$$\tilde{\delta}\cdot\tilde{\sigma}_x = \frac{\mu_1(r)}{\kappa(r)}\{x''E + y''F + x'^2\sigma_{xx}\cdot\sigma_x + 2x'y'\sigma_{xy}\cdot\sigma_x + y'^2\sigma_{yy}\cdot\sigma_x\} + y'\frac{\mu_2(r)}{\kappa(r)}\tilde{\kappa}_n. \tag{27}$$

Now, from equations (25) and (27), we obtain

$$\tilde{\delta}\cdot\tilde{\sigma}_x - \delta\cdot\sigma_x = y'\frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - \kappa_n). \tag{28}$$

Similarly, we can find

$$\tilde{\delta}\cdot\tilde{\sigma}_y - \delta\cdot\sigma_y = x'\frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - \kappa_n). \tag{29}$$

Now,

$$\begin{aligned}\delta\cdot B &= \delta\cdot(T\times N), \\ &= \delta\cdot\{(a\sigma_x + b\sigma_y)\times(\sigma_x\times\sigma_y)\}, \\ &= \delta\cdot\{a\sigma_x\times(\sigma_x\times\sigma_y) + b\sigma_y\times(\sigma_x\times\sigma_y)\}, \\ &= \delta\cdot\{a(\sigma_x\cdot\sigma_y)\sigma_x - a(\sigma_x\cdot\sigma_x)\sigma_y + b(\sigma_y\cdot\sigma_y)\sigma_x - b(\sigma_y\cdot\sigma_x)\sigma_y\}, \\ &= \delta\cdot\{aF\sigma_x - aE\sigma_y + bG\sigma_x - bF\sigma_y\}, \\ &= \delta\cdot\{(aF + bG)\sigma_x - (aE + bF)\sigma_y\}, \\ &= (\delta\cdot\sigma_x)(aF + bG) - (\delta\cdot\sigma_y)(aE + bF).\end{aligned} \tag{30}$$

Also, for the isometric images, in a similar manner, we obtain

$$\begin{aligned}\tilde{\delta}\cdot\tilde{B} &= (\tilde{\delta}\cdot\tilde{\sigma}_x)(a\tilde{F} + b\tilde{G}) - (\tilde{\delta}\cdot\tilde{\sigma}_y)(a\tilde{E} + b\tilde{F}), \\ &= (\tilde{\delta}\cdot\tilde{\sigma}_x)(aF + bG) - (\tilde{\delta}\cdot\tilde{\sigma}_y)(aE + bF).\end{aligned} \tag{31}$$

Thus, in view of (28), (29), (30), and (31), we obtain

$$\begin{aligned}\tilde{\delta}\cdot\tilde{B} - \delta\cdot B &= (\tilde{\delta}\cdot\tilde{\sigma}_x - \delta\cdot\sigma_x)(aF + bG) - (\tilde{\delta}\cdot\tilde{\sigma}_y - \delta\cdot\sigma_y)(aE + bF), \\ &= y'\frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - \kappa_n)(aF + bG) - x'\frac{\mu_2(r)}{\kappa(r)}(\tilde{\kappa}_n - \kappa_n)(aE + bF), \\ &= \frac{\mu_2(r)}{\kappa(r)}\{a(Fy' - Ex') + b(Gy' - Fx')\}(\tilde{\kappa}_n - \kappa_n).\end{aligned}$$

□

**Corollary 1.** *Consider two smooth surfaces $P$ and $\tilde{P}$ with $G : P \to \tilde{P}$ being an isometry. Let $\delta(r)$ be a normal curve on surface $P$. If the normal curvature $\kappa_n$ is invariant, then the component for the position vector of the normal curve $\delta(r)$ along the binormal $B$ is invariant under $G$.*

**Proof.** If normal curvature $\kappa_n$ is invariant, i.e., $\tilde{\kappa}_n = \kappa_n$, then from Theorem (4), we obtained $\tilde{\delta} \cdot \tilde{B} = \delta \cdot B$. □

**Corollary 2.** *Consider two smooth surfaces $P$ and $\tilde{P}$, with $G : P \to \tilde{P}$ being an isometry. Let $\delta(r)$ be a normal curve on the surface $P$. If the position vector of the normal curve $\delta(r)$ is in the normal direction of $\delta$, then the component for the position vector of the normal curve $\delta(r)$ along the binormal $B$ is invariant under $G$.*

**Proof.** If the position vector of the curve is in the normal direction, i.e., $\delta(r) = \mu_1(r)n(r)$, we see that $\mu_2 = 0$. Thus, from Theorem (4), we can find that for a normal curve, the component of the position vector along binormal is invariant under the isometry. □

## 3 Conclusion

We examined the adequate condition for the invariance of the image of the normal curve under isometry in this article. We found that the geodesic curvature and the second kind of Christoffel symbols are isometrically invariant under the isometry of the surfaces after deriving the expression for the normal and geodesic curvature of the normal curves. Moreover, we also obtained the condition for the component of the position vector of the normal curve along the binormal surface and their invariance under isometry.

In the future, one can discuss some other properties such as the normal and the tangential components of normal curves under different transformations, namely, conformal, homothetic, and isometric of the surfaces using the Darboux frame instead of the Frenet frame in Euclidean 3-space. Furthermore, one can also find out that these components are invariant under the isometry of the surfaces in Euclidean 4-space. These results can also be extended to check the behavior of the mean, Gaussian, and sectional curvature of the normal curve under isometry between the smooth surfaces.

**Acknowledgment:** The authors are extremely thankful to the referees and the editorial team for their valuable comments, suggestions, and contributions, which have greatly improved the quality of the article.

**Funding information**: The authors did not receive any specific funding to support this study.

**Author contributions**: All authors have contributed equally to the preparation of this article.

**Conflict of interest**: All authors have declared that there is no conflict of interest.

**Ethical approval**: No experiments involving human or animal subjects were conducted by any of the authors and are included in this article.

**Data availability statement**: The authors confirm that the data supporting this study's conclusions are included in the article.

## References

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Dover Publications, Mineola, New York, 2016.
[2] B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?* Amer. Math. Monthly **110** (2003), no. 2, 147–152.

[3] A. Pressley, *Elementary Differential Geometry*, Springer-Verlag, London, 2001.
[4] M. S. Lone, *Geometric invariants of normal curves under conformal transformation in $E^3$*, Tamkang J. Math. **53** (2022), no. 1, 75–87.
[5] A. I. Bobenko and C. Gunn, *DVD-Video PAL*, Springer VideoMATH, Heidelberg, Berlin, Germany, 2018, https://www.springer.com/us/book/9783319734736.
[6] B. Y. Chen and F. Dillen, *Rectfying curve as centrode and extremal curve*, Bull. Inst. Math. Acad. Sin. **33** (2005), no. 2, 77–90.
[7] S. Deshmukh, B. Y. Chen, and S. H. Alshammari, *On rectifying curves in Euclidean 3-space*, Turkish J. Math. **42** (2018), no. 2, 609–620.
[8] A. A. Shaikh, M. S. Lone, and P. R. Ghosh, *Conformal image of an osculating curve on a smooth immersed surface*, J. Geom. Phys. **151** (2020), 103625.
[9] A. A. Shaikh, M. S. Lone, and P. R. Ghosh, *Rectifying curves under conformal transformation*, J. Geom. Phys. **163** (2021), 104117.
[10] A. A. Shaikh, M. S. Lone, and P. R. Ghosh, *Normal curves on a smooth immersed surface*, Indian J. Pure Appl. Math. **51** (2020), 1343–1355.
[11] A. A. Shaikh and P. R. Ghosh, *Rectifying curves on a smooth surface immersed in the Euclidean space*, Indian J. Pure Appl. Math. **50** (2019), no. 4, 883–890.
[12] A. A. Shaikh and P. R. Ghosh, *Rectifying and osculating curves on a smooth surface*, Indian J. Pure Appl. Math. **51** (2020), no. 1, 67–75.
[13] M. He, D. B. Goldgof, and C. Kambhamettu, *Variation of Gaussian curvature under conformal mapping and its application*, Comput. Math. Appl. **26** (1993), no. 1, 63–74.
[14] K. Ilarslan and E. Nešović, *Timelike and null normal curves in Minkowski space $\mathbb{E}_1^3$*, Indian J. Pure Appl. Math. **35** (2004), 881–888.
[15] C. Camci, L. Kula, and K. Ilarslan, *Characterizations of the position vector of a surface curve in Euclidean 3-space*, An. St. Univ. Ovidius Constanta **19** (2011), no. 3, 59–70.
[16] F. Schwarz, *Transformation to canonical form*, Algorithmic Lie Theory for Solving Ordinary Differential Equations, Chapman and Hall/CRC, New York, 2007, pp. 257–320.
[17] A. A. Shaikh and P. R. Ghosh, *Curves on a smooth surface with position vectors lie in the tangent plane*, Indian J. Pure Appl. Math. **51** (2020), no. 3, 1097–1104.
[18] S. Sharma and K. Singh, *Some aspects of rectifying curves on regular surfaces under different transformations*, Int. J. Anal. Appl. **21** (2023), 78.
[19] D. Fuchs, I. Izmestiev, M. Raffaelli, G. Szewieczek, and S. Tabachnikov, *Differential geometry of space curves: forgotten chapters*, Math. Intelligencer **46** (2024), 9–21.
[20] A. A. Shaikh and P. R. Ghosh, *On the position vector of surface curves in the Euclidean space*, Publ. Inst. Math. **126** (2022), 95–101.
[21] K. Ilarslan and E. Nesovic, *Some characterizations of osculating curves in the Euclidean spaces*, Demonstr. Math. **41** (2008), no. 4, 931–939.