Research Article

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Periodic and fixed points for mappings in extended *b*-gauge spaces equipped with a graph

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Abstract: This article presents the notions of *extended b-gauge space* $(U, Q_{\varphi;\Omega})$ and *extended* $\mathcal{J}_{\varphi;\Omega}$*-families* of *generalized extended pseudo-b-distances* on U. Furthermore, we look at these *extended* $\mathcal{J}_{\varphi;\Omega}$*-families* on U and define the *extended* $\mathcal{J}_{\varphi;\Omega}$-sequential completeness. We also look into some fixed and periodic point theorems for set-valued mappings in the new space with a graph that does not meet the completeness condition of the space. Additionally, this article includes some examples to explain the corresponding results and highlights some important consequences of our obtained results.

Keywords: generalized extended pseudo-*b*-distances, *G*-contraction, alpha-contraction, extended *b*-gauge space, fixed point, periodic point, graph

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1 Introduction

In 1922, Banach [1] achieved a pivotal milestone in fixed-point theory, presenting a result of profound significance. Subsequently, notable generalizations, including those by Hardy and Rogers [2], Mustafa et al. [3], and Khan et al. [4], have significantly advanced the field of fixed-point theory.

Jachymaski [5] also generalized the Banach contraction to Banach G-contraction by defining graph $G = (V, E)$ in $U \times U$, where V (the set of vertices) coincides with the nonempty set U and E (the set of edges) contains the diagonal but has no parallel edge. He proved that the conclusion of Banach result remained valid if the contraction condition holds for those ordered pairs that belong to the set of edges.

Later on, many researchers extended Banach G-contraction and obtained fixed-point results for mappings defined on complete metric spaces (Samreen and Kamran [6–8], Tiammee and Suantai [9], Nicolae et al. [10], Bojor [11], Asl et al. [12], Ahmad et al. [13], Petrusel and Petrusel [14], and Jiddah et al. [15]).

The study of gauge spaces was initiated by Dugundji [16], which generalizes metric spaces. Gauge spaces have the characteristic that the distance between two different points of the space may be zero. This simple characterization has been fascinating for many researchers around the world. For more definitions and

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results in gauge spaces, we recommend the researchers to refer to Agarwal et al. [17], Frigon [18], Chifu and Petrusel [19], Chis and Precup [20], Cherichi et al. [21], Cherichi and Samet [22], and Lazar and Petrusel [23].

Given a quasi-gauge space (U, Q), Wlodarczyk and Plebaniak [24] have introduced the notion of left (right) $\mathcal{J}$-families of generalized quasi-pseudodistances on U. These families generated by quasi-gauge Q determine a structure on U, which is more general than the structure on U determined by Q, and provide useful tools to obtain more general results with weaker assumptions, which can be seen in [25–31].

The aim of this article is to introduce the definition of an extended b-gauge space and an extended $\mathcal{J}_{\varphi;\Omega}$-family produced by extended b-gauge space $(U, Q_{\varphi;\Omega})$. Next, inspired by the idea of Ali et al. [32], Ali and Din [33], new G-contraction and α-contraction conditions with respect to this new extended family $\mathcal{J}_{\varphi;\Omega}$ of distances have been defined, and novel fixed and periodic point theorems are proved. Our results do not need completeness of the spaces $(U, Q_{\varphi;\Omega})$. Moreover, they provide information about periodic points as well and substantially generalize and improve the famous theorems of G-contractions and α-contractions in the proceeding of fixed points (in particular, see [32,33]). Multiple examples explaining ideas, definitions, and results are given.

2 Preliminaries

This section aims to recollect the relevant background material needed throughout this article.

Throughout this article, U is a non-void set, 2^U symbolizes the power set of the space U excluding the empty set, and $\mathbb{R}^+$ indicates the set of nonnegative real numbers. The collection of all fixed points of a multi-valued mapping $T : U \to 2^U$ symbolized by $F(T)$ is defined by $F(T) = \{u \in U : u \in T(u)\}$, and the collection of all periodic points of T symbolized by $P(T)$ is defined by $P(T) = \{u \in U : u \in T^{[k]}(u) \text{ for some } k \text{ in } \mathbb{N}\}$, where $T^{[k]} = T \circ T \circ T \circ \dots \circ T$ (k-times).

In 1966, Dugundji [16] initiated the idea of gauge spaces that generalizes metric spaces (or more generally pseudo-metric spaces). Here, we discuss the topology induced by gauge spaces and the condition in which these spaces are Hausdorff.

Definition 2.1. A map $q : U \times U \to [0, \infty)$ is a pseudo-metric, if for all $e, f, g \in U$, it satisfies
(a) $q(e, e) = 0$;
(b) $q(e, f) = q(f, e)$; and
(c) $q(e, g) \le q(e, f) + q(f, g)$.

The pair (U, q) is said to be pseudo-metric space.

Definition 2.2. Each family $Q = \{q_\beta : \beta \in \Omega\}$ of pseudo-metrics $q_\beta : U \times U \to [0, \infty)$ for $\beta \in \Omega$ is said to be gauge on U.

Definition 2.3. The family $Q = \{q_\beta : \beta \in \Omega\}$ of pseudo-metrics $q_\beta : U \times U \to [0, \infty)$ for $\beta \in \Omega$ is called to be separating if for each pair (e, f) where $e \ne f$, there is $q_\beta \in Q$ such that $q_\beta(e, f) > 0$.

Definition 2.4. Let $Q = \{q_\beta : \beta \in \Omega\}$ be a family of pseudo-metrics on U. The topology $\mathcal{T}(Q)$ on U whose subbase is defined by the family $\mathcal{B}(Q) = \{B(e, \varepsilon_\beta) : e \in U, \varepsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(e, \varepsilon_\beta) = \{f \in U : q_\beta(e, f) < \varepsilon_\beta\}$ is called the induced topology.

Definition 2.5. A topological space $(U, \mathcal{T})$ is called a gauge space if there exists gauge Q on U with $\mathcal{T} = \mathcal{T}(Q)$. The pair $(U, \mathcal{T}(Q))$ denotes the gauge space and is Hausdorff if Q is separating.

Example 2.6. Let $U = \mathbb{R}^2$ and let $q_1, q_2 : U \times U \to [0, \infty)$ be defined for all $(e_1, f_1), (e_2, f_2) \in \mathbb{R}^2$ by

$$q_1((e_1, f_1), (e_2, f_2)) = |e_2 - e_1| \quad \text{and} \quad q_2((e_1, f_1), (e_2, f_2)) = |f_2 - f_1|.$$

Then, q_1 and q_2 are pseudo-metrics on U.

Note that $q_1((2,3),(2,5)) = |2-2| = 0$, but $(2,3)$ and $(2,5)$ are distinct points. Also, $q_2((3,6),(5,6)) = |6-6| = 0$, but $(3,6)$ and $(5,6)$ are the distinct points. Therefore, q_1 and q_2 are not metrics on U.

Let the family $Q = \{q_1, q_2\}$ be a gauge on U. We now look for the topology $\mathcal{T}(Q)$ induced by gauge Q in the following manner.

First, finding balls $B(e, \varepsilon_1)$ for q_1, where $e = (e_1, f_1) \in U$ and $\varepsilon_1 > 0$:

$$\begin{aligned} B((e_1, f_1), \varepsilon_1) &= \{(e_2, f_2) \in U : q_1((e_1, f_1), (e_2, f_2)) < \varepsilon_1\} \\ &= \{(e_2, f_2) \in U : e_2 \in (-\varepsilon_1 + e_1, \varepsilon_1 + e_1)\}. \end{aligned}$$

Thus, $B((e_1, f_1), \varepsilon_1)$ contains all verticle strips in the plane.

Similarly,

$$\begin{aligned} B((e_1, f_1), \varepsilon_2) &= \{(e_2, f_2) \in U : q_2((e_1, f_1), (e_2, f_2)) < \varepsilon_2\} \\ &= \{(e_2, f_2) \in U : f_2 \in (-\varepsilon_2 + f_1, \varepsilon_2 + f_1)\}. \end{aligned}$$

Thus, $B((e_1, f_1), \varepsilon_2)$ contains all horizontal strips in the plane.

The subbase $\mathcal{B}(Q)$ for induced topology $\mathcal{T}(Q)$ is the collection of all vertical and horizontal infinite open strips. Their intersections are open rectangles that form the base of induced topology. The induced topology is thus the usual topology on $\mathbb{R}^2$. Therefore, (U, Q) is a gauge space.

In order to generalize metric spaces, Bakhtin [34] introduced the notion of b-metric spaces in 1989, which was formally presented by Czerwik [35] in 1993 in the following way.

Definition 2.7. A map $q : U \times U \to \mathbb{R}^+$ is said to be a b-metric, if for each $e, f, g \in U$, there exists $s \geq 1$ such that it satisfies
(a) $q(e, f) = 0 \Leftrightarrow e = f$;
(b) $q(e, f) = q(f, e)$;
(c) $q(e, g) \leq s\{q(e, f) + q(f, g)\}$.

The pair (U, q) is said to be a b-metric space.

Recently, Ali et al. [32] introduced the notion of b-gauge spaces and thus extended the idea of gauge spaces in the locale of b-metric spaces. We note down the following definitions of their work.

Definition 2.8. A map $q : U \times U \to [0, \infty)$ is a b-pseudo-metric, if there is $s \geq 1$ satisfying for all $e, f, g \in U$ the following conditions:
(a) $q(e, e) = 0$;
(b) $q(e, f) = q(f, e)$; and
(c) $q(e, g) \leq s\{q(e, f) + q(f, g)\}$.

For prescribed b-pseudo-metric q, (U, q) is called the b-pseudo-metric space.

Definition 2.9. Each family $Q = \{q_\beta : \beta \in \Omega\}$ of b-pseudo-metrics $q_\beta : U \times U \to [0, \infty)$, is called the b-gauge on U.

Definition 2.10. The family $Q = \{q_\beta : \beta \in \Omega\}$ is separating if for each pair (e, f), where $e \neq f$, there is $q_\beta \in Q$ such that $q_\beta(e, f) > 0$.

Definition 2.11. Let $Q = \{q_\beta : \beta \in \Omega\}$ be the family of b-pseudo-metrics on U. The topology $\mathcal{T}(Q)$ on U whose subbase is defined by the family $\mathcal{B}(Q) = \{B(e, \varepsilon_\beta) : e \in U, \varepsilon_\beta > 0, \beta \in \Omega\}$, where $B(e, \varepsilon_\beta) = \{f \in U : q_\beta(e, f) < \varepsilon_\beta\}$ is called the topology induced by Q. The pair $(U, \mathcal{T}(Q))$ is called to be a b-gauge space and is Hausdorff if Q is separating.

Ali et al. [32] presented the following example to show that b-pseudo-metric space (in fact, b-gauge space) is the generalization of metric space, pseudo-metric space (in fact, gauge space), and b-metric space.

Example 2.12. [32] Suppose $U = C([0, \infty), \mathbb{R})$ and describe $q : U \times U \to [0, \infty)$ by

$$q(u(t), v(t)) = \max_{t \in [0,1]}(u(t) - v(t))^2.$$

Then, d is a b-pseudo-metric, but not a metric, pseudo-metric or b-metric.

In 2017, Kamran et al. [36] enriched the notion of b-metric space by amending the triangular inequality and presented the following definition of extended b-metric space in view of generalizing b-metric space.

Definition 2.13. Let $\varphi : U \times U \to [1, \infty)$. A map $q : U \times U \to \mathbb{R}^+$ is said to be an extended b-metric, if for all $e, f, g \in U$, it satisfies
(a) $q(e, f) = 0 \Leftrightarrow e = f$;
(b) $q(e, f) = q(f, e)$;
(c) $q(e, g) \le \varphi(e, g)\{q(e, f) + q(f, g)\}$.

The pair (U, q) or simply U is called an extended b-metric space.

3 Main results

In order to introduce extended b-gauge spaces, we start here the introduction of the notion of extended pseudo-b metric.

Definition 3.1. [37] A map $q : U \times U \to \mathbb{R}^+$ is an extended pseudo-b-metric, if for all $e, f, g \in U$, there exists $\varphi : U \times U \to [1, \infty)$ satisfying the following conditions:
(a) $q(e, e) = 0$;
(b) $q(e, f) = q(f, e)$; and
(c) $q(e, g) \le \varphi(e, g)\{q(e, f) + q(f, g)\}$.

The pair (U, q) is said to be an extended pseudo-b-metric space.

Example 3.2. Suppose $U = [0, 1]$. Define $q : U \times U \to \mathbb{R}^+$ and $\varphi : U \times U \to [1, \infty)$ by

$$q(e, f) = (e - f)^2$$

and

$$\varphi(e, f) = e + f + 2,$$

for all $e, f \in U$.

Then, q is an extended pseudo-b-metric on U. Indeed, $q(e, e) = 0$ and $q(e, f) = q(f, e)$ for all $e, f \in U$. Furthermore, for all $e, f, g \in U$, $q(e, g) \le \varphi(e, g)\{q(e, f) + q(f, g)\}$ holds.

Example 3.3. Let $U = \{e, f, g\}$. Define $q : U \times U \to \mathbb{R}^+$ and $\varphi : U \times U \to [1, \infty)$ for all $e, f, g \in U$ by

$$q(e, e) = 0,$$

$$q(e, f) = q(f, e) = 1,$$

$$q(f, g) = q(g, f) = \frac{1}{2},$$

$$q(g,e) = q(e,g) = 2,$$

and

$$\varphi(e,f) = |e| + |f| + 2.$$

Furthermore, $q(e,g) \leq \varphi(e,g)\{q(e,f) + q(f,g)\}$ is satisfied. Indeed, q is an extended pseudo-b-metric on U. Note that q is not a pseudo-metric on U as $\frac{3}{2} = q(e,f) + q(f,g) < q(e,g) = 2$.

This example illustrates that an extended pseudo-b-metric serves as a generalization or extension of a pseudo-metric.

Definition 3.4. The family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ of extended pseudo-b-metrics $q_\beta : U \times U \to [0,\infty)$, $\beta \in \Omega$, is said to be an extended b-gauge on U (Ω-index set).

Definition 3.5. The family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ of extended pseudo-b-metrics $q_\beta : U \times U \to [0,\infty)$, $\beta \in \Omega$ is called to be separating if for every pair (e,f), where $e \neq f$, there exists $q_\beta \in Q_{\varphi;\Omega}$ such that $q_\beta(e,f) \neq 0$.

Definition 3.6. Let the family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ be an extended b-gauge on U. The topology $\mathcal{T}(Q_{\varphi;\Omega})$ on U whose subbase is defined by the family $\mathcal{B}(Q_{\varphi;\Omega}) = \{B(e,\varepsilon_\beta) : e \in U, \varepsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(e,\varepsilon_\beta) = \{f \in U : q_\beta(e,f) < \varepsilon_\beta\}$ is called the topology induced by $Q_{\varphi;\Omega}$. The topological space $(U, \mathcal{T}(Q_{\varphi;\Omega}))$ is an extended b-gauge space, denoted by $(U, Q_{\varphi;\Omega})$. We note that $(U, Q_{\varphi;\Omega})$ is Hausdorff if $Q_{\varphi;\Omega}$ is separating.

Remark 3.7. For $s_\beta = 1$, for each $\beta \in \Omega$, each gauge space is a b_s-gauge space, and for $\varphi_\beta(u,v) = s$, for each $\beta \in \Omega$, where $s \geq 1$, each b-gauge space is an extended b-gauge space. Hence, extended b-gauge space is the largest general space.

Next, we establish the idea of extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances on U (which are called extended $\mathcal{J}_{\varphi;\Omega}$-families on U, for short). These extended $\mathcal{J}_{\varphi;\Omega}$-families generalize extended b-gauges.

Definition 3.8. The family $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0,\infty)$, $\beta \in \Omega$, is called an extended $\mathcal{J}_{\varphi;\Omega}$-family of generalized extended pseudo-b-distances on $(U, Q_{\varphi;\Omega})$ if the following statements hold for all $u, v, w \in U$ and for all $\beta \in \Omega$:

(J1) $J_\beta(u,w) \leq \varphi_\beta(u,w)\{J_\beta(u,v) + J_\beta(v,w)\}$;

(J2) for each sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in N)$ in U fulfilling

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(u_m, u_n) = 0 \tag{3.1}$$

and

$$\lim_{m\to\infty} J_\beta(v_m, u_m) = 0, \tag{3.2}$$

the following holds:

$$\lim_{m\to\infty} q_\beta(v_m, u_m) = 0. \tag{3.3}$$

We denote

$$\mathbb{J}_{(U,Q_{\varphi;\Omega})} = \{\mathcal{J}_{\varphi;\Omega} : \mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}\}.$$

Also, we denote

$$U^0_{\mathcal{J}_{\varphi;\Omega}} = \{u \in U : J_\beta(u,u) = 0, \text{ for all } \beta \in \Omega\}$$

and

$$U^{+}_{\mathcal{J}_{\varphi;\Omega}} = \{u \in U : J_\beta(u,u) > 0, \text{ for all } \beta \in \Omega\}.$$

Thus, $U = U^{0}_{\mathcal{J}_{\varphi;\Omega}} \cup U^{+}_{\mathcal{J}_{\varphi;\Omega}}$.

Example 3.9. Let U contain at least two distinct elements, and suppose $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ is an extended b-gauge on U. Thus, $(U, Q_{\varphi;\Omega})$ is an extended b-gauge space.

Let there be at least two distinct but arbitrary and fixed elements in a set $F \subset U$. Let $a_\beta \in (0,\infty)$ satisfy $\delta_\beta(F) < a_\beta$, where $\delta_\beta(F) = \sup\{q_\beta(e,f) : e, f \in F\}$, for all $\beta \in \Omega$. Let $J_\beta : U \times U \to [0,\infty)$ for all $e, f \in U$ be defined as

$$J_\beta(e,f) = \begin{cases} q_\beta(e,f), & \text{if } F \cap \{e,f\} = \{e,f\}, \\ a_\beta, & \text{if } F \cap \{e,f\} \neq \{e,f\}. \end{cases} \tag{3.4}$$

Then, $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\} \in \mathbb{J}_{(U,Q)}$.

Note that $J_\beta(e,g) \le \varphi_\beta(e,g)\{J_\beta(e,f) + J_\beta(f,g)\}$, for all $e, f, g \in U$; thus, $(\mathcal{J}_1)$ is satisfied. Indeed, $(\mathcal{J}_1)$ will not satisfied only when there is some $e, f, g \in U$ such that $J_\beta(e,g) = a_\beta$, $J_\beta(e,f) = q_\beta(e,f)$, $J_\beta(f,g) = q_\beta(f,g)$, and $\varphi_\beta(e,g)\{q_\beta(e,f) + q_\beta(f,g)\} \le a_\beta$. However, then this gives rise to an element $h \in \{e,g\}$ with $h \notin F$, and on other hand, $e, f, g \in F$, which is impossible.

Next, assume that (3.1) and (3.2) hold by the sequences (u_m) and (v_m) in U. Then, (3.2) yields that for all $0 < \varepsilon < a_\beta$, there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that

$$J_\beta(v_m, u_m) < \varepsilon \quad \text{for all } m \ge m_1, \text{ for all } \beta \in \Omega. \tag{3.5}$$

By (3.5) and (3.4), denoting $m_2 = \min\{m_1(\beta) : \beta \in \Omega\}$, we have

$$F \cap \{v_m, u_m\} = \{v_m, u_m\}, \text{ for all } m \ge m_2$$

and

$$q_\beta(v_m, u_m) = J_\beta(v_m, u_m) < \varepsilon.$$

Thus, (3.3) is satisfied. Therefore, $\mathcal{J}_{\varphi;\Omega}$ is a $\mathcal{J}_{\varphi;\Omega}$-family on U.

We now state few trivial characteristics of extended $\mathcal{J}_{\varphi;\Omega}$-families on U.

Remark 3.10. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. Then, the following hold:

(i) $Q_{\varphi;\Omega} \in \mathbb{J}_{(U,Q_{\varphi;\Omega})}$.

(ii) Let $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}_{(U,Q_{\varphi;\Omega})}$. If for all $\beta \in \Omega$ and for all $u, v \in U\backslash$ $J_\beta(v,v) = 0$ and $J_\beta(u,v) = J_\beta(v,u)$, then for each $\beta \in \Omega$, J_β is an extended pseudo-b-metric.

(iii) Several examples of $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}_{(U,Q_{\varphi;\Omega})}$, which show that the maps J_β, $\beta \in \Omega$ are not the extended pseudo-b-metrics can be found in the literature (see [37], Example 3).

Proposition 3.11. *Let $(U, Q_{\varphi;\Omega})$ be a Hausdorff extended b-gauge space and the family $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be an extended $\mathcal{J}_{\varphi;\Omega}$-family on U. Then, there exists $\beta \in \Omega$ such that*

$$e \neq f \Rightarrow J_\beta(e,f) > 0 \vee J_\beta(f,e) > 0,$$

for each $e, f \in U$.

Proof. Let that there be $e \neq f$, $e, f \in U$ such that $J_\beta(e,f) = 0 = J_\beta(f,e)$ for all $\beta \in \Omega$. Then, $J_\beta(e,e) = 0$, for all $\beta \in \Omega$; using property $(\mathcal{J}1)$ in Definition 3.8, it follows that $J_\beta(e,e) \le \varphi_\beta(e,e)\{J_\beta(e,f) + J_\beta(f,e)\} = 0$, for all $\beta \in \Omega$.

Defining sequences (u_m) and (v_m) in U by $u_m = f$ and $v_m = e$, we see that Conditions (3.1) and (3.2) of property $(\mathcal{J}2)$ are satisfied, and therefore, Condition (3.3) holds, which implies that $q_\beta(e, f) = 0$, for all $\beta \in \Omega$. But this denies the fact that $(U, Q_{\varphi;\Omega})$ is a Hausdorff extended b-gauge space. Therefore, our supposition is wrong, and there exists $\beta \in \Omega$ such that for all $e, f \in U$,

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0. \qquad \square$$

We now define extended $\mathcal{J}_{\varphi;\Omega}$-completeness in an extended b-gauge space $(U, Q_{\varphi;\Omega})$, using extended $\mathcal{J}_{\varphi;\Omega}$-families on U.

Definition 3.12. Let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be an extended $\mathcal{J}_{\varphi;\Omega}$-family on the extended b-gauge space $(U, Q_{\varphi;\Omega})$.

(A) A sequence $(v_m : m \in \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U if

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(v_m, v_n) = 0, \quad \text{for all } \beta \in \Omega.$$

(B) The sequence $(v_m : m \in \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$-convergent to $v \in U$ if $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v$, where

$$\lim_{m\to\infty}^{\mathcal{J}_{\varphi;\Omega}} v_m = v \Leftrightarrow \lim_{m\to\infty} J_\beta(v, v_m) = 0 = \lim_{m\to\infty} J_\beta(v_m, v), \quad \text{for all } \beta \in \Omega.$$

(C) If $S^{\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} \neq \emptyset$, where

$$S^{\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} = \{v \in U : \lim_{m\to\infty}^{\mathcal{J}_{\varphi;\Omega}} v_m = v\},$$

then $(v_m : m \in \mathbb{N})$ in U is an extended $\mathcal{J}_{\varphi;\Omega}$-convergent sequence in U.

(D) The space $(U, Q_{\varphi;\Omega})$ is called extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete, if every extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U, extended $\mathcal{J}_{\varphi;\Omega}$-converges in U.

Remark 3.13. There exist examples of extended b-gauge space $(U, Q_{\varphi;\Omega})$ and $\mathcal{J}_{\varphi;\Omega}$-family on U with $\mathcal{J}_{\varphi;\Omega} \neq Q_{\varphi;\Omega}$ such that $(U, Q_{\varphi;\Omega})$ is $\mathcal{J}_{\varphi;\Omega}$-sequential complete but not $Q_{\varphi;\Omega}$-sequential complete (see [38], Example 3.10).

Definition 3.14. Let $T : U \to 2^U$ be a set-valued map. The map $T^{[k]}$ is said to be an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, if for each sequence $(w_m : m \in \mathbb{N})$ in $T^{[k]}(U)$, which is extended $Q_{\varphi;\Omega}$-convergent in U, one has $S^{Q_{\varphi;\Omega}}_{(w_m:m\in\mathbb{N})} \neq \emptyset$ and its subsequences (f_m) and (z_m) satisfy

$f_m \in T^{[k]}(z_m)$, for all $m \in \mathbb{N}$

has the property that there exists $w \in S^{Q_{\varphi;\Omega}}_{(w_m:m\in\mathbb{N})}$ such that $w \in T^{[k]}(w)$.

Definition 3.15. Let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be an extended $\mathcal{J}_{\varphi;\Omega}$-family on extended b-gauge space $(U, Q_{\varphi;\Omega})$. A set $Y \in 2^U$ is $\mathcal{J}_{\varphi;\Omega}$-closed in U if $Y = cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)$, where $cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)$ is the $\mathcal{J}_{\varphi;\Omega}$-closure on Y in U, denotes the collection of all $u \in U$ for which there is a sequence $(u_m : m \in \mathbb{N})$ in Y such that it is $\mathcal{J}_{\varphi;\Omega}$-converges to u.

Define $Cl^{\mathcal{J}_{\varphi;\Omega}}(U) = \{Y \in 2^U : Y = cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)\}$. Thus, $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ indicates the set of all $\mathcal{J}_{\varphi;\Omega}$-closed subsets of U.

Definition 3.16. Let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be an extended $\mathcal{J}_{\varphi;\Omega}$-family on extended b-gauge space U, and let for all $e \in U$, for all $F \in 2^U$, and for all $\beta \in \Omega$,

$$J_\beta(e, F) = \inf\{J_\beta(e, g) : g \in F\}.$$

Define on $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ the distance $D_\beta^{\mathcal{J}_{\varphi;\Omega}}$ of Hausdorff type for all $\beta \in \Omega$ and for all $E, F \in Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$, where $D_\beta^{\mathcal{J}_{\varphi;\Omega}} : Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \times Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \to [0, \infty)$, $\beta \in \Omega$ is defined as follows:

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(E, F) = \begin{cases} \max\left\{\sup_{e\in E} J_\beta(e, F), \sup_{f\in F} J_\beta(f, E)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Throughout this article, Ω indicates an index-set and $(U, Q_{\varphi;\Omega})$ is an extended b-gauge space equipped with the graph $G = (V, E)$ such that V (the set of vertices) is the set U and E (the set of edges) includes $\{(v, v) : v \in V\}$. Also, suppose that G has no parallel edges.

Furthermore, note that for each theorem and corollary, it is assumed that $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete.

We first proof the following result.

Lemma 3.17. *Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space, and let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be an extended $\mathcal{J}_{\varphi;\Omega}$-family on U. Then,*

$$J_\beta(u, A) \leq \varphi_\beta(u, A)\{J_\beta(u, v) + J_\beta(v, A)\},$$

for all $u, v \in U$, for all $\beta \in \Omega$, and for all $A \subset U$, where

$$\varphi_\beta(u, A) = \inf\{\varphi_\beta(u, a) : a \in A\}.$$

Proof. From axioms of definition, we can write for all $\beta \in \Omega$,

$$\begin{aligned} J_\beta(u, a) &\leq \varphi_\beta(u, a)\{J_\beta(u, v) + J_\beta(v, a)\}, \qquad \text{for all } u, v, a \in U \\ J_\beta(u, a) &\leq \varphi_\beta(u, a)J_\beta(u, v) + \varphi_\beta(u, a)J_\beta(v, a). \end{aligned}$$

By taking infimum of both sides over A, we obtain for all $\beta \in \Omega$,

$$\begin{aligned} \inf_{a\in A} J_\beta(u, a) &\leq \inf_{a\in A} \varphi_\beta(u, a)J_\beta(u, v) + \inf_{a\in A} \varphi_\beta(u, a)\inf_{a\in A} J_\beta(v, a) \\ J_\beta(u, A) &\leq \varphi_\beta(u, A)J_\beta(u, v) + \varphi_\beta(u, A)J_\beta(v, A) \\ J_\beta(u, A) &\leq \varphi_\beta(u, A)\{J_\beta(u, v) + J_\beta(v, A)\}. \end{aligned}$$

□

Our main results for set-valued G-contractions are now given below.

Theorem 3.18. *Let the set-valued map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy*

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tu, Tv) \leq a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu), \tag{3.6}$$

for all $(u, v) \in E$, where $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ is such that $a_\beta + b_\beta + c_\beta + 2e_\beta\varphi_\beta(z^{m-1}, Tz^m) < 1$ and $\lim_{m,n\to\infty}\varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and each $z^0 \in U$, here $z^m \in T(z^{m-1})$, for $m \in \mathbb{N}$.

Moreover, let

(i) there is $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$;

(ii) if $(u, v) \in E$ and $x \in Tu$ and $y \in Tv$ such that $J_\beta(x, y) \leq J_\beta(u, v)$, for all $\beta \in \Omega$, then $(x, y) \in E$;

(iii) for any $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $u \in U$, there exists $v \in Tu$ such that

$$J_\beta(u, v) \leq r_\beta J_\beta(u, Tu), \quad \text{for all } \beta \in \Omega.$$

Then, the following assertions hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $Q_{\varphi;\Omega}$-convergent sequence in U; thus, $S^{Q_{\varphi;\Omega}}_{(z^m: m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that for some $k \in \mathbb{N}$, $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U. Then,

(a_1) $F(T^{[k]}) \neq \emptyset$ and

(a_2) there exists $z \in F(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m: m\in\{0\}\cup\mathbb{N})}$.

Proof. (I) Using supposition (i), there is $z^0, z^1 \in U$ such that $z^1 \in Tz^0$ and $(z^0, z^1) \in E$. Now, for each $\beta \in \Omega$, applying (3.6), we have

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tz^0, Tz^1) \le a_\beta J_\beta(z^0, z^1) + b_\beta J_\beta(z^0, Tz^0) + c_\beta J_\beta(z^1, Tz^1) + e_\beta J_\beta(z^0, Tz^1) + L_\beta J_\beta(z^1, Tz^0). \tag{3.7}$$

Now as $J_\beta(z^1, Tz^1) \le D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tz^0, Tz^1)$ and $J_\beta(z^0, Tz^1) \le \varphi_\beta(z^0, Tz^1)\{J_\beta(z^0, z^1) + J_\beta(z^1, Tz^1)\}$, therefore (3.7) implies

$$J_\beta(z^1, Tz^1) \le \frac{1}{\zeta_\beta} J_\beta(z^0, z^1), \tag{3.8}$$

where $\zeta_\beta = \frac{1 - c_\beta - e_\beta\varphi(z^0, Tz^1)}{a_\beta + b_\beta + e_\beta\varphi(z^0, Tz^1)} > 1$. Now using assumption ($iii$), we have $z^2 \in Tz^1$ such that

$$J_\beta(z^1, z^2) \le \sqrt{\zeta_\beta} J_\beta(z^1, Tz^1). \tag{3.9}$$

Combining (3.8) and (3.9), we can write

$$J_\beta(z^1, z^2) \le \frac{1}{\sqrt{\zeta_\beta}} J_\beta(z^0, z^1), \qquad \text{for all } \beta \in \Omega. \tag{3.10}$$

Assumption (ii) and (3.10) imply that $(z^1, z^2) \in E$. Following the same steps, we find a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ in U such that $(z^m, z^{m+1}) \in E$ and

$$J_\beta(z^m, z^{m+1}) \le \left(\frac{1}{\sqrt{\zeta_\beta}}\right)^m J_\beta(z^0, z^1), \qquad \text{for all } \beta \in \Omega. \tag{3.11}$$

For convenience, let $\mu_\beta = \frac{1}{\sqrt{\zeta_\beta}}$, for each $\beta \in \Omega$.

Now, by repeated use of ($\mathcal{J}1$) and (3.11) for all $\beta \in \Omega$ and for all $n > m$, where $m, n \in \mathbb{N}$, we obtain

$$\begin{aligned}
J_\beta(z^m, z^n) &\le \varphi_\beta(z^m, z^n)\mu_\beta^m J_\beta(z^0, z^1) + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\mu_\beta^{m+1} J_\beta(z^0, z^1) \\
&\quad + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\varphi_\beta(z^{m+2}, z^n)\mu_\beta^{m+2} J_\beta(z^0, z^1) \\
&\quad + \ldots + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n) \ldots \varphi_\beta(z^{n-1}, z^n)\mu_\beta^{n-1} J_\beta(z^0, z^1) \\
&\le J_\beta(z^0, z^1)[\varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n) \ldots \varphi_\beta(z^m, z^n)\mu_\beta^m \\
&\quad + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n) \ldots \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\mu_\beta^{m+1} \\
&\quad + \ldots + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n) \ldots \varphi_\beta(z^m, z^n) \ldots \varphi_\beta(z^{n-1}, z^n)\mu_\beta^{n-1}].
\end{aligned}$$

Since for some $\mu_\beta < 1$, $\lim_{n,m\to\infty}\varphi_\beta(z^{m+1}, z^n)\mu_\beta < 1$, by ratio test, the series $\sum_{m=1}^\infty \mu_\beta^m \prod_{i=1}^m \varphi_\beta(z^i, z^n)$ is convergent. Let $S = \sum_{m=1}^\infty \mu_\beta^m \prod_{i=1}^m \varphi_\beta(z^i, z^n)$ and $S_m = \sum_{j=1}^m \mu_\beta^j \prod_{i=1}^j \varphi_\beta(z^i, z^n)$.

This gives

$$J_\beta(z^m, z^n) \le J_\beta(z^0, z^1)[S_{n-1} - S_m].$$

This implies

$$\limsup_{m\to\infty, \, n>m} J_\beta(z^m, z^n) = 0, \quad \text{for all } \beta \in \Omega. \tag{3.12}$$

Now, since $(U, Q_{\varphi;\Omega})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$ convergent in U; thus for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$, we have

$$\lim_{m\to\infty} J_\beta(z, z^m) = 0, \quad \text{for all } \beta \in \Omega. \tag{3.13}$$

Thus, from (3.12) and (3.13), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup\mathbb{N})}$, taking $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$, and applying ($\mathcal{J}2$) to these sequences, we obtain

$$\lim_{m\to\infty} q_\beta(z, z^m) = \lim_{m\to\infty} q_\beta(v_m, u_m) = 0, \quad \text{for all } \beta \in \Omega.$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$.

(II) To show (a_1), let $z^0 \in U$ be fixed and arbitrary. Since $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$ and

$$z^{(m+1)k} \in T^{[k]}(z^{mk}), \qquad \text{for } m \in \{0\} \cup \mathbb{N}$$

thus describing $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we obtain

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m \in T^{[k]}(x_m), \quad \text{for all } m \in \mathbb{N},$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Thus, applying the fact below

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(y_m:m\in\mathbb{N})} \quad \text{and} \quad S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(x_m:m\in\mathbb{N})}$$

and the assumption that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, there exists $z \in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$.

Thus, (a_1) holds.

The statement (a_2) follows from (a_1) and the certainty that $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$. □

Let $T : U \to U$ be a single-valued mapping. We have the following result.

Theorem 3.19. *Let the single-valued map $T : U \to U$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy*

$$J_\beta(Tu, Tv) \leq a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu), \tag{3.14}$$

for all $(u, v) \in E$, where $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ is such that $a_\beta + b_\beta + c_\beta + 2e_\beta\varphi_\beta(z^{m-1}, Tz^m) < 1$ and $\lim_{m,n\to\infty}\varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and each $z^0 \in U$, here $z^m = T^{[m]}(z^0)$, where $m \in \mathbb{N}$.

Moreover,

(i) *there exists $z^0 \in U$ such that $(z^0, Tz^0) \in E$;*

(ii) *for $(u, v) \in E$, we have $(Tu, Tv) \in E$, given that $J_\beta(Tu, Tv) \leq J_\beta(u, v)$ for all $\beta \in \Omega$;*

(iii) *if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $(z^m, z^{m+1}) \in E$ and $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} z^m = z$, then $(z^m, z) \in E$ and $(z, z^m) \in E$.*

Then, the following assertions hold:

(I) *For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $Q_{\varphi;\Omega}$-convergent sequence in U; thus, $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$.*

(II) *Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$ and $\varphi_\beta(z, Tz)\{c_\beta + e_\beta \lim_{m\to\infty}\varphi_\beta(z^m, Tz)\} < 1$. Then,*

(a_1) *$F(T^{[k]}) \neq \varnothing$;*

(a_2) *there exists $z \in \mathrm{F}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and*

(a_3) *for all $z \in \mathrm{F}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.*

(III) Furthermore, let $F(T^{[k]}) \neq \varnothing$, *for some* $k \in \mathbb{N}$ *and* $(U, Q_{\varphi;\Omega})$ *is a Hausdorff space. Then,*
(b_1) $F(T^{[k]}) = F(T)$;
(b_2) *there is* $z \in \mathrm{F}(T)$ *such that* $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$; *and*
(b_3) *for all* $z \in \mathrm{F}(T)$, $J_\beta(z, z) = 0$, *for all* $\beta \in \Omega$.

Proof. In view of Theorem 3.18, it remains to prove assertions (a_3) and (b_1)–(b_3) of the aforementioned theorem.

To prove (a_3), the contrary, suppose that $J_{\beta_0}(z, Tz) > 0$ for some $\beta_0 \in \Omega$. Using $\mathcal{J}1$, assumption (*iii*), and inequality (3.14), we can write

$$\begin{aligned} J_{\beta_0}(z, Tz) &\le \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + J_{\beta_0}(z^{m+1}, Tz)\} \\ &= \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + J_{\beta_0}(Tz^m, Tz)\} \\ &\le \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + a_{\beta_0}J_{\beta_0}(z^m, z) + b_{\beta_0}J_{\beta_0}(z^m, Tz^m) + c_{\beta_0}J_{\beta_0}(z, Tz) + e_{\beta_0}J_{\beta_0}(z^m, Tz) \\ &\qquad + L_{\beta_0}J_{\beta_0}(z, Tz^m)\} \\ &\le \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + a_{\beta_0}J_{\beta_0}(z^m, z) + b_{\beta_0}J_{\beta_0}(z^m, z^{m+1}) + c_{\beta_0}J_{\beta_0}(z, Tz) \\ &\quad + e_{\beta_0}\varphi_{\beta_0}(z^m, Tz)\{J_{\beta_0}(z^m, z) + J_{\beta_0}(z, Tz)\} + L_{\beta_0}J_{\beta_0}(z, z^{m+1})\}. \end{aligned}$$

Letting $m \to \infty$, since $\lim_{m,n\to\infty}\varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and for each $z^m, z^n \in U$, $\varphi_\beta(z^m, z^n)$ is finite, and thus, we obtain

$$J_{\beta_0}(z, Tz) \le \varphi_{\beta_0}(z, Tz)\{c_{\beta_0} + e_{\beta_0}\lim_{m\to\infty}\varphi_{\beta_0}(z^m, Tz)\}J_{\beta_0}(z, Tz).$$

Now, since $\varphi_\beta(z, Tz)\{c_\beta + e_\beta \lim_{m\to\infty}\varphi_\beta(z^m, Tz)\} < 1$, we obtain

$$J_{\beta_0}(z, Tz) \le \varphi_{\beta_0}(z, Tz)\{c_{\beta_0} + e_{\beta_0}\lim_{m\to\infty}\varphi_{\beta_0}(z^m, Tz)\}J_{\beta_0}(z, Tz) < J_{\beta_0}(z, Tz),$$

which is impossible. Thus, $J_\beta(z, Tz) = 0$, for all $\beta \in \Omega$.

Next, we prove that $J_\beta(Tz, z) = 0$, for all $\beta \in \Omega$.

$$\begin{aligned} J_\beta(Tz, z) &\le \varphi_\beta(Tz, z)\{J_\beta(Tz, z^{m+1}) + J_\beta(z^{m+1}, z)\} \\ &= \varphi_\beta(Tz, z)\{J_\beta(Tz, Tz^m) + J_\beta(z^{m+1}, z)\} \\ &\le \varphi_\beta(Tz, z)\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, Tz^m) + e_\beta J_\beta(z, Tz^m) + L_\beta J_\beta(z^m, Tz) + J_\beta(z^{m+1}, z)\} \\ &\le \varphi_\beta(Tz, z)\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, z^{m+1}) + e_\beta J_\beta(z, z^{m+1}) + L_\beta\varphi_\beta(z^m, Tz)\{J_\beta(z^m, z) + J_\beta(z, Tz)\} \\ &\qquad + J_\beta(z^{m+1}, z)\}. \end{aligned}$$

Letting $m \to \infty$, since $\lim_{m,n\to\infty}\varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and for each $z^m, z^n \in U$, $\varphi_\beta(z^m, z^n)$ is finite, and thus, we have

$$J_\beta(Tz, z) \le \varphi_\beta(Tz, z)\{b_\beta + L_\beta \lim_{m\to\infty}\varphi_\beta(z^m, Tz)\}J_\beta(z, Tz) \quad \text{for all } \beta \in \Omega.$$

Also, since we have proved that $J_\beta(z, Tz) = 0$ for all $\beta \in \Omega$, we obtain $J_\beta(Tz, z) = 0$ for all $\beta \in \Omega$. Hence, assertion (a_3) holds.

(III) Since $(U, Q_{\varphi;\Omega})$ is a Hausdorff space, using Proposition (3.11), assertion (a_3) suggests that for $z \in \mathrm{F}(T^{[k]})$, we have $z = T(z)$. This gives $z \in \mathrm{F}(T)$. Hence, (b_1) is true.

Assertions (a_2) and (b_1) imply (b_2).

To prove assertion (b_3), consider ($\mathcal{J}1$) and use (a_3) and (b_1), we have for all $z \in \mathrm{F}(T) = \mathrm{F}(T^{[k]})$ and for all $\beta \in \Omega$

$$J_\beta(z, z) \le \varphi(z, z)\{J_\beta(z, T(z)) + J_\beta(T(z), z)\} = 0.$$

□

Example 3.20. Suppose that $U = [0, 5]$ and $Q_{\varphi;\Omega} = \{q\}$, where $q : U \times U \to \mathbb{R}^+$ is an extended pseudo-b-metric on U given by

$$q(e, f) = |e - f|^2, \quad \text{for all } e, f \in U, \tag{3.15}$$

and $\varphi : U \times U \to [1, \infty)$ is defined by

$$\varphi(e, f) = |e| + |f| + 2.$$

Suppose $F = \left[\frac{1}{8}, 1\right] \subset U$ and let $J : U \times U \to [0, \infty)$ for all $e, f \in U$ be defined as

$$J(e, f) = \begin{cases} q(e, f), & \text{if } F \cap \{e, f\} = \{e, f\}, \\ 4, & \text{if } F \cap \{e, f\} \neq \{e, f\}, \end{cases} \tag{3.16}$$

where $\varphi(e, f) = |e| + |f| + 2$.

Let the graph $G = (V, E)$ be such that $V = U$ and

$$E = \{(e, f) : e \leq f\} \cup \{(e, e) : e \in U\}.$$

The single-valued map T is defined by

$$T(e) = \frac{e + 1}{5}, \quad \text{for all } e \in U. \tag{3.17}$$

(I.1) $(U, Q_{\varphi;\Omega})$ is an extended b-gauge space (Example 3.2), which is also Hausdorff.
(I.2) The family $\mathcal{J}_{\varphi;\Omega} = \{J\}$ is an extended $\mathcal{J}_{\varphi;\Omega}$-family on U (Example 3.9).
(I.3) $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequential complete.

For this, let $\{v_m : m \in \mathbb{N}\}$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence. We may suppose, without losing generality that for all $0 < \varepsilon_1 < \frac{1}{64}$ there exists $k_0 \in \mathbb{N}$ such that for all $n \geq m \geq k_0$ where $n, m \in \mathbb{N}$, we have

$$J(v_m, v_n) < \varepsilon_1 < \frac{1}{64}. \tag{3.18}$$

Then, using (3.16, (3.15), and (3.18), for all $0 < \varepsilon_1 < \frac{1}{64}$, there exists $k_0 \in \mathbb{N}$ such that for all $n \geq m \geq k_0$, where $n, m \in \mathbb{N}$, we obtain

$$J(v_m, v_n) = q(v_m, v_n) = |v_m - v_n|^2 < \varepsilon_1 < \frac{1}{64}. \tag{3.19}$$

$$v_m \in F = \left[\frac{1}{8}, 1\right]. \tag{3.20}$$

Rewriting (3.19), for all $0 < \varepsilon < \frac{1}{8}$, there exists $k_0 \in \mathbb{N}$ such that for all $n \geq m \geq k_0$, where $n, m \in \mathbb{N}$, we have

$$|v_m - v_n| < \varepsilon < \frac{1}{8}, \qquad \text{where} \quad \varepsilon = \sqrt{\varepsilon_1}.$$

Now, since $(\mathbb{R}, |\,.\,|)$ is complete, $F = \left[\frac{1}{8}, 1\right]$ is closed in $\mathbb{R}$, also by (3.20), $\{v_m \in F = \left[\frac{1}{8}, 1\right]\}$ for all $m \in \mathbb{N}$, $m \geq k_0$ and $\{v_m : m \in \mathbb{N}\}$ is Cauchy with respect to $|.|$, so there is $v \in F$ such that for all $0 < \varepsilon < \frac{1}{8}$, there exists $k_1 \in \mathbb{N}$ such that for all $m \geq k_1$ where $m \in \mathbb{N}$, we have

$$|v - v_m| < \varepsilon.$$

Hence, $\{v_m : m \in \mathbb{N}\}$ is extended $\mathcal{J}_{\varphi;\Omega}$-convergent to v.

This implies (U, Q) is extended $\mathcal{J}_{\varphi;\Omega}$-sequential complete.

(I.4) Next, we show that T satisfies Condition (3.10).

It is obvious that Condition (3.10) holds for $a = \frac{1}{5}$ and $b = c = e = L = 0$.

(I.5) Assumptions (i), (ii), and (iii) of Theorem 3.19 holds.

For $z^0 = 0$ and $z^1 = Tz^0 = \frac{1}{5}$, we have $(z^0, Tz^0) \in E$. Also for $(u, v) \in E$ we have $(Tu, Tv) \in E$, since T is non-decreasing. Furthermore, if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $(z^m, z^{m+1}) \in E$ and $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} z_m = z$, then $(z^m, z) \in E$ and $(z, z^m) \in E$.
$(I.6)$ Finally, we show that T is an extended $Q_{\varphi;\Omega}$-closed map on U.

For this let $(z_m : m \in \mathbb{N})$ be a sequence in $T(x) = [\frac{1}{5}, \frac{6}{5}]$ which is extended $Q_{\varphi;\Omega}$-convergent to each point of $S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$. Let $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ be its subsequences satisfying $v_m = T(u_m)$ for all $m \in \mathbb{N}$.

Let $z \in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})}$, then for all $\varepsilon_1 > 0$, there exists $k \in \mathbb{N}$ such that for all $m \geq k$, where $m \in \mathbb{N}$, we have

$$q(z, z_m) < \varepsilon_1.$$

As a result, for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m \geq k$, where $m \in \mathbb{N}$, we have

$$[q(z, z_m) = |z - z_m| < \varepsilon] \wedge [q(z, u_m) = |z - u_m| < \varepsilon] \wedge [q(z, v_m) = |z - v_m| < \varepsilon] \wedge [v_m = T(u_m)],$$

where $\varepsilon = \sqrt{\varepsilon_1}$.

We can also write, for all $0 < \varepsilon < \frac{1}{8}$, there exists $k \in \mathbb{N}$ such that for all $m \geq k$, where $m \in \mathbb{N}$, we have

$$[|z - z_m| < \varepsilon] \wedge [|z - u_m| < \varepsilon] \wedge [|z - v_m| < \varepsilon] \wedge [v_m = T(u_m)].$$

This, in particular, implies that for all $0 < \varepsilon < \frac{1}{8}$, there exists $k \in \mathbb{N}$ such that for all $m \geq k$, where $m \in \mathbb{N}$, we have

$$|z - u_m| = |z - 5v_m + 1| = |5z - 4z - 5v_m + 1| = \left|4\left(\frac{1}{4} - z\right) - 5(v_m - z)\right| < \varepsilon,$$

and we obtain

$$4\left|\frac{1}{4} - z\right| < \varepsilon + 5|v_m - z|.$$

Now, since $|z - v_m| \to 0$, when $m \to \infty$, we have $|\frac{1}{4} - z| < \varepsilon_2$ where $\varepsilon_2 = \frac{\varepsilon}{4} < \frac{1}{32}$. This gives $S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} = \{\frac{1}{4}\}$, and there exists $z = \frac{1}{4} \in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})}$ such that $\frac{1}{4} = T(\frac{1}{4})$.

Hence, T is an extended $Q_{\varphi;\Omega}$-closed map on U.
$(I.7)$ As all the suppositions of Theorem 3.19 hold, we obtain

$$\mathrm{F}(T) = \frac{1}{4},$$

$$\lim^{Q_{\varphi;\Omega}}_{m\to\infty} z^m = \frac{1}{4},$$

$$J\left(\frac{1}{4}, \frac{1}{4}\right) = 0.$$

Before moving ahead to the next results, we first define the family Ψ_φ of mappings $\psi : [0, \infty) \to [0, \infty)$ that are non-decreasing and satisfying the following conditions:
(i) $\psi(0) = 0$;
(ii) $\psi(\eta t) = \eta\psi(t) < \eta t$, for each $\eta, t > 0$,
(iii) $\sum_{i=1}^{\infty} r^i \psi^i(t) \prod_{m=1}^{i} \varphi(z^m, z^n) < \infty$,
where $r \geq 1$.

Theorem 3.21. *Let the set-valued map* $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ *and* $\varphi_\beta : U \times U \to [1, \infty)$ *for each* $\beta \in \Omega$ *and* $(u, v) \in E$ *satisfy*

$$D^{\mathcal{J}_{\varphi;\Omega}}_{\beta}(Tu, Tv) \leq \psi_\beta(J_\beta(u, v)), \tag{3.21}$$

where $\psi_\beta \in \Psi_\varphi$.

Moreover, let

(i) *there are* $z^0 \in U$ *and* $z^1 \in Tz^0$ *such that* $(z^0, z^1) \in E$;

(ii) *if* $(u, v) \in E$ *and* $x \in Tu$ *and* $y \in Tv$ *such that* $\frac{1}{r_\beta}J_\beta(x, y) < J_\beta(u, v)$, *for all* $\beta \in \Omega$, *where* $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$, *then* $(x, y) \in E$;

(iii) *for each* $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ *and* $x \in U$, *there exists* $y \in Tx$ *such that*

$$J_\beta(x, y) \leq r_\beta J_\beta(x, Tx), \quad \text{for all } \beta \in \Omega.$$

Then, the following assertions hold:

(*I*) *For each* $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ *is an extended* $Q_{\varphi;\Omega}$-*convergent sequence in U; thus,* $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(*II*) *Moreover, suppose that* $T^{[k]}$ *is an extended* $Q_{\varphi;\Omega}$-*closed map on U, for some* $k \in \mathbb{N}$. *Then,*

(c_1) $F(T^{[k]}) \neq \emptyset$;

(c_2) *there exists* $z \in \mathrm{F}(T^{[k]})$ *such that* $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.

Proof. (I) Using supposition (i), there is $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now for each $\beta \in \Omega$, applying (3.21) we have

$$J_\beta(z^1, Tz^1) \leq D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tz^0, Tz^1) \leq \psi_\beta(J_\beta(z^0, z^1)). \tag{3.22}$$

Now, using assumption (iii), for $z^1 \in U$, there is $z^2 \in Tz^1$ such that

$$J_\beta(z^1, z^2) \leq r_\beta J_\beta(z^1, Tz^1) \leq r_\beta\psi_\beta(J_\beta(z^0, z^1)), \quad \text{for all } \beta \in \Omega. \tag{3.23}$$

Applying ψ_β, we obtain

$$\psi(J_\beta(z^1, z^2)) \leq \psi r_\beta\psi_\beta(J_\beta(z^0, z^1)) = r_\beta\psi_\beta^2(J_\beta(z^0, z^1)), \quad \text{for all } \beta \in \Omega.$$

Using assumption (ii), from (3.23), it follows that $(z^1, z^2) \in E$. Now again for each $\beta \in \Omega$, using (3.21), we can write

$$J_\beta(z^2, Tz^2) \leq D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tz^1, Tz^2) \leq \psi_\beta(J_\beta(z^1, z^2)).$$

Using assumption (iii), for $z^2 \in U$, there exists $z^3 \in Tz^2$ such that

$$J_\beta(z^2, z^3) \leq r_\beta J_\beta(z^2, Tz^2) \leq r_\beta\psi_\beta(J_\beta(z^1, z^2)) \leq r_\beta^2\psi_\beta^2(J_\beta(z^0, z^1)), \quad \text{for all } \beta \in \Omega.$$

It is obvious that $(z^2, z^3) \in E$. Proceeding in the similar fashion, we find a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ such that $(z^m, z^{m+1}) \in E$ and

$$J_\beta(z^m, z^{m+1}) \leq r_\beta^m\psi_\beta^m(J_\beta(z^0, z^1)), \quad \text{for all } \beta \in \Omega. \tag{3.24}$$

Now, by the repeated use of ($\mathcal{J}1$) and (3.24) for all $\beta \in \Omega$ and for all $n > m$, where $m, n \in \mathbb{N}$, we obtain

$$\begin{aligned}
J_\beta(z^m, z^n) \leq{} & \varphi_\beta(z^m, z^n)r_\beta^m\psi_\beta^m(J_\beta(z^0, z^1)) + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)r_\beta^{m+1}\psi_\beta^{m+1}(J_\beta(z^0, z^1)) \\
& + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\varphi_\beta(z^{m+2}, z^n)r_\beta^{m+2}\psi_\beta^{m+2}(J_\beta(z^0, z^1)) \\
& + \ldots + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n) \ldots \varphi_\beta(z^{n-1}, z^n)r_\beta^{n-1}\psi_\beta^{n-1}(J_\beta(z^0, z^1)) \\
\leq{} & \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n) \ldots \varphi_\beta(z^m, z^n)r_\beta^m\psi_\beta^m(J_\beta(z^0, z^1)) \\
& + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n) \ldots \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)r_\beta^{m+1}\psi_\beta^{m+1}(J_\beta(z^0, z^1)) \\
& + \ldots + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n) \ldots \varphi_\beta(z^m, z^n) \ldots \varphi_\beta(z^{n-1}, z^n)r_\beta^{n-1}\psi_\beta^{n-1}(J_\beta(z^0, z^1)).
\end{aligned}$$

Let $S_m = \sum_{j=1}^m r_\beta^j\psi_\beta^j(J_\beta(z^0, z^1))\prod_{i=1}^j\varphi_\beta(z^i, z^n)$, and we can write

$$J_\beta(z^m, z^n) \le (S_{n-1} - S_m).$$

Since $S_m < \infty$, we can write

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0, \quad \text{for all } \beta \in \Omega. \tag{3.25}$$

Now, consider the proof of Theorem 3.18 and follow the steps ahead from inequality (3.12), we can easily complete the proof of this theorem. □

Let us consider $T : U \to U$, and we obtain the result given below for a single-valued mapping.

Theorem 3.22. *Let the map $T : U \to U$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy*

$$J_\beta(Tu, Tv) \le \psi_\beta(J_\beta(u, v)), \quad \text{for all } (u, v) \in E, \tag{3.26}$$

where $\psi_\beta \in \Psi_\varphi$.

Moreover, let

(*i*) *there exists $z^0 \in U$ such that $(z^0, Tz^0) \in E$;*
(*ii*) *for $(u, v) \in E$ we have $(Tu, Tv) \in E$, provided $J_\beta(Tu, Tv) \le J_\beta(u, v)$, for all $\beta \in \Omega$;*
(*iii*) *if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $(z^m, z^{m+1}) \in E$ and $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} z^m = z$, then $(z^m, z) \in E$ and $(z, z^m) \in E$.*

Then, the following statements are satisfied:

(*I*) *For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus, $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \ne \varnothing$.*
(*II*) *Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then,*
(c_1) *$F(T^{[k]}) \ne \varnothing$;*
(c_2) *there exists $z \in \mathrm{F}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and*
(c_3) *for all $z \in \mathrm{F}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.*
(III) *Furthermore, let $F(T^{[k]}) \ne \varnothing$ for some $k \in \mathbb{N}$ and $(U, Q_{\varphi;\Omega})$ is a Hausdorff space. Then,*
(d_1) *$F(T^{[k]}) = F(T)$;*
(d_2) *there exists $z \in \mathrm{F}(T)$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and*
(d_3) *for all $z \in \mathrm{F}(T^{[k]}) = \mathrm{F}(T)$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.*

Proof. Since every single-valued mapping can be viewed as a multi-valued mapping, it remains to prove assertion (c_3) and assertions (d_1)–(d_3) of the aforementioned theorem.

To prove (c_3), use $\mathcal{J}1$, assumption (*iii*) and inequality (3.26), we obtain for all $\beta \in \Omega$,

$$\begin{aligned} J_\beta(z, Tz) &\le \varphi_\beta(z, Tz)\{J_\beta(z, z^{m+1}) + J_\beta(z^{m+1}, Tz)\} \\ &\le \varphi_\beta(z, Tz)\{J_\beta(z, z^{m+1}) + J_\beta(Tz^m, Tz)\} \\ &\le \varphi_\beta(z, Tz)\{J_\beta(z, z^{m+1}) + \psi_\beta(J_\beta(z^m, z))\}. \end{aligned}$$

Letting $m \to \infty$, we have

$$J_\beta(z, Tz) = 0, \ \text{ for all } \beta \in \Omega.$$

Similarly, we can show that $J_\beta(Tz, z) = 0 \ \forall \beta \in \Omega$.

(III) Since $(U, Q_{\varphi;\Omega})$ is a Hausdorff space, using Proposition (3.11), assertion (c_3) suggests that for $z \in \mathrm{F}(T^{[k]})$, we have $z \in T(z)$. This gives $z \in \mathrm{F}(T)$. Hence, (d_1) is true.

Assertions (c_2) and (d_1) imply (d_2).

To prove assertion (d_3), consider ($\mathcal{J}1$) and use (c_3) and (d_1); we have for all $z \in \mathrm{F}(T^{[k]}) = \mathrm{F}(T)$,

$$J_\beta(z, z) \le \varphi(z, z)\{J_\beta(z, T(z)) + J_\beta(T(z), z)\} = 0, \ \text{ for all } \beta \in \Omega.$$

□

Remark 3.23.
(a) Our results are novel generalization of the results in [32], which requires the completeness of the space. Our results provide strong assertions with weak assumptions.
(b) Our results extend and improve the results in [32] as they give information about the periodic points as well.

4 Consequences

This section consists of important and fascinating consequences of the theorems proved in the previous section.

We set up some periodic and fixed point results for mappings fulfilling contraction inequalities involving function α.

Recall that U is a non-void set and the graph $G \coloneqq (V, E)$ is defined as

$$V \coloneqq U \quad \text{and} \quad E \coloneqq \{(a, b) \in U \times U : \alpha(a, b) \geq 1\},$$

where $\alpha : U \times U \to [0, \infty)$.

Corollary 4.1. *Let the map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy*

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tu, Tv) \leq a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu), \tag{4.1}$$

for all $\alpha(u, v) \geq 1$, where $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ is such that $a_\beta + b_\beta + c_\beta + 2e_\beta\varphi_\beta(z^{m-1}, Tz^m) < 1$ and $\lim_{m,n\to\infty}\varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and each $z^0 \in U$, here, $z^m \in T(z^{m-1})$, where $m \in \mathbb{N}$.

Moreover, let
(*a*) *there exists $z^0 \in U$ and $z^1 \in Tz^0$ such that $\alpha(z^0, z^1) \geq 1$;*
(*b*) *if $\alpha(u, v) \geq 1$ and $x \in Tu$ and $y \in Tv$ such that $J_\beta(x, y) \leq J_\beta(u, v)$, for all $\beta \in \Omega$, then $\alpha(x, y) \geq 1$;*
(*c*) *for any $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $u \in U$, there exists $v \in Tu$ such that*

$$J_\beta(u, v) \leq r_\beta J_\beta(u, Tu), \ \text{for all } \beta \in \Omega.$$

Then, the following assertions hold:
(*I*) *For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $Q_{\varphi;\Omega}$-convergent sequence in U; thus, $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.*
(*II*) *Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then,*
(*c_1*) *$F(T^{[k]}) \neq \emptyset$;*
(*c_2*) *There exists $z \in F(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.*

Proof. Consider the graph $G = (V, E)$ and define the map $\alpha : U \times U \to [0, \infty)$ for some $\rho \geq 1$ as:

$$\alpha(u, v) = \begin{cases} \rho, & \text{if } (u, v) \in E, \\ 0, & \text{otherwise}. \end{cases} \tag{4.2}$$

Hence, inequality (4.1) can be written as

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tu, Tv) \leq a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu), \tag{4.3}$$

for all $(u, v) \in E$. This yields that T satisfies inequality (3.6). Also conditions (a), (b), and (c) imply conditions (i), (ii), and (iii) of Theorem 3.18. The proof now follows from Theorem 3.18. □

Remark 4.2.

(a) Following the pattern of statement and proof of Corollary 4.1, we can state and prove the corollaries for Theorems 3.19, 3.21 and 3.22 in the same manner.
(b) The results in an extended b-gauge space are novel generalizations and improved versions of the results in [33], in which assumptions are weak and assertions are strong.
(c) Observed that in case, α greater than or equal to one is used in any contraction inequality, bringing it back to the regular contractive inequality without α (see for instance inequality (4.3)). Therefore, it seems that α-function plays no role in proving that a mapping has a fixed point. Hence, when the underlying space is enriched with the graph, the fixed-point theorems involving function α can easily be obtained.
(d) Since we have stated a few theorems for contraction inequalities involving function α, some more analogues of the aforementioned results for contraction inequalities involving function α can simply be derived from results in [6,7,39].

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