

Research Article

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Klein-Gordon potential in characteristic coordinates

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Abstract: By the Klein-Gordon potential, we call a convolution-type integral with a kernel, which is the fundamental solution of the Klein-Gordon equation and also a solution of the Cauchy problem to the same equation. An interesting question having several important applications (in general) is what boundary condition can be imposed on the Klein-Gordon potential on the boundary of a given domain so that the Klein-Gordon equation with initial conditions complemented by this “transparent” boundary condition would have a unique solution within that domain still given by the Klein-Gordon potential. It amounts to finding the trace of the Klein-Gordon potential to the boundary of the given domain. In this article, we analyze this question and construct a novel initial boundary-value problem for the Klein-Gordon equation in characteristic coordinates.

Keywords: Klein-Gordon equation, wave equation, Cauchy problem, characteristic coordinates

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1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, consider the Newton potential

$$u(x) = \varepsilon_n * f = \int_{\Omega} \varepsilon_n(x-y)f(y)dy, \quad \text{supp } f \subset \Omega, \quad (1.1)$$

where

$$\varepsilon_2(x-y) = -\frac{1}{2\pi} \ln|x-y|, \quad \varepsilon_n(x-y) = \frac{1}{(n-2)\sigma_n} |x-y|^{2-n}, \quad n \geq 3,$$

and σ_n is the surface area of the unit n -sphere. It is known that the Newton potential satisfies the Poisson equation, i.e.,

$$-\Delta u(x) = f(x), \quad x \in \Omega. \quad (1.2)$$

In [1], the boundary condition of the Newton potential (1.1) was constructed in the following form:

$$\frac{1}{2}u(x) = -\int_{\Omega} \varepsilon_n(x-y) \frac{\partial u(y)}{\partial n_y} dS_y + \int_{\partial\Omega} \frac{\partial \varepsilon_n(x-y)}{\partial n_y} u(y) dS_y, \quad x \in \partial\Omega. \quad (1.3)$$

Hence, if we consider equation (1.2) with boundary condition (1.3), then we find a unique solution to this boundary-value problem in the form (1.1). This phenomenon is also called Kac's principle of not feeling the boundary and history goes back to Kac's celebrated work [2]. For elliptic and hypoelliptic operators, further

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results in answering these questions were investigated in [3–5]. For different applications (in spectral theory), recent works [6–10] may also be of interest to readers. We also refer to works [11–13] on related topics.

We shall note that following these ideas in [14] for the Schrödinger equation in a bounded domain proposed new formulations of boundary conditions that have the physical meaning, that is, the property of suppressing waves reflected from the boundary. Moreover, it was shown that in the bounded domain, this solution coincides with the solution of the problem posed in the unbounded domain with the Sommerfeld radiation condition at infinity. Hence, the open problem from [15] was solved in [14].

Motivated by the previous developments in the field, the main purpose of this article is to construct similar “transparent” boundary formulae for the Klein-Gordon potential and use these conditions to construct as well as study a nonlocal initial boundary-value problem for the Klein-Gordon equation.

In the characteristic coordinates without the dissipation term, the Klein-Gordon equation is stated in the form

$$\frac{\partial^2 u(\xi, \eta)}{\partial \xi \partial \eta} + \frac{m^2}{4} u(\xi, \eta) = f(\xi, \eta). \quad (1.4)$$

When $m = 0$, it gives the wave equation. The Klein-Gordon equation is widely used to model various phenomena such as scalar field theory, relativistic quantum mechanics, and wave propagation in elastic media, among others. The dynamics of scalar and spinor fields share remarkable similarities in terms of characteristic coordinates, as demonstrated in [16]. Moreover, the solution of (1.4) is obeyed by plane wave solutions of the Dirac equation.

Let Ω be a square restricted with A_0B_0 : $\eta = \xi$, $0 < \xi < 1/2$, A_0A : $\eta = -\xi$, $0 < \xi < 1/2$, AB : $\eta = \xi - 1$, $1/2 < \xi < 1$, and B_0B : $\eta = 1 - \xi$, $1/2 < \xi < 1$.

It is known that the Riemann function (see, for example, [17, p. 247]) $R(\xi, \eta, \xi_1, \eta_1)$ of equation (1.4) can be represented by

$$R(\xi, \eta, \xi_1, \eta_1) = J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}), \quad (1.5)$$

where J_0 is the order zero Bessel function.

A fundamental solution of the Cauchy problem for equation (1.4) is presented as

$$\varepsilon(\xi_1, \eta_1) = \theta(\xi - \xi_1)\theta(\eta_1 - \eta)R(\xi, \eta, \xi_1, \eta_1). \quad (1.6)$$

By a Klein-Gordon potential in Ω , we call the integral

$$\begin{aligned} u(\xi, \eta) &= \int_{\Omega} \varepsilon(\xi - \xi_1, \eta_1 - \eta) f(\xi_1, \eta_1) d\xi_1 d\eta_1 \\ &= \int_{\eta}^{\xi} d\xi_1 \int_{\xi_1}^{\eta} R(\xi, \eta, \xi_1, \eta_1) f(\xi_1, \eta_1) d\eta_1 \\ &= \int_{\eta}^{\xi} d\xi_1 \int_{\xi_1}^{\eta} J_0(\sqrt{m^2(\xi - \xi_1)(\eta_1 - \eta)}) f(\xi_1, \eta_1) d\eta_1. \end{aligned} \quad (1.7)$$

It is easy to check that the Klein-Gordon potential satisfies the Cauchy condition for $\eta = \xi$, $0 < \xi < 1/2$, that is,

$$u|_{\eta=\xi} = 0, \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) u|_{\eta=\xi} = 0 \quad (1.8)$$

and equation (1.4). Thus, in this article:

- We establish trace formulae for the Klein-Gordon potential operator on A_0A : $\eta = -\xi$, and B_0B : $\eta = 1 - \xi$. We use this to introduce a new initial boundary-value problem for the Klein-Gordon equation in the square Ω . Our result shows that the unique solution of the new initial boundary-value problem coincides with the solution of the Cauchy problem for the Klein-Gordon equation. This means that the obtained boundary conditions have a property that the wave should pass through the lateral boundaries $\eta = -\xi$ and $\eta = 1 - \xi$ undisturbed, that is, it should remain undisturbed if the wave is carrying “specific information.”

In Section 2, we construct conditions on the characteristics and establish an initial boundary-value problem for the Klein-Gordon equation in Theorem 1.

2 Main results

Our main result for the Klein-Gordon potential operator is the following trace formulae, which in turn give a new initial boundary-value problem for the Klein-Gordon equation. Let Ω be square (domain) as in Section 1.

Theorem 1. *Let $f \in C^1(\overline{\Omega})$, $\text{supp} f \subset \Omega$ then the Klein-Gordon potential $u \in C^2(\overline{\Omega})$ satisfies the following boundary conditions:*

$$\begin{aligned} N[u]|_{A_0A} = N[u]|_{\eta=-\xi} &= \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, -\xi_1)}{\partial \eta_1} d\xi_1 \\ &+ \frac{m^2}{2} \int_0^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, -\xi_1) d\xi_1 = 0, \end{aligned} \quad (2.1)$$

$$\begin{aligned} N[u]|_{B_0B} = N[u]|_{\eta=1-\xi} &= - \int_{\frac{1}{2}}^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, 1 - \xi_1)}{\partial \xi_1} d\xi_1 \\ &+ \frac{m^2}{2} \int_{\frac{1}{2}}^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1 = 0, \end{aligned} \quad (2.2)$$

where J_0 is the zero-order Bessel function and J_1 is the first-order Bessel function.

Conversely, if $u \in C^2(\overline{\Omega})$ is a solution of the Klein-Gordon equation (1.4) satisfying the Cauchy condition (1.8) and the boundary conditions (2.1) and (2.2), then u coincides with the Klein-Gordon potential (1.7).

We shall note that for $m = 0$, the lateral boundary conditions of the Klein-Gordon potential coincide with the ones of the one-dimensional wave potential which is given [18]. This also shows that the lateral boundary conditions (2.1) and (2.2) are local if and only if $m = 0$. It follows from Theorem 1 that the kernel ε given by (1.6), which is a fundamental solution of the Klein-Gordon equation, is the Green function of the nonlocal initial boundary-value problem

$$\frac{\partial^2 u(\xi, \eta)}{\partial \xi \partial \eta} + \frac{m^2}{4} u(\xi, \eta) = f(\xi, \eta), \quad f \in C^1(\overline{\Omega}), \quad \text{supp} f \subset \Omega,$$

with the Cauchy conditions

$$u|_{\eta=\xi} = 0, \quad \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) u|_{\eta=\xi} = 0$$

and nonlocal boundary conditions

$$\begin{aligned} N[u]|_{A_0A} = N[u]|_{\eta=-\xi} &= \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, -\xi_1)}{\partial \eta_1} d\xi_1 \\ &+ \frac{m^2}{2} \int_0^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} u(\xi_1, -\xi_1) d\xi_1 = 0, \end{aligned}$$

$$\begin{aligned}
N[u]|_{B_0B} &:= N[u]|_{\eta=1-\xi} = - \int_0^{\xi} (\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, 1 - \xi_1)}{\partial \xi_1} d\xi_1 \\
&\quad + \frac{m^2}{2} \int_{\frac{1}{2}}^{\xi} \frac{J_1((\sqrt{-m^2(\xi - \xi_1)^2}))}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1 = 0.
\end{aligned}$$

Therefore, this initial nonlocal boundary-value problem can serve as an example of an explicitly solvable initial boundary-value problem for the Klein-Gordon equation for any $m^2 > 0$.

Proof. We continue f by zero outside of the square Ω , that is, $f \equiv 0$ in $\mathbb{R}^2 \setminus \Omega$. Then, the Klein-Gordon potential

$$u(\xi, \eta) = \int_{\eta}^{\xi} \int_{\xi_1}^{\eta} R(\xi, \eta, \xi_1, \eta_1) f(\xi_1, \eta_1) d\eta_1$$

is a solution of equation (1.4) for all $(\xi, \eta) \in \mathbb{R}^2$ and $u(\xi, \eta) \in C^2(\overline{\Omega})$ satisfying the homogeneous Cauchy condition (1.8) on the line $\xi = \eta$, $-\infty < \xi < +\infty$. Moreover, the value of the function u at the point (ξ, η) is determined by the value of $f(\xi_1, \eta_1)$ in the characteristic triangle $\Delta_{\xi, \eta} = \{\eta \leq \xi_1 \leq \xi, \eta \leq \eta_1 \leq \xi_1\}$.

Thus, the function $u(\xi, \eta)$ on A_0A , that is, $u|_{A_0A} = u(\xi, -\xi)$ is defined by the formula

$$u(\xi, -\xi) = \int_{-\xi}^{\xi} \int_{\xi_1}^{-\xi} R(\xi, -\xi, \xi_1, \eta_1) f(\xi_1, \eta_1) d\eta_1. \quad (2.3)$$

Since $f \equiv 0$ outside of $\xi_1 \leq 0$, $\eta_1 \leq -\xi_1$, the integral (2.3) takes the form

$$\begin{aligned}
u|_{A_0A} &= u(\xi, -\xi) \\
&= \int_0^{\xi} \int_{\xi_1}^{-\xi} R(\xi, -\xi, \xi_1, \eta_1) f(\xi_1, \eta_1) d\eta_1 \\
&= \int_0^{\xi} \int_{\xi_1}^{-\xi} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) f(\xi_1, \eta_1) d\eta_1.
\end{aligned} \quad (2.4)$$

Now in (2.4) instead of $f(\xi_1, \eta_1)$ let us plug in $\frac{\partial^2 u(\xi_1, \eta_1)}{\partial \xi_1 \partial \eta_1} + \frac{m^2}{4} u(\xi_1, \eta_1)$, that is,

$$u(\xi, -\xi) = \int_0^{\xi} \int_{\xi_1}^{-\xi} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \left(\frac{\partial^2 u}{\partial \xi_1 \partial \eta_1} + \frac{m^2}{4} u \right) d\eta_1 = I_{1,0}^- + I_1^-, \quad (2.5)$$

where

$$\begin{aligned}
I_{1,0}^- &= m^2 \int_0^{\xi} \int_{\xi_1}^{-\xi} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) u(\xi_1, \eta_1) d\eta_1, \\
I_1^- &= \int_0^{\xi} \int_{\xi_1}^{-\xi} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial^2 u}{\partial \xi_1 \partial \eta_1}(\xi_1, \eta_1) d\eta_1.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
I_1^- &= \int_0^{\xi} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial u}{\partial \xi_1} \Big|_{\xi_1}^{-\xi_1} d\xi_1 - \int_0^{\xi} d\xi_1 \int_{\xi_1}^{-\xi} \frac{\partial}{\partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial u}{\partial \xi_1} d\eta_1 \\
&= \int_0^{\xi} J_0(\sqrt{m^2(\xi - \xi_1)(\xi_1 - \xi)}) \frac{\partial u(\xi_1, -\xi_1)}{\partial \xi_1} d\xi_1 - \int_0^{\xi} J_0(\sqrt{m^2(\xi - \xi_1)(\xi_1 - \xi_1)}) \frac{\partial u(\xi_1, \xi_1)}{\partial \xi_1} d\xi_1 \\
&\quad - \int_0^{\xi} d\xi_1 \int_{\xi_1}^{-\xi} \frac{\partial}{\partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial u(\xi_1, \eta_1)}{\partial \xi_1} d\eta_1.
\end{aligned}$$

By the Cauchy condition (1.8), we have $\frac{\partial u(\xi_1, \xi_1)}{\partial \xi_1} = 0$. Using this the latter equality can be represented in the form

$$I_1^- = \int_0^\xi J_0(\sqrt{m^2(\xi - \xi_1)(\xi_1 - \xi)}) \frac{\partial u(\xi_1, -\xi_1)}{\partial \xi_1} d\xi_1 - I_{1,2}^-,$$

where

$$I_{1,2}^- = \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \frac{\partial}{\partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial u(\xi_1, \eta_1)}{\partial \xi_1} d\eta_1. \quad (2.6)$$

Changing the integration order, we obtain

$$\begin{aligned} I_{1,2} &= \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \frac{\partial}{\partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial u(\xi_1, \eta_1)}{\partial \xi_1} d\eta_1 \\ &= \int_{-\xi}^0 d\eta_1 \int_{\xi}^{-\eta_1} \frac{\partial}{\partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial u(\xi_1, \eta_1)}{\partial \xi_1} d\xi_1 \\ &\quad + \int_0^\xi d\eta_1 \int_{\xi}^{\eta_1} \frac{\partial}{\partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) \frac{\partial u(\xi_1, \eta_1)}{\partial \xi_1} d\xi_1. \end{aligned}$$

By using the formula $\frac{\partial}{\partial z} J_0(z) = -J_1(z)$ with $z = \sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}$, we arrive at

$$\begin{aligned} \frac{\partial_1}{\partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) &= \frac{\partial}{\partial z} J_0(z) \frac{\partial z}{\partial \eta_1} \\ &= -J_1(z) \frac{\partial}{\partial \eta_1} (\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) = \frac{m^2}{2} J_1(z) \frac{(\xi - \xi_1)}{\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}}. \end{aligned} \quad (2.7)$$

Hence integrating by parts, we obtain

$$\begin{aligned} I_{1,2} &= \frac{m^2}{2} \int_{-\xi}^0 \left[\frac{J_1(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)})}{\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}} (\xi - \xi_1) u(\xi_1, \eta_1) \right] \bigg|_{\xi}^{-\eta_1} d\eta_1 \\ &\quad + \frac{m^2}{2} \int_0^\xi \left[\frac{J_1(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)})}{\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}} (\xi - \xi_1) u(\xi_1, \eta_1) \right] \bigg|_{\xi}^{\eta_1} d\eta_1 \\ &\quad - \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \frac{\partial^2}{\partial \xi_1 \partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) u(\xi_1, \eta_1) d\eta_1 \\ &= \frac{m^2}{2} \int_{-\xi}^0 \frac{J_1(\sqrt{-m^2(\xi + \eta_1)(\xi + \eta_1)})}{\sqrt{-m^2(\xi + \eta_1)^2}} (\xi + \eta_1) u(-\eta_1, \eta_1) d\eta_1 \\ &\quad - \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \frac{\partial^2}{\partial \xi_1 \partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(-\xi - \eta_1)}) u(\xi_1, \eta_1) d\eta_1. \end{aligned} \quad (2.8)$$

Taking $-\eta_1 = \xi_1$ in the first integral of (2.8), we obtain

$$\begin{aligned} I_{1,2} &= \frac{m^2}{2} \int_0^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, -\xi_1) d\xi \\ &\quad - \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \frac{\partial^2}{\partial \xi_1 \partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}) u(\xi_1, \eta_1) d\eta_1. \end{aligned} \quad (2.9)$$

By using relations (2.6) and (2.9) from (2.5), it implies that

$$\begin{aligned} u|_{A_0A} = u(\xi, -\xi) &= \int_0^\xi \sqrt{-m^2(\xi - \xi_1)^2} \frac{\partial u(\xi_1, -\xi_1)}{\partial \xi_1} d\xi_1 - \frac{m^2}{2} \int_0^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, -\xi_1) d\xi_1 \\ &+ \int_0^\xi d\xi_1 \int_{\xi_1}^{-\xi_1} \left[\frac{\partial^2}{\partial \xi_1 \partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}) + m^2 J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}) u(\xi_1, -\eta_1) d\eta_1 \right]. \end{aligned} \quad (2.10)$$

Since $J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)})$ is the Riemann function of equation (1.4), we have

$$\frac{\partial^2}{\partial \xi_1 \partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}) + \frac{m^2}{4} J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}) = 0. \quad (2.11)$$

Using equality (2.5) from (2.10), it follows that

$$\begin{aligned} u|_{A_0A} = u|_{\eta=\xi} = u(\xi, -\xi) &= \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, -\xi_1)}{\partial \xi_1} d\xi_1 \\ &- \frac{m^2}{2} \int_0^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, -\xi_1) d\xi_1. \end{aligned} \quad (2.12)$$

It is easy to see that $\frac{du(\xi_1, -\xi_1)}{d\xi_1} = \left(\frac{\partial u}{\partial \xi_1} - \frac{\partial u}{\partial \eta_1} \right) \Big|_{\eta_1=-\xi_1}$ and this implies

$$\frac{\partial u(\xi_1, \eta_1)}{\partial \xi_1} \Big|_{\eta_1=-\xi_1} = \frac{du(\xi_1, -\xi_1)}{d\xi_1} + \frac{\partial u(\xi_1, \eta_1)}{\partial \eta_1} \Big|_{\eta_1=-\xi_1}. \quad (2.13)$$

Now integrating by parts in (2.12), we obtain

$$\begin{aligned} \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi, -\xi_1)}{\partial \xi_1} d\xi_1 &= \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \left(\frac{du(\xi_1, -\xi_1)}{d\xi_1} + \frac{\partial u(\xi_1, \eta_1)}{\partial \eta_1} \Big|_{\eta_1=-\xi_1} \right) d\xi_1 \\ &= \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, \eta_1)}{\partial \eta_1} \Big|_{\eta_1=-\xi_1} d\xi_1 \\ &+ J_0(\sqrt{-m^2(\xi - \xi_1)^2}) u(\xi_1, -\xi_1) \Big|_{\xi_1=\xi} - \int_0^\xi \frac{\partial}{\partial \xi_1} J_0(\sqrt{-m^2(\xi - \xi_1)^2}) u(\xi_1, -\xi_1) d\xi_1 \\ &= m^2 \int_0^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, -\xi_1) d\xi_1 + u(\xi, -\xi) \\ &+ \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, \eta_1)}{\partial \eta_1} \Big|_{\eta_1=-\xi_1} d\xi_1, \end{aligned}$$

since $J_0(0) = 1$.

This equality implies that

$$u(\xi, -\xi) = u|_{A_0A} + N[u]|_{A_0A} = u(\xi, -\xi) + N[u]|_{\eta=-\xi}, \quad (2.14)$$

that is,

$$\begin{aligned} N[u]|_{A_0A} = N[u]|_{\eta=-\xi} &= \int_0^\xi J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, -\xi_1)}{\partial \eta_1} d\xi_1 \\ &+ \frac{m^2}{2} \int_0^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, -\xi_1) d\xi_1 = 0. \end{aligned} \quad (2.15)$$

The boundary condition (2.15) is the lateral boundary condition of the Klein-Gordon potential on A_0A : $\eta = -\xi$, $0 < \xi < 1/2$.

Now it remains to find a lateral boundary condition on B_0B .

Since $f \equiv 0$ outside of the square $\Delta_{\xi,1-\xi} = \{1 - \xi \leq \xi_1 \leq \xi, \eta_1 = 1 - \xi, \eta_1 = 1 - \xi_1, 1/2 \leq \xi \leq 1, \eta_1 = \xi_1, 1 - \xi \leq \xi \leq 1/2\}$, using (1.7) and changing the integration order in the following integral we obtain

$$\begin{aligned} u|_{B_0B} &= u|_{\eta=1-\xi} = \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} R(\xi, \eta, \xi_1, \eta_1) \bigg|_{\eta=1-\xi} f(\xi_1, \eta_1) d\xi_1 \\ &= \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(1 - \xi - \eta_1)}) \frac{\partial^2 u}{\partial \xi_1 \partial \eta_1} d\xi_1 \\ &\quad + \frac{m^2}{4} \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(1 - \xi - \eta_1)}) u(\xi_1, \eta_1) d\xi_1 = I_1^+ + I_2^+, \end{aligned} \quad (2.16)$$

where

$$I_2^+ = \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(1 - \xi - \eta_1)}) \frac{\partial^2 u}{\partial \xi_1 \partial \eta_1} d\xi_1 \quad (2.17)$$

and

$$I_1^+ = \frac{m^2}{4} \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(1 - \xi - \eta_1)}) u(\xi_1, \eta_1) d\xi_1. \quad (2.18)$$

Integrating by parts in (2.17), we obtain

$$\begin{aligned} I_2^+ &= \int_{1-\xi}^{\frac{1}{2}} J_0 \left(\sqrt{m^2(\xi - \xi_1)(1 - \xi - \eta_1)} \frac{\partial u(\xi_1, \eta_1)}{\partial \eta_1} \right) \bigg|_{1-\eta_1}^{\eta_1} d\eta_1 - \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} \frac{\partial J_0}{\partial \xi_1} \frac{\partial u}{\partial \eta_1} d\xi_1 \\ &= \int_{1-\xi}^{\frac{1}{2}} \left(J_0(\sqrt{m^2(\xi - \eta_1)(1 - \xi - \eta_1)}) \frac{\partial u(\eta_1, \eta_1)}{\partial \eta_1} - J_0(\sqrt{m^2(\xi - (1 - \eta_1))(1 - \xi - \eta_1)}) \frac{\partial u(1 - \eta_1, \eta_1)}{\partial \eta_1} d\xi \right) d\eta_1 \\ &\quad - \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} \frac{\partial J_0}{\partial \xi_1} \frac{\partial u}{\partial \eta_1} d\xi_1. \end{aligned}$$

From the Cauchy condition (1.8), it follows that $\frac{\partial u(\eta_1, \eta_1)}{\partial \eta_1} = 0$.

Thus, the last integral can be represented in the form

$$I_2^+ = - \int_{1-\xi}^{\frac{1}{2}} J_0(\sqrt{m^2(\xi + \eta_1 - 1)(1 - \eta_1 - \xi)}) \frac{\partial u(1 - \eta_1, \eta_1)}{\partial \eta_1} d\eta_1 - \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} \frac{\partial J_0}{\partial \xi_1} \frac{\partial u}{\partial \eta_1} d\xi_1 = I_{2,1}^+ + I_{2,2}^+, \quad (2.19)$$

where

$$I_{2,1}^+ = - \int_{1-\xi}^{\frac{1}{2}} J_0(\sqrt{m^2(\xi + \eta_1 - 1)(1 - \eta_1 - \xi)}) \frac{\partial u(1 - \eta_1, \eta_1)}{\partial \eta_1} d\eta_1 \quad (2.20)$$

and

$$I_{2,2}^+ = - \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} \frac{\partial J_0}{\partial \xi_1} \frac{\partial u}{\partial \eta_1} d\xi_1. \quad (2.21)$$

Setting $1 - \eta_1 = \xi_1$, the integral $I_{2,1}^+$ is formulated as

$$I_{2,1}^+ = - \int_{1-\xi}^{\frac{1}{2}} J_0(\sqrt{m^2(\xi + \eta_1 - 1)(1 - \eta_1 - \xi)}) \frac{\partial u(1 - \eta_1, \eta_1)}{\partial \eta_1} d\eta_1 = - \int_{\frac{1}{2}}^{\xi} J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, 1 - \xi_1)}{\partial \eta_1} d\xi_1. \quad (2.22)$$

Changing the integration order in $I_{2,2}^+$, we obtain

$$\begin{aligned} I_{2,2}^+ &= - \int_{1-\xi}^{\frac{1}{2}} d\xi_1 \int_{\xi_1}^{1-\xi} \frac{\partial J_0}{\partial \xi_1} \frac{\partial u}{\partial \eta_1} d\eta_1 - \int_{\frac{1}{2}}^{\xi} d\xi_1 \int_{1-\xi_1}^{1-\xi} \frac{\partial J_0}{\partial \xi_1} \frac{\partial u}{\partial \eta_1} d\eta_1 \\ &= - \int_{1-\xi}^{\frac{1}{2}} \frac{\partial J_0}{\partial \xi_1} u \bigg|_{\xi_1}^{1-\xi} d\xi_1 + \int_{\frac{1}{2}}^{\xi} d\xi_1 \int_{\xi_1}^{1-\xi} \frac{\partial^2 J_0}{\partial \xi_1 \partial \eta_1} u(\xi_1, \eta_1) d\eta_1 \\ &\quad - \int_{\frac{1}{2}}^{\xi} \frac{\partial J_0}{\partial \xi_1} u(\xi_1, \eta_1) \bigg|_{1-\xi_1}^{1-\xi} d\xi_1 + \int_{\frac{1}{2}}^{\xi} d\xi_1 \int_{1-\xi_1}^{1-\xi} \frac{\partial^2 J_0}{\partial \xi_1 \partial \eta_1} u(\xi_1, \eta_1) d\eta_1. \end{aligned} \quad (2.23)$$

Using the equality $u(\xi_1, \xi_1) = 0$ and

$$\frac{\partial J_0}{\partial \xi_1} = \frac{\partial J_0(z)}{\partial z} \frac{\partial z_0}{\partial \xi_1} = m^2 \frac{J_1(z)}{z} (\eta - \eta_1)|_{\eta_1=1-\xi_1, \eta=1-\xi} = 0,$$

we obtain

$$\begin{aligned} I_{2,2}^+ &= \int_{1-\xi}^{\frac{1}{2}} d\xi_1 \int_{\xi_1}^{1-\xi} \frac{\partial^2 J_0}{\partial \xi_1 \partial \eta_1} u(\xi_1, \eta_1) d\eta_1 - \frac{m^2}{2} \int_{\frac{1}{2}}^{\xi} \frac{J_1(\sqrt{m^2(\xi - \xi_1)(1 - \xi - (1 - \xi_1))})}{\sqrt{-m^2(\xi_1 - \xi)^2}} (\xi - \xi_1) \\ &\quad \cdot u(\xi_1, 1 - \xi_1) d\xi + \int_{\frac{1}{2}}^{\xi} d\xi_1 \int_{1-\xi_1}^{1-\xi} \frac{\partial^2 J_0}{\partial \xi_1 \partial \eta_1} u(\xi_1, \eta_1) d\eta_1. \end{aligned} \quad (2.24)$$

Using (2.18), (2.19), (2.21), and (2.24) from (2.23) implies that

$$\begin{aligned} u|_{B_0 B} &= u|_{\eta=1-\xi} = - \int_{\frac{1}{2}}^{\xi} J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, 1 - \xi_1)}{\partial \eta_1} d\xi_1 \\ &\quad + \frac{m^2}{2} \int_{\frac{1}{2}}^{\xi} \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1 \\ &\quad + \int_{1-\xi}^{\frac{1}{2}} d\eta_1 \int_{1-\eta_1}^{\eta_1} \left(\frac{\partial^2 J_0}{\partial \xi_1 \partial \eta_1} + \frac{m^2}{4} J_0 \right) u(\xi_1, \eta_1) d\xi_1. \end{aligned} \quad (2.25)$$

Since $J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)})$ is the Riemann function, we have

$$\frac{\partial^2}{\partial \xi_1 \partial \eta_1} J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}) + \frac{m^2}{4} J_0(\sqrt{m^2(\xi - \xi_1)(\eta - \eta_1)}) \equiv 0.$$

It implies

$$\begin{aligned} u|_{B_0B} = u|_{\eta=1-\xi} &= - \int_{\frac{1}{2}}^{\xi} J_0(\sqrt{m^2(\xi - \xi_1)^2}) \frac{\partial u}{\partial \eta_1}(\xi_1, 1 - \xi_1) d\xi_1 \\ &\quad - \frac{m^2}{2} \int_{\frac{1}{2}}^{\xi} \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1. \end{aligned} \quad (2.26)$$

It is easy to see that

$$\frac{d}{d\xi_1} u(\xi_1, 1 - \xi_1) = \left[\frac{\partial u}{\partial \xi_1} - \frac{\partial u}{\partial \eta_1} \right] \bigg|_{\eta_1=1-\xi_1}$$

is the full derivative on the line $\eta_1 = 1 - \xi_1$. This gives

$$-\frac{\partial u}{\partial \eta_1} \bigg|_{\eta_1=1-\xi_1} = \frac{du}{d\xi_1} - \frac{\partial u}{\partial \xi_1} \bigg|_{\eta_1=1-\xi_1}. \quad (2.27)$$

Putting the right-hand side of (2.27) in (2.26) and integrating by parts, we obtain

$$\begin{aligned} u|_{B_0B} = u|_{\eta=1-\xi} &= \int_{\frac{1}{2}}^{\xi} J_0(\sqrt{-m^2(\xi - \xi_1)^2}) \left(\frac{du}{d\xi_1} - \frac{\partial u}{\partial \xi_1} \right) d\xi_1 \\ &\quad + \frac{m^2}{2} \int_{\frac{1}{2}}^{\xi} \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1 \\ &= u(\xi, 1 - \xi) - \int_{\frac{1}{2}}^{\xi} J_0(\sqrt{m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, 1 - \xi_1)}{\partial \xi_1} d\xi_1 \\ &\quad - \int_{\frac{1}{2}}^{\xi} u(\xi_1, 1 - \xi_1) \frac{\partial}{\partial \xi_1} J_0(\sqrt{m^2(\xi - \xi_1)^2}) d\xi_1 \\ &\quad + \frac{m^2}{2} \int_{\frac{1}{2}}^{\xi} \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1 = u(\xi, 1 - \xi) + N[u] \bigg|_{B_0B}, \end{aligned}$$

that is,

$$u|_{B_0B} = u(\xi, 1 - \xi) + N[u]|_{B_0B}, \quad (2.28)$$

where

$$\begin{aligned} N[u]|_{B_0B} &= - \int_{\frac{1}{2}}^{\xi} J_0(\sqrt{m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, 1 - \xi_1)}{\partial \xi_1} d\xi_1 \\ &\quad + \frac{m^2}{2} \int_{\frac{1}{2}}^{\xi} \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1. \end{aligned}$$

From (2.28), we see that the lateral boundary condition of the Klein-Gordon potential on B_0B is given in the form

$$\begin{aligned} N[u]|_{B_0B} = & - \int_0^\xi J_0(\sqrt{m^2(\xi - \xi_1)^2}) \frac{\partial u(\xi_1, 1 - \xi_1)}{\partial \xi_1} d\xi_1 \\ & + \frac{m^2}{2} \int_{\frac{1}{2}}^\xi \frac{J_1(\sqrt{-m^2(\xi - \xi_1)^2})}{\sqrt{-m^2(\xi - \xi_1)^2}} (\xi - \xi_1) u(\xi_1, 1 - \xi_1) d\xi_1 = 0. \end{aligned} \quad (2.29)$$

Thus, the lateral boundary conditions of the Klein-Gordon potential on A_0A and B_0B are given by formulae (2.15) and (2.29), respectively.

Let $\vartheta \in C^2(\overline{\Omega})$ satisfy equation (1.4) with the initial condition (1.8) and lateral boundary conditions (2.15) and (2.29). Let u be the Klein-Gordon potential defined by formula (1.7), then $\omega = u - \vartheta$ satisfies the homogeneous Klein-Gordon equation and the initial condition (1.8).

In this case the left-hand sides of (2.10) and (2.28) are equal to zero, that is,

$$\omega + N[\omega]|_{A_0A} = \omega|_{A_0A} = 0,$$

$$\omega + N[\omega]|_{B_0B} = \omega|_{B_0B} = 0.$$

Since $\omega|_{t=0} = 0$ from the uniqueness of a solution of the mixed Cauchy problem for the Klein-Gordon equation follows that $\omega = u - \vartheta \equiv 0$, that is, $u = \vartheta$. $\square$

3 Conclusion

In conclusion, our research has focused on investigating the Klein-Gordon potential, which is defined as a convolution-type integral with a kernel representing the fundamental solution of the Klein-Gordon equation. This kernel also serves as a solution to the Cauchy problem for the same equation. The primary question we have addressed in this study pertains to the imposition of boundary conditions on the Klein-Gordon potential at the boundary of a specified domain. These boundary conditions are designed to ensure the existence of a unique solution to the Klein-Gordon equation within the given domain while maintaining the solution in the form of the Klein-Gordon potential.

This research is of particular significance due to its potential applications across various fields. Understanding how to define these “transparent” boundary conditions and determine the trace of the Klein-Gordon potential at the domain’s boundary opens up possibilities in areas such as signal processing and wave propagation.

To tackle this question, we have conducted an in-depth analysis and introduced an approach in the form of an initial boundary-value problem for the Klein-Gordon equation within characteristic coordinates. Our findings contribute to advancing the theoretical foundations of boundary-value problems for the Klein-Gordon equation and provide practical tools for addressing real-world applications where such equations are relevant. Our future work in this direction will be about the multidimensional case.

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