Research Article

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On bivariate fractal interpolation for countable data and associated nonlinear fractal operator

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Abstract: Fractal interpolation has been conventionally treated as a method to construct a univariate continuous function interpolating a given finite data set with the distinguishing property that the graph of the interpolating function is the attractor of a suitable iterated function system. On the one hand, attempts have been made to extend the univariate fractal interpolation from a finite data set to a countably infinite set. On the other hand, fractal interpolation in higher dimensions, particularly the theory of fractal interpolation surfaces (FISs), has received increasing attention for more than a quarter century. This article targets a twofold extension of the notion of fractal interpolation by providing a general framework to construct FISs for a prescribed set consisting of countably infinite data on a rectangular grid. By using this as a crucial tool, we obtain a parameterized family of bivariate fractal functions simultaneously interpolating and approximating a prescribed bivariate continuous function. Some elementary properties of the associated nonlinear (not necessarily linear) fractal operators are established, thereby allowing the interaction of the notion of fractal interpolation with the theory of nonlinear operators.

Keywords: bivariate fractal interpolation, countable data set, alpha-fractal function, fractal operator, nonlinear operators, perturbation of operators

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1 Introduction

Over the past three decades, the subject of fractal interpolation has been one of the major research themes among the fractals community. The concept of fractal interpolation function (FIF) was introduced by Barnsley in his fundamental and pioneering work [1]. He proved the existence of a univariate continuous function interpolating a given finite data set, with its novelty lying in the fact that the graph of the constructed interpolant is a fractal, that is, the attractor of a suitable iterated function system (IFS) [2,3]. We do not attempt here to provide a complete list of references on fractal interpolation, as the field is quite large. Instead, we refer the reader to a few recent research on fractal interpolation [4–11] and the references therein. For a compendium of fractal interpolation and related topics, we refer the reader to the monograph [12].

As is well known, interpolation and approximation are two intimately related concepts. However, the interplay between these two theories is more subtle in the fractal setting. It is our opinion that the notion of

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α-fractal function [13,14] is mostly responsible for establishing interesting interconnections between interpolation and approximation theories of univariate fractal functions. In fact, α-fractal function provides a parameterized family of fractal functions that simultaneously interpolate and approximate a given univariate continuous function. The parameters can be adjusted so that the fractal functions share or modify the properties of the original function, for instance, smoothness and shape properties [15,16]. The analytical properties of the fractal operator that maps a function to its fractal counterpart, and new approximation classes of self-referential functions obtained by "fractalizing" various approximation classes of (non-fractal) functions (such as polynomials, rational functions, and trigonometric polynomials) have received considerable attention in the literature on univariate fractal approximation theory [13,14,17–19]. We stress here that the studies on the aforementioned univariate fractal operator are mostly confined to the realm of bounded linear operators on various function spaces.

As with any such salient idea, numerous questions and results based on fractal interpolation were spawned. One such question is related to its extension to the multivariate case. In this direction, several works have been done to interpolate a given bivariate data using bivariate FIFs (fractal surfaces); see, for instance, [20–28] and a few research works deal with the multivariate FIFs [29,30]. Among various constructions available in the literature, the general framework to construct fractal surfaces given in [31] aroused our interest, partially due to the fact that the construction thereat is amenable to obtain a bivariate analogue of the α-fractal function, which is a natural entry point to delve into the theory of bivariate fractal approximation, see also [32,33].

Much of the existing literature dealing with univariate and multivariate FIFs concentrates primarily on finite data sets. This is no happenstance, as fractal interpolation hinges on the Hutchinson's fundamental result about the existence of an invariant set (attractor) for a set-valued map induced by a finite number of contractions [2]. Attempts were made in literature to define these concepts in the setting of countably infinite number of maps. For instance, Secelean adapted the Hutchinson approach so as to handle a countable number of contractions [34,35] and deduced the existence of a univariate fractal function interpolating a data set consisting of countably infinite points [35,36]. Bivariate FIF for an infinite sequence of data is not studied hitherto, and naturally, we want to generalize the construction of fractal surfaces so as to accommodate countably infinite data sets. Let us note that this is practically important. For instance, in the theory of sampling and reconstruction, often one works with infinite sequence of data points, and a general approach focuses to seek an approximate rather than the perfect reconstruction on some restricted class of signals.

One of the objectives of this article is to unveil a method for constructing fractal interpolation surface (FIS) for a countably infinite data set over rectangular grids. We are inspired by the finite case treated in [31], and we will adapt this construction to the setting of countable data. This part of the current findings may be viewed also as a sequel to [36], where the univariate fractal interpolation for a countable data set has been studied. The countable bivariate scenario explored in this study necessitates certain extra considerations and assumptions, making the analysis more difficult. As hinted earlier, among various constructions of FISs, our choice of [31] is guided by the fact that it offers an efficient platform to obtain a parameterized family of fractal functions corresponding to a prescribed bivariate continuous function. Crucial to further development will be the aforementioned parameterized family consisting of approximate fractal reconstructions of the original function (seed function) defined on a rectangular domain. This family of fractal functions is obtained by sampling the seed function at infinite number of grid points in the domain and applying the countable FIS scheme developed in the first part of this article.

We define a fractal operator that sends each (non-fractal) seed function to its fractal approximant, and study some analytical properties of this operator, which, in general, is nonlinear. Though the bounded linear fractal operator has its origin in the theory of univariate α-fractal function (see, for example, [14]), our focus will be to its intriguing links with the perturbation theory of operators (not necessarily linear or bounded). Thus, the research reported here could open the door for an intense and fruitful interaction of two fields – fractal interpolation and the theory of nonlinear operators, and the potential applications lie, for instance, in the field of sampling and reconstruction. The present article is a corrected and improved version of the note available in [37].

2 Preliminaries

We list pertinent definitions and notation from the theory of countable IFS [34,36] and nonlinear operators for use in the subsequent parts of this article. The terminologies on perturbation theory of nonlinear operators assembled here can be found in [38], and many of these are obvious modifications to the notions present in the well known treatise for perturbation theory of linear operators by Kato [39].

2.1 Countable IFS

Definition 2.1. Let (X, d) be a compact metric space and $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of continuous self-maps on X. The system $\{X, (\omega_i)_{i\in\mathbb{N}}\}$ is said to be a countable iterated function system (CIFS). We say that the CIFS is hyperbolic if the functions ω_i, $i \in \mathbb{N}$, are contractions with respective contractivity factors r_i such that $\sup_{i\in\mathbb{N}} r_i < 1$.

Definition 2.2. A non-empty set $A \subseteq X$ is said to be a set fixed point of the CIFS $\{X, (\omega_i)_{i\in\mathbb{N}}\}$ if

$$A = \overline{\bigcup_{i\in\mathbb{N}} \omega_i(A)},$$

where the notation bar denotes the closure of the respective set. A non-empty set $A \subseteq X$ is said to be the attractor of the CIFS if A is compact, and it is the unique set fixed point of the CIFS.

Definition 2.3. Let $\mathcal{H}(X)$ denote the set of all non-empty closed (and hence compact) subsets of X endowed with the Hausdorff-Pompeiu metric [3]. The set-valued operator $\mathcal{W} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$\mathcal{W}(B) = \overline{\bigcup_{i\in\mathbb{N}} \omega_i(B)} \quad \forall B \in \mathcal{H}(X) \tag{2.1}$$

is said to be the Hutchinson operator associated with the CIFS $\{X, (\omega_i)_{i\in\mathbb{N}}\}$.

Theorem 2.4. [34] *Suppose that the CIFS $\{X, (\omega_i)_{i\in\mathbb{N}}\}$ is hyperbolic. Then the corresponding Hutchinson operator is a contraction on the complete metric space $\mathcal{H}(X)$, and consequently, by the Banach fixed point theorem, it has an attractor. That is, there exists a unique non-empty compact set $A \subseteq X$ such that*

$$A = \mathcal{W}(A) = \overline{\bigcup_{i\in\mathbb{N}} \omega_i(A)}.$$

Moreover,
1. *for every nonempty closed set $B \subseteq X$, the sequence $(\mathcal{W}^n(B))_{n\in\mathbb{N}}$, where $\mathcal{W}^n$ denotes the n-fold autocomposition of $\mathcal{W}$, converges to A.*
2. *the attractor A can be approximated with respect to the Hausdorff-Pompeiu metric by the attractors of the partial IFSs $\{X, (\omega_i)_{i=1}^n\}_{n\in\mathbb{N}}$.*

2.2 A few basics of linear and nonlinear operator theory

In this subsection, let X and Y be two normed linear spaces over the same field $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.5. An operator $A : D(A) \subseteq X \to Y$ is said to be closed if its graph $G := \{(x, A(x)) : x \in D(A)\}$ is a closed subset of $X \times Y$. That is, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$, $x_n \to x$ and $A(x_n) \to y \in Y$ implies $x \in D(A)$ and $A(x) = y$.

Definition 2.6. An operator $A : D(A) \subseteq X \to Y$ is said to be closable if it has a closed extension. That is, there exists a set $X_0, D(A) \subseteq X_0 \subseteq X$ and a closed operator $\tilde{A} : X_0 \subseteq X \to Y$ such that $\tilde{A}(x) = A(x)$ for all $x \in D(A)$.

Definition 2.7. Let $A : D(A) \subseteq X \to Y$ and $B : D(B) \subseteq X \to Y$ be two operators such that $D(B) \subseteq D(A)$. If for every sequence $(x_n)_{n\in\mathbb{N}}$ in $D(B)$ with $x_n \to x, B(x_n) \to y$ and $A(x_n) \to z$ imply $x \in D(A)$ and $A(x) = z$, then A is said to be B-closed.

Definition 2.8. Let $A : D(A) \subseteq X \to Y$ and $B : D(B) \subseteq X \to Y$ be two operators such that $D(B) \subseteq D(A)$. Assume that, for every pair of convergent sequences $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$ in $D(B)$ having a same limit $x \in X$, the sequences $(B(x_n))_{n\in\mathbb{N}}$ and $(B(x'_n))_{n\in\mathbb{N}}$ also converge to a same limit. If for every pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ having the same limit x, we have

$$A(x_n) \to z, A(x'_n) \to z' \Rightarrow x \in D(A) \quad \text{and} \quad z' = z,$$

then A is said to be B-closable.

Definition 2.9. Let X and Y be normed linear spaces over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with norms $||\cdot||_X$ and $||\cdot||_Y$ respectively. Let $A : D(A) \subseteq X \to Y$ be an operator. Define $p(A)$ by

$$p(A) = \max\left\{\sup_{x\in D(A), x\neq 0} \frac{||A(x)||_Y}{||x||_X}, ||A(0)||_Y\right\}.$$

Definition 2.10. If $p(A) < \infty$, then the operator A is said to be norm-bounded operator, and the quantity $p(A)$ is called the norm of A.

Definition 2.11. An operator $A : D(A) \subseteq X \to Y$ is said to be topologically bounded if it maps bounded sets to bounded sets.

Remark 2.12. Let $D(A) = X$. In contrast to the case of linear operators, here, the two notions – norm-boundedness and topologically boundedness – are not equivalent.

Definition 2.13. Let $A : D(A) \subseteq X \to Y$ and $B : D(B) \subseteq X \to Y$ be two operators such that $D(B) \subseteq D(A)$. Then we say that A is relatively (norm) bounded with respect to B or simply B-bounded if for some non-negative constants a and b, the following inequality holds:

$$||A(x)||_Y \leq a||x||_X + b||B(x)||_Y \quad \forall x \in D(B). \tag{2.2}$$

The infimum b_0 of all values of b for which the above inequality is satisfied is called the relative bound of A with respect to B or simply the B-bound of A.

Remark 2.14. If b is chosen very close to b_0, then the other constant a will, in general, have to be chosen very large; thus it is, in general, impossible to set $b = b_0$ in the inequality (2.2) in the aforementioned definition; see [39].

Definition 2.15. An operator $A : D(A) \subseteq X \to Y$ is said to be Lipschitz if there exists a constant $M > 0$ such that

$$||A(x) - A(y)||_Y \leq M||x - y||_X \quad \forall x, y \in D(A).$$

For a Lipschitz operator $A : D(A) \subseteq X \to Y$, the Lipschitz constant is defined by

$$|A| := \sup_{x\neq y} \frac{||A(x) - A(y)||_Y}{||x - y||_X}.$$

Definition 2.16. Let $A : D(A) \subseteq X \to Y$ and $B : D(B) \subseteq X \to Y$ be two operators such that $D(B) \subseteq D(A)$. We say that A is relatively Lipschitz with respect to B or simply B-Lipschitz if the following inequality holds for some non-negative constants M_1 and M_2.

$$\|A(x) - A(y)\|_Y \leq M_1\|x - y\|_X + M_2\|B(x) - B(y)\|_Y \quad \forall x, y \in D(A). \tag{2.3}$$

The infimum of all such values of M_2 is called the B-Lipschitz constant of A.

Remark 2.17. Similar to Remark 2.14, let us note that the inequality (2.3) may not hold when M_2 is replaced with the infimum of all values of M_2.

Definition 2.18. Let X, Y be normed linear spaces, X^*, Y^* be the corresponding dual spaces, and $T : X \to Y$ be a bounded linear operator. The adjoint or dual T^* of T is the unique map $T^* : Y^* \to X^*$ defined by

$$T^*(\psi) := \psi \circ T \quad \text{for all } \psi \in Y^*.$$

Definition 2.19. Given a Banach space X, the annihilator (or pre-annihilator) of a subset S of X^* is the subspace defined as follows:

$$S^{\perp} := \{x \in X : \psi(x) = 0 \quad \forall \psi \in S\}.$$

Further, given a Banach space X and a bounded linear map $T : X \to X$, we call a subspace Y of X invariant under T or T-invariant if $T(Y) \subseteq Y$. Further, if $Y \neq \emptyset$, $Y \neq X$, then Y is called a nontrivial invariant subspace.

Lemma 2.20. *Let X be a non-separable Banach space and T : X → X be a bounded linear operator. Then, for every non-zero element* $x \in X$*, the subspace* $\overline{\text{span}\{x, T(x), T^2(x), \ldots\}}$ *is a non-trivial closed invariant subspace. Here, the* span *denotes the linear span and bar denotes the closure.*

Lemma 2.21. *Let X be a Banach space and T : X → X be a bounded linear operator. If Y is a closed invariant subspace of the operator* $T^* : X^* \to X^*$*, then* $Y^{\perp}$ *is a closed invariant subspace of T.*

3 Construction of countable bivariate FIS

Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N}_n := \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. Let us first recall that corresponding to each double sequence $s : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ denoted by $s(i, j) = s_{i,j}$, there can be considered three important limits, namely,

$$\lim_{i,j\to\infty} s_{i,j}, \lim_{i\to\infty}\left(\lim_{j\to\infty} s_{i,j}\right) \quad \text{and} \quad \lim_{j\to\infty}\left(\lim_{i\to\infty} s_{i,j}\right).$$

Further, it is worth to recall that the existence of $\lim_{i,j\to\infty} s_{i,j}$ does not ensure, in general, the existence of the limits:

(1) $\lim_{i\to\infty} s_{i,j}$ for each fixed $j \in \mathbb{N}_0$.
(2) $\lim_{j\to\infty} s_{i,j}$ for each fixed $i \in \mathbb{N}_0$.
(3) iterated limits: $\lim_{i\to\infty}(\lim_{j\to\infty} s_{i,j})$ and $\lim_{j\to\infty}(\lim_{i\to\infty} s_{i,j})$.

Definition 3.1. A set $D = \{(x_i, y_j, z_{i,j}) : i, j \in \mathbb{N}_0\} \subset \mathbb{R}^3$ is said to be a bivariate countable system data (CSD) if
(1) the sequences $(x_i)_{i\in\mathbb{N}_0}$ and $(y_j)_{j\in\mathbb{N}_0}$ are strictly increasing and bounded,
(2) the double sequence $(z_{i,j})$ is convergent in the sense that $\lim_{i,j\to\infty} z_{i,j}$ exists and it is finite,
(3) $\lim_{j\to\infty} z_{i,j} < \infty$ for each fixed $i \in \mathbb{N}_0$, and $\lim_{i\to\infty} z_{i,j} < \infty$ for each fixed $j \in \mathbb{N}_0$.

Notation 3.2. For a bivariate CSD, let $M \coloneqq z_{\infty\infty} \coloneqq \lim_{i,j\to\infty} z_{i,j}$. Let $a \coloneqq x_0$, $b \coloneqq x_\infty \coloneqq \lim_{i\to\infty} x_i$, $c \coloneqq y_0$ and $d \coloneqq y_\infty \coloneqq \lim_{j\to\infty} y_j$. Set $I \coloneqq [a, b]$ and $J \coloneqq [c, d]$. Assume that K is a sufficiently large compact interval containing the set $\{z_{i,j} : i, j = 0, 1, 2, \ldots\}$ and $X \coloneqq I \times J \times K$.

3.1 Construction of FIS for countable data

Extensions of Barnsley's framework [1] in two ways – by considering a countably infinite one-dimensional data set [36] and a bivariate finite data set [31] – have been one of the directions of further studies in fractal interpolation. With an intent to generalize these two approaches and inspired by the two publications [31,36], we ask:

Question 3.3. Can we construct a continuous function $g : I \times J \to \mathbb{R}$ interpolating the bivariate CSD, that is, $g(x_i, y_j) = z_{i,j}$ for all $i, j \in \mathbb{N}_0$, satisfying the following properties?
(1) g is the fixed point of a suitable Read-Bajraktarevíc type operator,
(2) the graph of g can be realized as the attractor of an appropriate CIFS.

In what follows, we answer the aforementioned question in the affirmative.
For $i, j \in \mathbb{N}$, let $I_i \coloneqq [x_{i-1}, x_i]$ and $J_j \coloneqq [y_{j-1}, y_j]$. For $i \in \mathbb{N}_0$, let

$$s_i \coloneqq \frac{1 + (-1)^i}{2},$$

(i.e., $s_i = 0$ if i is odd, and $s_i = 1$ if i is even). Define $\tau : \mathbb{N} \times \{0, \infty\} \to \mathbb{N}$ by

$$\tau(i, 0) \coloneqq i - 1 + s_i \quad \text{and} \quad \tau(i, \infty) \coloneqq i - s_i. \tag{3.1}$$

For $i, j \in \mathbb{N}$, let $u_i : I \to I_i$ and $v_j : J \to J_j$ be contractive homeomorphisms satisfying

$$u_i(a) = s_{i-1}x_{i-1} + s_i x_i, \quad u_i(b) = s_i x_{i-1} + s_{i+1}x_i, \tag{3.2}$$

$$|u_i(x) - u_i(x')| \le a_i|x - x'| \quad \forall x, x' \in I, \tag{3.3}$$

$$v_j(c) = s_{j-1}y_{j-1} + s_j y_j, \quad v_j(d) = s_j y_{j-1} + s_{j+1}y_j, \tag{3.4}$$

$$|v_j(y) - v_j(y')| \le b_j|y - y'| \quad \forall y, y' \in J, \tag{3.5}$$

where a_i, b_j are positive constants such that $\sup_i a_i < 1$ and $\sup_j b_j < 1$. Let

$$c_{i,j} \coloneqq \max\{a_i, b_j\}.$$

Clearly $\sup_{i,j} c_{i,j} < 1$. By using (3.2) and (3.4), one can easily observe that for all $i, j \in \mathbb{N}$,

$$u_i^{-1}(x_i) = s_i a + s_{i+1}b = u_{i+1}^{-1}(x_i) \quad \text{and} \quad v_j^{-1}(y_j) = s_j c + s_{j+1}d = v_{j+1}^{-1}(y_j). \tag{3.6}$$

For every $(i, j) \in \mathbb{N} \times \mathbb{N}$, we consider the constants $\theta_i, \lambda_j, \alpha_{i,j}$ in $(0, \infty)$ such that the following assertions hold:

$$\lim_i \theta_i = \lim_j \lambda_j = 0; \tag{3.7}$$

$$\lim_j \alpha_{i,j} = 0 \quad \text{for all } i \in \mathbb{N}, \quad \lim_i \alpha_{i,j} = 0 \quad \text{for all } j \in \mathbb{N}, \quad \text{and} \quad \lim_{i,j} \alpha_{i,j} = 0; \tag{3.8}$$

$$\|\alpha\|_\infty = \sup_{i,j} \alpha_{i,j} < 1. \tag{3.9}$$

For $(i, j) \in \mathbb{N} \times \mathbb{N}$, let us now define the functions $F_{i,j} : X \to K$ such that for every pair of points (x, y, z) and (x', y', z') in $X = I \times J \times K$,

$$|F_{i,j}(x, y, z) - F_{i,j}(x', y', z)| \le \theta_i|x - x'| + \lambda_j|y - y'|, \tag{3.10}$$

$$|F_{i,j}(x, y, z) - F_{i,j}(x, y, z')| \le \alpha_{i,j}|z - z'|, \tag{3.11}$$

$$F_{i,j}(x_k, y_l, z_{kl}) = z_{\tau(i,k),\tau(j,l)} \quad \forall k, l \in \{0, \infty\}. \tag{3.12}$$

Remark 3.4. From (3.10), (3.11) and the following inequality

$$|F_{i,j}(x,y,z) - F_{i,j}(x',y',z')| \le |F_{i,j}(x,y,z) - F_{i,j}(x',y',z)| + |F_{i,j}(x',y',z) - F_{i,j}(x',y',z')|,$$

we deduce that, for each $i,j \in \mathbb{N}$, $F_{i,j}$ is a Lipschitz continuous function.

We note that with some mild conditions on the constants involved thereat, (3.7)–(3.11) imply that the maps $F_{i,j}$, $i,j \in \mathbb{N}$, are equi-Lipschitz.

Proposition 3.5. *Suppose that* $\sup_i \theta_i$ *and* $\sup_j \lambda_j$ *are real numbers. Denote*

$$\beta \coloneqq \sqrt{2}\max\left\{\sup_i \theta_i, \sup_j \lambda_j\right\},$$
$$\beta' \coloneqq \sqrt{3}\max\left\{\sup_i \theta_i, \sup_j \lambda_j, \sup_{i,j} \alpha_{i,j}\right\}.$$

Then

$$|F_{i,j}(x,y,z) - F_{i,j}(x',y',z)| \le \beta\|(x,y) - (x',y')\|, \quad \forall (x,y),(x',y') \in I \times J,\ z \in K,$$

and

$$|F_{i,j}(x,y,z) - F_{i,j}(x',y',z')| \le \beta'\|(x,y,z) - (x',y',z')\|, \quad \forall (x,y,z),(x',y',z') \in X,$$

where $\|\cdot\|$ *denotes the Euclidean norm on* $\mathbb{R}^2$ *or* $\mathbb{R}^3$.

Proof. We use the elementary inequality

$$|x - x'| + |y - y'| \le \sqrt{2}\sqrt{(x - x')^2 + (y - y')^2}$$

and (3.10) to obtain

$$|F_{i,j}(x,y,z) - F_{i,j}(x',y',z)| \le \beta\|(x,y) - (x',y')\|, \quad \forall (x,y),(x',y') \in I \times J,\ z \in K.$$

The inequality

$$|x - x'| + |y - y'| + |z - z'| \le \sqrt{3}\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

and (3.11) provide

$$|F_{i,j}(x,y,z) - F_{i,j}(x',y',z')| \le \beta'\|(x,y,z) - (x',y',z')\|, \quad \forall (x,y,z),(x',y',z') \in X,$$

completing the proof. □

Recall that $X = I \times J \times K$. Having defined the maps u_i, v_j and $F_{i,j}$, for each $(i,j) \in \mathbb{N} \times \mathbb{N}$, we define $W_{i,j} : X \to X$ by

$$W_{i,j}(x,y,z) \coloneqq (u_i(x), v_j(y), F_{i,j}(x,y,z)). \tag{3.13}$$

Then $\{X, (W_{i,j})_{(i,j)\in\mathbb{N}\times\mathbb{N}}\}$ is a CIFS. By definition, we have

$$W_{i,j}(x_k, y_l, z_{kl}) = (x_{\tau(i,k)}, y_{\tau(j,l)}, z_{\tau(i,k)\tau(k,l)}) \quad \text{for all } (k,l) \in \{0,\infty\}. \tag{3.14}$$

For $(x,y,z),(x',y',z') \in \mathbb{R}^3$ and $\delta > 0$, define the metric d_δ as follows:

$$d_\delta((x,y,z),(x',y',z')) \coloneqq \|(x,y) - (x',y')\| + \delta|z - z'|. \tag{3.15}$$

It can be easily verified that the metric d_δ defined earlier is equivalent to the Euclidean metric on $\mathbb{R}^3$ for all $\delta > 0$. This ensures that the metric space $(X, \delta_{|X})$ is compact.

Proposition 3.6. *The countable IFS* $\{X, W_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ *is hyperbolic with respect to the metric* d_δ *defined in* (3.15), *with*

$$\delta := \inf_{i,j} \frac{1 - c_{i,j}}{2\beta},$$

where β *is as in the previous proposition. Hence, it possesses an attractor, that is, there exists a unique non-empty compact set* $A \subseteq X$ *such that* $A = \mathcal{W}(A) = \overline{\bigcup_{i,j\geq 1} W_{i,j}(A)}$.

Proof. For $(x,y,z), (x',y',z') \in X$, we have

$$\begin{aligned} d_\delta(W_{i,j}(x,y,z), W_{i,j}(x',y',z')) &= \|(u_i(x), v_j(y)) - (u_i(x'), v_j(y'))\| + \delta|F_{i,j}(x,y,z) - F_{i,j}(x',y',z')| \\ &\leq \|(u_i(x), v_j(y)) - (u_i(x'), v_j(y'))\| + \delta[|F_{i,j}(x,y,z) \\ &\quad - F_{i,j}(x',y',z)| + |F_{i,j}(x',y',z) - F_{i,j}(x',y',z')|] \\ &\leq c_{i,j}\|(x,y) - (x',y')\| + \delta[\theta_i|x - x'| + \lambda_j|y - y'| + \|\alpha\|_\infty|z - z'|] \\ &\leq c_{i,j}\|(x,y) - (x',y')\| + \delta\left[\frac{\beta}{\sqrt{2}}|x - x'| + \frac{\beta}{\sqrt{2}}|y - y'| + \|\alpha\|_\infty|z - z'|\right] \\ &\leq (c_{i,j} + \beta\delta)\|(x,y) - (x',y')\| + \delta\|\alpha\|_\infty|z - z'| \\ &\leq \max\left\{\sup_{i,j}(c_{i,j} + \beta\delta), \|\alpha\|_\infty\right\}[\|(x,y) - (x',y')\| + \delta|z - z'|]. \end{aligned}$$

By the choice of δ, we have $\sup_{i,j}(c_{i,j} + \beta\delta) < 1$. Hence, for each $(i,j) \in \mathbb{N} \times \mathbb{N}$, $W_{i,j}$ is a contraction with respect to the metric d_δ. Consequently, the Hutchinson operator $\mathcal{W} : \mathcal{H}(X) \to \mathcal{H}(X)$ is a contraction map (see also Theorem 2.4). The rest of the proof follows from the Banach fixed point theorem. □

Assume that for each $(i,j) \in \mathbb{N} \times \mathbb{N}$, the function $F_{i,j} : X \to K$ further satisfies the following, the so-called matching conditions:

(1) for all $i \in \mathbb{N}$ and $x^* = u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i)$,

$$F_{i,j}(x^*, y, z) = F_{i+1,j}(x^*, y, z) \quad \forall y \in J, z \in K, \text{ and} \tag{3.16}$$

(2) for all $j \in \mathbb{N}$ and $y^* = v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j)$,

$$F_{i,j}(x, y^*, z) = F_{i,j+1}(x, y^*, z) \quad \forall x \in I, z \in K. \tag{3.17}$$

Furthermore, we assume that the following limits exist for each fixed $(x,y,z) \in X$.

$$\exists \lim_i F_{i,j}(x,y,z), \quad \forall j \in \mathbb{N} \quad \text{and} \quad \exists \lim_j F_{i,j}(x,y,z), \quad \forall i \in \mathbb{N}. \tag{3.18}$$

Let us denote by $C(I \times J)$ the set of all real-valued continuous functions defined on $I \times J$. Consider the set

$$C^*(I \times J) := \{h \in C(I \times J) : h(x_k, y_l) = z_{kl} \quad \text{for all } k, l \in \{0, \infty\}\},$$

endowed with the uniform metric. It is plain to see that:

Lemma 3.7. *The metric subspace* $C^*(I \times J)$ *is closed in the complete space* $C(I \times J)$, *and hence, it is complete.*

For each $h \in C^*(I \times J)$, we define $T(h) : I \times J \to \mathbb{R}$ as follows:

$$T(h)(x,y) := \begin{cases} F_{i,j}(u_i^{-1}(x), v_j^{-1}(y), h(u_i^{-1}(x), v_j^{-1}(y))): \\ \quad \text{if } (x,y) \in I_i \times J_j \text{ for some } (i,j) \in \mathbb{N} \times \mathbb{N}, \\ \lim\limits_{j\to\infty} F_{i,j}(u_i^{-1}(x), d, h(u_i^{-1}(x), d)): \\ \quad \text{if } x \in I_i \text{ for some } i \in \mathbb{N}, \text{ and } y = d, \\ \lim\limits_{i\to\infty} F_{i,j}(b, v_j^{-1}(y), h(b, v_j^{-1}(y))): \\ \quad \text{if } x = b, \text{ and } y \in J_j \text{ for some } j \in \mathbb{N}, \\ z_{\infty\infty}: \\ \quad \text{if } x = b \text{ and } y = d. \end{cases} \tag{3.19}$$

In what follows, we intend to show that $T : C^*(I \times J) \to C^*(I \times J)$ defined by $h \mapsto T(h)$ is well defined. We begin with a few comments on the definition of $T(h)$ above.

Remark 3.8. The existence of limits in the definition of $T(h)$ is guaranteed by (3.18). Furthermore, note that $x_i \in I_{i+1} \cap I_i$, it appears that $T(h)(x_i, d)$ receives two expressions, namely,

$$T(h)(x_i, d) = \lim_{j\to\infty} F_{i,j}(u_i^{-1}(x_i), d, h(u_i^{-1}(x_i), d)),$$

$$T(h)(x_i, d) = \lim_{j\to\infty} F_{i+1,j}(u_{i+1}^{-1}(x_i), d, h(u_{i+1}^{-1}(x_i), d)).$$

However, condition (3.16) enables to determine $T(h)(x_i, d)$ uniquely. Similarly, $T(h)(b, y_j)$ is defined univocally for each $j \in \mathbb{N}$.

Next we state an elementary result from analysis [40], crucial for the continuity arguments in the upcoming lemma, but omit the proof.

Lemma 3.9. *Let $f : (a,b) \times (c,d) \to \mathbb{R}$ and $\lambda \in (c,d)$. Then the following hold.*

(1) $\lim_{(x,y)\to(b,\lambda)} f(x,y) = L$ *if and only if for any non-decreasing sequence* (r_n) in (a,b), *where* $r_n \to b$, *and for any sequence* (t_n) *in* (c,d), *where* $t_n \to \lambda$, *the sequence* $(f(r_n, t_n))$ *converges to* L.

(2) $\lim_{(x,y)\to(b,\lambda)} f(x,y) = L$ *if and only if for any non-decreasing sequence* (r_n) *in* (a,b), *where* $r_n \to b$, *and for any monotone sequence* (t_n) *in* (c,d), *where* $t_n \to \lambda$, *the sequence* $(f(r_n, t_n))$ *converges to* L.

(3) $\lim_{(x,y)\to(b,d)} f(x,y) = L$ *if and only if for every pair of non-decreasing sequences* (r_n) *in* (a,b) *and* (t_n) *in* (c,d) *such that* $r_n \to b$, $t_n \to d$, *the sequence* $(f(r_n, t_n))$ *converges to* L.

Lemma 3.10. *For each $h \in C^*(I \times J)$, the function $T(h) : I \times J \to \mathbb{R}$ is well defined. Moreover, $T(h) \in C^*(I \times J)$ whenever $h \in C^*(I \times J)$. That is, the operator $T : C^*(I \times J) \to C^*(I \times J)$ is well defined.*

Proof. It follows from (3.16) and (3.17) that $T(h)$ is well defined on the boundary of $I_i \times J_j$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. For instance, let us note the following. Set $y_{j-1} \le y \le y_j$. Then $(x_{i-1}, y) \in (I_{i-1} \times J_j) \cap (I_i \times J_j)$. Treating (x_{i-1}, y) as an element in $I_{i-1} \times J_j$, we have

$$T(h)(x_{i-1}, y) = F_{i-1,j}(u_{i-1}^{-1}(x_{i-1}), v_j^{-1}(y), h(u_{i-1}^{-1}(x_{i-1}), v_j^{-1}(y))). \tag{3.20}$$

On the other hand, treating (x_{i-1}, y) as an element in $I_i \times J_j$, we obtain

$$T(h)(x_{i-1}, y) = F_{i,j}(u_i^{-1}(x_{i-1}), v_j^{-1}(y), h(u_i^{-1}(x_{i-1}), v_j^{-1}(y))). \tag{3.21}$$

From (3.6), one has $u_{i-1}^{-1}(x_{i-1}) = s_{i-1}a + s_i b = u_i^{-1}(x_{i-1})$. Hence, using (3.16), it follows, according to (3.20) and (3.21), that $T(h)(x_{i-1}, y)$ is uniquely determined.

It is obvious that $T(h)$ is continuous on $[a,b) \times [c,d)$. Next we will prove the continuity of $T(h)$ at (b,y), where $y \in J$. For this purpose, let us consider a sequence $((r_n, t_n))$ in $[a,b) \times [c,d)$ such that (r_n)

is monotonically increasing and $(r_n, t_n) \longrightarrow (b, y)$ as $n \to \infty$. We have to show that $T(h)(r_n, t_n) \to T(h)(b, y)$. Two cases can occur.

Case 1. Let $y < d$. Then $y \in J_{j_0} \subset [c, d)$ for some $j_0 \in \mathbb{N}$.
For each $n = 1, 2, \ldots,$ let $i_n, j_n \in \mathbb{N}$ be such that $(r_n, t_n) \in I_{i_n} \times J_{j_n}$.

Subcase 1.1. Assume that $y \in (y_{j_0-1}, y_{j_0})$. Since $t_n \to y$, one can find $N \in \mathbb{N}$ such that $t_n \in J_{j_0}$ for all $n \geq N$. Let $n \geq N$. Then

$$T(h)(r_n, t_n) = F_{i_n,j_0}(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n), h(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n))). \tag{3.22}$$

On the other hand,

$$T(h)(b, y) = \lim_{n\to\infty} F_{i_n,j_0}(b, v_{j_0}^{-1}(y), h(b, v_{j_0}^{-1}(y))). \tag{3.23}$$

In view of (3.18), equations (3.22) and (3.23) with some basic algebra of limits provide

$$\begin{aligned}
&\lim_{n\to\infty}|T(h)(r_n, t_n) - T(h)(b, y)| \\
&= \lim_{n\to\infty}|F_{i_n,j_0}(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n), h(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n))) - F_{i_n,j_0}(b, v_{j_0}^{-1}(y), h(b, v_{j_0}^{-1}(y)))| \\
&\leq \lim_{n\to\infty}[|F_{i_n,j_0}(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n), h(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n))) - F_{i_n,j_0}(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n), h(b, v_{j_0}^{-1}(y)))| \\
&\quad + |F_{i_n,j_0}(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n), h(b, v_{j_0}^{-1}(y))) - F_{i_n,j_0}(b, v_{j_0}^{-1}(y), h(b, v_{j_0}^{-1}(y)))|] \\
&\leq \lim_{n\to\infty}[\alpha_{i_n,j_0}|h(u_{i_n}^{-1}(r_n), v_{j_0}^{-1}(t_n)) - h(b, v_{j_0}^{-1}(y))| + \theta_{i_n}|u_{i_n}^{-1}(r_n) - b| + \lambda_{j_0}|v_{j_0}^{-1}(t_n) - v_{j_0}^{-1}(y)|] \\
&\leq \lim_{n\to\infty}[2\|h\|_\infty \alpha_{i_n,j_0} + (b-a)\theta_{i_n} + \lambda_{j_0}|v_{j_0}^{-1}(t_n) - v_{j_0}^{-1}(y)|],
\end{aligned} \tag{3.24}$$

where in the second step, we used the triangle inequality, the third step is consequent upon (3.10) and (3.11) and the last step is plain. Now using (3.7), (3.8), $t_n \to y$ as $n \to \infty$, and the continuity of $v_{j_0}^{-1}$, we have

$$\lim_{n\to\infty} \alpha_{i_n,j_0} = 0, \quad \lim_{n\to\infty} \theta_{i_n} = 0, \quad \lim_{n\to\infty}|v_{j_0}^{-1}(t_n) - v_{j_0}^{-1}(y)| = 0.$$

Consequently, (3.24) yields $\lim_{n\to\infty}|T(h)(r_n, t_n) - T(h)(b, y)| = 0$, and hence we infer that $T(h)$ is continuous at (b, y) for $y \in (y_{j_0-1}, y_{j_0})$.

Subcase 1.2. Let $y = y_{j_0-1}$. Let us suppose that (t_n) is monotonically increasing and $(r_n, t_n) \to (b, y)$. Then one can find $N_0 \in \mathbb{N}$ such that $t_n \in (y_{j_0-2}, y_{j_0-1}] \subset J_{j_0-1}$ for all $n \geq N_0$. Then computations similar to that in (3.24) provide

$$\begin{aligned}
&\lim_{n\to\infty}|T(h)(r_n, t_n) - T(h)(b, y)| \\
&= \lim_{n\to\infty}|F_{i_n,j_0-1}(u_{i_n}^{-1}(r_n), v_{j_0-1}^{-1}(t_n), h(u_{i_n}^{-1}(r_n), v_{j_0-1}^{-1}(t_n))) - F_{i_n,j_0-1}(b, v_{j_0-1}^{-1}(y), h(b, v_{j_0-1}^{-1}(y)))| \\
&\leq \lim_{n\to\infty}[|F_{i_n,j_0-1}(u_{i_n}^{-1}(r_n), v_{j_0-1}^{-1}(t_n), h(u_{i_n}^{-1}(r_n), v_{j_0-1}^{-1}(t_n))) \\
&\quad - F_{i_n,j_0-1}(u_{i_n}^{-1}(r_n), v_{j_0-1}^{-1}(t_n), h(b, v_{j_0-1}^{-1}(y)))| + |F_{i_n,j_0-1}(u_{i_n}^{-1}(r_n), v_{j_0-1}^{-1}(t_n), h(b, v_{j_0-1}^{-1}(y))) \\
&\quad - F_{i_n,j_0-1}(b, v_{j_0-1}^{-1}(y), h(b, v_{j_0-1}^{-1}(y)))|] \\
&\leq \lim_{n\to\infty}[\alpha_{i_n,j_0-1}|h(u_{i_n}^{-1}(r_n), v_{j_0-1}^{-1}(t_n)) - h(b, v_{j_0-1}^{-1}(y))| \\
&\quad + \theta_{i_n}|u_{i_n}^{-1}(r_n) - b| + \lambda_{j_0-1}|v_{j_0-1}^{-1}(t_n) - v_{j_0-1}^{-1}(y)|] \\
&\leq \lim_{n\to\infty}[2\|h\|_\infty \alpha_{i_n,j_0-1} + (b-a)\theta_{i_n} + \lambda_{j_0-1}|v_{j_0-1}^{-1}(t_n) - v_{j_0-1}^{-1}(y)|] \\
&= 0.
\end{aligned}$$

Similarly, if $(t_n)_{n\in\mathbb{N}}$ is monotonically decreasing, then there exists $N_0 \in \mathbb{N}$ such that $t_n \in [y_{j_0-1}, y_{j_0}) \subset J_{j_0}$ for all $n \geq N_0$. By using a similar argument as earlier, we can prove that

$$\lim_{n\to\infty}|T(h)(r_n, t_n) - T(h)(b, y)| = 0.$$

By combining the two aforementioned subcases, we conclude that $T(h)$ is continuous at (b, y) for any $y \in [c, d)$.

Case 2. Let $y = d$. In this case, due to the previous lemma, we can assume that both (r_n) and (t_n) are monotonically increasing. For each $n = 1, 2, \ldots$, let $i_n, j_n \in \mathbb{N}$ be such that $r_n \in I_{i_n}$, $t_n \in J_{j_n}$. By using (3.7), (3.11), and (3.12), we have as mentioned earlier

$$\begin{aligned}
&\lim_{n\to\infty}|T(h)(r_n, t_n) - T(h)(b, d)|\\
&\quad\leq \lim_{n\to\infty}[|F_{i_n,j_n}(u_{i_n}^{-1}(r_n), v_{j_n}^{-1}(t_n), h(u_{i_n}^{-1}(r_n), v_{j_n}^{-1}(t_n))) - F_{i_n,j_n}(b, d, h(u_{i_n}^{-1}(r_n), v_{j_n}^{-1}(t_n)))|\\
&\qquad + |F_{i_n,j_n}(b, d, h(u_{i_n}^{-1}(r_n), v_{j_n}^{-1}(t_n))) - F_{i_n,j_n}(b, d, h(b, d))| + |F_{i_n,j_n}(b, d, h(b, d)) - z_{\infty\infty}|]\\
&\quad\leq \lim_{n\to\infty}[\theta_{i_n}|u_{i_n}^{-1}(r_n) - b| + \lambda_{j_n}|v_{j_n}^{-1}(t_n) - d| + \alpha_{i_n,j_n}|h(u_{i_n}^{-1}(r_n), v_{j_n}^{-1}(t_n)) - h(b, d)|] + \lim_{n\to\infty}|F_{i_n,j_n}(b, d, z_{\infty\infty}) - z_{\infty\infty}|\\
&\quad\leq \lim_{n\to\infty}[\theta_{i_n}(b - a) + \lambda_{j_n}(d - c) + 2\alpha_{i_n,j_n}\|h\|_\infty] + \lim_{n\to\infty}|z_{\tau(i_n,\infty),\tau(j_n,\infty)} - z_{\infty\infty}| = 0.
\end{aligned}$$

Similarly, we can prove that $T(h)$ is continuous at (x, d) for all $x \in [a, b)$. We next prove that $T(h)$ interpolates the given bivariate CSD. To this end, let us note that for every $(i, j) \in \mathbb{N} \times \mathbb{N}$, by the condition on τ given in (3.1), we can choose $(k, l) \in \{0, \infty\} \times \{0, \infty\}$ such that $i = \tau(i, k)$ and $j = \tau(j, l)$. By (3.2), it follows that $x_k = u_i^{-1}(x_i)$ and $y_l = v_j^{-1}(y_j)$. Using (3.12) and (3.19),

$$T(h)(x_i, y_j) = F_{i,j}(x_k, y_l, h(x_k, y_l)) = F_{i,j}(x_k, y_l, z_{kl}) = z_{\tau(i,k),\tau(j,l)} = z_{i,j}.$$

The continuity of $T(h)$ further implies $T(h)(x_k, y_l) = z_{kl}$ for $k, l \in \{0, \infty\}$. Consequently, T maps $C^*(I \times J)$ into itself. This completes the proof. □

Theorem 3.11. *Let* $\{X, W_{i,j} : (i, j) \in \mathbb{N} \times \mathbb{N}\}$ *be the CIFS defined through* (3.1)–(3.13)*and* (3.16)–(3.18). *Then there exists a unique continuous function* $g : I \times J \to \mathbb{R}$ *such that* $g(x_i, y_j) = z_{i,j}$ *for all* $i, j \in \mathbb{N}_0 \times \mathbb{N}_0$, *and the graph of* g, *namely,*

$$G = \{(x, y, g(x, y)) : (x, y) \in I \times J\},$$

is the attractor of the CIFS defined earlier.

Proof. By the aforementioned lemma, we have $T : C^*(I \times J) \to C^*(I \times J)$ is well defined. We shall prove that it is, in fact, a contraction map. To this end, let $h_1, h_2 \in C^*(I \times J)$. For $(x, y) \in [a, b) \times [c, d)$, choose $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that $(x, y) \in I_i \times J_j$

$$\begin{aligned}
|T(h_1)(x, y) - T(h_2)(x, y)| &= |F_{i,j}(u_i^{-1}(x), v_j^{-1}(y), h_1(u_i^{-1}(x), v_j^{-1}(y))) - F_{i,j}(u_i^{-1}(x), v_j^{-1}(y), h_2(u_i^{-1}(x), v_j^{-1}(y)))|\\
&\leq \alpha_{i,j}|h_1(u_i^{-1}(x), v_j^{-1}(y)) - h_2(u_i^{-1}(x), v_j^{-1}(y))|\\
&\leq \|\alpha\|_\infty\|h_1 - h_2\|_\infty.
\end{aligned}$$

By the continuity of $T(h_1)$ and $T(h_2)$, the aforementioned inequality holds for all other $(x, y) \in I \times J$ as well. Therefore,

$$\|T(h_1) - T(h_2)\|_\infty \leq \|\alpha\|_\infty\|h_1 - h_2\|_\infty.$$

That is, T is a contraction map. In view of Lemma 3.7, the Banach fixed point theorem now ensures that there exists a unique $g \in C^*(I \times J)$ satisfying

$$T(g) = g. \tag{3.25}$$

Now, let $G = \{(x, y, g(x, y)) : (x, y) \in I \times J\}$ be the graph of g and $\tilde{G} := \{(x, y, g(x, y)) : (x, y) \in [a, b) \times [c, d)\} \subset G$. For $(x, y) \in [a, b) \times [c, d)$, there exist $i_0, j_0 \in \mathbb{N} \times \mathbb{N}$ such that $(x, y) \in I_{i_0} \times J_{j_0}$. Thus, by (3.25),

$$\begin{aligned}(x, y, g(x, y)) &= (x, y, F_{i_0 j_0}(u_{i_0}^{-1}(x), v_{j_0}^{-1}(y), g(u_{i_0}^{-1}(x), v_{j_0}^{-1}(y))))\\ &= W_{i_0 j_0}(u_{i_0}^{-1}(x), v_{j_0}^{-1}(y), g(u_{i_0}^{-1}(x), v_{j_0}^{-1}(y))) \in W_{i_0 j_0}(G).\end{aligned}$$

Hence, $\tilde{G} \subset \bigcup_{i,j\geq 1} W_{i,j}(G)$. Taking closure on both sides, we obtain $G \subset \overline{\bigcup_{i,j\geq 1} W_{i,j}(G)}$.

Conversely, for $(x, y) \in I \times J$ and $(i, j) \in \mathbb{N} \times \mathbb{N}$, we have $u_i(x) \in I_i$ and $v_j(y) \in J_j$. Therefore

$$W_{i,j}(x, y, g(x, y)) = (u_i(x), v_j(y), F_{i,j}(x, y, g(x, y))) = (u_i(x), v_j(y), g(u_i(x), v_j(y))) \in G.$$

Since G is closed, one has

$$\overline{\bigcup_{i,j\geq 1} W_{i,j}(G)} \subset G,$$

and so,

$$G = \overline{\bigcup_{i,j\geq 1} W_{i,j}(G)}.$$

This concludes the proof. □

Remark 3.12. As mentioned in Section 1, the motivation for the aforementioned construction of FIS corresponding to a countably infinite data set comes from the construction of FIS in the case of finite data set given by Ruan and Xu [31]. The interested reader is encouraged to compare the above construction with that in [31, Theorem 3.1], in particular, the conditions imposed on the maps $F_{i,j}$ and the definition of the Read-Bajraktarević operator T. In the setting of a countable bivariate data set, establishing the continuity of T is a more subtle matter, for which relatively stronger conditions that we imposed on $F_{i,j}$ came to our aid. On the other hand, for the construction of a parameterized family of fractal functions, which forms the subject matter of the second part of this article, these assumptions on $F_{i,j}$ are easier to come by.

3.2 Approximation of the attractor of countable IFS

Let (X, d) be a compact metric space. For $(i, j) \in \mathbb{N} \times \mathbb{N}$, let $\omega_{i,j} : X \to X$ be contraction maps with contractivity factor $r_{i,j}$ such that $\sup_{i,j} r_{i,j} < 1$. Consider the CIFS $\{X, (\omega_{i,j})_{i,j\in\mathbb{N}}\}$. Assume that $m, n \in \mathbb{N}$. Let us refer to $\{X, (\omega_{i,j})_{(i,j)\in\mathbb{N}_m\times\mathbb{N}_n}\}$ as the *partial IFS* associated with the CIFS $\{X, (\omega_{i,j})_{i,j\in\mathbb{N}}\}$. We will denote the attractor of the aforesaid partial IFS by $A_{m,n}$. Our aim is to prove that the double sequence $(A_{m,n})_{m,n\in\mathbb{N}}$ provides an approximation of the attractor of the CIFS in the space $\mathcal{H}(X)$ endowed with the Hausdorff-Pompeiu metric.

Let us begin with the following theorem:

Theorem 3.13. *Let (X, d) be a complete metric space and $(K_{m,n})_{m,n\in\mathbb{N}}$ be a double sequence of compact subsets of X.*

(1) *Assume that $K_{m,n} \subset K_{m+1,n}$ and $K_{m,n} \subset K_{m,n+1}$ for all $m, n \in \mathbb{N}$. If the set $K := \bigcup_{m,n\geq 1} K_{m,n}$ is relatively compact, then*

$$\overline{K} = \overline{\bigcup_{m,n\geq 1} K_{m,n}} = \lim_{m,n\to\infty} K_{m,n}.$$

(2) *If $K_{m+1,n} \subset K_{m,n}$ and $K_{m,n+1} \subset K_{m,n}$ for all $m, n \in \mathbb{N}$, then*

$$C := \bigcap_{m,n\geq 1} K_{m,n} = \lim_{m,n\to\infty} K_{m,n}.$$

Proof. It can be proved similar to [34, Lemma 2.4]. □

Lemma 3.14. *For every $m, n \in \mathbb{N}$, we have $A_{m,n} \subset A_{m+1,n}$, $A_{m,n} \subset A_{m,n+1}$ and $A_{m,n} \subset A_{m+1,n+1}$.*

Proof. Choose $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, m\}, j_1, j_2, \ldots, j_q \in \{1, 2, \ldots, n\}$. Define the functions $\omega_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q} : X \to X$ by

$$\omega_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}(x) = \omega_{i_1 j_1} \circ \ldots \circ \omega_{i_p j_q}(x).$$

Obviously, $\omega_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$ is a contraction map with contractivity factor at most $\prod_{k=1}^{p}\prod_{l=1}^{q} r_{i_k j_l} < 1$. Let $a_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$ be its unique fixed point, then following [2,§ 3.1 (3)], we have

$$A_{m,n} = \overline{\{a_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q} : i_1, i_2, \ldots, i_p \in \mathbb{N}_m, j_1, j_2, \ldots, j_q \in \mathbb{N}_n\}}.$$

The proof follows immediately from the aforementioned observation. □

Lemma 3.15. [34, Lemma 2.2] *If $(E_\lambda)_{\lambda\in\Lambda}$ is a family of subsets of a topological space, then*

$$\overline{\bigcup_{\lambda\in\Lambda} \overline{E_\lambda}} = \overline{\bigcup_{\lambda\in\Lambda} E_\lambda}.$$

The following theorem is analogous to [34, Prop. 3.6].

Theorem 3.16. *The set $A = \overline{\bigcup_{m,n\geq 1} A_{m,n}}$ in $\mathcal{H}(X)$ is the attractor of the CIFS $\{X, (\omega_{i,j})_{i,j\in\mathbb{N}}\}$.*

Proof. For every $m, n \in \mathbb{N}$, we have

$$\begin{aligned}\overline{\bigcup_{i,j\geq 1} \omega_{i,j}(A_{m,n})} &= \overline{\bigcup_{i=1}^{m}\bigcup_{j=1}^{n} \omega_{i,j}(A_{m,n})} \cup \overline{\bigcup_{i=m+1}^{\infty}\bigcup_{i=n+1}^{\infty} \omega_{i,j}(A_{m,n})} \\ &= A_{m,n} \cup \overline{\bigcup_{i=m+1}^{\infty}\bigcup_{i=n+1}^{\infty} \omega_{i,j}(A_{m,n})}.\end{aligned} \tag{3.26}$$

By Lemma 3.15, we have

$$\begin{aligned}\overline{\bigcup_{i,j\geq 1}\bigcup_{m,n\geq 1} \omega_{i,j}(A_{m,n})} &= \overline{\bigcup_{m,n\geq 1}\bigcup_{i,j\geq 1} \omega_{i,j}(A_{m,n})} \\ &= \overline{\bigcup_{m,n\geq 1}\overline{\bigcup_{i,j\geq 1} \omega_{i,j}(A_{m,n})}} \\ &= \overline{\bigcup_{m,n\geq 1}\left(A_{m,n} \cup \overline{\bigcup_{i\geq m+1}\bigcup_{j\geq n+1} \omega_{i,j}(A_{m,n})}\right)} \\ &= \overline{\bigcup_{m,n\geq 1} A_{m,n}} \cup \overline{\bigcup_{m,n\geq 1}\bigcup_{i\geq m+1}\bigcup_{j\geq n+1} \omega_{i,j}(A_{m,n})}.\end{aligned} \tag{3.27}$$

By Lemma 3.14, we obtain

$$\overline{\bigcup_{m,n\geq 1}\bigcup_{i\geq m+1}\bigcup_{j\geq n+1} \omega_{i,j}(A_{m,n})} \subseteq \overline{\bigcup_{m,n\geq 1} A_{m,n}}. \tag{3.28}$$

By combining (3.27) and (3.28), we obtain

$$\overline{\bigcup_{i,j\geq 1}\bigcup_{m,n\geq 1} \omega_{i,j}(A_{m,n})} = \overline{\bigcup_{m,n\geq 1} A_{m,n}}. \tag{3.29}$$

Now, using the continuity of $\omega_{i,j}$ for each $i, j \in \mathbb{N}$,(3.29) and Lemma 3.15, we have

$$\begin{aligned}\overline{\bigcup_{i,j\geq 1} \omega_{i,j}\left(\overline{\bigcup_{m,n\geq 1} A_{m,n}}\right)} &\subseteq \overline{\bigcup_{i,j\geq 1} \overline{\omega_{i,j}\left(\bigcup_{m,n\geq 1} A_{m,n}\right)}} \\ &= \overline{\bigcup_{i,j\geq 1} \omega_{i,j}\left(\bigcup_{m,n\geq 1} A_{m,n}\right)} \\ &= \overline{\bigcup_{i,j\geq 1}\bigcup_{m,n\geq 1} \omega_{i,j}(A_{m,n})} = \overline{\bigcup_{m,n\geq 1} A_{m,n}}.\end{aligned} \tag{3.30}$$

Conversely, by using (3.29), we have

$$\begin{aligned}\overline{\bigcup_{m,n\geq 1} A_{m,n}} &= \overline{\bigcup_{i,j\geq 1}\bigcup_{m,n\geq 1}\omega_{i,j}(A_{m,n})}\\ &= \overline{\bigcup_{i,j\geq 1}\omega_{i,j}\left(\bigcup_{m,n\geq 1} A_{m,n}\right)}\\ &\subseteq \overline{\bigcup_{i,j\geq 1}\omega_{i,j}\left(\overline{\bigcup_{m,n\geq 1} A_{m,n}}\right)}.\end{aligned} \tag{3.31}$$

Equations (3.30) and (3.31) together yield the following:

$$\overline{\bigcup_{i,j\geq 1}\omega_{i,j}\left(\overline{\bigcup_{m,n\geq 1} A_{m,n}}\right)} = \overline{\bigcup_{m,n\geq 1} A_{m,n}}.$$

Since X is compact, and $A = \overline{\cup_{m,n\geq 1} A_{m,n}} \subseteq X$ is closed, it follows that A is the attractor. □

The aforementioned theorem in conjunction with Theorem 3.13 and Lemma 3.14 provides the promised approximation of the attractor of CIFS by the attractors of the partial IFSs. To be more precise, we have the following result.

Theorem 3.17. *Let A be the attractor of the CIFS $\{X, (\omega_{i,j})_{i,j\in\mathbb{N}}\}$ and $(A_{m,n})$ be the double sequence of attractors of the associated partial IFSs $\{X, (\omega_{i,j})_{(i,j)\in\mathbb{N}_m\times\mathbb{N}_n}\}$. Then*

$$A = \lim_{m,n\to\infty} A_{m,n},$$

where the limit is taken with respect to the Hausdorff-Pompeiu metric.

As a special case of the previous theorem, we have the following.

Corollary 3.18. *Let us consider the CIFS $\{X, (W_{i,j})_{i,j\in\mathbb{N}}\}$ defined in the construction of our countable FIS (see* (3.13)*) and denote by G its attractor obtained in Proposition* 3.6*. For $m, n \in \mathbb{N}$, let $G_{m,n}$ be the attractor of the partial IFS $\{X, (W_{i,j})_{(i,j)\in\mathbb{N}_m\times\mathbb{N}_n}\}$. Then*

$$\lim_{m,n\to\infty} G_{m,n} = G,$$

where the limit is taken with respect to the Hausdorff-Pompeiu metric.

4 A parameterized family of bivariate fractal functions and associated fractal operator

As mentioned in the introduction, in order to explore some approximation theoretic aspects, we consider here a special case of the countable bivariate FIF constructed in the previous section. This is influenced by the notion of α-fractal function [13], an offspring of the univariate FIF for a finite data set.

Definition 4.1. Let $I \times J = [a, b] \times [c, d] \subset \mathbb{R}^2$. We say that $\Delta = \{x_i : i = 0, 1, 2, \dots\} \times \{y_j : j = 0, 1, 2, \dots\} \subset I \times J$ is a partition of $I \times J$ if

(1) the sequences $(x_i)_{i\in\mathbb{N}_0}$ and $(y_j)_{i\in\mathbb{N}_0}$ are strictly increasing such that $x_0 = a$ and $y_0 = c$,

(2) $\lim_{i\to\infty} x_i = b := x_\infty$ and $\lim_{j\to\infty} y_j = d := y_\infty$.

Let $\mathrm{Lip}(I \times J) \subset C(I \times J)$ denote the set of all Lipschitz continuous real-valued functions defined on $I \times J$. That is, $f \in \mathrm{Lip}(I \times J)$ if there exists a constant $l_f > 0$ such that

$$|f(x,y) - f(s,t)| \le l_f \|(x,y) - (s,t)\| \quad \forall (x,y),(s,t) \in I \times J.$$

Definition 4.2. Let $\alpha \in \mathrm{Lip}(I \times J)$ and Δ be a partition of $I \times J$. For each $(i,j) \in \mathbb{N} \times \mathbb{N}$, define $I_i = [x_{i-1}, x_i]$, $J_j = [y_{j-1}, y_j]$ and set

$$\alpha_{i,j} = \|\alpha\|_{\infty, I_i \times J_j} := \sup_{(x,y) \in I_i \times J_j} |\alpha(x,y)|.$$

We say that α is a scaling function if the following conditions are satisfied:
(1) $\lim_{i,j} \alpha_{i,j} = \lim_i (\sup_j \alpha_{i,j}) = \lim_j (\sup_i \alpha_{i,j}) = 0$;
(2) $\|\alpha\|_\infty := \sup_{(x,y) \in I \times J} |\alpha(x,y)| = \sup_{i,j} \alpha_{i,j} < 1$.

Remark 4.3. The aforementioned conditions imposed on the scaling function α coincide with those in (3.8).

Fix $f \in \mathrm{Lip}(I \times J)$. We refer to f as the *germ function* or *seed function*. Consider the countable data set

$$D = \{(x_i, y_j, f(x_i, y_j)) : i, j \in \mathbb{N}_0\},$$

where $\Delta = \{(x_i, y_j) : i, j \in \mathbb{N}_0\}$ is a partition of $I \times J$. We construct a class of bivariate FIFs corresponding to the countable data $D = \{(x_i, y_j, f(x_i, y_j)) : i, j \in \mathbb{N}_0\}$ by choosing appropriate maps u_i, v_j and $F_{i,j}$ that constitute the countable IFS defined in the previous section.

For $i \in \mathbb{N} \cup \{0\}$, as mentioned earlier, let us denote $s_i := \frac{1 + (-1)^i}{2}$.
(1) Consider the affine function $u_i : I \to I_i$ satisfying (3.2). That is,

$$u_i(a) = s_{i-1} x_{i-1} + s_i x_i,\ u_i(b) = s_i x_{i-1} + s_{i+1} x_i,$$
$$u_i(x) = a_i x + c_i \quad \forall x \in I.$$

(2) For $j \in \mathbb{N}$, we consider the affine function $v_j : J \to J_j$ satisfying (3.3), that is,

$$v_j(c) = s_{j-1} y_{j-1} + s_j y_j,\ v_j(d) = s_j y_{j-1} + s_{j+1} y_j,$$
$$v_j(y) = b_j y + d_j \quad \forall y \in J.$$

(3) Assume that $L : \mathrm{Lip}(I \times J) \to \mathrm{Lip}(I \times J)$ is an operator satisfying the boundary conditions

$$L(f)(x_k, y_l) = f(x_k, y_l) \quad \text{for all } k, l \in \{0, \infty\}. \tag{4.1}$$

Let K be a sufficiently large compact interval containing the set $\{f(x_i, y_j) : i, j = 0, 1, 2, \ldots\}$ and $X = I \times J \times K$. Choose $k > 0$ such that

$$|z| \le k \quad \forall z \in K.$$

For $i, j \in \mathbb{N}$, define $F_{i,j} : X \to K$ by

$$F_{i,j}(x,y,z) := \alpha(u_i(x), v_j(y))z + f(u_i(x), v_j(y)) - \alpha(u_i(x), v_j(y)) L(f)(x,y). \tag{4.2}$$

Let us consider the CIFS $\{X, W_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$, where

$$W_{i,j}(x,y,z) = (u_i(x), v_j(y), F_{i,j}(x,y,z))$$

as in the previous section. In what follows, we will denote by l_g the Lipschitz constant of a function $g \in \mathrm{Lip}(I \times J)$.

Theorem 4.4. *Assume that the partition Δ, scaling function α and operator L are fixed. Then corresponding to each $f \in \mathrm{Lip}(I \times J)$, there exists a unique continuous function $f^* : I \times J \to \mathbb{R}$ such that*
(1) *f^* interpolates f at the points in Δ, that is, $f^*(x_i, y_j) = f(x_i, y_j)$ for all $(x_i, y_j) \in \Delta$,*
(2) *the graph of f^* is the attractor of the CIFS $\{X, W_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ defined earlier.*

Furthermore, f^* *satisfies*

$$f^*(x,y) = \begin{cases} f(x,y) + \alpha(x,y)(f^* - L(f))(u_i^{-1}(x), v_j^{-1}(y)) \\ \quad \text{if } (x,y) \in I_i \times J_j \text{ for some } (i,j) \in \mathbb{N} \times \mathbb{N}, \\ f(x,d) + \alpha(x,d)(f^* - L(f))(u_i^{-1}(x), d) \\ \quad \text{if } x \in I_i \text{ for some } i \in \mathbb{N}, \text{ and } y = d, \\ f(b,y) + \alpha(b,y)(f^* - L(f))(b, v_j^{-1}(y)), \\ \quad \text{if } x = b, \text{ and } y \in J_j \text{ for some } j \in \mathbb{N}, \\ f(b,d), \\ \quad \text{if } x = b \text{ and } y = d. \end{cases} \tag{4.3}$$

Proof. Consider the countable data set

$$D = \{(x_i, y_j, f(x_i, y_j)) : i, j \in \mathbb{N}_0\}.$$

We shall first prove that the functions $F_{i,j}$ satisfy (3.10)–(3.11), and the matching conditions (3.16)–(3.17) prescribed in Theorem 3.11. To this end, let $(x,y,z), (x',y',z) \in X = I \times J \times K$, we have

$$\begin{aligned} |F_{i,j}(x,y,z) - F_{i,j}(x',y',z)| &= |[\alpha(u_i(x), v_j(y))z + f(u_i(x), v_j(y)) - \alpha(u_i(x), v_j(y))L(f)(x,y)] \\ &\quad - [\alpha(u_i(x'), v_j(y'))z + f(u_i(x'), v_j(y')) - \alpha(u_i(x'), v_j(y'))L(f)(x',y')]| \\ &\le \sqrt{2}kl_\alpha[|a_i||x-x'| + |b_j||y-y'|] + \sqrt{2}l_f[|a_i||x-x'| + |b_j||y-y'|] \\ &\quad + \sqrt{2}\alpha_{i,j}l_{L(f)}[|x-x'| + |y-y'|] + \sqrt{2}\|L(f)\|_\infty l_\alpha[|a_i||x-x'| + |b_j||y-y'|] \\ &\le [\sqrt{2}|a_i|(kl_\alpha + l_f + l_\alpha\|L(f)\|_\infty) + \sqrt{2}l_{L(f)}\sup_j \alpha_{i,j}]|x-x'| \\ &\quad + [\sqrt{2}|b_j|(kl_\alpha + l_f + l_\alpha\|L(f)\|_\infty) + \sqrt{2}l_{L(f)}\sup_i \alpha_{i,j}]|y-y'|. \end{aligned}$$

Thus, the map $F_{i,j}$ given in (4.2) satisfies:

(1) condition (3.10) with

$$\theta_i = \sqrt{2}|a_i|(kl_\alpha + l_f + l_\alpha\|L(f)\|_\infty) + \sqrt{2}l_{L(f)}\sup_j \alpha_{i,j},$$

$$\lambda_j = \sqrt{2}|b_j|(kl_\alpha + l_f + l_\alpha\|L(f)\|_\infty) + \sqrt{2}l_{L(f)}\sup_i \alpha_{i,j};$$

(2) condition (3.11) with

$$\alpha_{i,j} = \|\alpha\|_{\infty, I_i \times J_j} := \sup_{(x,y)\in I_i \times J_j} |\alpha(x,y)|;$$

(3) conditions required in (3.7), since by the definition of scaling function

$$\lim_{i,j} \alpha_{i,j} = \sup_i \lim_j \alpha_{i,j} = \sup_j \lim_i \alpha_{i,j} = 0;$$

and since $|a_i| = \frac{x_i - x_{i-1}}{b-a}$, $|b_j| = \frac{y_j - y_{j-1}}{d-c}$ provide

$$\lim_i |a_i| = \lim_j |b_j| = 0;$$

(4) interpolation condition at the four vertices of the rectangle $I \times J$, given in (3.12), with

$$F_{i,j}(x_k, y_l, f(x_k, y_l)) = f(x_{\tau(i,k)}, y_{\tau(j,l)}) \quad \forall k, l \in \{0, \infty\};$$

(5) matching conditions (3.16)–(3.17);
(6) the existence of limits in (3.18).

Consequently, by Theorem 3.11, there exists a continuous function f^* interpolating the data set D, and the graph of f^* is the attractor of the CIFS $\{X, (W_{i,j})_{i,j\in\mathbb{N}}\}$. The self-referential equation for f^* also follows from Theorem 3.11. □

Definition 4.5. Emphasizing its dependence on the function α, the function f^* is referred to as the (countable bivariate) α-fractal function associated to the germ function f, with respect to the parameters α, Δ and L. To indicate its dependence with parameters α, Δ and L, we shall denote f^* by $f^{\alpha}_{\Delta,L}$.

Remark 4.6. In fact, we obtain a family of fractal functions $\{f^{\alpha}_{\Delta,L}\}$ corresponding to each germ function f, obtained for different choices of parameters α, Δ, and L. Note that each member of the family $\{f^{\alpha}_{\Delta,L}\}$ interpolates the germ function f at points in Δ, that is,

$$f^{\alpha}_{\Delta,L}(x_i, y_j) = f(x_i, y_j) \quad \forall i, j \in \mathbb{N}_0, \quad \text{and} \quad f^{\alpha}_{\Delta,L}(x_k, y_l) = f(x_k, y_l) \quad \text{for } k, l \in \{0, \infty\}.$$

Remark 4.7. The fundamental impetus for the definition of parameterized family of fractal functions reported above was Navascues's construction of the α-fractal function, which is widely recognized in the approximation theory of univariate fractal functions (see, for instance, [13,14]). The difference here is that the germ function is bivariate and that the sampling of the germ function is done at a countably infinite number of points in the domain. Also, we do not make a standing assumption that L is a bounded linear operator.

As in the context of univariate α-fractal function, we consider the self-referential function $f^* = f^{\alpha}_{\Delta,L}$ as the image of a given (non-fractal) function f under an operator $\mathcal{F}^{\alpha}_{\Delta,L}$, thus associating with f its self-referential counterpart $f^{\alpha}_{\Delta,L}$. More precisely, we have

Definition 4.8. Let α, Δ, and L be fixed. The operator

$$\mathcal{F}^{\alpha}_{\Delta,L} : \mathrm{Lip}(I \times J) \subset C(I \times J) \to C(I \times J), \quad \mathcal{F}^{\alpha}_{\Delta,L}(f) \coloneqq f^{\alpha}_{\Delta,L},$$

which assigns to each $f \in \mathrm{Lip}(I \times J)$ its self-referential counterpart $f^{\alpha}_{\Delta,L}$ is called the α-fractal operator on $\mathrm{Lip}(I \times J)$.

The following results, which point to the error committed in "fractalizing" the germ function f, is well known in the univariate setting [13].

Proposition 4.9. *Let $f \in \mathrm{Lip}(I \times J)$ be the germ function and $f^{\alpha}_{\Delta,L}$ be the α-fractal function associated with f corresponding to the partition Δ, scale function α, and parameter map L. Then we have the following inequality*:

$$\|f^{\alpha}_{\Delta,L} - f\|_{\infty} \le \|\alpha\|_{\infty} \|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty}.$$

Proof. Choose $(x, y) \in I_i \times J_j$ for some $(i, j) \in \mathbb{N} \times \mathbb{N}$. From the self-referential equation (4.3), we have

$$|f^{\alpha}_{\Delta,L}(x, y) - f(x, y)| = |\alpha(x, y)||(f^{\alpha}_{\Delta,L} - L(f))(u_i^{-1}(x), v_j^{-1}(y))| \le \|\alpha\|_{\infty} \|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty}.$$

If $x \in I_i$ for some $i \in \mathbb{N}$ and $y = d$, then

$$|f^{\alpha}_{\Delta,L}(x, d) - f(x, d)| = |\alpha(x, d)||(f^{\alpha}_{\Delta,L} - L(f))(u_i^{-1}(x), d)| \le \|\alpha\|_{\infty} \|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty}.$$

Similarly, if $x = b$ and $y \in J_j$ for some $j \in \mathbb{N}$, one has

$$|f^{\alpha}_{\Delta,L}(b, y) - f(b, y)| \le \|\alpha\|_{\infty} \|f - L(f)\|_{\infty}.$$

Also, $f^{\alpha}_{\Delta,L}(b, d) = f(b, d)$. Consequently,

$$\|f^{\alpha}_{\Delta,L} - f\|_{\infty} \le \|\alpha\|_{\infty} \|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty},$$

completing the proof. □

Corollary 4.10. *Let $f \in \mathrm{Lip}(I \times J)$ be the germ function and $f^{\alpha}_{\Delta,L}$ be the α-fractal function associated with f with respect to the parameters α, Δ, and L. Then we have the following inequality*:

$$\|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty} \le \frac{1}{1 - \|\alpha\|_{\infty}} \|f - L(f)\|_{\infty}.$$

In particular, if $L = Id$, the identity operator on $\mathrm{Lip}(I \times J)$, then $\mathcal{F}^{\alpha}_{\Delta,L} = Id$.

Proof. We have

$$\begin{aligned}\|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty} &= \|f^{\alpha}_{\Delta,L} - f + f - L(f)\|_{\infty}\\ &\le \|f^{\alpha}_{\Delta,L} - f\|_{\infty} + \|f - L(f)\|_{\infty}\\ &\le \|\alpha\|_{\infty}\|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty} + \|f - L(f)\|_{\infty}.\end{aligned}$$

Therefore,

$$\|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty} \le \frac{1}{1 - \|\alpha\|_{\infty}} \|f - L(f)\|_{\infty},$$

completing the proof. □

Corollary 4.11. *Let $f \in \mathrm{Lip}(I \times J)$ be the germ function and $f^{\alpha}_{\Delta,L}$ be the α-fractal function associated with f corresponding to α, Δ, and L. Then we have the following inequality*:

$$\|f^{\alpha}_{\Delta,L} - f\|_{\infty} \le \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} \|f - L(f)\|_{\infty}.$$

Proof. From Proposition 4.9 and the triangle inequality

$$\|f^{\alpha}_{\Delta,L} - f\|_{\infty} \le \|\alpha\|_{\infty}\|f^{\alpha}_{\Delta,L} - L(f)\|_{\infty} \le \|\alpha\|_{\infty}(\|f^{\alpha}_{\Delta,L} - f\|_{\infty} + \|f - L(f)\|_{\infty})$$

proving the claim. □

As an immediate consequence of the previous corollary, we obtain sequences of fractal functions converging uniformly to a prescribed bivariate Lipschitz continuous function, as specified in the upcoming result.

Corollary 4.12. *Let $f \in \mathrm{Lip}(I \times J)$ be the germ function.*

(1) *Assume that the partition Δ and scale function α are fixed. Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of operators on $\mathrm{Lip}(I \times J)$ such that $L_n(f)(x_k, y_l) = f(x_k, y_l)$ for $k, l \in \{0, \infty\}$ and for each $n \in \mathbb{N}$. We further assume that $L_n(f) \to f$ uniformly. Then the sequence $(f^{\alpha}_{\Delta,L_n})_{n\in\mathbb{N}}$ of α-fractal functions associated to f converges uniformly to f.*

(2) *Assume that the partition Δ and the operator $L : \mathrm{Lip}(I \times J) \to \mathrm{Lip}(I \times J)$ are fixed. Let $(\alpha^n)_{n\in\mathbb{N}}$ be a sequence of scale functions such that $\|\alpha^n\|_{\infty} \to 0$ as $n \to \infty$. Then the sequence $(f^{\alpha^n}_{\Delta,L})_{n\in\mathbb{N}}$ converges uniformly to f.*

The proof of the following proposition is similar to its univariate counterpart in [13], but included here at the referee's behest.

Proposition 4.13. *If $L : \mathrm{Lip}(I \times J) \to \mathrm{Lip}(I \times J)$ is a linear operator, then $\mathcal{F}^{\alpha}_{\Delta,L}$ is a linear operator.*

Proof. Let $f, g \in \mathrm{Lip}(I \times J)$ and $c_1, c_2 \in \mathbb{R}$. By using the functional equation for $f^{\alpha}_{\Delta,L}$ and $g^{\alpha}_{\Delta,L}$, we have

$$f^{\alpha}_{\Delta,L}(x, y) = f(x, y) + \alpha(x, y)(f^{\alpha}_{\Delta,L} - L(f))(u_i^{-1}(x), v_j^{-1}(y)),$$

$$g^{\alpha}_{\Delta,L}(x, y) = g(x, y) + \alpha(x, y)(g^{\alpha}_{\Delta,L} - L(f))(u_i^{-1}(x), v_j^{-1}(y)),$$

for all $(x, y) \in I_i \times J_j$. On multiplying the aforementioned expressions by c_1 and c_2, respectively, and adding, we obtain

$$(c_1 f^{\alpha}_{\Delta,L} + c_2 g^{\alpha}_{\Delta,L})(x,y) = (c_1 f + c_2 g)(x,y) + \alpha(x,y)(c_1 f^{\alpha}_{\Delta,L} + c_2 g^{\alpha}_{\Delta,L} - L(c_1 f + c_2 g))(u_i^{-1}(x), v_j^{-1}(y)),$$

for all $(x, y) \in I_i \times J_j$, where $(i, j) \in \mathbb{N} \times \mathbb{N}$. Similar expressions for the other points $(x, y) \in I \times J$. This shows that $(c_1 f^{\alpha}_{\Delta,L} + c_2 g^{\alpha}_{\Delta,L})$ is the fixed point of the RB-operator associated with the construction of $(c_1 f + c_2 g)^{\alpha}_{\Delta,L}$. Hence, by the uniqueness of the fixed point of the RB-operator, we obtain

$$(c_1 f^{\alpha}_{\Delta,L} + c_2 g^{\alpha}_{\Delta,L}) = (c_1 f + c_2 g)^{\alpha}_{\Delta,L},$$

as desired. □

Remark 4.14. As noted earlier, when the germ function is univariate and Δ is finite, the notion of α-fractal function and associated fractal operator are well studied. Further, in the literature, the fractal operator is studied with the standing assumption that the parameter map L is a bounded linear operator [13,14,18,19,33]. Consequently, the fractal operator is widely investigated only within the confines of the theory of bounded linear operators. Here, we do not assume L to be linear or bounded, thereby enhancing the scope of the fractal operator.

Proposition 4.15. *Let* $\mathrm{Lip}(I \times J) \subset C(I \times J)$ *be endowed with the uniform norm. If* $L : \mathrm{Lip}(I \times J) \to \mathrm{Lip}(I \times J)$ *is a continuous operator (not necessarily linear), then so is the operator* $\mathcal{F}^{\alpha}_{\Delta,L}$.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Lipschitz functions such that $f_n \to f$ in $\mathrm{Lip}(I \times J)$. By using the self-referential equation and by routine computations, we have

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f_n) - \mathcal{F}^{\alpha}_{\Delta,L}(f)\|_{\infty} \le \frac{1}{1 - \|\alpha\|_{\infty}}\|f_n - f\|_{\infty} + \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}}\|L(f_n) - L(f)\|_{\infty}.$$

The aforementioned inequality in conjunction with the convergence of $(f_n)_{n\in\mathbb{N}}$ and the continuity of L establishes that $\mathcal{F}^{\alpha}_{\Delta,L}(f_n) \to \mathcal{F}^{\alpha}_{\Delta,L}(f)$. This guarantees the continuity of $\mathcal{F}^{\alpha}_{\Delta,L}$. □

Proposition 4.16. *The α-fractal operator* $\mathcal{F}^{\alpha}_{\Delta,L} : \mathrm{Lip}(I \times J) \to C(I \times J)$ *is an L-bounded operator with the L-bound not exceeding* $\frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}}$.

Proof. According to Corollary 4.11, we have

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f) - f\|_{\infty} \le \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}}(\|f\|_{\infty} + \|L(f)\|_{\infty}).$$

Thus,

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f)\|_{\infty} \le \frac{1}{1 - \|\alpha\|_{\infty}}\|f\|_{\infty} + \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}}\|L(f)\|_{\infty},$$

proving the claim. □

Corollary 4.17. *If L is a norm-bounded nonlinear operator, that is, $p(L) < \infty$, then* $\mathcal{F}^{\alpha}_{\Delta,L}$ *is also norm-bounded.*

Proof. From the previous proposition, we have

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f)\|_{\infty} \le \frac{1}{1 - \|\alpha\|_{\infty}}\|f\|_{\infty} + \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}}\|L(f)\|_{\infty}.$$

Since L is norm-bounded, we have

$$\sup_{f \in \mathrm{Lip}(I \times J), f \neq 0} \frac{\|L(f)\|_{\infty}}{\|f\|_{\infty}} < p(L) < \infty.$$

Consequently,

$$\sup_{f\in \mathrm{Lip}(I\times J), f\neq 0} \frac{\|\mathcal{F}^{\alpha}_{\Delta,L}(f)\|_\infty}{\|f\|_\infty} \leq \frac{1}{1-\|\alpha\|_\infty} + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} p(L).$$

Hence, we have

$$p(\mathcal{F}^{\alpha}_{\Delta,L}) := \max\left\{ \sup_{f\in \mathrm{Lip}(I\times J), f\neq 0} \frac{\|\mathcal{F}^{\alpha}_{\Delta,L}(f)\|_\infty}{\|f\|_\infty}, \|\mathcal{F}^{\alpha}_{\Delta,L}(0)\|_\infty \right\} < \infty,$$

as required. □

Similarly, one can prove the following.

Corollary 4.18. *If L is a topologically bounded operator, then so is the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$.*

Proposition 4.19. *The operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is L-Lipschitz with the L-Lipschitz constant less than or equal to $\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}$.*

Proof. By using similar computations as in Proposition 4.15, one obtains

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f) - \mathcal{F}^{\alpha}_{\Delta,L}(g)\|_\infty \leq \frac{1}{1-\|\alpha\|_\infty}\|f-g\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|L(f)-L(g)\|_\infty, \tag{4.4}$$

proving the claim. □

Corollary 4.20. *If $L : \mathrm{Lip}(I\times J) \subset C(I\times J) \to \mathrm{Lip}(I\times J)$ is a Lipschitz operator, then so is the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$. Furthermore,*

$$|\mathcal{F}^{\alpha}_{\Delta,L}| \leq \frac{1+\|\alpha\|_\infty |L|}{1-\|\alpha\|_\infty}.$$

Proof. The assertion follows clearly from the aforementioned computations. □

Proposition 4.21. *If $L : \mathrm{Lip}(I\times J) \subset C(I\times J) \to \mathrm{Lip}(I\times J)$ is a Cauchy-continuous operator* (*that is, L maps Cauchy sequences to Cauchy sequences*), *then $\mathcal{F}^{\alpha}_{\Delta,L}$ is not a closed operator.*

Proof. Let us assume on the contrary that the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is closed. Choose a function $f\in C(I\times J)\backslash \mathrm{Lip}(I\times J)$, and consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Lip}(I\times J)$ such that $f_n \to f$ as $n\to\infty$. Since L is a Cauchy-continuous operator and $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Lip}(I\times J)$, it follows that $(L(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. By using (4.4) with $f=f_m$ and $g=f_n$, one can infer that $(\mathcal{F}^{\alpha}_{\Delta,L}(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $C(I\times J)$. Assume that $\mathcal{F}^{\alpha}_{\Delta,L}(f_n)\to g$. By the fact that $\mathcal{F}^{\alpha}_{\Delta,L}$ is closed, it follows that $f\in \mathrm{Lip}(I\times J)$ and $g=\mathcal{F}^{\alpha}_{\Delta,L}(f)$, which contradicts the choice of f. □

Proposition 4.22. *If $L : \mathrm{Lip}(I\times J) \to \mathrm{Lip}(I\times J)$ is a closed operator, then the operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is L-closed.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Lipschitz functions such that $f_n\to f$, $L(f_n)\to g$ and $\mathcal{F}^{\alpha}_{\Delta,L}(f_n)\to h$ as $n\to\infty$. Since L is a closed operator, $f_n\to f$, $L(f_n)\to g$ together imply $f\in \mathrm{Lip}(I\times J)$ and $g=L(f)$. By (4.4), we have

$$\|\mathcal{F}^{\alpha}_{\Delta,L}(f_n) - \mathcal{F}^{\alpha}_{\Delta,L}(f)\|_\infty \leq \frac{1}{1-\|\alpha\|_\infty}\|f_n-f\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|L(f_n)-L(f)\|_\infty.$$

Consequently, $\mathcal{F}^{\alpha}_{\Delta,L}(f_n)\to\mathcal{F}^{\alpha}_{\Delta,L}(f)$ as $n\to\infty$. By the uniqueness of the limit, $h=\mathcal{F}^{\alpha}_{\Delta,L}(f)$ and hence, the assertion. □

Similarly, one can prove the following result.

Proposition 4.23. *If $L : \mathrm{Lip}(I\times J) \to \mathrm{Lip}(I\times J)$ is a closable operator, then the operator $\mathcal{F}^{\alpha}_{\Delta,L}$ is L-closable.*

5 Extension of fractal operator and some properties

In this short section, we extend the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ to the whole of $C(I \times J)$, and we shall refer to this extension operator as the α-fractal operator on $C(I \times J)$. The following lemma is a standard result [41].

Lemma 5.1. *Let $A : D(A) \subset X \to Y$ be a Lipschitz operator with Lipschitz constant $|A|$, where X, Y are metric spaces, the second one being complete. Then there exists a Lipschitz extension $\tilde{A} : \overline{D(A)} \subset X \to Y$ of A such that $|\tilde{A}| = |A|$.*

The operators $L : \mathrm{Lip}(I \times J) \to C(I \times J)$ and $\mathcal{F}^{\alpha}_{\Delta,L} : \mathrm{Lip}(I \times J) \to C(I \times J)$ are densely defined. Therefore, if L is a Lipschitz operator, then, by the aforementioned lemma and Corollary 4.20, we have Lipschitz extensions $\tilde{L} : C(I \times J) \to C(I \times J)$ of L and $\tilde{\mathcal{F}}^{\alpha}_{\Delta,L} : C(I \times J) \to C(I \times J)$ of $\mathcal{F}^{\alpha}_{\Delta,L}$ preserving their respective Lipschitz constants. By a slight abuse of notation, we denote this extension of the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ also by $\mathcal{F}^{\alpha}_{\Delta,L}$. This observation is formally recorded in the following proposition.

Proposition 5.2. *If $L : \mathrm{Lip}(I \times J) \subset C(I \times J) \to \mathrm{Lip}(I \times J)$, is a Lipschitz operator, then the α-fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : \mathrm{Lip}(I \times J) \subset C(I \times J) \to C(I \times J)$ has a Lipschitz extension $\mathcal{F}^{\alpha}_{\Delta,L} : C(I \times J) \to C(I \times J)$ with Lipschitz constant*

$$|\mathcal{F}^{\alpha}_{\Delta,L}| \leq \frac{1 + \|\alpha\|_{\infty}|L|}{1 - \|\alpha\|_{\infty}}.$$

Definition 5.3. For a prescribed *germ* function $f \in C(I \times J)$, the function $f^{\alpha}_{\Delta,L} = \mathcal{F}^{\alpha}_{\Delta,L}(f)$, where $\mathcal{F}^{\alpha}_{\Delta,L}$ is the Lipschitz extension in the previous proposition, is called the (countable bivariate) α-fractal function associated to the germ function f with respect to the parameters α, Δ, and L.

Remark 5.4. If L is a bounded linear operator, then $\mathcal{F}^{\alpha}_{\Delta,L} : C(I \times J) \to C(I \times J)$ is also a bounded linear operator. In the setting of univariate functions, the bounded linear fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : C(I \times J) \to C(I \times J)$ is well studied [13,14,16,19].

The notion of invariant subspace is fundamental in operator theory. We next provide a class of proper closed invariant subspaces for the bounded linear fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : C(I \times J) \to C(I \times J)$. Recall that the dual of $C([a,b] \times [c,d])$ is isomorphic to the space of all Borel measures equipped with total variation norm, and it is a non-separable Banach space; see [42, §IV.6.3, Theorem 3] for details. Consider the non-zero linear functional $\psi_{a,c} \in (C([a,b] \times [c,d]))^*$ given by $\psi_{(a,c)}(f) = f(a,c)$ for all $f \in C([a,b] \times [c,d])$.

Theorem 5.5. *The subspace $W_{(x_i,y_j)} := \{f \in C([x_0, x_\infty] \times [y_0, y_\infty]) : f(x_i, y_j) = 0\}$ is invariant for the fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$ for all $i, j \in \mathbb{N}_0 \cup \{\infty\}$, for permissible choices of α, Δ and bounded linear operator L.*

Proof. For any partition Δ, scale vector α, and bounded linear operator L and any $f \in C([x_0, x_\infty] \times [y_0, y_\infty])$, we have

$$\begin{aligned}
(\mathcal{F}^{\alpha}_{\Delta,L})^*(\psi_{(x_i,y_j)})(f) &= \psi_{(x_i,y_j)} \circ \mathcal{F}^{\alpha}_{\Delta,L}(f) \\
&= \psi_{(x_i,y_j)}(f^{\alpha}_{\Delta,L}) \\
&= f^{\alpha}_{\Delta,L}(x_i, y_j) \\
&= f(x_i, y_j) \\
&= \psi_{(x_i,y_j)}(f).
\end{aligned}$$

Consequently,

$$((\mathcal{F}^{\alpha}_{\Delta,L})^*)^m(\psi_{(x_i,y_j)}) = \psi_{(x_i,y_j)} \quad \forall m \in \mathbb{N},$$

and hence, by Lemma 2.20,

$$\overline{\operatorname{span}\{\psi_{(x_i,y_j)}, (\mathcal{F}^{\alpha}_{\Delta,L})^*(\psi_{(x_i,y_j)}), ((\mathcal{F}^{\alpha}_{\Delta,L})^*)^2(\psi_{(x_i,y_j)}), \dots\}} = \operatorname{span}\{\psi_{(x_i,y_j)}\}$$

is a non-trivial closed invariant subspace of $(C([x_0, x_\infty] \times [y_0, y_\infty]))^*$. Now, from Lemma 2.21, it follows that

$$(\operatorname{span}\{\psi_{(x_i,y_j)}\})^{\perp} = W_{(x_i,y_j)}$$

is a non-trivial closed invariant subspace for $\mathcal{F}^{\alpha}_{\Delta,L}$. □

Remark 5.6. Along with the proof of the previous theorem, it is perhaps worth recalling that the closed invariant subspace $Y^{\perp}$ in Lemma 2.21 can be trivial. For example, consider the space $X = \ell_1$ and take $Y = c_0$, the space of all real sequences convergent to zero. Then, Y is a closed invariant subspace of the identity operator $Id : \ell_\infty \to \ell_\infty$. However, the preannihilator of Y is zero.

Theorem 5.7. *Let $\Delta = \{x_i : i \in \mathbb{N}_0\} \times \{y_j : j \in \mathbb{N}_0\}$ be a partition of the rectangle $[a, b] \times [c, d]$, and L be a bounded linear operator. Then*

$$W \coloneqq \{f \in C([a, b] \times [c, d]) : f(x_i, y_j) = 0, \quad \forall (i, j) \in \mathbb{N}_0 \times \mathbb{N}_0, \quad f(a, c) = f(a, d) = f(b, c) = f(b, d) = 0\}$$

is a non-trivial closed invariant subspace for the α- fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$.

Proof. By the above theorem, we have $W_{(x_k,y_l)}$ are non-trivial closed invariant subspaces of $\mathcal{F}^{\alpha}_{\Delta,L}$ for $k, l \in \{0, \infty\}$. The result follows immediately by taking the intersection of these subspaces. □

6 Concluding remarks

Fractal interpolation is one of the few methods of interpolation that can produce both smooth and nonsmooth functions interpolating a prescribed finite set of univariate data. The apparent intricacy of the definition of fractal interpolation belies the remarkable richness of its structure so much so that the theory of fractal interpolation has developed at a rapid pace and continues to flourish. Extensions of Barnsley's original framework of FIF in two ways – considering a countably infinite one-dimensional data set and considering a finite bivariate data set – have been one of the topics of further research studies in fractal interpolation. On the other hand, another area of study in the theory of fractal interpolation is the concept of α-fractal function, which is an offspring of the FIF. The α-fractal formalism paved the way for fruitful interactions between fractal interpolation and other branches of mathematics. By combining these three lines of research, in this article, we investigated fractal interpolation for a bivariate countably infinite data set for the first time in the literature and studied the associated notion of α-fractal function. The concept of fractal operator that emerges quite naturally with the notion of α-fractal function is studied in a more general setting of nonlinear operators. In contrast to the countable univariate and finite bivariate counterparts, the countable bivariate situation considered in this article required some additional considerations and assumptions, making the analysis less straightforward. Overall, the present article makes some modest contributions to the field of fractal approximation theory by studying certain generalizations of fractal interpolation. As in the case of univariate fractal interpolation and fractal surfaces, it is felt that the countable bivariate fractal interpolants developed in the present work can find applications in the approximation theory, the theory of sampling and reconstruction, and in the interface of fractal interpolation and operator theory.

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