DE GRUYTER Demonstratio Mathematica 2024; 57: 20240013

Research Article

Lotfi Jlali* and Jamel Benameur

Long time decay of incompressible convective Brinkman-Forchheimer in $L^2(\mathbb{R}^3)$

https://doi.org/10.1515/dema-2024-0013
received November 3, 2023; accepted March 19, 2024

Abstract: In this article, we study the global existence, uniqueness, and continuity for the solution of incompressible convective Brinkman-Forchheimer on the whole space $\mathbb{R}^3$ when $4\mu\beta \geq 1$. Additionally, we give an asymptotic type of convergence of the global solution.

Keywords: the convective Brinkman-Forchheimer, global solution, blow-up

MSC 2020: 35Q35, 35A02, 35B44

1 Introduction

The incompressible convective Brinkman-Forchheimer equations on the whole space $\mathbb{R}^3$ are given by

$$(\mathrm{CBF}) \quad \begin{cases} \partial_t u - \mu\Delta u + u \cdot \nabla u + \alpha u + \beta\, |u|^{r-1} u + \nabla p = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0, x) = u^0(x), & \text{in} \quad \mathbb{R}^3, \end{cases}$$

where $u(x, t) = (u_1, u_2, u_3)$ denotes the velocity field, and the scalar function $p(x, t)$ represents the pressure. The constant μ denotes the positive Brinkman coefficient (effective viscosity). Besides, α and β, which are two positive constants, denote, respectively, the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients. The exponent r can be greater than or equal to 1.

The equations Convective Brinkman Forchheimer (CBF) describe the motion of incompressible fluid flows in a saturated porous medium. This model has been used in connection with some real-world phenomena, e.g., in the theory of non-Newtonian fluids (see e.g., Shenoy [1]) or in tidal dynamics (see, e.g., Gordeev [2]; Likhtarnikov [3]).

In what follows, we will consider only the critical homogenous CBF equations when $r = 3$:

$$(\mathrm{S}) \quad \begin{cases} \partial_t u - \mu\Delta u + u \cdot \nabla u + \alpha u + \beta\, |u|^2 u + \nabla p = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0, x) = u^0(x), & \text{in} \quad \mathbb{R}^3. \end{cases}$$

The first equation of system (S) has the same scaling as the Navier-Stokes equations (NSE) only when the permeability coefficient α is equal to zero. In this case, when $(\alpha = 0)$ is called damped NSE where the damping term is $\beta\, |u|^{r-1} u$. The resistance to the motion of the flow, which is caused by physical factors such as porous

*** Corresponding author: Lotfi Jlali**, Department of Mathematics and Statistics, College of Science Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11566, Saudi Arabia, e-mail: Lmjlali@imamu.edu.sa
Jamel Benameur: Department of Mathematics, Faculty of Sciences, University of Gabes, Zrig Eddakhlania, Tunisia, e-mail: jamelbenameur@gmail.com

media flow, drag or friction, or other dissipative mechanisms, is shaped by the damping term $\beta\,|u|^{r-1}u$. Several authors studied the damped NSE among them Cai and Jiu [4] proved the global existence of weak solution in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{r+1}(\mathbb{R}^+, L^{r+1}(\mathbb{R}^3)).$$

Benameur [5] brought new findings to the field, which is the model damping $a(e^{b|u|^2} - 1)u$. He used the Friedrich method and some new tools to prove that there is a global solution u in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b,$$

where $\mathcal{E}_b = \{f \in L^4(\mathbb{R}^+ \times \mathbb{R}^3) : (e^{b|f|^2} - 1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}$.

Before treating the global existence, we consider the definition of weak solutions.

Definition 1.1. The function pair (u, p) is called a weak solution of the problem (S) if for any $T > 0$, the following conditions are satisfied:

(1) $u \in L^\infty([0, T]; L^2_\sigma(\mathbb{R}^3)) \cap L^2([0, T]; \dot{H}^1(\mathbb{R}^3)) \cap L^4([0, T], L^4(\mathbb{R}^3))$.

(2) $\partial_t u - \mu\Delta u + u\cdot\nabla u + \alpha u + \beta\,|u|^2 u = -\nabla p$ in $\mathcal{D}'([0, T] \times \mathbb{R}^3)$: for any $\Phi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ such that $\operatorname{div}\Phi(t, x) = 0$ for all $(t, x) \in [0, T] \times \mathbb{R}^3$ and $\Phi(T) = 0$, we have

$$-\int_0^T (u; \partial_t\Phi)_{L^2} + \int_0^T (\nabla u; \nabla\Phi)_{L^2} + \alpha\int_0^T (u; \Phi)_{L^2} + \int_0^T ((u\cdot\nabla)u; \Phi)_{L^2} + \beta\int_0^T (|u|^2 u; \Phi)_{L^2} = (u^0; \Phi(0))_{L^2}.$$

(3) $\operatorname{div} u(x, t) = 0$ for a.e. $(t, x) \in [0, T] \times \mathbb{R}^3$. ($(\ ;\ )_{L^2}$ is the inner product in $L^2(\mathbb{R}^3)$.)

Theorem 1.1. *Let $u^0 \in L^2(\mathbb{R}^3)$ be a divergence-free vector fields and $4\mu\beta \geq 1$, then there is a unique global solution $u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$ of (S). Moreover, we have*

$$\|u(t)\|^2_{L^2} + 2\mu\int_0^t \|\nabla u\|^2_{L^2} + 2\alpha\int_0^t \|u\|^2_{L^2} + 2\beta\int_0^t \|u\|^4_{L^4} d \leq \|u^0\|^2_{L^2}, \quad \forall t \geq 0, \tag{1.1}$$

$$e^{2\alpha t}\|u(t)\|^2_{L^2} + 2\mu\int_0^t e^{2\alpha z}\|\nabla u(z)\|^2_{L^2}\mathrm{d}z + 2\beta\int_0^t e^{2\alpha z}\|u(z)\|^4_{L^4}\mathrm{d}z \leq \|u^0\|^2_{L^2}, \quad \forall t \geq 0. \tag{1.2}$$

Remark 1.1.

(1) Unlike in other blow-up problems (see [6–8]) when the nonlinear capacity methods is used, in this work, the proof of existence and uniqueness is based on the Friedrich method. The continuity and uniqueness are proved by new tools.

(2) The proofs of (1.1) and (1.2) are given by the Friedrich method and Cantor diagonal process, respectively.

(3) The uniqueness and inequality (1.2) imply that $(t \mapsto e^{2\alpha t}\|u(t)\|^2_{L^2})$ is decreasing. Indeed, for $0 \leq t_1 \leq t_2$, by uniqueness of this solution, we obtain

$$e^{2\alpha(t_2 - t_1)}\|u(t_2)\|^2_{L^2} = e^{2\alpha(t_2 - t_1)}\|u(t_1 + (t_2 - t_1))\|^2_{L^2} \leq \|u(t_1)\|^2_{L^2}$$

and

$$e^{2\alpha t_2}\|u(t_2)\|^2_{L^2} \leq e^{2\alpha t_1}\|u(t_1)\|^2_{L^2}.$$

(4) The last property does not imply that $\lim_{t\to\infty} e^{\alpha t}\|u(t)\|_{L^2} = 0$.

(5) The idea of looking for an asymptotic type of convergence $e^{\alpha t}\|u(t)\|_{L^2}$ to zero is due to the following: the solution of linear system

$$(SL)\quad \begin{cases} \partial_t f - \mu\Delta f + \alpha f = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} f = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ f(0, x) = u^0(x), & \text{in} \quad \mathbb{R}^3, \end{cases}$$

has the following form:

$$f(t,x) = e^{-\alpha t} e^{\mu t \Delta} u^0.$$

We have

$$e^{\alpha t}\|f(t)\|_{L^2} = \|e^{\mu t\Delta} u^0\|_{L^2}.$$

The dominated convergence theorem implies that

$$\lim_{t\to\infty} e^{\alpha t}\|f(t)\|_{L^2} = 0.$$

We are now ready to state the second result.

Theorem 1.2. *Let $u^0 \in L^2(\mathbb{R}^3)$ be a divergence-free vector fields and $4\mu\beta \geq 1$. If $u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$ is the unique global solution of (S) given by Theorem 1.1, then*

$$\|u(t)\|_{L^2} = o(e^{-\alpha t}), \;\; t \to \infty. \tag{1.3}$$

The remainder of this article is organized as follows. In Second 2, we give some notations, definitions, and preliminary results. Section 3 is devoted to prove Theorem 1.1, and this proof is done in two steps. In the first, we give a general result of equicontinuity, while in the second, we apply the Friedritch method to construct a global solution of (S). Moreover, uniqueness and the continuity of solution are also mentioned. In Section 4, we study the large time decay.

2 Notations and preliminary results

2.1 Notations

- For a function $f: \mathbb{R}^3 \longrightarrow \bar{\mathbb{R}}$ and $R > 0$, the Friedrich operator J_R is defined by

$$J_R(D) = \mathcal{F}^{-1}(\chi_{B_R}\widehat{f}),$$

  where B_R is the ball of center 0 and radius R.
- $L^2_\sigma(\mathbb{R}^3) = \{f \in (L^2(\mathbb{R}^3))^3;\ \mathrm{div} f = 0\}$,
- The Leray projection $\mathbb{P} : (L^2(\mathbb{R}^3))^3 \longrightarrow (L^2(\mathbb{R}^3))^3$ is defined by

$$\mathcal{F}^{-1}(\mathbb{P}f) = \widehat{f}\,(\xi) - \left(\widehat{f}(\xi)\cdot \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} = M(\xi)\widehat{f}(\xi),$$

  where $M(\xi)$ is the matrix $\left(\delta_{k,\ell} - \frac{\xi_k\xi_\ell}{|\,\xi\,|^2}\right)_{1\leq k,\ell\leq 3}$.

2.2 Preliminary results

In this section, we collect some classical results, and we give some lemmas that are well suited to the study of system (S). We start by the following elementary inequality.

Proposition 2.1. [9] *Let H be Hilbert space.*

(1) *If (x_n) is a bounded sequence of elements in H, then there is a subsequence $(x_{\varphi(n)})$ such that*

$$(x_{\varphi(n)}|y) \to (x|y), \;\; \forall y \in H.$$

(2) *If $x \in H$ and (x_n) is a bounded sequence of elements in H such that*

$$(x_n|y) \to (x|y), \;\; \forall y \in H,$$

then $\|x\| \leq \liminf_{n\to\infty}\|x_n\|$.

(3) *If* $x \in H$ *and* (x_n) *is a bounded sequence of elements in* H *such that*

$$(x_n|y) \to (x|y), \quad \forall y \in H$$

and

$$\limsup_{n\to\infty} \|x_n\| \leq \|x\|,$$

then $\lim_{n\to\infty}\|x_n - x\| = 0$.

Lemma 2.1. [10] *Let* s_1 *and* s_2 *be two real numbers.*

(1) *If* $s_1 < 3/2$ *and* $s_1 + s_2 > 0$, *there exists a constant* $C_1 = C_1(s_1, s_2)$, *such that if* $f, g \in \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3)$, *then* $f.\,g \in \dot{H}^{s_1+s_2-\frac{3}{2}}(\mathbb{R}^3)$ *and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}} \leq C_1(\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}}\|g\|_{\dot{H}^{s_1}}).$$

(2) *If* $s_1, s_2 < 3/2$ *and* $s_1 + s_2 > 0$, *there exists a constant* $C_2 = C_2(s_1, s_2)$ *such that: if* $f \in \dot{H}^{s_1}(\mathbb{R}^3)$ *and* $g \in \dot{H}^{s_2}(\mathbb{R}^3)$, *then* $f.\,g \in \dot{H}^{s_1+s_2-\frac{3}{2}}(\mathbb{R}^3)$ *and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}} \leq C_2\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}}.$$

Lemma 2.2. [11] *For all* $x, y \in \mathbb{R}^3$, *we have*

$$(|x|^2x - |y|^2)y(x-y) \geq \frac{1}{2}(|x|^2 + |y|^2)|x-y|^2.$$

Lemma 2.3. [5] *For all* $s > \frac{d}{2}$: $L^1(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$. *Moreover, we have*

$$\|f\|_{H^{-s}(\mathbb{R}^d)} \leq \sigma_{s,d}\|f\|_{L^1(\mathbb{R}^d)}, \quad \forall f \in H^{-s}(\mathbb{R}^d),$$

where $\sigma_{s,d} = \left(\int_{\mathbb{R}^d}(1 + |\xi|^2)^{-s}\mathrm{d}\xi\right)^{1/2}$.

Lemma 2.4. [5] *Let* $p \in (1, \infty)$ *and* $\Omega \neq \emptyset$ *be an open subset of* $\mathbb{R}^4$. *If* (f_n) *is a bounded sequence in* $L^2(\Omega) \cap L^p(\Omega)$ *and* $f \in L^2(\Omega)$ *such that*

$$\lim_{n\to\infty}(f_n|g)_{L^2} = (f|g)_{L^2}, \quad \forall g \in L^2(\Omega).$$

Then, $f \in L^p(\Omega)$ *and*

$$\|f\|_{L^2} \leq \liminf_{n\to\infty}\|f_n\|_{L^2},$$
$$\|f\|_{L^p} \leq \liminf_{n\to\infty}\|f_n\|_{L^p}.$$

3 Proof of Theorem 1.1

This proof is done in two steps.

3.1 Step 1

In this step, we prove a general result for bounded sequence in energy space of system (S).

Proposition 3.1. *Let* $\nu_1, \nu_2, \nu_3 \in [0, \infty)$, $r_1, r_2, r_3 \in (0, \infty)$, *and* $f_0 \in L^2_\sigma(\mathbb{R}^3)$. *For* $n \in \mathbb{N}$, *let* $F_n : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ *be a measurable function in* $C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ *such that*

$$A_n(D)F_n = F_n, \ \ F_n(0, x) = A_n(D)f_0(x)$$

and

$$(E1) \qquad \partial_t F_n + \sum_{k=1}^{3} \nu_k \, |D_k|^{2r_k} F_n + A_n(D) \operatorname{div} (F_n \otimes F_n) + \alpha F_n + \beta A_n(D)[|F_n|^2 F_n] = 0,$$

$$(E2) \qquad \|F_n(t)\|^2_{L^2} + 2 \sum_{k=1}^{3} \nu_k \int_{t_1}^{t_2} \||D_k|^{r_k} F_n\|^2_{L^2} + 2\alpha \int_0^t \|F_n\|^2_{L^2} + 2\beta \int_0^t \|F_n\|^4_{L^4} \leq \|f_0\|^2_{L^2}.$$

Then, for every $\varepsilon > 0$, *there is* $\delta = \delta(\varepsilon, \alpha, \beta, \|f_0\|_{L^2}) > 0$ *such that for all* $t_1, t_2 \in \mathbb{R}^+$, *we have*

$$(|t_2 - t_1| < \delta \Rightarrow \|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} < \varepsilon), \ \ \forall n \in \mathbb{N}, \tag{3.1}$$

with $s_0 = \max\{3, 2\max_{1\leq i\leq 3} r_i\}$.

Proof. This proof is inspired from [5]-Proposition 3.1. Integrating $(E1)$ on the interval $[t_1, t_2] \subset \mathbb{R}^+$ and taking the inner product in H^{-s_0}, we obtain

$$\|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} \leq I_{1,n}(t_1, t_2) + I_{2,n}(t_1, t_2) + I_{3,n}(t_1, t_2) + I_{4,n}(t_1, t_2),$$

with

$$I_{1,n}(t_1, t_2) = \sum_{k=1}^{3} \nu_k \int_{t_1}^{t_2} \||D_k|^{2r_k} F_n\|_{H^{-s_0}},$$

$$I_{2,n}(t_1, t_2) = \int_{t_1}^{t_2} \|A_n(D) \operatorname{div}(F_n \otimes F_n)\|_{H^{-s_0}},$$

$$I_{3,n}(t_1, t_2) = \alpha \int_{t_1}^{t_2} \|F_n\|_{H^{-s_0}},$$

$$I_{4,n}(t_1, t_2) = \beta \int_{t_1}^{t_2} \|A_n(D)[|F_n|^2 F_n]\|_{H^{-s_0}}.$$

Let $\varepsilon > 0$ be a positive real, let us find a positive real $\delta > 0$ such that if $|t_2 - t_1| < \delta$, we obtain

$$I_{k,n} < \frac{\varepsilon}{4}, \quad \forall k = \{1, 2, 3, 4\}.$$

- To estimate $I_{1,n}(t_1, t_2)$, we write

$$\begin{aligned}
I_{1,n}(t_1, t_2) &= \sum_{k=1}^{3} \nu_k \int_{t_1}^{t_2} \||D_k|^{2r_k} F_n\|_{H^{-s_0}} \\
&\leq \left(\sum_{k=1}^{3} \nu_k\right) \int_{t_1}^{t_2} \|F_n\|_{H^{2r_k - s_0}} \\
&\leq \left(\sum_{k=1}^{3} \nu_k\right) \int_{t_1}^{t_2} \|F_n(\tau)\|_{H^0} d\tau \\
&\leq \left(\sum_{k=1}^{3} \nu_k\right) \int_{t_1}^{t_2} \|F_n(\tau)\|_{L^2} d\tau \\
&\leq \left(\sum_{k=1}^{3} \nu_k\right) \|f_0\|_{L^2}(t_2 - t_1).
\end{aligned}$$

Then, if $|t_2 - t_1| < \delta_1 = \frac{\varepsilon}{4\left(\sum_{k=1}^{3} \nu_k\right)\|f_0\|_{L^2} + 4}$, we obtain

$$I_{1,n}(t_1, t_2) < \frac{\varepsilon}{4}.$$

- To estimate $I_{2,n}(t_1, t_2)$, from Lemma (2.3), we have

$$\begin{aligned}
I_{2,n}(t_1, t_2) &= \int_{t_1}^{t_2} \|A_n(D)\operatorname{div}(F_n \otimes F_n)\|_{H^{-s_0}} \\
&\le \int_{t_1}^{t_2} \|\operatorname{div}(F_n \otimes F_n)(\tau)\|_{H^{-3}} \mathrm{d}\tau \\
&\le \int_{t_1}^{t_2} \|(F_n \otimes F_n)(\tau)\|_{H^{-2}} \mathrm{d}\tau \\
&\le \sigma_{2,3} \int_{t_1}^{t_2} \|(F_n \otimes F_n)(\tau)\|_{L^1} \mathrm{d}\tau \\
&\le \sigma_{2,3} \int_{t_1}^{t_2} \|F_n(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
&\le \sigma_{2,3} \|f_0\|_{L^2}^2 (t_2 - t_1).
\end{aligned}$$

Then, if $|t_2 - t_1| < \delta_2 = \frac{\varepsilon}{4\sigma_{2,3}\|f_0\|_{L^2}^2 + 4}$, we obtain

$$I_{2,n}(t_1, t_2) < \frac{\varepsilon}{4}.$$

- To estimate $I_{3,n}(t_1, t_2)$, we write

$$\begin{aligned}
I_{3,n}(t_1, t_2) &= \alpha \int_{t_1}^{t_2} \|F_n\|_{H^{-s_0}} \\
&\le \alpha \int_{t_1}^{t_2} \|F_n(\tau)\|_{H^0} \mathrm{d}\tau \\
&\le \alpha \int_{t_1}^{t_2} \|F_n(\tau)\|_{L^2} \mathrm{d}\tau \\
&\le \alpha \int_{t_1}^{t_2} \|F_n(\tau)\|_{L^2} \mathrm{d}\tau \\
&\le \alpha \|f_0\|_{L^2} (t_2 - t_1).
\end{aligned}$$

Then, if $|t_2 - t_1| < \delta_3 = \frac{\varepsilon}{4\alpha\|f_0\|_{L^2} + 4}$, we obtain

$$I_{3,n}(t_1, t_2) < \frac{\varepsilon}{4}.$$

- To estimate $I_{4,n}(t_1, t_2)$, from Lemma (2.3), we have

$$I_{4,n}(t_1, t_2) \le \beta \int_{t_1}^{t_2} \|A_n(D)[|F_n|^2 F_n]\|_{H^{-3}} \le \sigma_{3,3}\ \beta \int_{t_1}^{t_2} \|(|F_n|^2 F_n)(\tau)\|_{L^1} \mathrm{d}\tau.$$

Now, for $(n, R, t) \in \mathbb{N} \times (0, +\infty) \times [0, +\infty)$, we consider the following subset of $\mathbb{R}^3$:

$$X_n(R, t) = \{x \in \mathbb{R}^3 / |F_n(t, x)| \le R\},$$

$$
\begin{aligned}
I_{4,n}(t_1,t_2) &\leq \sigma_{3,3}\,\beta\int_{t_1}^{t_2}\int_{X_n(R,t)}|F_n(\tau,x)|^3\mathrm{d}x\mathrm{d}\tau + \sigma_{3,3}\,\beta\int_{t_1}^{t_2}\int_{X_n(R,t)^c}|F_n(\tau,x)|^3\mathrm{d}x\mathrm{d}\tau\\
&\leq \sigma_{3,3}\,\beta\int_{t_1}^{t_2}\int_{X_n(R,t)}R|F_n(\tau,x)|^2\mathrm{d}x\mathrm{d}\tau + \sigma_{3,3}\,\beta\int_{t_1}^{t_2}\int_{X_n(R,t)^c}\frac{|F_n(\tau,x)|^4}{R}\mathrm{d}x\mathrm{d}\tau\\
&\leq \sigma_{3,3}\,\beta R\int_{t_1}^{t_2}\int_{X_n(R,t)}|F_n(\tau,x)|^2\mathrm{d}x\mathrm{d}\tau + \frac{\sigma_{3,3}\,\beta}{R}\int_{t_1}^{t_2}\int_{X_n(R,t)^c}|F_n(\tau,x)|^4\mathrm{d}x\mathrm{d}\tau\\
&\leq \sigma_{3,3}\,\beta R\int_{t_1}^{t_2}\|F_n(\tau)\|_{L^2}^2\mathrm{d}\tau + \frac{\sigma_{3,3}\,\beta}{R}\int_{t_1}^{t_2}\|F_n(\tau)\|_{L^4}^4\mathrm{d}\tau.
\end{aligned}
$$

Using $(E2)$, we obtain

$$
I_{4,n}(t_1,t_2) \leq \sigma_{3,3}\,\beta R\|f_0\|_{L^2}^2(t_2-t_1) + \frac{\sigma_{3,3}\,\beta}{2\alpha R}\|f_0\|_{L^2}^2.
$$

Hence, with the choices $R = R_\varepsilon = \frac{4\ \sigma_{3,3}\|f_0\|_{L^2}^2+4}{\varepsilon}$ and $\delta_4 = \frac{\varepsilon}{8\alpha\sigma_{3,3}\,\beta\||f_0|^2\|_{L^2}^2+8}$, we obtain

$$
|t_2-t_1| < \delta_4 \Rightarrow I_{4,n}(t_1,t_2) < \frac{\varepsilon}{4}.
$$

To conclude, it suffices to take $\delta = \min\{\delta_1,\delta_2,\delta_3,\delta_4\}$. □

3.2 Step 2

In this step, we construct a global solution of (S), where we use a method inspired by [12]. For this, consider the approximate system with the parameter $n \in \mathbb{N}$:

$$
(S_n)\begin{cases}
\partial_t u - \mu\Delta J_n u + J_n(J_n u\cdot\nabla J_n u) + \alpha J_n u + \beta J_n[|J_n u|^2 J_n u] = -\nabla P_n, & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^3\\
P_n = (-\Delta)^{-1}(\mathrm{div}\,J_n(J_n u\cdot\nabla J_n u + \beta\mathrm{div}\,J_n[|J_n u|^2 J_n u])) & \\
\mathrm{div}\ u = 0, & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^3,\\
u(0,x) = J_n u^0(x), & \text{in}\quad \mathbb{R}^3.
\end{cases}
$$

- By the Cauchy-Lipschitz theorem, we obtain a unique solution $u_n \in C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ of (S_n), with the following properties:

$$
\mathrm{div} u_n = 0, J_n u_n = u_n.
$$

  Moreover, u_n satisfies

$$
\begin{cases}
\partial_t u_n - \mu\Delta u_n + J_n(u_n.\,\nabla u_n) + \alpha u_n + \beta J_n[|u_n|^2 u_n] = -\nabla p_n, & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^3,\\
p_n = (-\Delta)^{-1}(\mathrm{div}\,J_n(u_n.\,\nabla u_n + \beta\mathrm{div}\,J_n[|u_n|^2 u_n])), & \\
\mathrm{div}\ u_n = 0, & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^3,\\
u_n(0,x) = J_n u^0(x), & \text{in}\quad \mathbb{R}^3.
\end{cases}
$$

  and the following energy estimate

$$
\|u_n(t)\|_{L^2}^2 + 2\mu\int_0^t\|\nabla u_n\|_{L^2}^2 + 2\alpha\int_0^t\|u_n\|_{L^2}^2 + 2\beta\int_0^t\|u_n\|_{L^4}^4 \leq \|u^0\|_{L^2}^2,\ \ \forall t\geq 0, \tag{3.2}
$$

$$
e^{2\alpha t}\|u_n(t)\|_{L^2}^2 + 2\mu\int_0^t e^{2\alpha z}\|\nabla u_n(z)\|_{L^2}^2\mathrm{d}z + 2\beta\int_0^t e^{2\alpha z}\|u_n(z)\|_{L^4}^4\mathrm{d}z \leq \|u^0\|_{L^2}^2,\ \ \forall t\geq 0. \tag{3.3}
$$

- By Inequality (3.2), we obtain (u_n) that is bounded in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+, H^1(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3)).$$

  Using Proposition 3.1, we deduce that

$$\text{the sequence } (u_n) \text{ is equicontinuous in } C_b(\mathbb{R}^+, H^{-1}(\mathbb{R}^3)). \tag{3.4}$$

- Let $(T_q)_q \in (0,\infty)^{\mathbb{N}}$ such that $T_q < T_{q+1}$ and $T_q \to \infty$ as $q \to \infty$. Let $(\theta_q)_{q\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^3)$ such that: for all $q \in \mathbb{N}$,

$$\begin{cases} \theta_q(x) = 1, \ \ \forall x \in B\left(0, q+1+\dfrac{1}{4}\right), \\ \theta_q(x) = 0, \ \ \forall x \in B(0, q+2)^c, \\ 0 \le \theta_q \le 1. \end{cases}$$

  Using (3.2)–(3.4) and classical argument by combining Ascoli's theorem and the Cantor diagonal process, we obtain a nondecreasing $\varphi : \mathbb{N} \to \mathbb{N}$ and

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3)),$$

  such that for all $q \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \|\theta_q(u_{\varphi(n)} - u)\|_{L^\infty([0,T_q],H^{-4})} = 0. \tag{3.5}$$

  Combining the aforementioned inequalities, we obtain (1.1) and (1.2).
- It remains to show that u is a solution of system (S). Using the same idea in [5], we prove that u satisfies

$$\partial_t u - \mu\Delta u + u\cdot\nabla u + \alpha u + \beta|u|^2 u = -\nabla p, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^3).$$

  Then, u is a solution of our system (S).

3.3 Continuity of the solution in $L^2(\mathbb{R}^3)$

- By Inequality (1.1), we have

$$\limsup_{t\to 0} \|u(t)\|_{L^2} \le \|u^0\|_{L^2}.$$

  Then, Proposition 2.1 implies that

$$\limsup_{t\to 0} \|u(t) - u^0\|_{L^2} = 0.$$

  This ensures the continuity of the solution u at 0.
- To prove the continuity on $(0,\infty)$, consider the function

$$v_{n,\varepsilon}(t) = u_{\varphi(n)}(t+\varepsilon), \quad p_{n,\varepsilon}(t) = p_{\varphi(n)}(t+\varepsilon), \ \text{ for } n \in \mathbb{N} \text{ and } \varepsilon > 0.$$

  We have

$$\begin{aligned} &\partial_t u_{\varphi(n)} - \mu\Delta u_{\varphi(n)} + J_{\varphi(n)}(u_{\varphi(n)}\cdot\nabla u_{\varphi(n)}) + \alpha u_{\varphi(n)} + \beta J_{\varphi(n)}(|u_{\varphi(n)}|^2 u_{\varphi(n)}) + \nabla p_{\varphi(n)} = 0, \\ &\partial_t v_{n,\varepsilon} - \mu\Delta v_{n,\varepsilon} + J_{\varphi(n)}(v_{n,\varepsilon}\cdot\nabla v_{n,\varepsilon}) + \alpha v_{n,\varepsilon} + \beta J_{\varphi(n)}(|v_{n,\varepsilon}|^2 v_{n,\varepsilon}) + \nabla p_{n,\varepsilon} = 0. \end{aligned}$$

  The function $w_{n,\varepsilon} = u_{\varphi(n)} - v_{n,\varepsilon}$ fulfills the following:

$$\begin{aligned} &\partial_t w_{n,\varepsilon} - \mu\Delta w_{n,\varepsilon} + \alpha w_{n,\varepsilon} + \beta J_{\varphi(n)}(|u_{\varphi(n)}|^2 u_{\varphi(n)} - |v_{n,\varepsilon}|^2 v_{n,\varepsilon}) + \nabla(p_{\varphi(n)} - p_{n,\varepsilon}) \\ &\quad = J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla w_{n,\varepsilon}) - J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla u_{\varphi(n)}) - J_{\varphi(n)}(u_{\varphi(n)}\cdot\nabla w_{n,\varepsilon}). \end{aligned}$$

Taking the scalar product with $w_{n,\varepsilon}$ in $L^2(\mathbb{R}^3)$ and using the fact that $\langle w_{n,\varepsilon}\cdot\nabla w_{n,\varepsilon};\, w_{n,\varepsilon}\rangle = 0$ and $\operatorname{div} w_{n,\varepsilon} = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\|w_{n,\varepsilon}(t)\|^2_{L^2} + \mu\|\nabla w_{n,\varepsilon}(t)\|^2_{L^2} + \alpha\|w_{n,\varepsilon}(t)\|^2_{L^2} + \beta\langle J_{\varphi(n)}(|u_{\varphi(n)}|^2u_{\varphi(n)} - |v_{n,\varepsilon}|^2v_{n,\varepsilon});\, w_{n,\varepsilon}\rangle_{L^2}\\ = -\langle J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla u_{\varphi(n)});\, w_{n,\varepsilon}\rangle_{L^2}.\end{aligned} \tag{3.6}$$

From Lemma 2.2, we have

$$\begin{aligned}\langle J_{\varphi(n)}(|u_{\varphi(n)}|^2u_{\varphi(n)} - |v_{n,\varepsilon}|^2v_{n,\varepsilon});\, w_{n,\varepsilon}\rangle_{L^2} &= \langle(|u_{\varphi(n)}|^2u_{\varphi(n)} - |v_{n,\varepsilon}|^2v_{n,\varepsilon});\, J_{\varphi(n)}w_{n,\varepsilon}\rangle_{L^2}\\ &= \langle(|u_{\varphi(n)}|^2u_{\varphi(n)} - |v_{n,\varepsilon}|^2v_{n,\varepsilon});\, w_{n,\varepsilon}\rangle_{L^2}\\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}(|u_{\varphi(n)}|^2 + |v_{n,\varepsilon}|^2)|w_{n,\varepsilon}|^2,\end{aligned}$$

which implies

$$\beta\langle J_{\varphi(n)}(|u_{\varphi(n)}|^2u_{\varphi(n)} - |v_{n,\varepsilon}|^2v_{n,\varepsilon});\, w_{n,\varepsilon}\rangle_{L^2} \geq \frac{\beta}{2}\int_{\mathbb{R}^3}(|u_{\varphi(n)}|^2 + |v_{n,\varepsilon}|^2)|w_{n,\varepsilon}|^2. \tag{3.7}$$

Taking into account the equality

$$I_{n,\varepsilon} = \langle J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla u_{\varphi(n)});\, w_{n,\varepsilon}\rangle_{L^2} = \langle J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla v_{n,\varepsilon});\, w_{n,\varepsilon}\rangle_{L^2}$$

and, combining the Cauchy-Schwarz and Young's inequalities, we obtain, for $f = v_{n,\varepsilon}$ or $f = u_{\varphi(n)}$,

$$\begin{aligned}|\langle J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla f);\, w_{n,\varepsilon}\rangle_{L^2}| &\leq \int_{\mathbb{R}^3}|w_{n,\varepsilon}||f||\nabla w_{n,\varepsilon}|\\ &= \left(\int_{\mathbb{R}^3}|w_{n,\varepsilon}|^2|f|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|\nabla w_{n,\varepsilon}|^2\right)^{\frac{1}{2}}\\ &\leq \beta\int_{\mathbb{R}^3}|w_{n,\varepsilon}|^2|f|^2 + \frac{1}{4\beta}\|\nabla w_{n,\varepsilon}\|^2_{L^2}.\end{aligned} \tag{3.8}$$

This inequality is true for $f = v_{n,\varepsilon}$ and $f = u_{\varphi(n)}$, then using the elementary inequality

$$\min(a, b) \leq \frac{1}{2}(a + b),$$

with $a = \int_{\mathbb{R}^3}|w_{n,\varepsilon}|^2|v_{n,\varepsilon}|^2$ and $b = \int_{\mathbb{R}^3}|w_{n,\varepsilon}|^2|u_{\varphi(n)}|^2$, we obtain

$$|I_{n,\varepsilon}| \leq \frac{\beta}{2}\int_{\mathbb{R}^3}|w_{n,\varepsilon}|^2(|v_{n,\varepsilon}|^2 + |u_{\varphi(n)}|^2) + \frac{1}{4\beta}\|\nabla w_{n,\varepsilon}\|^2_{L^2}.$$

Combining Inequalities (3.7) and (3.8), we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{n,\varepsilon}(t)\|^2_{L^2} + \left(\mu - \frac{1}{4\beta}\right)\|\nabla w_{n,\varepsilon}(t)\|^2_{L^2} + \alpha\|w_{n,\varepsilon}(t)\|^2_{L^2} \leq 0. \tag{3.9}$$

We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{n,\varepsilon}(t)\|^2_{L^2} + \left(\mu - \frac{1}{4\beta}\right)\|\nabla w_{n,\varepsilon}(t)\|^2_{L^2} \leq 0.$$

Hence, we obtain

$$\|w_{n,\varepsilon}(t)\|^2_{L^2} \leq \|w_{n,\varepsilon}(0)\|^2_{L^2}$$

and

$$\|u_{\varphi(n)}(t+\varepsilon) - u_{\varphi(n)}(t)\|^2_{L^2} \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|^2_{L^2}.$$

For $t_0 > 0$ and $\varepsilon \in (0, t_0/2)$, we have

$$\|u_{\varphi(n)}(t_0+\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 \le \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2,$$
$$\|u_{\varphi(n)}(t_0-\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 \le \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2.$$

Thus,

$$\begin{aligned}\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 &= \|J_{\varphi(n)}u_{\varphi(n)}(\varepsilon) - J_{\varphi(n)}u_{\varphi(n)}(0)\|_{L^2}^2\\ &= \|J_{\varphi(n)}(u_{\varphi(n)}(\varepsilon) - u^0)\|_{L^2}^2\\ &\le \|u_{\varphi(n)}(\varepsilon) - u^0\|_{L^2}^2\\ &\le \|u_{\varphi(n)}(\varepsilon)\|_{L^2}^2 + \|u^0\|_{L^2}^2 - 2\mathrm{Re}\langle u_{\varphi(n)}(\varepsilon); u^0\rangle_{L^2}\\ &\le 2\|u^0\|_{L^2}^2 - 2\mathrm{Re}\langle u_{\varphi(n)}(\varepsilon); u^0\rangle.\end{aligned}$$

Since $\liminf_{n\to\infty}\langle u_{\varphi(n)}(\varepsilon); u^0\rangle_{L^2} = \langle u(\varepsilon); u^0\rangle_{L^2}$,

$$\liminf_{n\to\infty}\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \le 2\|u^0\|_{L^2}^2 - 2\mathrm{Re}\langle u(\varepsilon); u^0\rangle_{L^2}.$$

Moreover, for all $q, N \in \mathbb{N}$,

$$\begin{aligned}\|J_N(\theta_q(u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0)))\|_{L^2}^2 &\le \|\theta_q(u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2}^2\\ &\le \|u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.\end{aligned}$$

For q big enough using (3.5), we obtain

$$\|J_N(\theta_q(u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0)))\|_{L^2}^2 \le \liminf_{n\to\infty}\|u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2.$$

Then,

$$\|J_N(\theta_q(u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0)))\|_{L^2}^2 \le 2(\|u^0\|_{L^2}^2 - \mathrm{Re}\langle u(\varepsilon); u^0\rangle_{L^2}).$$

By applying the Monotone convergence theorem in the order N and q, we obtain

$$\|u(t_0\pm\varepsilon) - u(t_0)\|_{L^2}^2 \le 2(\|u^0\|_{L^2}^2 - \mathrm{Re}\langle u(\varepsilon); u^0\rangle_{L^2}).$$

Using the continuity at 0 and make $\varepsilon \longrightarrow 0$, we obtain the continuity at t_0.

3.4 Uniqueness of u in $L^2(\mathbb{R}^3)$

Let u and v be two solutions of (S) in the space

$$C(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3)).$$

Set $w = u - v$. Then, the function w satisfies the following:

$$\partial_t w - \mu\Delta w + \alpha w + \beta(|u|^2u - |v|^2v) + \nabla(p_u - p_v) = w\cdot\nabla w - w\cdot\nabla u - u\cdot\nabla w.$$

Taking the scalar product in L^2 with w, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \mu\|\nabla w\|_{L^2}^2 + \alpha\|w\|_{L^2}^2 + \beta\langle(|u|^2u - |v|^2v); w\rangle_{L^2} = -\langle w\cdot\nabla u; w\rangle_{L^2}.$$

By adapting the same method for the proof of the continuity of such solution in $L^2(\mathbb{R}^3)$, with u, v, and w instead of $u_{\varphi(n)}$, $v_{n,\varepsilon}$, and $w_{n,\varepsilon}$ in order, we find

$$\beta\langle(|u|^2u - |v|^2v); w\rangle_{L^2} \ge \frac{\beta}{2}\int_{\mathbb{R}^3}(|u|^2 + |v|^2)|w|^2$$

and

$$
\begin{aligned}
|\langle w\cdot\nabla u;\, w\rangle_{L^2}| &= |\langle w\cdot\nabla v;\, w\rangle_{L^2}| \\
&\leq \begin{cases} \beta \int_{\mathbb{R}^3} |w|^2|u|^2 + \frac{1}{4\beta}\|\nabla w\|_{L^2}^2 \\ \beta \int_{\mathbb{R}^3} |w|^2|v|^2 + \frac{1}{4\beta}\|\nabla w\|_{L^2}^2 \end{cases} \\
&\leq \frac{\beta}{2}\min\left(\int_{\mathbb{R}^3} |w|^2|u|^2, \int_{\mathbb{R}^3} |w|^2|v|^2\right) + \frac{1}{4\beta}\|\nabla w\|_{L^2}^2 \\
&\leq \frac{\beta}{2}\int_{\mathbb{R}^3} |w|^2(|u|^2+|v|^2) + \frac{1}{4\beta}\|\nabla w\|_{L^2}^2.
\end{aligned}
$$

Combining the aforementioned inequalities, we find the following energy estimate:

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \left(\mu - \frac{1}{4\beta}\right)\|\nabla w\|_{L^2}^2 + \alpha\|w\|_{L^2}^2 \leq 0.
$$

Using the fact $\mu - \frac{1}{4\beta} \geq 0$ and $\alpha > 0$, we obtain

$$
\|w(t)\|_{L^2}^2 \leq \|w^0\|_{L^2}^2.
$$

As $w^0 = 0$, then $w = 0$ and $u = v$, which implies the uniqueness. And the proof of Theorem 1.1 is completed.

4 Proof of Theorem (1.2)

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be a regular function, such that

$$
\begin{cases} 0 \leq \chi \leq 1, & \\ \chi = 1, & \text{on } B(0,1), \\ \chi = 0, & \text{on } B(0,2)^c. \end{cases}
$$

Also, for $\delta > 0$, put the following functions χ_δ and ψ_δ defined by

$$
\chi_\delta(\xi) = \chi(\delta\xi), \;\; \psi_\delta = \mathcal{F}^{-1}(\chi_\delta).
$$

We have

$$
\|\psi_\delta\|_{L^1} = \|\psi_1\|_{L^1}, \;\; \forall \delta > 0.
$$

In addition, we define the operator S_δ by

$$
S_\delta(f) = \chi_\delta(D)f = \psi_\delta * f.
$$

For $\delta > 0$, which will be fixed later, put the following functions:

$$
v_\delta = \psi_\delta * u = \mathcal{F}^{-1}(\chi_\delta(\xi)\hat{u}), \quad w_\delta = u - v_\delta = \mathcal{F}^{-1}((1-\chi_\delta(\xi))\hat{u}).
$$

Let $\varepsilon > 0$ be a fixed real.
• Estimate of v_δ: We have

$$
e^{2\alpha t}\|v_\delta(t)\|_{L^2}^2 + 2\mu\int_0^t e^{2\alpha z}\|\nabla v_\delta(z)\|_{L^2}^2 \mathrm{d}z \leq A_\delta(t) + B_\delta(t), \tag{4.1}
$$

where

$$\begin{aligned} A_\delta(t) &= 2\int_0^t e^{2\alpha z}|\langle S_\delta(u\cdot\nabla u)(z); v_\delta(z)\rangle_{L^2}|\mathrm{d}z,\\ B_\delta(t) &= 2\beta\int_0^t e^{2\alpha z}|\langle S_\delta(|u|^2u)(z); v_\delta(z)\rangle_{L^2}|\mathrm{d}z. \end{aligned}$$

To estimate A_δ, we write

$$\begin{aligned} A_\delta(t) &= \int_0^t e^{2\alpha z}|\langle S_\delta(u\cdot\nabla u)(z); v_\delta(z)\rangle_{L^2}|\mathrm{d}z\\ &\le \int_0^t e^{2\alpha z}\|S_\delta(u\cdot\nabla u)(z)\|_{L^2}\|v_\delta(z)\|_{L^2}\mathrm{d}z\\ &\le c_0\int_0^t e^{2\alpha z}\left(\int_{|\xi|<2\delta}|\xi|^2|\mathcal{F}(u\otimes u)(z,\xi)|^2\mathrm{d}\xi\right)^{1/2}\|v_\delta(z)\|_{L^2}\mathrm{d}z\\ &\le \sqrt{2}c_0\delta^{1/2}\int_0^t e^{2\alpha z}\left(\int_{|\xi|<\delta}|\xi||\mathcal{F}(u\otimes u)(z,\xi)|^2\mathrm{d}\xi\right)^{1/2}\|v_\delta(z)\|_{L^2}\mathrm{d}z\\ &\le \sqrt{2}c_0\delta^{1/2}\int_0^t e^{2\alpha z}\|(u\otimes u)(z)\|_{\dot H^{1/2}}\|v_\delta(z)\|_{L^2}\mathrm{d}z\\ &\le C\delta^{1/2}\int_0^t e^{2\alpha z}\|\nabla u(z)\|_{L^2}^2\|v_\delta(z)\|_{L^2}\mathrm{d}z. \end{aligned}$$

Then,

$$A_\delta(t)\le C\delta^{1/2}\|u^0\|_{L^2}^3\ \ \forall t\ge 0. \tag{4.2}$$

To estimate B_δ, by the Hölder inequality, we obtain

$$\begin{aligned} B_\delta(t) &= \int_0^t e^{2\alpha z}|\langle S_\delta(|u|^2u)(z); v_\delta(z)\rangle_{L^2}|\mathrm{d}z\\ &\le \int_0^t e^{2\alpha z}\|S_\delta(|u|^2u)(z)\|_{L^{4/3}}\|v_\delta(z)\|_{L^4}\mathrm{d}z\\ &\le \int_0^t e^{2\alpha z}\|\psi_\delta*(|u|^2u)(z)\|_{L^{4/3}}\|v_\delta(z)\|_{L^4}\mathrm{d}z\\ &\le \int_0^t e^{2\alpha z}\|\psi_\delta\|_{L^1}\|(|u|^2u)(z)\|_{L^{4/3}}\|v_\delta(z)\|_{L^4}\mathrm{d}z\\ &\le \|\psi_1\|_{L^1}\int_0^t e^{2\alpha z}\|(|u|^2u)(z)\|_{L^{4/3}}\|v_\delta(z)\|_{L^4}\mathrm{d}z\\ &\le \|\psi_1\|_{L^1}\int_0^\infty e^{2\alpha z}\|u(z)\|_{L^4}^3\|v_\delta(z)\|_{L^4}\mathrm{d}z\\ &\le \|\psi_1\|_{L^1}\left(\int_0^\infty e^{2\alpha z}\|u(z)\|_{L^4}^4\mathrm{d}z\right)^{3/4}\left(\int_0^\infty e^{2\alpha z}\|v_\delta(z)\|_{L^4}^4\mathrm{d}z\right)^{1/4}. \end{aligned}$$

Using (1.2), we obtain

$$B_\delta(t) \le \|\psi_1\|_{L^1}\left(\frac{\|u^0\|_{L^2}^2}{2\beta}\right)^{3/4}\left(\int_0^\infty e^{2\alpha z}\|v_\delta(z)\|_{L^4}^4 dz\right)^{1/4}.$$

By interpolation, we obtain

$$\begin{aligned}\|v_\delta(z)\|_{L^4}^4 &\le C\|v_\delta(z)\|_{L^2}\|\nabla v_\delta(z)\|_{L^2}^3\\ &\le 2C\delta\|v_\delta(z)\|_{L^2}^2\|\nabla v_\delta(z)\|_{L^2}^2\\ &\le 2C\delta\|u^0\|_{L^2}^2\|\nabla v_\delta(z)\|_{L^2}^2\\ &\le 2C\delta\|u^0\|_{L^2}^2\|\nabla u(z)\|_{L^2}^2.\end{aligned}$$

Then,

$$\int_0^\infty e^{2\alpha z}\|v_\delta(z)\|_{L^4}^4 dz \le 2C\delta\|u^0\|_{L^2}^2\int_0^\infty e^{2\alpha z}\|\nabla u(z)\|_{L^2}^2 dz \le C\mu^{-1}\|u^0\|_{L^2}^4\delta$$

and

$$B_\delta(t) \le \|\psi_1\|_{L^1}\left(\frac{\|u^0\|_{L^2}^2}{2\beta}\right)^{3/4}(C\mu^{-1})^{1/4}\|u^0\|_{L^2}\delta^{1/4}. \tag{4.3}$$

Combining Inequalities (4.1)–(4.3), we obtain

$$e^{2\alpha t}\|v_\delta(t)\|_{L^2}^2 \le C(u^0)\delta^{1/4}(1+\delta^{1/4}),$$

which implies the existence of $\delta_0 > 0$, such that

$$\sup_{t\ge 0} e^{2\alpha t}\|v_\delta(t)\|_{L^2}^2 < (\varepsilon/2)^2, \quad \forall \delta \in (0, \delta_0]. \tag{4.4}$$

- Prove that $w_{\delta_0} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^3))$. We have

$$\begin{aligned}\int_0^\infty e^{2\alpha t}\|w_{\delta_0}(t)\|_{L^2}^2 dt &= c_0\int_0^\infty e^{2\alpha z}\|\widehat{w}_{\delta_0}(t)\|_{L^2}^2 dt\\ &= c_0\int_0^\infty\int_{|\xi|>\delta_0} e^{2\alpha z}\,|\hat{u}(t,\xi)|^2 d\xi dt\\ &\le c_0\delta_0^{-2}\int_0^\infty e^{2\alpha z}\int_{|\xi|>\delta_0}|\xi|^2|\hat{u}(t,\xi)|^2 d\xi dt\\ &\le c_0\delta_0^{-2}\int_0^\infty e^{2\alpha z}\|\widehat{\nabla u}(t)\|_{L^2}^2 dt\\ &\le \delta_0^{-2}\int_0^\infty e^{2\alpha z}\|\nabla u(t)\|_{L^2}^2 dt\\ &\le 2\mu^{-1}\delta_0^{-2}\|u^0\|_{L^2}^2.\end{aligned}$$

Then, $(t \mapsto e^{2\alpha t}\|w_{\delta_0}(t)\|_{L^2}^2) \in C(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, L^2(\mathbb{R}^3))$.
Put the following subset of $\mathbb{R}^+$:

$$F_\varepsilon = \{t \ge 0 : \; e^{2\alpha t}\|w_{\delta_0}(t)\|_{L^2}^2 \ge (\varepsilon/2)^2\}.$$

As $(t \mapsto e^{2\alpha t}\|w_{\delta_0}(t)\|_{L^2}^2) \in C(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, L^2(\mathbb{R}^3))$, then there exists $t_\varepsilon \in \mathbb{R}^+\backslash F_\varepsilon$ such that

$$e^{2\alpha t_\varepsilon}\|w_{\delta_0}(t_\varepsilon)\|_{L^2}^2 < (\varepsilon/2)^2. \tag{4.5}$$

- Combining (4.4)–(4.5), we obtain

$$e^{2\alpha t_\varepsilon}\|u(t_\varepsilon)\|_{L^2} < \varepsilon. \tag{4.6}$$

- Now, put the following system:

$$(S)\quad \begin{cases} \partial_t f - \mu\Delta f + f\cdot\nabla f + \alpha f + \beta\,|f|^2 f + \nabla q = 0 & \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3\\ \operatorname{div} f = 0 & \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3\\ f(0,x) = u(t_\varepsilon, x) & \text{in} \quad \mathbb{R}^3. \end{cases}$$

By the first step, there is a unique solution $f_\varepsilon \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$ of (2) satisfying for all $t \geq 0$:

$$e^{2\alpha t}\|f_\varepsilon(t)\|_{L^2}^2 + 2\mu\int_0^t e^{2\alpha z}\|\nabla f_\varepsilon(z)\|_{L^2}^2 dz + 2\beta\int_0^t e^{2\alpha z}\|f_\varepsilon(z)\|_{L^4}^4 dz \leq \|u(t_\varepsilon)\|_{L^2}^2.$$

The uniqueness of the solution implies that $f(t) = u(t_\varepsilon + t)$ for all $t \geq 0$ and

$$e^{2\alpha t}\|u(t_\varepsilon + t)\|_{L^2}^2 \leq \|u(t_\varepsilon)\|_{L^2}^2,$$

which implies that

$$e^{2\alpha t}\|u(t)\|_{L^2}^2 \leq e^{2\alpha t_\varepsilon}\|u(t_\varepsilon)\|_{L^2}^2 < \varepsilon^2, \quad \forall t \geq t_\varepsilon.$$

Then, the proof of Theorem 1.2 is completed.

5 Conclusion

This work deals with a detailed study of the global existence, uniqueness, and continuity for the solution of incompressible convective Brinkman-Forchheimer on the whole space $\mathbb{R}^3$ when $4\mu\beta \geq 1$. An asymptotic type of convergence of the global solution associated with the aforementioned problem is also established.

Acknowledgments: The authors appreciate the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) for supporting and supervising this project.

Funding information: The authors extend their appreciation to the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia for funding this research through the Project Number IFP-IMSIU-2023018.

Author contributions: All authors contributed equally in this work. All authors have accepted responsibility for entire content of the manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: Data sharing is not applicable to the article as no datasets were generated or analyzed during this study.

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