Research Article

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Generalized Yosida inclusion problem involving multi-valued operator with XOR operation

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Abstract: In this article, we study a generalized Yosida variational inclusion problem involving multi-valued operator with XOR operation. It is shown that the generalized Yosida variational inclusion problem involving multi-valued operator with XOR operation is equivalent to a fixed point equation. We have proved that the generalized Yosida approximation operator is Lipschitz continuous. Finally, we prove an existence and convergence result for our problem.

Keywords: Yosida, XOR, existence, Lipschitz, convergence

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1 Introduction

It is well known that variational principles conned a prime role in many branches of pure and applied sciences, e.g., general theory of relativity, gauge field theory related to modern particle physics and solitary wave theory. From some time back, variational principles are used as powerful tools for solving problems occurring in mathematical and engineering sciences. These principles have been simplified by the theory of variational inequalities.

It is proved by Baiocchi and Capelo [1] in 1971 that fluid through a porous media can be studied using the tools of variational inequalities. The traffic equilibrium problem dealt by Smith [2] as an inequality problem but later on Dafermos [3] clarified that it is a variational inequality problem. Many problems of physical sciences related to real life can also be studied in the framework of variational inequalities.

First variational inequality problem is converted into a fixed point problem, and then, one can apply several well-known iterative methods for solving variational inequalities (see, e.g., [4–8]).

The generalized form of variational inequality known as variational inclusion were introduced by Hassouni and Moudafi [9]. Variational inclusions are mathematical models for various optimization problems in finance, economics, transportation, network analysis, engineering and technology, etc. (see [10–18]). Variational inclusions are reduced to fixed point equations using the concept of resolvent operator of the form $[I + \lambda M]^{-1}$, where $\lambda > 0$ is a constant. If M possesses some monotonicity property, then the resolvent of M has full domain and is firmly non-expansive.

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To transform the monotone operators into single-valued operators that possess Lipschitzian property in Hilbert spaces, Yosida approximation operator method is applicable. Since Yosida approximation operator depends on resolvent operator, it is obvious that they are applied to solve several problems related to variational analysis. Yosida approximation operators are used to study wave equations, heat equations, heat flow, linear equations related to coupled sound, etc. (see [19–23]).

The XOR logical operation, exclusive or, takes two boolean operands and returns true if and only if the operands are different. Conversely, it returns false if the two operands have the same value.

Let us discuss some examples of XOR operation. Imagine two people walking toward each other down a hallway wide enough for two people. If they are both walking on the same side of the hallway, they will be in each other's way. If they both walk on opposite sides of the hallway, they will be able to pass each other. Whether they can pass each other or will get in each other's way depends on the XOR of which side each is on.

If we have a pair of polarizing filters, such as the lenses of polarizing sunglasses. Hold the filter up to the lenses of polarizing sunglasses. Holding the filters up to the light so that we are looking through both filters in series at the light. If the filters are aligned, light will pass through. If we rotate one of them 90 degrees, the combination will block the light. This process also behaves like XOR logic.

Due to importance and applications of the above-discussed concepts, in this article, we consider a generalized Yosida inclusion problem involving multi-valued operator with XOR operation. Section 2 is the general section, based on required definitions and results to achieve our goal. In Section 3, we state our problem and have shown its equivalence with a fixed point equation. On the basis of fixed point equation, we construct an iterative algorithm. Section 4 is our main part of this article in which we obtain an existence and convergence result.

2 Basic definitions and notions

We denote a real ordered Hilbert space by $\widehat{H}$ equipped with norm $||\cdot||$ and inner product $\langle\cdot,\cdot\rangle$. Let d be the metric induced by the norm $||\cdot||$, $2^{\widehat{H}}$ be the family of nonempty subsets of $\widehat{H}$, and $\widetilde{C}(\widehat{H})$ be the family of nonempty compact subsets of $\widehat{H}$.

In the following, we mention some known concepts and results to prove our main result.

Definition 2.1. [24] The set $\widetilde{C}_{\widehat{H}}$ is a cone if $\hat{a} \in \widetilde{C}_{\widehat{H}}$ implies $\lambda\hat{a} \in \widetilde{C}_{\widehat{H}}$, for every positive scalar λ.

Definition 2.2. [24] A cone $\widetilde{C}_{\widehat{H}}$ is called a normal cone, if and only if there exists a constant $\lambda_{\widetilde{N}_{\widehat{H}}} > 0$ such that $0 \le \hat{a} \le \hat{b}$ implies

$$||\hat{a}|| \le \lambda_{\widetilde{N}_{\widehat{H}}}||\hat{b}||,$$

where $\lambda_{\widetilde{N}_{\widehat{H}}}$ is called the normal constant.

Definition 2.3. Let $\widetilde{C}_{\widehat{H}}$ be a cone. For arbitrary elements $\hat{a}, \hat{b} \in \widehat{H}, \hat{a} \le \hat{b}$ holds if and only if $\hat{a} - \hat{b} \in \widetilde{C}_{\widehat{H}}$. Then, the relation "≤" in $\widehat{H}$ is called partial order relation.

Additionally, if $\hat{a} \le \hat{b} (or\, \hat{b} \le \hat{a})$ holds, then $\hat{a}$ and $\hat{b}$ are said to be comparable to each other (denoted by $\hat{a} \propto \hat{b}$).

Definition 2.4 up to Definition 2.5, Propositions 2.1 and 2.2 can be found in [25–31].

Definition 2.4. For the set $\{\hat{a}, \hat{b}\}$, by $\mathrm{lub}\{\hat{a}, \hat{b}\}$ and $\mathrm{glb}\{\hat{a}, \hat{b}\}$, we mean the least upper bound and the greatest lower bound. Suppose they exist, then some binary operations are defined as follows:

(i) $\hat{a} \vee \hat{b} = \mathrm{lub}\{\hat{a}, \hat{b}\}$, where $\vee$ is known as *OR* operation,

(ii) $\hat{a} \wedge \hat{b} = \text{glb}\{\hat{a}, \hat{b}\}$, where $\wedge$ is known as *AND* operation,
(iii) $\hat{a} \oplus \hat{b} = (\hat{a} - \hat{b}) \vee (\hat{b} - \hat{a})$, where $\oplus$ is called *XOR* operation,
(iv) $\hat{a} \odot \hat{b} = (\hat{a} - \hat{b}) \wedge (\hat{b} - \hat{a})$, where $\odot$ is called *XNOR* operation.

Proposition 2.1. *Let $\oplus$ and $\odot$ be the XOR operation and XNOR operation, respectively. Then, the following axioms are true*:
(i) $\hat{a} \odot \hat{a} = 0, \hat{a} \odot \hat{b} = \hat{b} \odot \hat{a} = -(\hat{a} \oplus \hat{b}) = -(\hat{b} \oplus \hat{a})$,
(ii) *if* $\hat{a} \propto 0$, *then* $-\hat{a} \oplus 0 \leq \hat{a} \leq \hat{a} \oplus 0$,
(iii) $(\lambda\hat{a}) \oplus (\lambda\hat{b}) = |\lambda|(\hat{a} \oplus \hat{b})$,
(iv) $0 \leq \hat{a} \oplus \hat{b}$, *if* $\hat{a} \propto \hat{b}$,
(v) *if* $\hat{a} \propto \hat{b}$, *then* $\hat{a} \oplus \hat{b} = 0$, *if and only if* $\hat{a} = \hat{b}$.

Proposition 2.2. *Let $\widetilde{C}_{\widehat{H}}$ be a normal cone in $\widehat{H}$ with normal constant $\lambda_{\widetilde{N}_{\widehat{H}}}$, then for each $\hat{a}, \hat{b} \in \widehat{H}$, the following relations are true*:
(i) $||0 \oplus 0|| = ||0|| = 0$,
(ii) $||\hat{a} \vee \hat{b}|| = ||\hat{a}|| \vee ||\hat{b}|| \leq ||\hat{a}|| + ||\hat{b}||$,
(iii) $||\hat{a} \oplus \hat{b}|| \leq ||\hat{a} - \hat{b}|| \leq \lambda_{\widetilde{N}}||\hat{a} \oplus \hat{b}||$,
(iv) *if* $\hat{a} \propto \hat{b}$, *then* $||\hat{a} \oplus \hat{b}|| = ||\hat{a} - \hat{b}||$.

Definition 2.5. Let $B : \widehat{H} \to \widehat{H}$ be a single-valued mapping and $M : \widehat{H} \to 2^{\widehat{H}}$ be a multi-valued mapping. Then,
(i) B is called $\hat{\xi}$-order non-extended mapping if there exists a constant $\xi > 0$ such that $\hat{\xi}(\hat{a} \oplus \hat{b}) \leq B(\hat{a}) \oplus B(\hat{b})$, for all $\hat{a}, \hat{b} \in \widehat{H}$,
(ii) B is called comparison mapping if $\hat{a} \propto \hat{b}$, then $B(\hat{a}) \propto B(\hat{b}), \hat{a} \in B(\hat{a})$ and $\hat{b} \propto B(\hat{b})$, for all $\hat{a}, \hat{b} \in \widehat{H}$,
(iii) B is called strongly comparison mapping if B is a comparison mapping and $B(\hat{a}) \propto B(\hat{b})$ if and only if $\hat{a} \propto \hat{b}$, for all $\hat{a}, \hat{b} \in \widehat{H}$,
(iv) M is called weak-comparison mapping if $p_{\hat{a}} \in M(\hat{a}), \hat{a} \propto p_{\hat{a}}$, and if $\hat{a} \propto \hat{b}$, then there exists $p_{\hat{b}} \in M(\hat{b})$ such that $p_{\hat{a}} \propto p_{\hat{b}}$, for all $\hat{a}, \hat{b} \in \widehat{H}$,
(v) M is called $\tilde{\alpha}_B$-weak non-ordinary difference mapping with respect to B, if it is weak comparison and for each $\hat{a}, \hat{b} \in \widehat{H}$, $p_{\hat{a}} \in M(B(\hat{a}))$ and $p_{\hat{b}} \in M(B(\hat{b}))$ such that

$$(p_{\hat{a}} \oplus p_{\hat{b}}) \oplus \tilde{\alpha}_B(B(\hat{a}) \oplus B(\hat{b})) = 0, \text{ where } \tilde{\alpha}_B > 0 \text{ is a constant,}$$

(vi) M is called $\tilde{\rho}$-order different weak-comparison mapping with respect to B, if for all $\hat{a}, \hat{b} \in \widehat{H}$, there exists $p_{\hat{a}} \in M(B(\hat{a})), p_{\hat{b}} \in M(B(\hat{b}))$ such that

$$\tilde{\rho}(p_{\hat{a}} - p_{\hat{b}}) \propto \hat{a} - \hat{b}, \text{ where } \tilde{\rho} > 0 \quad \text{is a constant,}$$

(vii) Weak comparison mapping M is called $(\tilde{\alpha}_B, \tilde{\rho})$-weak ANODD, if it is an $\tilde{\alpha}$-weak-non ordinary difference mapping and $\tilde{\rho}$-order different weak-comparison mapping associated with B and $[B + \tilde{\rho}M](\widehat{H}) = \widehat{H}$.

Definition 2.6. The mapping $B : \widehat{H} \to \widehat{H}$ is said to be Lipschitz continuous, if

$$||B(\hat{a}) - B(\hat{b})|| \leq \lambda_B||\hat{a} - \hat{b}||, \quad \text{for all } \hat{a}, \hat{b} \in \widehat{H}, \quad \text{where } \lambda_B > 0 \text{ is a constant}.$$

Definition 2.7. [32] The multi-valued mapping $S : \widehat{H} \to C(\widehat{H})$ is called D-Lipschitz continuous if there exists a constant $\lambda_{S_D} > 0$ such that

$$D(S(\hat{a}), S(\hat{b})) \leq \lambda_{S_D}||\hat{a} - \hat{b}||, \quad \text{for all } \hat{a}, \hat{b} \in \widehat{H},$$

where $D(\cdot,\cdot)$ is the Hausdörff metric on $\widetilde{C}_{\widehat{H}}$.

Definition 2.8. [32] The mapping $N : \widehat{H} \times \widehat{H} \to \widehat{H}$ is Lipschitz continuous in the first argument if

$$\|N(\hat{a}, \cdot) - N(\hat{b}, \cdot)\| \leq \lambda_{N_1}\|\hat{a} - \hat{b}\|, \text{ for all } \hat{a}, \hat{b} \in \widehat{H}, \quad \text{where } \lambda_{N_1} > 0 \text{ is a constant}.$$

Similarly, we can define Lipschitz continuity of N in the second argument.

Definition 2.9. [24] Let B be $\hat{\xi}$-ordered non-extended mapping and M be $\widetilde{\alpha}_B$-weak non-ordinary difference mapping with respect to B. We define the generalized resolvent operator $R^M_{B,\widetilde{\rho}} : \widehat{H} \to \widehat{H}$ as:

$$R^M_{B,\widetilde{\rho}} = [B + \widetilde{\rho} M]^{-1}(\hat{a}), \quad \text{for all } \hat{a} \in \widehat{H}, \quad \text{where } \widetilde{\rho} > 0 \text{ is a constant}. \tag{1}$$

Lemma 2.1. [24] *Let $M : \widehat{H} \to 2^{\widehat{H}}$ be an $(\widetilde{\alpha}_B, \widetilde{\rho})$-weak ANODD mapping and $B : \widehat{H} \to \widehat{H}$ be a $\hat{\xi}$-ordered non-extended mapping associated with $R^M_{B,\widetilde{\rho}}$. Then, for $\widetilde{\alpha}_B > \frac{1}{\widetilde{\rho}}$, the following relation holds*:

$$\begin{aligned} & R^M_{B,\widetilde{\rho}}(\hat{a}) \oplus R^M_{B,\widetilde{\rho}}(\hat{b}) \leq \frac{1}{\widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1)}(\hat{a} \oplus \hat{b}), \quad \textit{for all } \hat{a}, \hat{b} \in \widehat{H}, \\ \textit{or} \quad & R^M_{B,\widetilde{\rho}}(\hat{a}) \oplus R^M_{B,\widetilde{\rho}}(\hat{b}) \leq \frac{1}{\theta}(\hat{a} \oplus \hat{b}), \end{aligned} \tag{2}$$

where $\theta = \widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1)$.

Definition 2.10. The generalized Yosida approximation operator is defined as

$$Y^M_{B,\widetilde{\rho}}(\hat{a}) = \frac{1}{\widetilde{\rho}}[B - R^M_{B,\widetilde{\rho}}](\hat{a}), \quad \text{for all } \hat{a} \in \widehat{H}. \tag{3}$$

Lemma 2.2. *The generalized Yosida approximation operator $Y^M_{B,\widetilde{\rho}}$ is Lipschitz continuous, provided $\hat{a} \propto \hat{b}$, $R^M_{B,\widetilde{\rho}}(\hat{a}) \propto R^M_{B,\widetilde{\rho}}(\hat{b})$ and B is λ_B-Lipschitz continuous.*

Proof. For all $\hat{a}, \hat{b} \in \widehat{H}$, we have

$$\begin{aligned} \|Y^M_{B,\widetilde{\rho}}(\hat{a}) - Y^M_{B,\widetilde{\rho}}(\hat{b})\| &= \left\| \frac{1}{\widetilde{\rho}}[B - R^M_{B,\widetilde{\rho}}](\hat{a}) - \frac{1}{\widetilde{\rho}}[B - R^M_{B,\widetilde{\rho}}](\hat{b}) \right\| \\ &\leq \frac{1}{\widetilde{\rho}}\|B(\hat{a}) - B(\hat{b})\| + \frac{1}{\widetilde{\rho}}\|R^M_{B,\widetilde{\rho}}(\hat{a}) - R^M_{B,\widetilde{\rho}}(\hat{b})\|. \end{aligned} \tag{4}$$

Since B is λ_B-Lipschitz continuous, $R^M_{B,\widetilde{\rho}}(\hat{a}) \propto R^M_{B,\widetilde{\rho}}(\hat{b})$, using (*iv*) of Proposition 2.2, we have

$$\|Y^M_{B,\widetilde{\rho}}(\hat{a}) - Y^M_{B,\widetilde{\rho}}(\hat{b})\| \leq \frac{1}{\widetilde{\rho}}\lambda_B\|\hat{a} - \hat{b}\| + \frac{1}{\widetilde{\rho}}\|R^M_{B,\widetilde{\rho}}(\hat{a}) \oplus R^M_{B,\widetilde{\rho}}(\hat{b})\|. \tag{5}$$

Using comparability of $\hat{a}, \hat{b}$ and (2), (5) becomes

$$\begin{aligned} \|Y^M_{B,\widetilde{\rho}}(\hat{a}) - Y^M_{B,\widetilde{\rho}}(\hat{b})\| &\leq \frac{1}{\widetilde{\rho}}\lambda_B\|\hat{a} - \hat{b}\| + \frac{1}{\widetilde{\rho}}\frac{1}{\widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1)}\|\hat{a} \oplus \hat{b}\| \\ &= \frac{1}{\widetilde{\rho}}\lambda_B\|\hat{a} - \hat{b}\| + \frac{1}{\widetilde{\rho}}\frac{1}{\widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1)}\|\hat{a} - \hat{b}\|. \end{aligned}$$

Thus, we have

$$\|Y^M_{B,\widetilde{\rho}}(\hat{a}) - Y^M_{B,\widetilde{\rho}}(\hat{b})\| \leq \frac{1}{\widetilde{\rho}}\left[\lambda_B + \frac{1}{\widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1)}\right]\|\hat{a} - \hat{b}\|,$$

i.e.,

$$\|Y^M_{B,\widetilde{\rho}}(\hat{a}) - Y^M_{B,\widetilde{\rho}}(\hat{b})\| \leq Y(\widetilde{\theta})\|\hat{a} - \hat{b}\|,$$

where $Y(\widetilde{\theta}) = \frac{1}{\widetilde{\rho}}\Big[\lambda_B + \frac{1}{\theta}\Big]$, where $\theta = \widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1), \widetilde{\alpha}_B > \frac{1}{\widetilde{\rho}}$. □

3 Statement of the problem and iterative algorithm

Let $\widehat{H}$ be an ordered real Hilbert space and $A, B, \widetilde{f} : \widehat{H} \to \widehat{H}, N : \widehat{H} \times \widehat{H} \to \widehat{H}$ be the mappings. Suppose $M : \widehat{H} \to 2^{\widehat{H}}$ is the generalized Yosida approximation operator. Let $S, T : \widehat{H} \to \widetilde{C}(\widehat{H})$ be the multi-valued mappings. We study the following problem:

Find $\hat{a} \in \widehat{H}, u \in S(\hat{a}), v \in T(\hat{a})$ such that

$$0 \in A(Y^M_{B,\widetilde{\rho}}(\hat{a})) + N(u, v) \oplus M(\widetilde{f}(\hat{a})). \tag{6}$$

It is easy to obtain many previously studied problem from (6) for suitable choices of operators.

Now, we will show that Problem (6) is equivalent to a fixed point equation.

Lemma 3.1. *Problem* (6) *admits a solution* $\hat{a} \in \widehat{H}, u \in S(\hat{a}), v \in T(\hat{a})$ *if and only if it satisfies the equation*:

$$\widetilde{f}(\hat{a}) = R^M_{B,\widetilde{\rho}}[B(\widetilde{f}(\hat{a})) + \widetilde{\rho}\{A(Y^M_{B,\widetilde{\rho}}(\hat{a})) + N(u, v)\}]. \tag{7}$$

Proof. Let us suppose that $\hat{a} \in \widehat{H}, u \in S(\hat{a}), v \in T(\hat{a})$ satisfies equation (7) and using the definitions of generalized resolvent operator $R^M_{B,\widetilde{\rho}}$, we obtain

$$\widetilde{f}(\hat{a}) = [B + \widetilde{\rho}M]^{-1}[B(\widetilde{f}(\hat{a})) + \widetilde{\rho}\{A(Y^M_{B,\widetilde{\rho}}(\hat{a})) + N(u, v)\}],$$

i.e.,

$$B(\widetilde{f}(\hat{a})) + \widetilde{\rho}M(\widetilde{f}(\hat{a})) = B(\widetilde{f}(\hat{a})) + \widetilde{\rho}\{A(Y^M_{B,\widetilde{\rho}}(\hat{a})) + N(u, v)\},$$
$$M(\widetilde{f}(\hat{a})) = \{A(Y^M_{B,\widetilde{\rho}}(\hat{a})) + N(u, v)\},$$
$$M(\widetilde{f}(\hat{a})) \oplus M(\widetilde{f}(\hat{a})) = A(Y^M_{B,\widetilde{\rho}}(\hat{a})) + N(u, v) \oplus M(\widetilde{f}(\hat{a})).$$

Thus, we have

$$0 \in A(Y^M_{B,\widetilde{\rho}}(\hat{a})) + N(u, v) \oplus M(\widetilde{f}(\hat{a})).$$ □

Based on Lemma 3.1, we define the following iterative algorithm for solving problem (6).

Iterative Algorithm 3.1. For initial vectors $\hat{a}_0 \in \widehat{H}, u_0 \in S(\hat{a}_0)$ and $v_0 \in T(\hat{a}_0)$, let there exist $\hat{a}_1 \in \widehat{H}$ such that

$$\widetilde{f}(\hat{a}_1) = (1 - \alpha)\widetilde{f}(\hat{a}_0) + \alpha R^M_{B,\widetilde{\rho}}[B(\widetilde{f}(\hat{a}_0)) + \widetilde{\rho}\{A(Y^M_{B,\widetilde{\rho}}(\hat{a}_0)) + N(u_0, v_0)\}].$$

By Nadler [33], there exist $u_1 \in S(\hat{a}_1)$ and $v_1 \in T(\hat{a}_1)$ such that

$$||u_1 - u_0|| \le D(S(\widehat{a}_0), S(\hat{a}_1)),$$
$$||v_1 - v_0|| \le D(T(\widehat{a}_0), T(\hat{a}_1)).$$

Inductively, we can obtain the sequences $\{\hat{a}_n\}, \{u_n\}, \{v_n\}$ by the following scheme:

$$\widetilde{f}(\hat{a}_{n+1}) = (1 - \alpha)\widetilde{f}(\hat{a}_n) + \alpha R^M_{B,\widetilde{\rho}}[B(\widetilde{f}(\hat{a}_n)) + \widetilde{\rho}\{A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n)) + N(u_n, v_n)\}], \tag{8}$$

$$u_n \in S(\hat{a}_n), ||u_{n+1} - u_n|| \le D(S(\hat{a}_{n+1}), S(\hat{a}_n)), \tag{9}$$

$$v_n \in T(\hat{a}_n), ||v_{n+1} - v_n|| \le D(T(\hat{a}_{n+1}), T(\hat{a}_n)), \tag{10}$$

where $0 \le \alpha \le 1, \widetilde{\rho} > 0$ is a constant and $n = 0, 1, 2, \ldots$.

4 Existence and convergence result

In this section, we prove an existence and convergence result for the generalized Yosida inclusion problem involving multi-valued operator with XOR operation (6).

Theorem 4.1. *Let $\widehat{H}$ be a real-ordered Hilbert space and $\widetilde{C}_{\widehat{H}}$ be a normal cone in $\widehat{H}$ with constant $\lambda_{\widetilde{N}_{\widehat{H}}} > 0$. Let $A, B, \widetilde{f} : \widehat{H} \to \widehat{H}$ and $N : \widehat{H} \times \widehat{H} \to \widehat{H}$ be the mappings. Let $S, T : \widehat{H} \to \widetilde{C}(\widehat{H}), M : \widehat{H} \to 2^{\widehat{H}}$ be the multi-valued mappings and $Y^M_{B,\widetilde{\rho}} : \widehat{H} \to \widehat{H}$ be the Yosida approximation operator. Suppose that the following conditions are satisfied:*

- (*i*) *The mapping A is Lipschitz continuous with constant λ_A and the mapping B is Lipschitz continuous with constant λ_B.*
- (*ii*) *The mapping $\widetilde{f}$ is Lipschitz continuous with constant $\lambda_{\widetilde{f}}$ and strongly monotone with constant $\delta_{\widetilde{f}}$.*
- (*iii*) *The mapping N is Lipschitz continuous in both arguments with constants λ_{N_1} and λ_{N_2}, respectively.*
- (*iv*) *The generalized resolvent operator $R^M_{B,\widetilde{\rho}}$ is $\frac{1}{\theta}$-Lipschitz continuous.*
- (*v*) *The generalized Yosida approximation operator $Y^M_{B,\widetilde{\rho}}$ is $Y(\widetilde{\theta})$-Lipschitz continuous.*
- (*vi*) *The multi-valued mappings S and T are D-Lipschitz continuous with constants λ_{S_D} and λ_{T_D}, respectively.*

If $\hat{a}_{n+1} \propto \hat{a}_n$, $\widetilde{f}(\hat{a}_{n+1}) \propto \widetilde{f}(\hat{a}_n)$, $B(\widetilde{f}(\hat{a}_{n+1})) \propto B(\widetilde{f}(\hat{a}_n))$, $A(Y^M_{B,\widetilde{\rho}}(\hat{a}_{n+1})) \propto A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n))$, $N(u_n, v_n) \propto N(u_{n-1}, v_{n-1})$ *for* $n = 0, 1, 2, \ldots$ *and the following condition is satisfied:*

$$\theta(1-\alpha)\lambda_{\widetilde{f}} + \alpha[(\lambda_B\lambda_{\widetilde{f}} + \lambda_A Y(\widetilde{\theta})) + \widetilde{\rho}(\lambda_{N_1}\lambda_{S_D} + \lambda_{N_2}\lambda_{T_D})] < \theta\delta_{\widetilde{f}}, \tag{11}$$

where $\theta = \widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1)$, $Y(\widetilde{\theta}) = \frac{1}{\widetilde{\rho}}\left[\lambda_B + \frac{1}{\widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho}-1)}\right]$, $\widetilde{\alpha}_B > \frac{1}{\widetilde{\rho}}$.

Then, there exist $\hat{a} \in \widehat{H}$, $u \in S(\hat{a})$, and $v \in T(\hat{a})$, the solution of the generalized Yosida inclusion problem involving multi-valued operator with XOR operation (6) *and the sequences $\{a_n\}, \{u_n\}$, and $\{v_n\}$ generated by Algorithm* 3.1 *converge strongly to a, u, and v, respectively.*

Proof. Using (8) of iterative Algorithm 3.1 and (*iv*) of Proposition 2.1, we have

$$\begin{aligned}
0 &\le \widetilde{f}(\hat{a}_{n+1}) \oplus \widetilde{f}(\hat{a}_n)\\
&= [(1-\alpha)\widetilde{f}(\hat{a}_n) + \alpha R^M_{B,\widetilde{\rho}}\{B(\widetilde{f}(\hat{a}_n)) + \widetilde{\rho}(A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n)) + N(u_n, v_n))\}]\\
&\quad \oplus [(1-\alpha)\widetilde{f}(\hat{a}_{n-1}) + \alpha R^M_{B,\widetilde{\rho}}\{B(\widetilde{f}(\hat{a}_{n-1})) + \widetilde{\rho}(A(Y^M_{B,\widetilde{\rho}}(\hat{a}_{n-1})) + N(u_{n-1}, v_{n-1}))\}].
\end{aligned}$$

It follows from (*iii*) of Proposition 2.2 and Lipschitz continuity of the generalized resolvent operator $R^M_{B,\widetilde{\rho}}$, that

$$\begin{aligned}
\|\widetilde{f}(\hat{a}_{n+1}) \oplus \widetilde{f}(\hat{a}_n)\| &\le \lambda_{\widetilde{N}_{\widehat{H}}}\|(1-\alpha)(\widetilde{f}(\hat{a}_n) \oplus \widetilde{f}(\hat{a}_{n-1}))\|\\
&\quad + \lambda_{\widetilde{N}_{\widehat{H}}}\|R^M_{B,\widetilde{\rho}}\{B(\widetilde{f}(\hat{a}_n)) + \widetilde{\rho}(A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n)) + N(u_n, v_n))\\
&\quad \oplus R^M_{B,\widetilde{\rho}}\{B(\widetilde{f}(\hat{a}_{n-1})) + \widetilde{\rho}(A(Y^M_{B,\widetilde{\rho}}(\hat{a}_{n-1})) + N(u_{n-1}, v_{n-1}))\}\|\\
&\le \lambda_{\widetilde{N}_{\widehat{H}}}(1-\alpha)\|\widetilde{f}(\hat{a}_n) \oplus \widetilde{f}(\hat{a}_{n-1})\|\\
&\quad + \lambda_{\widetilde{N}_{\widehat{H}}}\alpha\frac{1}{\theta}\|(B(\widetilde{f}(\hat{a}_n)) + \widetilde{\rho}(A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n)) + N(u_n, v_n)))\\
&\quad \oplus (B(\widetilde{f}(\hat{a}_{n-1})) + \widetilde{\rho}(A(Y^M_{B,\widetilde{\rho}}(\hat{a}_{n-1})) + N(u_{n-1}, v_{n-1})))\|\\
&\le \lambda_{\widetilde{N}_{\widehat{H}}}(1-\alpha)\|\widetilde{f}(\hat{a}_n) \oplus \widetilde{f}(\hat{a}_{n-1})\| + \lambda_{\widetilde{N}_{\widehat{H}}}\alpha\frac{1}{\theta}\|B(\widetilde{f}(\hat{a}_n)) \oplus B(\widetilde{f}(\hat{a}_{n-1}))\|\\
&\quad + \lambda_{\widetilde{N}_{\widehat{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\|A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n)) \oplus A(Y^M_{B,\widetilde{\rho}}(\hat{a}_{n-1}))\|\\
&\quad + \lambda_{\widetilde{N}_{\widehat{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\|N(u_n, v_n) \oplus N(u_{n-1}, v_{n-1})\|.
\end{aligned}$$

Since $\widetilde{f}(\hat{a}_n) \propto \widetilde{f}(\hat{a}_{n-1})$, $B(\widetilde{f}(\hat{a}_n)) \propto B(\widetilde{f}(\hat{a}_{n-1}))$, $A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n)) \propto A(Y^M_{B,\widetilde{\rho}}(\hat{a}_{n-1}))$, $N(u_n, v_n) \propto N(u_{n-1}, v_{n-1})$, using (*iv*) of Proposition 2.2, we obtain

$$\begin{aligned}\|\widetilde{f}(\hat{a}_{n+1}) \oplus \widetilde{f}(\hat{a}_n)\| \le{}& \lambda_{\widetilde{N}_{\widetilde{H}}}(1-\alpha)\|\widetilde{f}(\hat{a}_n) - \widetilde{f}(\hat{a}_{n-1})\| + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\|B(\widetilde{f}(\hat{a}_n)) - B(\widetilde{f}(\hat{a}_{n-1}))\| \\ &+ \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\,\|A(Y^M_{B,\widetilde{\rho}}(\hat{a}_n)) - A(Y^M_{B,\widetilde{\rho}}(\hat{a}_{n-1}))\| \\ &+ \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\,\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\|.\end{aligned} \tag{12}$$

Since $\widetilde{f}$ is $\lambda_{\widetilde{f}}$-Lipschitz continuous, B is λ_B-Lipschitz continuous, and A is λ_A-Lipschitz continuous, from (12), we have

$$\begin{aligned}\|\widetilde{f}(\hat{a}_n) \oplus \widetilde{f}(\hat{a}_{n-1})\| \le{}& \lambda_{\widetilde{N}_{\widetilde{H}}}(1-\alpha)\lambda_{\widetilde{f}}\|\hat{a}_n - \hat{a}_{n-1}\| + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\lambda_B\lambda_{\widetilde{f}}\|\hat{a}_n - \hat{a}_{n-1}\| \\ &+ \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\lambda_A\|Y^M_{B,\lambda}(\hat{a}_n) - Y^M_{B,\lambda}(\hat{a}_{n-1})\| \\ &+ \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\,\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\|.\end{aligned} \tag{13}$$

Using Lipschitz continuity of N in both the arguments, we evaluate

$$\begin{aligned}\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\| &= \|N(u_n, v_n) - N(u_{n-1}, v_n) + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\| \\ &\le \|N(u_n, v_n) - N(u_{n-1}, v_n)\| + \|N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\| \\ &\le \lambda_{N_1}\|u_n - u_{n-1}\| + \lambda_{N_2}\|v_n - v_{n-1}\| \\ &\le \lambda_{N_1}D(S(\hat{a}_n), S(\hat{a}_{n-1})) + \lambda_{N_2}D(T(\hat{a}_n), T(\hat{a}_{n-1})) \\ &\le \lambda_{N_1}\lambda_{S_D}\|\hat{a}_n - \hat{a}_{n-1}\| + \lambda_{N_2}\lambda_{T_D}\|\hat{a}_n - \hat{a}_{n-1}\| \\ &= (\lambda_{N_1}\lambda_{S_D} + \lambda_{N_2}\lambda_{T_D})\|\hat{a}_n - \hat{a}_{n-1}\|.\end{aligned} \tag{14}$$

Applying Lemma 2.2 and (14), (13) becomes

$$\begin{aligned}\|\widetilde{f}(\hat{a}_{n+1}) \oplus \widetilde{f}(\hat{a}_n)\| \le{}& \lambda_{\widetilde{N}_{\widetilde{H}}}(1-\alpha)\lambda_{\widetilde{f}}\|\hat{a}_n - \hat{a}_{n-1}\| + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\lambda_B\lambda_{\widetilde{f}}\|\hat{a}_n - \hat{a}_{n-1}\| \\ &+ \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\lambda_A Y(\widetilde{\theta})\|\hat{a}_n - \hat{a}_{n-1}\| + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}(\lambda_{N_1}\lambda_{S_D} + \lambda_{N_2}\lambda_{T_D})\|\hat{a}_n - \hat{a}_{n-1}\| \\ ={}& \left[\lambda_{\widetilde{N}_{\widetilde{H}}}(1-\alpha)\lambda_{\widetilde{f}} + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\lambda_B\lambda_{\widetilde{f}} + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}\lambda_A Y(\widetilde{\theta}) + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}(\lambda_{N_1}\lambda_{S_D} + \lambda_{N_2}\lambda_{T_D})\right]\|\hat{a}_n - \hat{a}_{n-1}\| \\ ={}& \left[\lambda_{\widetilde{N}_{\widetilde{H}}}(1-\alpha)\lambda_{\widetilde{f}} + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}(\lambda_B\lambda_{\widetilde{f}} + \lambda_A Y(\widetilde{\theta})) + \lambda_{\widetilde{N}_{\widetilde{H}}}\alpha\frac{1}{\theta}\widetilde{\rho}(\lambda_{N_1}\lambda_{S_D} + \lambda_{N_2}\lambda_{T_D})\right]\|\hat{a}_n - \hat{a}_{n-1}\| \\ ={}& \lambda_{\widetilde{N}_{\widetilde{H}}}\left[(1-\alpha)\lambda_{\widetilde{f}} + \alpha\frac{1}{\theta}[(\lambda_B\lambda_{\widetilde{f}} + \lambda_A Y(\widetilde{\theta})) + \widetilde{\rho}(\lambda_{N_1}\lambda_{S_D} + \lambda_{N_2}\lambda_{T_D})]\right]\|\hat{a}_n - \hat{a}_{n-1}\|,\end{aligned}$$

where $\theta = \widetilde{\xi}(\widetilde{\alpha}_B\widetilde{\rho} - 1)$, $Y(\widetilde{\theta}) = \frac{1}{\widetilde{\rho}}\left[\lambda_{\widetilde{g}} + \frac{1}{\theta}\right]$ and $\widetilde{\alpha}_B > \frac{1}{\widetilde{\rho}}$.

Thus, we have

$$\|\widetilde{f}(\hat{a}_{n+1}) \oplus \widetilde{f}(\hat{a}_n)\| \le \lambda_{\widetilde{N}_{\widetilde{H}}}\xi(\theta)\|\hat{a}_n - \hat{a}_{n-1}\|,$$

where $\xi(\theta) = (1-\alpha)\lambda_{\widetilde{f}} + \alpha\frac{1}{\theta}[(\lambda_B\lambda_{\widetilde{f}} + \lambda_A Y(\widetilde{\theta})) + \widetilde{\rho}(\lambda_{N_1}\lambda_{S_D} + \lambda_{N_2}\lambda_{T_D})]$.

Since $\widetilde{f}(\hat{a}_{n+1}) \propto \widetilde{f}(\hat{a}_n)$, using (iv) of Proposition 2.2, we have

$$\|\widetilde{f}(\hat{a}_{n+1}) - \widetilde{f}(\hat{a}_n)\| \le \lambda_{\widetilde{N}_{\widetilde{H}}}\xi(\theta)\|\hat{a}_n - \hat{a}_{n-1}\|. \tag{15}$$

As $\widetilde{f}$ is strongly monotone, we have

$$\|\widetilde{f}(\hat{a}_{n+1}) - \widetilde{f}(\hat{a}_n)\| \ge \delta_{\widetilde{f}}\|\hat{a}_{n+1} - \hat{a}_n\|,$$

which implies that

$$\|\hat{a}_{n+1} - \hat{a}_n\| \le \frac{1}{\delta_{\widetilde{f}}}\|\widetilde{f}(\hat{a}_{n+1}) - \widetilde{f}(\hat{a}_n)\|. \tag{16}$$

Combining (15) and (16), we have

$$||\hat{a}_{n+1} - \hat{a}_n|| \le \frac{1}{\delta_{\widetilde{f}}}||\widetilde{f}\,(\hat{a}_{n+1}) - \widetilde{f}\,(\hat{a}_n)|| \le \lambda_{\widetilde{N}_{\widehat{H}}} \nu ||\hat{a}_n - \hat{a}_{n-1}||.$$

Thus, we have

$$||\hat{a}_{n+1} - \hat{a}_n|| \le \lambda_{\widetilde{N}_{\widehat{H}}} \nu^n ||\hat{a}_1 - \hat{a}_0||,$$

where $\nu = \frac{\xi(\theta)}{\delta_{\widetilde{f}}}$.

Hence, for $m > n > 0$, we have

$$||\hat{a}_m - \hat{a}_n|| \le \sum_{i=n}^{m-1} ||\hat{a}_{i+1} - \hat{a}_i|| \le ||\hat{a}_1 - \hat{a}_0|| \sum_{i=n}^{m-1} \nu^i.$$

It follows from Condition (11) that $0 < \nu < 1$, and thus, $||\hat{a}_m - \hat{a}_n|| \to 0$, as $n \to \infty$ and so $\{\hat{a}_n\}$ is a Cauchy sequence in $\widehat{H}$. Since $\widehat{H}$ is complete, there exists $\hat{a}^* \in \widehat{H}$ such that $\hat{a}_n \to \hat{a}^*$, as $n \to \infty$. It is clear from D-Lipschitz continuity of S and T, (9), (10) of Algorithm 3.1 that

$$u_{n+1} \in S(\hat{a}_{n+1}) : ||u_{n+1} - u_n|| \le D(S(\hat{a}_{n+1}), S(\hat{a}_n)) \le \lambda_{S_D} ||\hat{a}_{n+1} - \hat{a}_n||,$$
$$v_{n+1} \in T(\hat{a}_{n+1}) : ||v_{n+1} - v_n|| \le D(T(\hat{a}_{n+1}), T(\hat{a}_n)) \le \lambda_{T_D} ||\hat{a}_{n+1} - \hat{a}_n||,$$

i.e., $\{u_n\}$ and $\{v_n\}$ are also the Cauchy sequences in $\widehat{H}$; thus, there exist u and v such that $u_n \to u$ and $v_n \to v$. This completes the proof. □

5 Conclusion

In this study, we have considered an application-oriented problem, i.e., generalized Yosida inclusion problem involving multi-valued operator with XOR operation. Since Yosida approximation operator and XOR operation both have several applications in modern sciences and technologies, engineers and other scientists can use our results for solving their practical problems. Our results can also be extended in higher-dimensional spaces.

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