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Research Article

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Fuzzy fractional delay integro-differential equation with the generalized Atangana-Baleanu fractional derivative

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Abstract: In this work, we consider a class of fuzzy fractional delay integro-differential equations with the generalized Caputo-type Atangana-Baleanu (ABC) fractional derivative. By using the monotone iterative method, we not only obtain the existence and uniqueness of the solution for the given problem with the initial condition but also give the monotone iteration sequence converging to the unique solution of the problem. Furthermore, we also give the continuous dependence of the unique solution on initial value. Finally, an example is presented to illustrate the main results obtained. The results presented in this study are new and open a new avenue of research for fuzzy fractional delay integro-differential equations with the generalized ABC fractional derivative.

Keywords: fuzzy fractional delay integro-differential equation, generalized Atangana-Baleanu fractional derivative, Monotone iterative method

MSC 2020: 26A33, 34A08

1 Introduction

In the last few decades, the theory and application of fractional calculus have developed and spread to an extent never experienced before. In particular, fractional calculus has been used in modeling several complex phenomena in the fields of science and engineering, such as aerodynamics and control systems, dynamics of earthquakes, material viscoelastic theory, bioengineering, biomathematics, signal processing and so on, please see [1–6] and references therein.

The theory of fuzzy calculus and fuzzy differential equations has become one of the most important subjects in the mathematical analysis area. More specifically, Agarwal et al. [7,8] initiated the concept of fuzzy-type Riemann-Liouville differentiability based on the Hukuhara differentiability. Subsequently, by applying the Hausdorff measure of non-compactness, these authors established some existence results to a class of fuzzy fractional integral equations under appropriate compactness-type conditions. Allahviranloo et al. [9,10] first proposed the concept of the generalized Hukuhara fractional Riemann-Liouville and Caputo differentiability of fuzzy-valued functions, then they used the Mittag-Leffler function to give the analytic

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solutions of the fuzzy fractional differential equations. On the basis of generalized Hukuhara differentiability, many excellent achievements can be found in [11–15].

Recently, based on the power function kernel under the space of the real-valued functions, Caputo and Fabrizio [16] introduced a new concept of fractional derivative in the Caputo background as follows:

$$ {}^{CF}\mathcal{D}^{\rho}_{e^+}\mathcal{M}(t) = \frac{N(\rho)}{1-\rho}\int_e^t \exp\left(-\frac{\gamma}{1-\rho}(t-s)\right)\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}(s)\mathrm{d}s, \tag{1.1} $$

where $\rho \in (0,1)$, $N(\rho)$ is a normalization constant such that $N(0) = N(1) = 1$.

Atangana and Baleanu [17] proposed two generalized fractional derivatives in Caputo and Riemann-Liouville sense. The kernel in the Atangana-Baleanu (AB) fractional derivative is based on the generalized Mittag-Leffler function which is non-singular and non-local. In addition, the AB fractional derivative satisfies all properties of fractional derivatives. Inspired by the above results, Baleanu and Fernandez [18] proposed a new concept of the Caputo-type Atangana-Baleanu (ABC) fractional derivative.

$$ {}^{ABC}\mathcal{D}^{\rho}_{e^+}\mathcal{M}(t) = \frac{N(\rho)}{1-\rho}\int_e^t E_{\rho,1}\left(-\frac{\rho}{1-\rho}(t-s)^{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}(s)\mathrm{d}s, \tag{1.2} $$

where $E_{\rho,1}\left(-\frac{\rho}{1-\rho}(t-s)^{\rho}\right) = \Sigma_{k=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^k \frac{(t-s)^{\rho k}}{\Gamma(\rho k+1)}$ and $\exp(z)$ in (1.1) transform into $E_{\rho,1}(z)$. One can go through the articles [19–25] for more properties about the AB derivative and its applications in various branches of science.

In 2020, Vu et al. [26] introduced the generalized Mittag-Leffler function, which replaced the Mittag-Leffler function kernel in (1.2) as follows:

$$ {}^{ABC}\mathcal{D}^{\rho,h}_{e^+}\mathcal{M}(t) = \frac{N(\rho)}{1-\rho}\int_e^t E_{\rho,1}\left(\frac{\rho}{1-\rho}(h(t)-h(s))^{\rho}\right)\mathcal{M}'(s)\mathrm{d}s. $$

Subsequently, they studied the fuzzy fractional differential equations with the generalized ABC fractional derivative as follows:

$$ \begin{cases} {}^{ABC}\mathcal{D}^{\rho,h}_{e^+}\mathcal{M}(t) = G(t,\mathcal{M}(t)) \quad t \in (e,f], \\ \mathcal{M}(t) = \mathcal{M}(e) \in E, \end{cases} \tag{1.3} $$

where $G : (e,f] \times E \to E$ is continuous, $\rho \in (0,1)$ and E is the space of all fuzzy numbers. Meanwhile, they established the existence and uniqueness results of (1.3) by using the method of successive approximation and the fixed point theorem.

To the best of our knowledge, no one has till now investigated the fuzzy fractional delay integro-differential equation with the generalized ABC fractional derivative. Inspired by the above results, we investigate the following fuzzy fractional delay integro-differential equation:

$$ \begin{cases} {}^{ABC}\mathcal{D}^{\rho,h}_{e^+}\mathcal{M}(t) = F(t,\mathcal{M}(t),\mathcal{M}_t) + \int_e^t G(t,s,\mathcal{M}(s),\mathcal{M}_s)\mathrm{d}s, \quad t \in [e,f], \\ \mathcal{M}(t) = \psi(t-e), \quad t \in [e-\nu, e](\nu > 0), \end{cases} \tag{1.4} $$

where $\rho \in (0,1)$, $\mathcal{Z} = \{(t,s)|e < s \le t \le f\}$, $\psi \in C_\nu$ ($C_\nu = C([-\nu,0],E)$ denotes the space of all continuous mappings from $[-\nu,0]$ to E), $\mathcal{M}_t(s) = \mathcal{M}(t+s)$ for $t \in [e,f]$, $s \in [-\nu,0]$. $F : [e,f] \times E \times C_\nu \to E$, $G : \mathcal{Z} \times E \times C_\nu \to E$ are continuous.

It is well known that the monotone iterative method is found to be an important and efficient tool to obtain monotone iterative sequences of approximate solutions for initial and boundary value problems. For some applications of this technique to nonlinear fractional differential equations, see [27–33]. According to the monotone iterative method, we not only prove the existence and uniqueness of the solution of (1.4) but also provide a monotone iterative sequence of the approximate solution of the initial problem (1.4). Furthermore, we also give the continuous dependence of the unique solution on initial value.

In recent years, due to the unique advantages and irreplaceability of fractional calculus, more and more people have been attracted to study it, although some research results have been achieved, see [34–40]. However, many new research directions have been proposed, such as fuzzy fractional differential equations, fuzzy transformations and pulse control.

Fuzzy fractional differential equation is a fractional differential equation defined on a specific fuzzy set. It has the advantages of non-locality and memory of fractional derivatives, as well as the advantages of describing uncertainty, and has a wide range of applications, and is often used to deal with some problems in dynamical systems affected by imprecision and ambiguity, so it has attracted extensive attention. To the best of our knowledge, no one has investigated the fuzzy fractional delay integro-differential equation with the generalized ABC fractional derivative. Inspired by the above results, we use the monotone iterative method to study fuzzy fractional delay integro-differential equations with generalized ABC derivatives. It is worth pointing out that our approach employed is also novel, the results presented in this work are new and opens a new avenue of research for fuzzy fractional delay integro-differential equation with the generalized ABC fractional derivative.

The work is organized as follows. In Section 2, we recall some basic definitions and notations. In Section 3, we present the existence and uniqueness of the solution of fuzzy fractional delay integro-differential equation with the ABC fractional derivative by using the monotone iterative method. Furthermore, we also provide a monotone iterative sequence of the approximate solution and give the continuous dependence of the unique solution on initial value. In Section 4, as an application, an example is given to illustrate the main results.

2 Some definitions and notations

This section gives some basic definitions and notations to prepare for the subsequent process.

Let $[e,f] \subseteq \mathbb{R}_+$ and $\mathcal{K}^h_{\rho-1}(t-s) = h'(s)(h(t)-h(s))^{\rho-1}$. The space of real-valued continuously differentiable functions is denoted by $C^1([e,f],\mathbb{R})$ on $[e,f]$. Let T_K be a set of real functions $h \in C^1([e,f],\mathbb{R}_+)$, where h is increasing, $h'(t) \neq 0$, $\forall t \in (e,f)$ and $h'(t)$ is positive-defined on (e,f). For the generalization of some other results, see [18].

We denote $L([e,f],E)$ the space of $\mathcal{M} \in E$ which satisfies $D_0[\mathcal{M}(t),\mathbf{0}] \in L^1[e,f]$ and the space of continuous fuzzy functions is denoted by $C([e,f],E)$ on $[e,f]$.

Definition 2.1. [26] Let $\rho \in (0,1)$ and $h \in T_K$. The generalized fuzzy fractional integral concerning the kernel h-function of $\mathcal{M} : [e,f] \to E$ is defined by

$$\mathfrak{I}^{\rho,h}_{e^+}\mathcal{M}(t) = \frac{1}{\Gamma(\rho)}\int_e^t \mathcal{K}^h_{\rho-1}(t-s)\mathcal{M}(s)\mathrm{d}s. \tag{2.1}$$

Definition 2.2. [26] Let $\rho \in (0,1)$, $h \in T_K$ and $\mathcal{M} \in L([e,f],E)$. The generalized Riemann-Liouville-type Atangana-Baleanu (ABR) fractional derivative of $\mathcal{M}$ is defined by

$${}^{ABR}\mathfrak{I}^{\rho,h}_{e^+}\mathcal{M}(t) = \frac{N(\rho)}{(1-\rho)}\frac{1}{h'(t)}\left[\int_e^t h'(s)E_{\rho,1}\left(\frac{\rho}{1-\rho}(h(t)-h(s))^\rho\right)\mathcal{M}(s)\mathrm{d}s\right].$$

Definition 2.3. [26] Suppose that $\rho \in (0,1)$, $h \in T_K$ and $\mathcal{M}$ is d-monotone. If $\mathcal{M}' \in L([e,f],E)$, then the generalized ABC fractional derivative of $\mathcal{M}$ is defined by

$${}^{ABC}\mathcal{D}^{\rho,h}_{e^+}\mathcal{M}(t) = \frac{N(\rho)}{(1-\rho)}\int_e^t E_{\rho,1}\left(\frac{\rho}{1-\rho}(h(t)-h(s))^\rho\right)\mathcal{M}'(s)\mathrm{d}s.$$

Definition 2.4. Let $\rho \in (0,1), h \in T_K$ and $\mathcal{M} \in L([e,f],E)$. The generalized Atangana-Baleanu (GAB) fractional integral of $\mathcal{M}$ is defined by

$$^{AB}\mathfrak{I}_{e^+}^{\rho,h}\mathcal{M}(t) = \frac{1-\rho}{N(\rho)}\mathcal{M}(t) + \frac{\rho}{N(\rho)}\mathfrak{I}_{e^+}^{\rho,h}\mathcal{M}(t).$$

For other definitions and properties of the GAB fractional calculus under the fuzzy background, see [26].

Definition 2.5. [41] A fuzzy number is a fuzzy set $z : \mathbb{R} \to [0,1]$ which satisfies the following conditions (1)–(4):
(1) z is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $z(x_0) = 1$.
(2) z is fuzzy convex in $\mathbb{R}$, i.e., for $0 \le \mu \le 1$, $x_1, x_2 \in \mathbb{R}$

$$z(\mu x_1 + (1-\mu)x_2) \ge \min\{z(x_1), z(x_2)\}.$$

(3) z is upper semicontinuous on $\mathbb{R}$.
(4) $[z]^0 = \{m \in \mathbb{R} | z(m) > 0\}$ is compact.

We denote by E the space of all fuzzy numbers on $\mathbb{R}$ which equipments a metric

$$D_0[z_1, z_2] = \sup_{0\le r\le 1} H([z_1]^r, [z_2]^r), \quad \forall z_1, z_2 \in E,$$

where $H([z_1]^r, [z_2]^r) = \max\{|\underline{z}_1(r) - \underline{z}_2(r)|, |\overline{z}_1(r) - \overline{z}_2(r)|\}$. We denote by (E, D_0) a complete metric space and define $[z]^r = \{m \in \mathbb{R} | z(m) > r\}$ as the r-level of z, with $r \in (0,1]$. Meanwhile, we define the diameter of the r-level set of z as $\mathrm{diam}[z]^r = \overline{z}(r) - \underline{z}(r)$.

A triangular fuzzy number defined as a fuzzy set in E, is specified by an ordered triple $z = (m_1, m_2, m_3)$ with $m_1 \le m_2 \le m_3$ such that $[z]^r = [\underline{z}(r), \overline{z}(r)] = [m_1 + (m_2 - m_1)r, m_3 - (m_3 - m_2)r]$.

Definition 2.6. [42] The generalized Hukuhara difference of two fuzzy numbers $z_1, z_2 \in E$ is defined as follows:

$$z_1 \ominus_{gH} z_2 = z_3 \Leftrightarrow \begin{cases} (i) \quad z_1 = z_2 + z_3, & \text{if } \mathrm{diam}([z_1]^r) \ge \mathrm{diam}([z_2]^r), \\ (ii) \quad z_2 = z_1 + (-1)z_3, & \text{if } \mathrm{diam}([z_1]^r) \le \mathrm{diam}([z_2]^r). \end{cases}$$

Suppose that $d([\mathcal{M}(t)]^r)$ is non-decreasing (non-increasing), then $\mathcal{M} : [e,f] \to E$ is d-increasing (d-decreasing) on $[e,f]$. If $\mathcal{M}$ is d-increasing or d-decreasing on $[e,f]$, then $\mathcal{M}$ is called as d-monotone on $[e,f]$.

For other operation rules and properties of fuzzy fractional calculus on E, please refer to [41–44].

3 Main results

In this section, we establish the existence and uniqueness of the solution of the initial value problem (1.4) and the main results are as follows.

Lemma 3.1. *Let G, H be continuous. A d-monotone fuzzy function $\mathcal{M}$ is a solution of* (1.4) *if and only if $\mathcal{M}$ satisfies*

$$\begin{cases} \mathcal{M}(t) = \psi(t-e), & t \in [e-\nu, e], \\ \mathcal{M}(t) \ominus_{gH} \psi(0) = \dfrac{1-\rho}{N(\rho)}\left[G(t, \mathcal{M}(t), \mathcal{M}_t) + \displaystyle\int_e^t H(t,s,\mathcal{M}(s),\mathcal{M}_s)\mathrm{d}s\right] & \\ + \dfrac{\rho}{N(\rho)\Gamma(\rho)}\displaystyle\int_e^t \mathcal{K}_{\rho-1}^h(t-s)\left[G(s,\mathcal{M}(s),\mathcal{M}_s) + \displaystyle\int_e^s H(s,\tau,\mathcal{M}(\tau),\mathcal{M}_\tau)\mathrm{d}\tau\right]\mathrm{d}s, & t \in [e,f], \end{cases} \tag{3.1}$$

and the fuzzy function ${}^{AB}\mathfrak{I}_{e^+}^{\rho,h}G_{\mathcal{M}}(t)$ is d-increasing on $[e,f]$, where $G_{\mathcal{M}}(t) = G(t, \mathcal{M}(t), \mathcal{M}_t) + \int_e^t H(t, s, \mathcal{M}(s), \mathcal{M}_s)\mathrm{d}s$.

Proof. Let $\mathcal{M}$ be a d-monotone solution of (1.4) and $\mathcal{M}(t) = \mathcal{M}(t) \ominus_{gH} \mathcal{M}(a)$. It ensures that $\mathfrak{U}(t)$ is d-increasing (see Remark 2.3 in [26]). From Theorem 2.12 in [26] and (1.4), we obtain

$$ {}^{AB}\mathfrak{I}_{e^+}^{\rho,h}({}^{ABC}\mathcal{D}_{e^+}^{\rho,h}\mathcal{M})(t) = \mathcal{M}(t) \ominus_{gH} \psi(0) = \mathcal{M}(t) \ominus_{gH} \mathcal{M}(a). \tag{3.2} $$

According to (1.4) and the continuity of the functions G and H, we obtain

$$ \begin{aligned} {}^{AB}\mathfrak{I}_{e^+}^{\rho,h}({}^{ABC}\mathcal{D}_{e^+}^{\rho,h}\mathcal{M})(t) &= {}^{AB}\mathfrak{I}_{e^+}^{\rho,h}\Big[G(t, \mathcal{M}(t), \mathcal{M}_t) + \int_e^t H(t, s, \mathcal{M}(s), \mathcal{M}_s)\mathrm{d}s\Big] \\ &= \frac{1-\rho}{N(\rho)}G_{\mathcal{M}}(t) + \frac{\rho}{N(\rho)\Gamma(\rho)}\int_e^t \mathcal{K}_{\rho-1}^{h}(t-s)G_{\mathcal{M}}(s)\mathrm{d}s. \end{aligned} \tag{3.3} $$

Hence, we have

$$ \mathcal{M}(t) \ominus_{gH} \psi(0) = \mathcal{M}(t) \ominus_{gH} \mathcal{M}(a) = \frac{1-\rho}{N(\rho)}G_{\mathcal{M}}(t) + \frac{\rho}{N(\rho)\Gamma(\rho)}\int_e^t \mathcal{K}_{\rho-1}^{h}(t-s)G_{\mathcal{M}}(s)\mathrm{d}s, \tag{3.4} $$

i.e., $\mathcal{M}$ satisfies (3.1). Further, since $\mathcal{M}(t)$ is d-increasing, ${}^{AB}\mathfrak{I}_{e^+}^{\rho,h}G_{\mathcal{M}}(t)$ is also d-increasing on $[e,f]$.

Conversely, we suppose that $\mathcal{M}$ is d-monotone function satisfying (3.1) and ${}^{AB}\mathfrak{I}_{e^+}^{\rho,h}G_{\mathcal{M}}(t)$ is d-increasing, then by using operator ${}^{ABR}\mathcal{D}_{e^+}^{\rho,h}$ and Proposition 2.1 in [26], we obtain

$$ {}^{ABR}\mathcal{D}_{e^+}^{\rho,h}[\mathcal{M}(\cdot) \ominus_{gH} \mathcal{M}(a)](t) = G(t, \mathcal{M}(t), \mathcal{M}_t) + \int_e^t H(t, s, \mathcal{M}(s), \mathcal{M}_s)\mathrm{d}s = G_{\mathcal{M}}(t). \qquad \square $$

Next we define the set $B_\eta = B(\eta, \mathcal{M}^0) = \{\mathcal{M} \in C([e,f], E) : D_0[\mathcal{M}(t), \mathcal{M}^0] \le \eta\}, \eta > 0$.

Theorem 3.1. *Let $h \in T_K$ be bijective and the following conditions hold:*

(N1) *There exist the bounded integrable functions $L_1(t)$ and $P_1(t)$, such that*

$$ D_0[G(t, X(t), X_t), G(t, Y(t), Y_t)] \le L_1(t)D_0[X(t), Y(t)], \quad t \in [e,f] \tag{3.5} $$

and

$$ D_0[G(t, X(t), X_t), \mathbf{0}] \le P_1(t), \quad t \in [e,f], \tag{3.6} $$

for $G : [e,f] \times B_\eta \times B_\eta \to E$, $X(t) \ge Y(t)$, $X_t \ge Y_t$.

(N2) *There exist the bounded integrable functions $L_2(s)$ and $P_2(s)$, such that*

$$ D_0[H(t, s, X(s), X_s), H(t, s, Y(s), Y_s)] \le L_2(s)D_0[X(s), Y(s)], \quad (t,s) \in \mathcal{Z} \tag{3.7} $$

and

$$ D_0[H(t, s, X(s), X_s), \mathbf{0}] \le P_2(s), \quad (t,s) \in \mathcal{Z}, \tag{3.8} $$

for $H : \mathcal{Z} \times B_\eta \times B_\eta \to E$, $X(s) \ge Y(s)$, $X_s \ge Y_s$.

Then, the equation (1.4) *has a unique solution on $[e,\tilde{f}]$ provided that*

$$ h(t) - h(e) < \left(\frac{N(\rho)\Gamma(\rho)}{L_1 + L_2(e-f)} - \Gamma(\rho)(1-\rho)\right)^{1/\rho}, \quad t \in [e,\tilde{f}], \tag{3.9} $$

where

$$ \tilde{f} := \min\left\{f, h^{-1}\left(h(e) + \left[\frac{\eta\Gamma(\rho)N(\rho)}{P_1 + (e-f)P_2} + \Gamma(\rho)(\rho-1)\right]^{1/\rho}\right)\right\}. \tag{3.10} $$

Moreover, the unique solution can be given by the following iterative sequence:

$$\mathcal{M}^n(t) \ominus_{gH} \psi(0) = \frac{1-\rho}{N(\rho)} G_{U^{n-1}}(t) + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}^h_{\rho-1}(t-s) G_{U^{n-1}}(s) \mathrm{d}s, \quad t \in [e, \tilde{f}].$$

Proof. To prove the theorem, we will divide it into two steps as follows:

Step 1: Suppose that $\mathfrak{B}$ be a set of continuous functions such that $\mathcal{M}(t) = \psi(t-e)$ and $\mathcal{M} \in B_\eta$. We note that the sequence $\{\mathcal{M}^n\}$ is given by $\mathcal{M}^0(t) = \mathcal{M}^0$, where

$$\begin{cases} \mathcal{M}^0(t) = \psi(t-e), & t \in [e-\nu, e], \\ \mathcal{M}^0(t) = \psi(0), & t \in [e, \tilde{f}], \end{cases} \tag{3.11}$$

and

$$\begin{cases} \mathcal{M}^n(t) = \psi(t-e), & t \in [e-\nu, e], \\ \mathcal{M}^n(t) \ominus_{gH} \psi(0) = \dfrac{1-\rho}{N(\rho)} G_{U^{n-1}}(t) + \dfrac{\rho}{N(\rho)\Gamma(\rho)} \displaystyle\int_e^t \mathcal{K}^h_{\rho-1}(t-s) G_{\mathcal{M}^{n-1}}(s) \mathrm{d}s, & t \in [e, \tilde{f}], \end{cases} \tag{3.12}$$

where for $n = 1, 2, 3, \ldots$, $G_{\mathcal{M}^{n-1}}(t) = G(t, \mathcal{M}^{n-1}(t), \mathcal{M}^{n-1}_t) + \int_e^t H(t, s, \mathcal{M}^{n-1}(s), \mathcal{M}^{n-1}_s) \mathrm{d}s$. For $n \geq 0$, we have $\mathcal{M}^n(t) \in B_\eta$ if and only if $\mathcal{M}^n(t) \ominus_{gH} \psi(0) \in B(\eta, 0)$. By mathematical induction, we will prove $\mathcal{M}^n(t) \in \mathfrak{B}$. Obviously, when $n = 0$, $\mathcal{M}^0(t) \in \mathfrak{B}$. Now, we suppose $\mathcal{M}^{n-1}(t) \in \mathfrak{B}$ and for $n \geq 2$,

$$\begin{aligned} D_0[\mathcal{M}^n(t), \mathcal{M}^0(t)] &= \frac{1-\rho}{N(\rho)} D_0[G_{\mathcal{M}^{n-1}}(t), \mathbf{0}] + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}^h_{\rho-1}(t-s) D_0[G_{\mathcal{M}^{n-1}}(s), \mathbf{0}] \mathrm{d}s \\ &= \frac{1-\rho}{N(\rho)} \left(D_0[G(t, \mathcal{M}^{n-1}(t), \mathcal{M}^{n-1}_t), \mathbf{0}] + \int_e^t D_0[H(t, s, \mathcal{M}^{n-1}(s), \mathcal{M}^{n-1}_s), \mathbf{0}] \mathrm{d}s \right) \\ &\quad + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}^h_{\rho-1}(t-s) \left(D_0[G(t, \mathcal{M}^{n-1}(t), \mathcal{M}^{n-1}_t), \mathbf{0}] \right. \\ &\quad \left. + \int_e^s D_0[G(s, \tau, \mathcal{M}^{n-1}(\tau), \mathcal{M}^{n-1}_\tau), \mathbf{0}] \mathrm{d}\tau \right) \mathrm{d}s \\ &\leq \frac{1-\rho}{N(\rho)} [P_1 + (e-f)P_2] + \frac{(h(\tilde{b}) - h(e))^\rho}{N(\rho)\Gamma(\rho)} P_1 \\ &\quad + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t h'(s)(h(\tilde{b}) - h(e))^{\rho-1} \int_e^t P_2(\tau) \mathrm{d}\tau \mathrm{d}s \\ &\leq \frac{1-\rho}{N(\rho)} [P_1 + (e-f)P_2] + \frac{(h(\tilde{b}) - h(e))^\rho}{N(\rho)\Gamma(\rho)} [P_1 + (e-f)P_2] \\ &= \left[\frac{1-\rho}{N(\rho)} + \frac{(h(\tilde{b}) - h(e))^\rho}{N(\rho)\Gamma(\rho)} \right] [P_1 + (e-f)P_2] \\ &\leq \eta, \end{aligned} \tag{3.13}$$

where $L_1, L_2, P_1, P_2 > 0$. Therefore, we obtain $D_0 = [\mathcal{M}^n(t), \mathcal{M}^0(t)] \leq \eta$, which implies $U^n(t) \in \mathfrak{B}$ and $\{\mathcal{M}^n\}$ is well-defined.

In addition, applying induction, for $n = 1$, we have

$$D_0[\mathcal{M}^1(t), \mathcal{M}^0(t)] \leq \left[\frac{1-\rho}{N(\rho)} + \frac{(h(t) - h(e))^{\rho-1}}{N(\rho)\Gamma(\rho)} \right] P^*,$$

where $P^* = P_1 + (e - f)P_2$. For $n = k$, we assume

$$D_0 = [\mathcal{M}^k(t), \mathcal{M}^{k-1}(t)] \le P^* \left[\frac{1-\rho}{N(\rho)} + \frac{(h(t) - h(e))^{\rho-1}}{N(\rho)\Gamma(\rho)}\right]^k \tilde{L}^{k-1},$$

where $\tilde{L} = L_1 + L_2(e - f)$. When $n = k + 1$, we have

$$\begin{aligned}
D_0[\mathcal{M}^{k+1}(t), \mathcal{M}^k(t)] \le{}& \frac{1-\rho}{N(\rho)}\Bigg(D_0[G(t, \mathcal{M}^k(t), \mathcal{M}_t^k), G(t, \mathcal{M}^{k-1}(t), \mathcal{M}_t^{k-1})] \\
&+ \int_e^t D_0[H(t, s, \mathcal{M}^k(s), \mathcal{M}_s^k), H(t, s, \mathcal{M}^{k-1}(s), \mathcal{M}_s^{k-1})]\mathrm{d}s\Bigg) \\
&+ \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}_{\rho-1}^h(t-s)\Bigg(D_0[G(t, \mathcal{M}^k(s), \mathcal{M}_s^k), G(t, \mathcal{M}^{k-1}(s), \mathcal{M}_s^{k-1})] \\
&+ \int_e^t D_0[H(s, \tau, \mathcal{M}^k(\tau), \mathcal{M}_\tau^k), H(t, s, \mathcal{M}^{k-1}(\tau), \mathcal{M}_\tau^{k-1})]\mathrm{d}\tau\Bigg)\mathrm{d}s \\
\le{}& \frac{1-\rho}{N(\rho)}(L_1(t)D_0[\mathcal{M}^k(t), \mathcal{M}^{k-1}(t)] + L_2(e-f)D_0[\mathcal{M}^k(t), \mathcal{M}^{k-1}(t)]) \\
&+ \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}_{\rho-1}^h(t-s)(L_1(t)D_0[\mathcal{M}^k(s), \mathcal{M}^{k-1}(s)] \\
&+ L_2(e-f)D_0[\mathcal{M}^k(s), \mathcal{M}^{k-1}(s)])\mathrm{d}s \\
\le{}& \tilde{L}\frac{1-\rho}{N(\rho)}D_0[\mathcal{M}^k(t), \mathcal{M}^{k-1}(t)] + \tilde{L}\frac{\rho}{N(\rho)\Gamma(\rho)}\int_e^t \mathcal{K}_{\rho-1}^h(t-s)D_0[\mathcal{M}^k(s), \mathcal{M}^{k-1}(s)]\mathrm{d}s \\
\le{}& P^*\frac{1-\rho}{N(\rho)}\left[\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^{\rho-1}}{N(\rho)\Gamma(\rho)}\right]^k \tilde{L}^k \\
&+ P^*\tilde{L}^k \frac{\rho}{N(\rho)\Gamma(\rho)}\int_e^t \mathcal{K}_{\rho-1}^h(t-s)\left[\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^{\rho-1}}{N(\rho)\Gamma(\rho)}\right]^k ds \\
\le{}& P^*\left[\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^{\rho-1}}{N(\rho)\Gamma(\rho)}\right]^{k+1} \tilde{L}^k,
\end{aligned} \tag{3.14}$$

where h is increasing. Hence, for $n = 1, 2, 3 \cdots$, we have

$$D_0[\mathcal{M}^n(t), \mathcal{M}^{n-1}(t)] \le P^*\left[\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^{\rho-1}}{N(\rho)\Gamma(\rho)}\right]^n \tilde{L}^{n-1}. \tag{3.15}$$

From (3.15), we can obtain

$$\begin{aligned}
D_0[\mathcal{M}^{n+1}(t), \mathcal{M}^n(t)] ={}& \tilde{L}\frac{1-\rho}{N(\rho)}D_0[\mathcal{M}^n(t), \mathcal{M}^{n-1}(t)] + \tilde{L}\frac{\rho}{N(\rho)\Gamma(\rho)}\int_e^t \mathcal{K}_{\rho-1}^h(t-s)D_0[\mathcal{M}^n(s), \mathcal{M}^{n-1}(s)]\mathrm{d}s \\
\le{}& \tilde{L}\frac{1-\rho}{N(\rho)}D_0[\mathcal{M}^n(n), \mathcal{M}^{n-1}(t)] + \tilde{L}\frac{(h(t)-h(e))^\rho}{N(\rho)\Gamma(\rho)}D_0[\mathcal{M}^n(t), \mathcal{M}^{n-1}(t)] \\
={}& \tilde{L}\left(\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^\rho}{N(\rho)\Gamma(\rho)}\right)D_0[\mathcal{M}^n(t), \mathcal{M}^{n-1}(t)] \\
={}& PD_0[\mathcal{M}^n(t), \mathcal{M}^{n-1}(t)] \\
\le{}& P^2D_0[\mathcal{M}^{n-1}(t), \mathcal{M}^{n-2}(t)] \\
\le{}& \cdots \\
\le{}& P^nD_0[\mathcal{M}^1(t), \mathcal{M}^0(t)],
\end{aligned} \tag{3.16}$$

where $P \equiv \tilde{L}\left(\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^{\rho}}{N(\rho)\Gamma(\rho)}\right)$. It is obvious that $P \in (0,1)$.

Thus, for any positive integers m and n, one obtains

$$\begin{aligned} D_0[\mathcal{M}^{n+m}(t), \mathcal{M}^n(t)] &\le D_0[\mathcal{M}^{n+m}(t), \mathcal{M}^{n-m-1}(t)] + D_0[\mathcal{M}^{n+m-1}(t), \mathcal{M}^{n+m-2}(t)] \\ &\quad + \cdots + D_0[\mathcal{M}^{n+1}(t), \mathcal{M}^n(t)] \\ &\le (P^{n+m-1} + P^{n+m-2} + \cdots + P^n) D_0[\mathcal{M}^1(t), \mathcal{M}^0(t)] \\ &\le \frac{P^n}{1-P} D_0[\mathcal{M}^1(t), \mathcal{M}^0(t)]. \end{aligned} \tag{3.17}$$

According to (3.17), we can show that $\{\mathcal{M}^n\}$ is a Cauchy sequence. Then, we have $\mathcal{M}^n \to \tilde{\mathcal{M}}$ and $D_0[\mathcal{M}^{n+m}(t), \mathcal{M}^n(t)] \to 0$ as $n \to \infty$.

Next let $F_{\mathcal{M}^n}(t) = G(t, \mathcal{M}^n(t), \mathcal{M}_t^n) + \int_e^t H(t, s, \mathcal{M}^n(s), \mathcal{M}_s^n)ds$, we shall show $G_{\mathcal{M}^n} \to G_{\tilde{\mathcal{M}}}$ uniformly on $[e, \tilde{f}]$. Since $\mathcal{M}^n \to \tilde{\mathcal{M}} \in C([e, \tilde{f}], B_\eta)$ as $n \to \infty$ for $\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$ such that

$$D_0[\mathcal{M}^n(t), \tilde{\mathcal{M}}(t)] < \frac{\varepsilon}{\tilde{L}}, n \ge N(\varepsilon).$$

Hence, we have

$$D_0[F_{\mathcal{M}^n}(t), F_{\tilde{\mathcal{M}}}(t)] \le \tilde{L} D_0[\mathcal{M}^n(t), \tilde{\mathcal{M}}(t)] < \varepsilon, n \ge N(\varepsilon),$$

which implies $G_{\mathcal{M}^n} \to G_{\tilde{\mathcal{M}}}$ uniformly on $[e, \tilde{f}]$. Further, from (3.12), we can obtain

$$\begin{aligned} &D_0\left[\mathcal{M}^n(t) \ominus_{gH} \mathcal{M}^0(t), \frac{1-\rho}{N(\rho)} G_{\tilde{\mathcal{M}}}(t) + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}_{\rho-1}^h(t-s) G_{\tilde{\mathcal{M}}}(s)ds\right] \\ &\le \frac{1-\rho}{N(\rho)} D_0[G_{\mathcal{M}^n}(t), G_{\tilde{\mathcal{M}}}(t)] + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}_{\rho-1}^h(t-s) D_0[G_{\mathcal{M}^n}(s), G_{\tilde{\mathcal{M}}}(s)]ds. \end{aligned} \tag{3.18}$$

In view of $G_{\mathcal{M}^n} \to G_{\tilde{\mathcal{M}}}$, the right-hand side of (3.18) converges to 0 as $n \to \infty$. Hence, taking the limit in (3.12) as $n \to \infty$, one obtains

$$\tilde{\mathcal{M}}(t) \ominus_{gH} U^0(t) = \frac{1-\rho}{N(\rho)} G_{\tilde{\mathcal{M}}}(s) + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}_{\rho-1}^h(t-s) G_{\tilde{\mathcal{M}}}(s)ds. \tag{3.19}$$

Step 2: We shall show the uniqueness of the solution. Suppose that $\tilde{\mathcal{M}}$ and $\tilde{\sigma}$ are two solutions of (1.4) such that $\tilde{\mathcal{M}}(a) = \tilde{\sigma}(a)$. Let

$$D[\tilde{\mathcal{M}}, \tilde{\sigma}] = \sup_{a \le t \le \tilde{b}} D_0[\tilde{\mathcal{M}}(t), \tilde{\sigma}(t)].$$

Then, from (3.5) and (3.7), we obtain

$$\begin{aligned} D[\tilde{\mathcal{M}}, \tilde{\sigma}] &\le \tilde{L} \frac{1-\rho}{N(\rho)} D[\tilde{\mathcal{M}}, \tilde{\sigma}] + \tilde{L} \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}_{\rho-1}^h(t-s) D[\tilde{\mathcal{M}}, \tilde{\sigma}]ds \\ &= \tilde{L} \frac{1-\rho}{N(\rho)} D[\tilde{\mathcal{M}}, \tilde{\sigma}] + \tilde{L} \frac{(h(t)-h(e))^\rho}{N(\rho)\Gamma(\rho)} D[\tilde{\mathcal{M}}, \tilde{\sigma}] \\ &= \tilde{L}\left(\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^\rho}{N(\rho)\Gamma(\rho)}\right) D[\tilde{\mathcal{M}}, \tilde{\sigma}]. \end{aligned}$$

So, we know that

$$\left[1 - \tilde{L}\left(\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^\rho}{N(\rho)\Gamma(\rho)}\right)\right] D[\tilde{\mathcal{M}}, \tilde{\sigma}] \le 0.$$

According to (3.9), it is easy to obtain $D[\tilde{\mathcal{M}}, \tilde{\sigma}] = 0$. Thus, $\tilde{\mathcal{M}}(t) = \tilde{\sigma}(t)$. □

Based on Theorem 3.1, we give the continuous dependence of the solution.

Theorem 3.2. *Suppose that σ is a solution of the following equation*:

$$\begin{cases} {}^{ABC}\mathcal{D}_{e^+}^{\rho,h}\sigma(t) = G(t,\sigma(t),\sigma_t) + \int_e^t H(t,s,\sigma(s),\sigma_s)\mathrm{d}s, & t \in [e,f], \\ \sigma(t) = \psi(t-e), & t \in [e-\nu, e], \end{cases} \tag{3.20}$$

where $\mathcal{M}(a) \neq \sigma(a)$, then the following estimate holds:

$$\sup_{a\le t\le \tilde{b}} D_0[\mathcal{M}(t),\sigma(t)] \le \frac{1}{1-P} D_0[\mathcal{M}(a),\sigma(a)]. \tag{3.21}$$

Proof. Suppose that $\tilde{\varrho}(t)$ is a solution of (3.20) and $\tilde{\varrho}(a) = \sigma^0(t) = \sigma(a)$. Similar to **Step 1** in Theorem 3.1, we can find

$$\varrho(t) \ominus_{gH} \sigma^0(t) = \frac{1-\rho}{N(\rho)} G_{\tilde{\varrho}}(t) + \frac{\rho}{N(\rho)\Gamma(\rho)} \int_e^t \mathcal{K}_{\rho-1}^h(t-s) G_{\tilde{\varrho}}(s)\mathrm{d}s.$$

Similar to **Step 2** in Theorem 3.1, one obtains

$$D[\tilde{\mathcal{M}},\tilde{\varrho}] \le D_0[\mathcal{M}^0(t),\sigma^0(t)] + \tilde{L}\left[\frac{1-\rho}{N(\rho)} + \frac{(h(t)-h(e))^\rho}{N(\rho)\Gamma(\rho)}\right] D[\tilde{\mathcal{M}},\tilde{\varrho}],$$

which implies

$$D[\tilde{\mathcal{M}},\tilde{\varrho}] \le \frac{1}{1-p} D_0[\mathcal{M}^0(t),\sigma^0(t)] = \frac{1}{1-p} D_0[\mathcal{M}(a),\sigma(a)],$$

which means (3.21) holds. □

4 An illustrative example

In this section, we will present an illustrative example.

Example 4.1. Suppose that we have the following fuzzy fractional integral differential equation with the generalized ABC fractional derivative:

$$\begin{cases} {}^{ABC}\mathcal{D}_{e^+}^{\rho,h}\mathcal{M}(t) = \frac{t^6 - 8t^3 e + 4e^2}{4}(c_1,c_2,c_3)(t^3-e)^\gamma + \frac{3t^6}{4}\mathcal{M}(t) - \frac{2((t-1)^3 - e^3)}{\rho+1}\mathcal{M}_t \\ \quad + \int_e^t \left[\frac{3s^2}{5}(c_1,c_2,c_3)(s-e)^\gamma + \frac{2s^2}{5}\mathcal{M}(s) + 6(s-1)^2\mathcal{M}_s\right]\mathrm{d}s, & t \in [0,1], \\ \mathcal{M}(t) = \psi(t), & t \in [-1,0], \end{cases} \tag{4.1}$$

and where $h(t) = t^3$, $\mathcal{M}(t) = (c_1,c_2,c_3)(h(t)-h(e))^\gamma$, $\mathcal{M}_t = (c_1,c_2,c_3)(h(t-1)-h(e))^\gamma$, $h \in T_K$, $\gamma \ge 0$, $e \ge 0$, $c_1 < c_2 < c_3$. Obviously,

$$\begin{cases} G(t,\mathcal{M}(t),\mathcal{M}_t) = \frac{t^6 - 8t^3 e + 4e^2}{4}(c_1,c_2,c_3)(t^3-e)^\gamma + \frac{3t^6}{4}\mathcal{M}(t) - \frac{2((t-1)^3-e^3)}{\gamma+1}\mathcal{M}_t, \\ \int_e^t H(t,s,\mathcal{M}(s),\mathcal{M}_s)\mathrm{d}s = \int_e^t \left[\frac{3s^2}{5}(c_1,c_2,c_3)(s-e)^\gamma + \frac{2s^2}{5}\mathcal{M}(s) + 6(s-1)^2\mathcal{M}_s\right]\mathrm{d}s. \end{cases}$$

Furthermore, it is easy to obtain

$$D_0[G(t,\mathcal{M}(t),\mathcal{M}_t), G(t,\sigma(t),\sigma_t)] \le \frac{3t^6}{4} D_0[\mathcal{M}(t),\sigma(t)]$$

and

$$D_0[G(t,\mathcal{M}(t),\mathcal{M}_t),\mathbf{0}] \le [c_3+(c_3-c_2)r]\left(t^{3\gamma+6}-\frac{2}{\rho+1}(t-1)^{3\gamma}\right),$$

where $\mathcal{M}(t)\ge\sigma(t)$, $\sigma_t\ge\mathcal{M}_t$, $L_1(t)=\frac{3t^6}{4}$, and $P_1(t)=[c_3+(c_3-c_2)r]\left(t^{3\gamma+6}-\frac{2}{\rho+1}(t-1)^{3\gamma}\right)$.

We also have

$$D_0[H(t,s,\mathcal{M}(s),\mathcal{M}_s),H(t,s,\sigma(s),\sigma_s)] \le \frac{2s^2}{5}D_0[\mathcal{M}(s),\sigma(s)]$$

and

$$D_0[H(t,s,\mathcal{M}(s),\mathcal{M}_s),\mathbf{0}] \le [c_3+(c_3-c_2)r](s^{6\gamma}-6(s-1)^{6\gamma}),$$

where $\mathcal{M}(s)\ge\sigma(s)$, $\sigma_s\ge\mathcal{M}_s$, $L_2(s)=\frac{2s^2}{5}$, and $P_2(s)=[c_3+(c_3-c_2)r](s^{6\gamma}-6(s-1)^{6\gamma})$. Then, conditions $(N1)$ and $(N2)$ hold.

Next we consider that

$$\begin{aligned}
{}^{AB}\mathfrak{J}^{\rho,h}_{e^+}({}^{ABC}\mathcal{D}^{\rho,h}_{e^+}\mathcal{M}(t)) &= (c_1,c_2,c_3)\left\{\frac{1-\rho}{N(\rho)}\left[\frac{t^6-8t^3e+4e^2}{4}(t^3-e)^\gamma+\frac{3t^6}{4}\mathcal{M}(t)\right.\right.\\
&\quad \left.-\frac{2((t-1)^3-e^3)}{\rho+1}\mathcal{M}_t+\int_e^t\left[\frac{3s^2}{5}(s-e)^\gamma+\frac{2s^2}{5}\mathcal{M}(s)+6(s-1)^2\mathcal{M}_s\right]ds\right]\\
&\quad +\frac{\rho}{N(\rho)\Gamma(\rho)}\int_e^t\mathcal{K}^h_{\rho-1}(t-s)\left[\frac{s^6-8s^3e+4e^2}{4}(s^3-e)^\gamma+\frac{3s^6}{4}\mathcal{M}(s)\right.\\
&\quad \left.\left.-\frac{2((s-1)^3-e^3)}{\gamma+1}\mathcal{M}_s+\int_e^s\left[\frac{3\tau^2}{5}(\tau-e)^\gamma+\frac{2\tau^2}{5}\mathcal{M}(\tau)+6(\tau-1)^2\mathcal{M}_\tau\right]\right]ds\right\}\\
&= (c_1,c_2,c_3)\left\{\frac{1-\rho}{N(\rho)}\left[t^{3\gamma+6}+\frac{1}{3(\gamma+1)}t^{3\gamma+3}\right]+\frac{4\rho}{3N(\rho)}\frac{\Gamma(\gamma+2)(t)^{\rho+\gamma+1}}{\Gamma(\rho+\gamma+2)}\right\}.
\end{aligned} \tag{4.2}$$

Thus, using Lemma 3.1, we can obtain the solution of (4.1)

$$\mathcal{M}(t)\ominus_{gH}\psi(0)=(c_1,c_2,c_3)\left\{\frac{1-\rho}{N(\rho)}\left[t^{3\gamma+6}+\frac{1}{3(\gamma+1)}t^{3\gamma+3}\right]+\frac{4\rho}{3N(\rho)}\frac{\Gamma(\gamma+2)(t)^{\rho+\gamma+1}}{\Gamma(\rho+\gamma+2)}\right\}. \tag{4.3}$$

The d-monotone solution of (4.1) on $[0,1]$ is illustrated in Figures 1 and 2, where $N(\rho)=1-\rho+\frac{\rho}{\Gamma(\rho)}$, $(c_1,c_2,c_3)=(1,1.5,2)$, $\psi(0)=(1,2,3)$ and $[e,f]=[0,1]$.

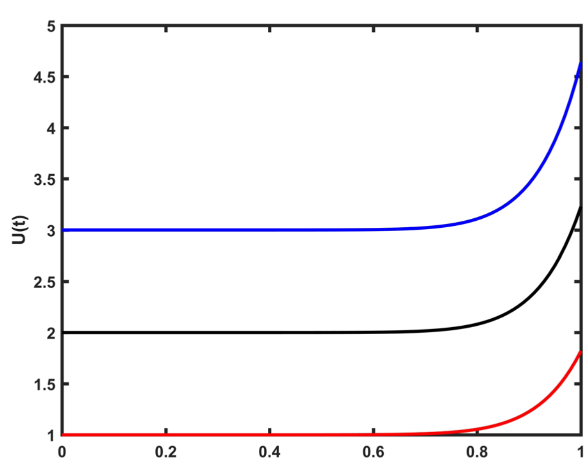

Figure 1: Plot of the d-increasing solutions of (4.1) for $\rho=0.5$, $\gamma=2$ and $t\in[0,1]$.

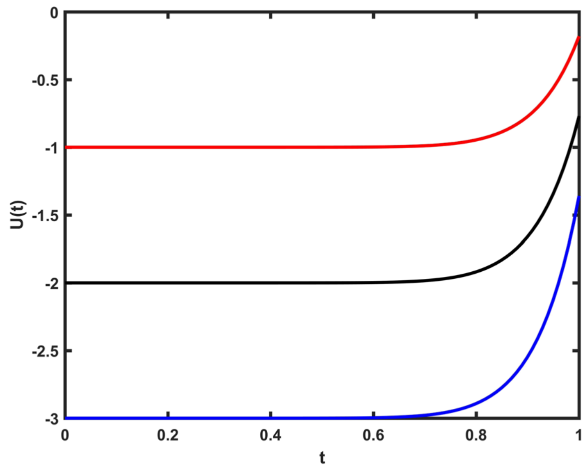

Figure 2: Plot of the d-decreasing solutions of (4.1) for $\rho = 0.5$, $\gamma = 2$ and $t \in [0, 1]$.

5 Conclusion

In this work, we consider a class of fuzzy fractional delay integro-differential equations with the generalized ABC fractional derivative. By using the monotone iterative method, we not only obtain the uniqueness and existence of the solution for the given problem with initial condition, but also give the monotone iterative sequence converging to the unique solution of the problem. Furthermore, we also give the continuous dependence of the unique solution on initial value. Finally, an example is presented to illustrate the main results obtained. The results presented in this work are new and opens a new avenue of research for fuzzy fractional delay integro-differential equation with the generalized ABC fractional derivative. We believe that the research in this work will be useful for the investigation of topics related to the fuzzy fractional delay integro-differential equation.

Look forward to the future. Although there have been a lot of excellent results on the initial (boundary) value problem of fractional differential equations, there is not much work on the initial value problem of fuzzy fractional differential equations. In particular, there has not been a lot of research and development on the boundary value problem. In this work, we mainly study the uniqueness and existence of the solution of a class of fuzzy fractional delay integro-differential equations. There are many interesting topics worth exploring in the future. For example, one can study the stability and controllability of fuzzy fractional delay integro-differential equation, and also try to apply deep learning methods to find deep learning solutions of such equations. We believe that further research on fuzzy fractional delay integro-differential equation will lead to many practical applications.

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