Research Article

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On Kantorovich variant of Brass-Stancu operators#

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Abstract: In this study, we deal with Kantorovich-type generalization of the Brass-Stancu operators. For the sequence of these operators, we study L^p-convergence and give some upper estimates for the L^p-norm of the approximation error via first-order averaged modulus of smoothness and the first-order K-functional. Moreover, we show that the Kantorovich generalization of each Brass-Stancu operator satisfies variation detracting property and is bounded with respect to the norm of the space of functions of bounded variation on $[0, 1]$. Finally, we present graphical and numerical examples to compare the convergence of these operators to given functions under different parameters.

Keywords: Brass-Stancu-Kantorovich operators, L^p-convergence, averaged modulus of smoothness, K-functional, variation detracting property

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1 Introduction

For a non-negative integer parameter r, and any natural number n such that $n > 2r$, Stancu generalized Bernstein's fundamental polynomial $p_{n,k}(x)$ is given by

$$p_{n,k}(x) = \begin{cases} \binom{n}{k} x^k(1-x)^{n-k}; & 0 \le k \le n \\ 0; & k < 0 \text{ or } k > n \end{cases}, \quad x \in [0,1], \tag{1}$$

as

$$w_{n,k,r}(x) := \begin{cases} (1-x)p_{n-r,k}(x); & 0 \le k < r \\ (1-x)p_{n-r,k}(x) + xp_{n-r,k-r}(x); & r \le k \le n-r, \\ xp_{n-r,k-r}(x); & n-r < k \le n \end{cases} \quad x \in [0,1], \tag{2}$$

and therefore, constructed and studied Bernstein-type positive linear operators $L_{n,r} : C[0,1] \to C[0,1]$ defined by

$$L_{n,r}(f;\, x) := \sum_{k=0}^{n} w_{n,k,r}(x) f\left(\frac{k}{n}\right) \tag{3}$$

Dedicated to victims who suffered from Earthquakes in 2023.

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for $f \in C[0,1]$, and $x \in [0,1]$ ([1] and [2]). From (2), the operators can be written as

$$L_{n,r}(f;\, x) := \sum_{k=0}^{n-r} p_{n-r,k}(x)\left[(1-x)f\left(\frac{k}{n}\right) + xf\left(\frac{k+r}{n}\right)\right]. \tag{4}$$

For the cases $r = 0$ and $r = 1$, Stancu's operators correspond to the well-known Bernstein operators and satisfy $L_{n,r}(f;0) = f(0)$ and $L_{n,r}(f;\,1) = f(1)$. There are several papers related to these operators. To mention a few, we first refer to Agratini's work [3], in which, using divided differences, the author proved monotonicity of $\{L_{n,r}(f;\,x)\}_{n\in\mathbb{N}}$ when f is convex. Gonska [4] obtained an upper bound for the rate of the approximation by Stancu's operators in terms of the second modulus of smoothness of real valued continuous functions on $[0,1]$. Yun and Xiang [5] obtained that each $L_{n,r}$ preserves Lipschitz' constant and order of a Lipschitz continuous function and that the sequence $\{L_{n,r}(f;\,x)\}_{n\in\mathbb{N}}$ is monotonically non-increasing when the function f is convex on $[0,1]$. Moreover, the authors also proved simultaneous approximation, in other words, if $f \in C^p[0,1]$, $p \in \mathbb{N}$, then $\lim_{n\to\infty} L_{n,r}^{(p)}(f) = f^{(p)}$ uniformly on [0,1].

The operators $L_{n,r}$ are usually called as *Stancu operators* in the literature. As Stancu noted in his paper [2], for the case $r = 2$, the operators $L_{n,2}$ were constructed and studied earlier by Brass [6]. Thus, as in [4], we call the operators $L_{n,r}$ as *Brass-Stancu operators*.

Recall that for $f \in L^p[0,1]$, $1 \le p < \infty$, the well-known Kantorovich operators [7] are defined by

$$K_n(f;\, x) := \sum_{k=0}^{n} p_{n,k}(x)\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t\right), \tag{5}$$

where $n \in \mathbb{N}$, $x \in [0,1]$, and $p_{n,k}(x)$ are given by (1). It is well-known that L^p-convergence by the sequence of Kantorovich operators was due to Lorentz [8]:

Theorem 1.1. [8] *Let $f \in L^p[0,1]$, $1 \le p < \infty$. Then*

$$\lim_{n\to\infty} K_n(f) = f$$

in $L^p[0,1]$.

For our main objectives, we need to recall here the well-known shape property for a positive linear operator, called as *variation detracting property*.

Let, as usual, $V_{[0,1]}[f]$ denote the total variation of f on $[0,1]$ and $TV[0,1]$ denote the class of all functions of bounded variation on $[0,1]$. Let also $BV[0,1]$ denote the space of functions of bounded variation on $[0,1]$, with the norm $\|f\|_{\mathrm{BV}} := V_{[0,1]}[f] + |f(c)|$, where c is any fixed point in $[0,1]$. For Bernstein operators B_n, $n \in \mathbb{N}$, Lorentz [8, p. 23] proved that *if $f \in TV[0,1]$, then*

$$V_{[0,1]}[B_n f] \le V_{[0,1]}[f].$$

The operator B_n having this inequality is called as *variation detracting*. The second and third authors proved in [9] that each Brass-Stancu operator $L_{n,r}$ is variation detracting as well. Note that for the case $r = 2$, Brass proved earlier that each operator $L_{n,2}$ is variation detracting [6].

In [10, Proposition 3.3], Bardaro et al. proved that each Kantorovich operator K_n is variation detracting and bounded operator with respect to the BV-norm of $BV[0,1]$. Namely, the authors proved that

Proposition 1.1. *Let $f \in TV[0,1]$. Then,*

$$V_{[0,1]}[K_n(f)] \le V_{[0,1]}[f]$$

and

$$\|K_n(f)\|_{\mathrm{BV}} \le 2\,\|f\|_{\mathrm{BV}}$$

for $n \in \mathbb{N} \cup \{0\}$.

In [11], Stancu generalized the operators $L_{n,r}$ given by (4) so that the new operators depend on two non-negative integer parameters. A Kantorovich-type generalization of these two non-negative dependent generalized operators were constructed by Dimitriu in [12], where the author studied some global smoothness preservation properties of these Kantorovich variants. A-statistical convergence of continuous functions on $[0,1]$ by these operators was studied by Kajla [13].

As in the construction of the Kantorovich operators (5), by replacing the point evaluations $f\left(\frac{k}{n}\right)$ in (3) with the integral means $(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t$, we consider the following Kantorovich generalization of the Brass-Stancu operators:

$$K_{n,r}(f;\,x) := \sum_{k=0}^{n} w_{n,k,r}(x)\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t\right), \tag{6}$$

where $f \in L^p[0,1]$ and $w_{n,k,r}(x)$, $x \in [0,1]$, $0 \le k \le n$, are given by (2). Each $K_{n,r}$ is a positive linear operator from $L^p[0,1]$ into $C[0,1]$. We refer the operators $K_{n,r}$ as Brass-Stancu-Kantorovich operators throughout. Note that for a different kind of integral generalization of the Brass-Stancu operators, which is obtained by differentiating the operator $L_{n,r}(f;\,x)$ with respect to x, L^p-approximation was studied in [9]. As far as we have searched, neither L^p-approximation with rates nor variation detracting property have been obtained for the Brass-Stancu-Kantorovich operators given by (6) yet.

Motivated by the corresponding results for the Kantorovich operators K_n given by (5), the aim of this article is twofold. First, we present L^p-approximation and give some estimates for the rate of the L^p-approximation by the sequence of the Brass-Stancu-Kantorovich operators. For the estimates, we use τ-modulus and K-functional of the first-order in L^p-norm. Second, as in Kantorovich operators, we show that each Brass-Stancu-Kantorovich operator is variation detracting and is bounded with respect to the norm of the space of functions of bounded variation on $[0,1]$.

From the definition of the functions, $w_{n,k,r}(x)$, $K_{n,r}(f;\,x)$ can be expressed as

$$K_{n,r}(f;\,x) = \sum_{k=0}^{n-r} p_{n-r,k}(x)(n+1)\left[(1-x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t + x\int_{\frac{k+r}{n+1}}^{\frac{k+r+1}{n+1}} f(t)\mathrm{d}t\right], \tag{7}$$

and in the cases $r=0$ and $r=1$, Brass-Stancu-Kantorovich operators reduce to the Kantorovich operators $K_{n,0} = K_{n,1} = K_n$.

2 L^p-approximation by the Brass-Stancu-Kantorovich operators

In this section, we study L^p-approximation, as well as the rate of the approximation, by the Brass-Stancu-Kantorovich operators. We adopt the standard notation for the monomials $e_\nu(t) := t^\nu$, $t \in [0,1]$, $\nu = 0,1,\ldots$.

Theorem 2.1. *Let r be a non-negative fixed integer and $f \in C[0,1]$. Then,*

$$\lim_{n\to\infty} K_{n,r}(f) = f$$

uniformly on $[0,1]$.

Proof. Making use of the first three moments of the Brass-Stancu operators $L_{n,r}$ given by

$$L_{n,r}(e_0;\,x) = 1, \quad L_{n,r}(e_1;\,x) = x, \quad L_{n,r}(e_2;\,x) = x^2 + \left[1 + \frac{r(r-1)}{n}\right]\frac{x(1-x)}{n},$$

in [2], it readily follows that

$$\begin{aligned} K_{n,r}(e_0;\ x) &= 1, \\ K_{n,r}(e_1;\ x) &= \frac{n}{n+1}x + \frac{1}{2(n+1)}, \\ K_{n,r}(e_2;\ x) &= \frac{n^2}{(n+1)^2}\left[x^2 + \left(1 + \frac{r(r-1)}{n}\right)\frac{x(1-x)}{n}\right] + \frac{3nx+1}{3(n+1)^2}. \end{aligned} \tag{8}$$

Hence, the result is obtained by using the well-known Korovkin theorem (e.g., [14]). □

Here we present the L^p-approximation.

Theorem 2.2. *Let r be a non-negative fixed integer and* $f \in L^p[0,1], 1 \le p < \infty$. *Then,*

$$\lim_{n\to\infty} K_{n,r}(f) = f$$

in $L^p[0,1]$.

Proof. Since the cases $r = 0, 1$ give Kantorovich operators K_n, we already have the result from Theorem 1.1. Denoting the operator norm of $K_{n,r}$, acting from $L^p[0,1]$ into itself by $\|K_{n,r}\|$, it is sufficient to show that there exist an $M > 0$ such that $\|K_{n,r}\| \le M$. Now, taking into accout the fact that $\varphi(t) = |t|^p, 1 \le p < \infty, t \in [0,1]$ is convex and that for $w_{n,k,r}(x) \ge 0, x \in [0,1]$, we have $\sum_{k=0}^{n} w_{n,k,r}(x) = 1$, by using Jensen's inequality and integral form of Jensen's inequality (e.g., [14]), we have

$$\begin{aligned} |K_{n,r}(f;\ x)|^p &\le \sum_{k=0}^{n} w_{n,k,r}(x)\left|(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t\right|^p \\ &\le \sum_{k=0}^{n} w_{n,k,r}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\mathrm{d}t \\ &= \left\{(1-x)\sum_{k=0}^{n-r} p_{n-r,k}(x) + x\sum_{k=r}^{n} p_{n-r,k-r}(x)\right\}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\mathrm{d}t. \end{aligned} \tag{9}$$

Integrating (9) over $[0,1]$, using the well-known Beta integral, we obtain

$$\begin{aligned} \int_0^1 |K_{n,r}(f;\ x)|^p\mathrm{d}x &\le \sum_{k=0}^{n-r}\binom{n-r}{k}\int_0^1 x^k(1-x)^{n-r-k+1}\mathrm{d}x(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\mathrm{d}t \\ &\quad + \sum_{k=r}^{n}\binom{n-r}{k-r}\int_0^1 x^{k-r+1}(1-x)^{n-k}\mathrm{d}x(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\mathrm{d}t \\ &= \frac{n+1}{n-r+2}\left\{\sum_{k=0}^{n-r}\frac{n-r-k+1}{n-r+1}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\mathrm{d}t + \sum_{k=r}^{n}\frac{k-r+1}{n-r+1}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p\mathrm{d}t\right\}. \end{aligned} \tag{10}$$

Since $n > 2r$, $r \in \mathbb{N} \cup \{0\}$, we have $n - r > r$. Thus, (10) can be expressed as

$$\int_0^1 |K_{n,r}(f;\, x)|^p \mathrm{d}x \le \frac{n+1}{n-r+2}\left\{\left(\sum_{k=0}^{r-1} + \sum_{k=r}^{n-r}\right)\frac{n-r-k+1}{n-r+1}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p \mathrm{d}t + \left(\sum_{k=r}^{n-r} + \sum_{k=n-r+1}^{n}\right)\frac{k-r+1}{n-r+1}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p \mathrm{d}t\right\}$$

$$= \frac{n+1}{n-r+2}\left\{\sum_{k=0}^{r-1}\frac{n-r-k+1}{n-r+1} + \sum_{k=r}^{n-r}\frac{n-2r+2}{n-r+1} + \sum_{k=n-r+1}^{n}\frac{k-r+1}{n-r+1}\right\}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p \mathrm{d}t.$$

Since $n - 2r + 1 < n - r + 1$, we have $n - 2r + 2 \le n - r + 1$. Thus, for $0 \le k \le r - 1$, we obtain $\frac{n-r-k+1}{n-r+1} \le 1$ and for $n - r + 1 \le k \le n$, we have $\frac{k-r+1}{n-r+1} \le 1$. Therefore, the last formula can be simplified as

$$\int_0^1 |K_{n,r}(f;\, x)|^p \mathrm{d}x \le \frac{n+1}{n-r+2}\left(\sum_{k=0}^{r-1} + \sum_{k=r}^{n-r} + \sum_{k=n-r+1}^{n}\right)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p \mathrm{d}t$$

$$= \frac{n+1}{n-r+2}\sum_{k=0}^{n}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p \mathrm{d}t = \frac{n+1}{n-r+2}\int_0^1 |f(t)|^p \mathrm{d}t.$$

Now, since, for fixed $r > 1$, we can take $\sup_{n>2r}\frac{n+1}{n+2-r} = \frac{2r+2}{r+3} =: M$, we obtain

$$\int_0^1 |K_{n,r}(f;\, x)|^p \mathrm{d}x \le M\int_0^1 |f(t)|^p \mathrm{d}t.$$

Passing to L^p-norm, we obtain $\|K_{n,r}(f)\|_p \le M^{\frac{1}{p}}\, \|f\|_p$ for every $f \in L^p[0,1]$. Namely, $K_{n,r}$ is a bounded operator with $\|K_{n,r}\|_p \le M^{\frac{1}{p}}$. Note that for $r = 0, 1$, we have $M = 1$ [14]. Thus, for any $\varepsilon > 0$, by the density of $C[0,1]$ in $L^p[0,1]$, there is a $g \in C[0,1]$ such that $\|f - g\|_p < \varepsilon$ and, by Theorem 2.1, there exists an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we obtain

$$\|K_{n,r}(g) - g\| < \varepsilon,$$

where $\|.\|$ is the usual sup-norm in $C[0,1]$. Using the inequality

$$\|K_{n,r}(f) - f\|_p \le M^{\frac{1}{p}}\, \|f - g\|_p + \|K_{n,r}(g) - g\|_p + \|g - f\|_p < (M^{\frac{1}{p}} + 2),$$

we arrive at the desired result. □

For the rate of the L^p-approximation by the sequence of the Brass-Stancu-Kantorovich operators, we first apply Popov's theorem (as represented in Theorem 2.3) to obtain an upper estimate in terms of the averaged modulus of smoothness of the first order. After, following Müller's approach in [15], by using some properties of the averaged modulus of smoothness, we give an upper bound for the space of functions such that $f' \in L^p[0,1]$. Moreover, as in [16], we also use the equivalence of the first-order K-functional with modulus of smoothness of first order to obtain an upper bound in L^p -norm. Now, we give some needful definitions.

Definition 2.1. [17] Let $f \in L^p[0,1]$, $1 \le p < \infty$, and $\delta > 0$. The usual modulus of smoothness of the first order and step δ for the function f in L^p-norm is defined as

$$\omega_1(f;\, \delta)_p = \sup_{0<h\le\delta}\left\{\int_0^{1-h} |f(x+h) - f(x)|^p \mathrm{d}x\right\}^{1/p}. \tag{11}$$

Here, let us consider the space

$$L^{p1}[0,1] := \{f \in L^p[0,1] \,|\, f \text{ is absolutely continuous on } [0,1] \text{ and } f' \in L^p[0,1],\ 1 \le p < \infty\}.$$

Below, we recall Peetre's K-functional, defined for functions in $L^p[0,1]$.

Definition 2.2. ([15] or [18]) Let $f \in L^p[0,1]$, $1 \le p < \infty$, and $\delta > 0$. Peetre's K-functional of the first order is defined as

$$K_{1,p}(f;\delta) := \inf_{g \in L^{p1}} \{\|f-g\|_p + \delta\,\|g'\|_p\}. \tag{12}$$

Johnen [19] proved that there are positive constants c_1 and c_2 which do not depend on f and p, such that

$$c_1\omega_1(f;\delta)_p \le K_{1,p}(f;\delta) \le c_2\omega_1(f;\delta)_p. \tag{13}$$

Let, as usual, $M[0,1]$ denote the space

$$M[0,1] = \{f \,|\, f \text{ is bounded and measurable on } [0,1]\}.$$

Recall the notion of averaged modulus of smoothness for functions from $M[0,1]$.

Definition 2.3. ([17] or [20]) Let $f \in M[0,1]$ and $\delta > 0$. Averaged modulus of smoothness (or τ-modulus) of the first order for step δ in L^p-norm, $1 \le p < \infty$, is given by

$$\tau_1(f;\delta)_p = \|\omega_1(f,.;\delta)\|_p,$$

where

$$\omega_1(f,x;\delta) = \sup\left\{|f(t+h)-f(t)| : t, t+h \in \left[x-\frac{\delta}{2}, x+\frac{\delta}{2}\right] \cap [0,1]\right\}$$

is the local modulus of smoothness of the first order for the function f at the point $x \in [0,1]$ and for step δ.

For $f \in M[0,1]$, $\delta, \delta', \lambda \in \mathbb{R}^+$, we need the following properties of the τ-modulus (e.g., [17]):

(τ1) $\tau_1(f;\delta)_p \le \tau_1(f;\delta')_p$, when $0 < \delta \le \delta'$.

(τ2) $\tau_1(f;\lambda\delta)_p \le (2\lfloor\lambda\rfloor + 2)^2\tau_1(f;\delta)_p$, where $\lfloor . \rfloor$ is the greatest integer that does not exceed the number.

(τ3) $\tau_1(f;\delta)_p \le \delta\,\|f'\|_p$ if $f \in L^{p1}[0,1]$.

For every Borel measurable and bounded function f defined on $[0,1]$, we already have the following results for the Kantorovich operators K_n, $n \in \mathbb{N}$,

$$\|K_n(f) - f\|_p \le 748\tau_1\left(f;\frac{1}{\sqrt{n+1}}\right)_p$$

([21, p. 335]) and

$$\|K_n(f) - f\|_p \le C\tau_1\left(f;\sqrt{\frac{3n+1}{12(n+1)^2}}\right)_p, \tag{14}$$

where C is a positive constant that does not depend on f (see the special case of Proposition 4.2 in [22]). In order to obtain an upper estimate for the L^p-norm of the approximation error in Theorem 2.2, we apply the following very useful theorem of Popov [23] for the special case that the interval is taken here as $[0,1]$.

Theorem 2.3. [23] *Let $L : M[0,1] \to M[0,1]$ be a positive linear operator, having the properties*

$$L(e_0;x) = 1, \quad L(e_1;x) = x + \alpha(x), \quad L(e_2;x) = x^2 + \beta(x), \quad x \in [0,1].$$

Let

$$A \coloneqq \sup\{|\beta(x) - 2x\alpha(x)| : \quad x \in [0,1]\} \le 1.$$

Then, for $f \in M[0,1]$ *and* $1 \le p < \infty$*, the following estimate holds*

$$\|L(f) - f\|_p \le C\tau_1(f; \sqrt{A})_p,$$

where C *is a positive constant which does not depend on the operator* L*, the function* f*, and the* L^p*-norm.*

Therefore, we present the following upper estimate for the L^p-norm of the approximation error by the Brass-Stancu-Kantorovich operators:

Theorem 2.4. *Let* r *be a non-negative fixed integer,* $f \in M[0,1]$ *and* $1 \le p < \infty$*. Then,*

$$\|K_{n,r}(f) - f\|_p \le C\tau_1(f; \sqrt{A_{n,r}})_p$$

for all $n \in \mathbb{N}$ *such that* $n > 2r$*, where*

$$A_{n,r} = \frac{3n + 1 + 3r(r-1)}{12(n+1)^2} \le 1, \tag{15}$$

and the positive constant C *does not depend on* f*.*

Proof. Let $f \in M[0,1]$ and $1 \le p < \infty$. According to Theorem 2.3, it suffices to show that

$$A_{n,r} \coloneqq \sup\{K_{n,r}((e_1 - xe_0)^2; x) : x \in [0,1]\} \le 1.$$

From the linearity of the operators and (8), for $x \in [0,1]$, we obtain

$$\begin{aligned} K_{n,r}((e_1 - xe_0)^2; x) &= \frac{n - 1 + r(r-1)}{(n+1)^2} x(1-x) + \frac{1}{3(n+1)^2} \\ &\le \frac{n - 1 + r(r-1)}{4(n+1)^2} + \frac{1}{3(n+1)^2} \\ &= A_{n,r}, \end{aligned}$$

where $A_{n,r}$ is given by (15). Note that the cases for $r = 0$ and $r = 1$ give the result (14) for Kantorovich operators. Now, it remains to show that $A_{n,r} \le 1$. Since $r \ge 0$ and $n > 2r$, we obtain $n - 1 \ge 2r$. Hence, we obtain

$$\begin{aligned} A_{n,r} &= \frac{3n + 1 + 3r(r-1)}{12(n+1)^2} \le \frac{3n + 1 + 3r^2}{12(n+1)^2} \\ &\le \frac{3n + 1 + \frac{3}{4}(n-1)^2}{12(n+1)} = \frac{3n^2 + 6n + 7}{48(n+1)^2} \le 1, \end{aligned}$$

which completes the proof. □

Now, as in [15, Theorem 2], we present the following estimate for L^p-norm of the error of approximation of $f \in L^{p1}[0,1]$.

Theorem 2.5. *Let* r *be a non-negative fixed integer,* $f \in L^{p1}[0,1]$*,* $1 \le p < \infty$*. Then,*

$$\|K_{n,r}(f) - f\|_p \le \frac{K_r}{\sqrt{n+1}} \|f'\|_p$$

for all $n \in \mathbb{N}$ *such that* $n > 2r$*, where* K_r *is a positive constant that may depend only on* r*.*

Proof. Since $n + 1 \geq 2(r + 1)$, from (15), we have

$$\begin{aligned} A_{n,r} &= \frac{1}{4(n+1)} + \frac{3r^2 - 3r - 2}{12(n+1)^2} \\ &\leq \left[\frac{1}{4} + \frac{3r^2 - 3r - 2}{24(r+1)}\right]\frac{1}{n+1} \end{aligned}$$

for $r > 1$ and $A_{n,r} < \frac{1}{4(n+1)}$ for $r = 0$ or 1. Thus, we can write

$$A_{n,r} \leq \frac{C_r}{n+1}, \tag{16}$$

where $C_r := \begin{cases} \frac{3r^2+3r+4}{24(r+1)}, & r > 1 \\ \frac{1}{4}, & r = 0, 1 \end{cases}$ is a positive constant. Therefore, for any $f \in L^{p1}[0, 1]$, combining Theorem 2.4 with the properties (τ1)–(τ3) of τ-modulus, we reach to

$$\begin{aligned} \|K_{n,r}(f) - f\|_p &\leq C\tau_1\left(f; \sqrt{\frac{C_r}{n+1}}\right)_p \\ &\leq (2\lfloor\sqrt{C_r}\rfloor + 2)^2 C\tau_1\left(f; \frac{1}{\sqrt{n+1}}\right)_p \\ &\leq \frac{K_r}{\sqrt{n+1}}\, \|f'\|_p, \end{aligned}$$

where $K_r = C(2\lfloor\sqrt{C_r}\rfloor + 2)^2$ is a positive constant and $1 \leq p < \infty$. □

Here similar to Kantorovich operators, which can be captured from Müller's paper [16, p. 246], we present an upper bound for the L^p-norm of the approximation error in terms of the first-order modulus of smoothness in L^p-norm.

Theorem 2.6. *Let r be a non-negative fixed integer, $f \in L^p[0, 1]$, $1 \leq p < \infty$. Then,*

$$\|K_{n,r}(f) - f\|_p \leq M_{r,p}\,\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right)_p$$

for all $n \in \mathbb{N}$ such that $n > 2r$, where $M_{r,p}$ is a positive constant that may depend on r and p, but independent of f.

Proof. Since $K_{n,r}$ is bounded, with $\|K_{n,r}\|_p \leq M^{1/p}$, we obtain

$$\|K_{n,r}(g) - g\|_p \leq (M^{1/p} + 1)\|g\|_p$$

for any $g \in L^p[0, 1]$, $1 \leq p < \infty$. On the other hand, in view of Theorem 2.5, we obtain

$$\|K_{n,r}(g) - g\|_p \leq \frac{K_r}{\sqrt{n+1}}\, \|g'\|_p$$

for $g \in L^{p1}[0, 1]$, where K_r is a positive constant. Therefore, for $f \in L^p[0, 1]$, it follows that

$$\begin{aligned} \|K_{n,r}(f) - f\|_p &\leq \|K_{n,r}(f - g) - (f - g)\|_p + \|K_{n,r}(g) - g\|_p \\ &\leq (M^{1/p} + 1)\left\{\|f - g\|_p + \frac{K_r}{\sqrt{n+1}}\, \|g'\|_p\right\}. \end{aligned}$$

Taking infimum over $g \in L^{p1}[0, 1]$, making use of the definition of K-functional given by (12) and relation between K-functional and usual modulus of smoothness given by (13), we finally obtain

$$\begin{aligned}
\|K_{n,r}(f)-f\|_p &\le (M^{1/p}+1)K_{1,p}\left(f;\ \frac{K_r}{\sqrt{n+1}}\right)\\
&\le c_2(M^{1/p}+1)\omega_1\left(f;\ \frac{K_r}{\sqrt{n+1}}\right)_p\\
&\le M_{r,p}\omega_1\left(f;\ \frac{1}{\sqrt{n+1}}\right)_p,
\end{aligned}$$

where $M_{r,p} \coloneqq c_2(M^{1/p}+1)(K_r+1)$, which is followed by the fact $\omega_1(f;\ \lambda\delta)_p \le (\lambda+1)\omega_1(f;\ \lambda\delta)_p$, $\lambda > 0$. We reach to the desired result. □

3 Variation detracting property

In this part, we show that each Brass-Stancu-Kantorovich operator $K_{n,r}$ is also variation detracting and bounded operator, with respect to BV-norm.

Theorem 3.1. *Let r be a non-negative fixed integer and $f \in TV[0,1]$. Then,*

$$V_{[0,1]}[K_{n,r}(f)] \le V_{[0,1]}[f] \tag{17}$$

and

$$\|K_{n,r}(f)\|_{\mathrm{BV}} \le 2\ \|f\|_{\mathrm{BV}} \tag{18}$$

for every $n \in \mathbb{N}$ such that $n > 2r$.

Proof. Since the cases $r = 0, 1$ give the result in Proposition 1.1, we consider the case where $r > 1$. Let us show first (17). Assume that $f \in TV[0,1]$. Then, it readily follows that $K_{n,r}(f) \in AC[0,1]$, the class of absolutely continuous functions on $[0,1]$. As in [10, Proposition 3.3], we set

$$F_{n,k} \coloneqq (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t = \int_0^1 f\left(\frac{k+u}{n+1}\right)\mathrm{d}u, \quad 0 \le k \le n.$$

Thus, from (7) $K_{n,r}(f;\ x)$ can be expressed as

$$K_{n,r}(f;\ x) = \sum_{k=0}^{n-r} p_{n-r,k}(x)[(1-x)F_{n,k} + xF_{n,k+r}].$$

Differentiating $K_{n,r}(f;\ x)$, we obtain

$$\begin{aligned}
(K_{n,r}(f;\ x))' = (n-r)&\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\{(1-x)[F_{n,k+1}-F_{n,k}] + x[F_{n,k+r+1}-F_{n,k+r}]\}\\
&+\sum_{k=0}^{n-r} p_{n-r,k}(x)[F_{n,k+r}-F_{n,k}].
\end{aligned} \tag{19}$$

From (19), we obtain

$$\begin{aligned}
V_{[0,1]}[K_{n,r}(f)] &= \int_0^1 |(K_{n,r}(f;\ x))'|\mathrm{d}x\\
&\le \frac{1}{n-r+1}\sum_{k=0}^{n-r-1}\{(n-r-k)|F_{n,k+1}-F_{n,k}| + (k+1)|F_{n,k+r+1}-F_{n,k+r}|\}\\
&\quad+\frac{1}{n-r+1}\sum_{k=0}^{n-r}|F_{n,k+r}-F_{n,k}|\\
&\eqqcolon \frac{1}{n-r+1}\{S^1_{n,r}+S^2_{n,r}\}.
\end{aligned} \tag{20}$$

Since $n > 2r$, we have $n - r > r$, which implies that $n - r - 1 \geq r$. As in [9], we can split the sums $S^i_{n,r}$, $i = 1, 2$, into three parts:

$$\begin{aligned} S^1_{n,r} &= \sum_{k=0}^{n-r-1} \{(n-r-k)|F_{n,k+1} - F_{n,k}| + (k+1)|F_{n,k+r+1} - F_{n,k+r}|\} \\ &= \left(\sum_{k=0}^{r-1} + \sum_{k=r}^{n-r-1}\right)(n-r-k)|F_{n,k+1} - F_{n,k}| + \left(\sum_{k=r}^{n-r-1} + \sum_{k=n-r}^{n-1}\right)(k-r+1)|F_{n,k+1} - F_{n,k}| \\ &= \sum_{k=0}^{r-1}(n-r-k)|F_{n,k+1} - F_{n,k}| + \sum_{k=r}^{n-r-1}(n-2r+1)|F_{n,k+1} - F_{n,k}| \\ &\quad + \sum_{k=n-r}^{n-1}(k-r+1)|F_{n,k+1} - F_{n,k}|. \end{aligned} \tag{21}$$

On the other hand, we can decompose the sum in $S^2_{n,r}$ into three parts as

$$\begin{aligned} S^2_{n,r} &= \sum_{k=0}^{n-r}|F_{n,k+r} - F_{n,k}| \\ &\leq \sum_{k=0}^{n-r}\{|F_{n,k+r} - F_{n,k+r-1}| + |F_{n,k+r-1} - F_{n,k+r-2}| + \cdots + |F_{n,k+2} - F_{n,k+1}| + |F_{n,k+1} - F_{n,k}|\} \\ &= \sum_{k=0}^{r-1}(k+1)|F_{n,k+1} - F_{n,k}| + r\sum_{k=r}^{n-r-1}|F_{n,k+1} - F_{n,k}| + \sum_{k=n-r}^{n-1}(n-k)|F_{n,k+1} - F_{n,k}|. \end{aligned} \tag{22}$$

Finally, combining (21) and (22) with (20), we arrive at

$$\begin{aligned} V_{[0,1]}[K_{n,r}(f)] &\leq \frac{1}{n-r+1}\left\{\sum_{k=0}^{r-1}(n-r-k) + \sum_{k=r}^{n-1-r}(n-2r+1) + \sum_{k=n-r}^{n-1}(k-r+1)\right. \\ &\quad \left. + \sum_{k=0}^{r-1}(k+1) + \sum_{k=r}^{n-1-r} r + \sum_{k=n-r}^{n-1}(n-k)\right\}|F_{n,k+1} - F_{n,k}| \\ &= \sum_{k=0}^{n-1}|F_{n,k+1} - F_{n,k}|. \end{aligned}$$

Now, it suffices to show that $\sum_{k=0}^{n-1}|F_{n,k+1} - F_{n,k}| \leq V_{[0,1]}[f]$. For this, as in the proof of Proposition 3.3 in [10], taking

$$u_k := \begin{cases} 0, & k = -1, \\ \dfrac{k+u}{n+1}, & 0 \leq k \leq n, \\ 1, & k = n+1, \end{cases}$$

we have, for the particular partition $\{u_k\}_{k=-1}^{n+1}$, the following:

$$\begin{aligned} \sum_{k=0}^{n-1}|F_{n,k+1} - F_{n,k}| &\leq \int_0^1 \sum_{k=0}^{n-1}|f(u_{k+1}) - f(u_k)|\mathrm{d}u \\ &\leq \int_0^1 \sum_{k=-1}^{n}|f(u_{k+1}) - f(u_k)|\mathrm{d}u \\ &\leq V_{[0,1]}[f]. \end{aligned}$$

Now, using (17), (18) follows from the following inequality:

$$\begin{aligned}\|K_{n,r}(f)\|_{\mathrm{BV}} &= V_{[0,1]}[K_{n,r}(f)] + |K_{n,r}(f;0)| \\ &\le V_{[0,1]}[f] + |f(0)| + |K_{n,r}(f;0) - f(0)| \\ &= \|f\|_{\mathrm{BV}} + |K_{n,r}(f;0) - f(0)| \\ &= \|f\|_{\mathrm{BV}} + (n+1)\left|\int_0^{\frac{1}{n+1}} [f(t) - f(0)]\mathrm{d}t\right|.\end{aligned}$$

Using a similar argument in [10, Proposition 3.3], we obtain that $\|K_{n,r}(f)\|_{\mathrm{BV}} \le 2\,\|f\|_{\mathrm{BV}}$, which completes the proof. □

4 Graphical and numerical examples

In this section, we will provide comparisons of the convergence of the operators we have used by giving illustrated examples.

Example 4.1. The first example gives us an idea about the effect on a non-negative integer parameter r, and any natural number n such that $n > 2r$. As mentioned earlier, for $r = 2$, our operators $K_{n,2}(f;\, x)$ are the Kantorovich-type generalization of the operators $L_{n,2}(f;\, x)$ defined by Brass [6]. For $f(x) = x(1-x)$, $n = 50$ and $r = 2$, special case of our operators approximate so closely to given function (Figure 1).

Example 4.2. In this example, we shall see how the change of n values affects the convergence for specific value $r = 2$. Let us consider again the specific function $f(x) = x(1-x)$ in the interval $[0, 1]$. One can see in Figure 2 the approximation process for the different values of n.

One can directly see from Figure 2 that the higher values of n yield better approximation. It may be useful to take a look at Table 1 to see the effect of n values numerically for specific points $x = 0.4$ and $x = 0.7$.

Now, we present Table 2 to illustrate how various values of x affect the function $f(x) = x(1-x)$.

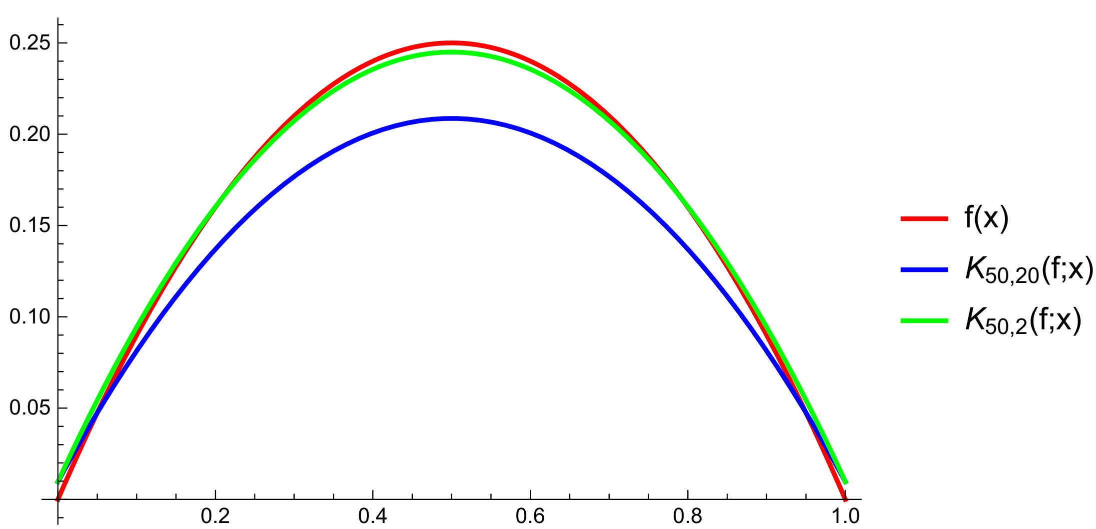

Figure 1: Approximation process of $K_{50,20}(f;\, x)$ and $K_{50,2}(f;\, x)$ to f.

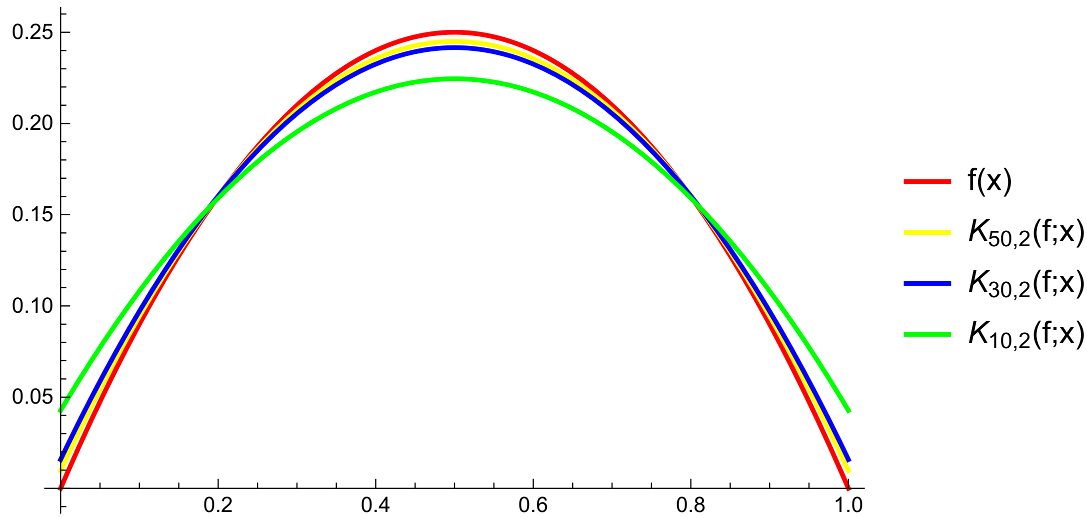

Figure 2: Approximation process of $K_{n,2}(f;\ x)$ for $n = 10, 30, 50$.

Table 1: Error of approximation by $K_{n,2}(f;\ 0.4)$ and $K_{n,2}(f;\ 0.7)$ for different n values

n	$\lvert K_{n,2}(f;\ 0.4) - f(0.4)\rvert$	$\lvert K_{n,2}(f;\ 0.7) - f(0.7)\rvert$
5	0.04592590	0.03092590
15	0.01505210	0.00942708
25	0.00895464	0.00549310
35	0.00636831	0.00386831
50	0.00444188	0.00267718
75	0.00295245	0.00176824
100	0.00221089	0.00131981

Table 2: Error of approximation by $K_{30,2}(f;\ x)$ for different x values

x	$\lvert K_{30,2}(f;\ x) - f(x)\rvert$
0.16	0.00277572
0.24	0.00186944
0.32	0.00527589
0.40	0.00744364
0.48	0.00837267
0.56	0.00806299
0.64	0.00651460
0.72	0.00372751
0.80	0.00029830
0.88	0.00556282

Example 4.3. It is observed that for sufficiently large n, the operators approximate better to the function regardless of the chosen suitable function. This is also another illustration to show the effects of n for $f(x) = \cos 2\pi x$ (Figure 3).

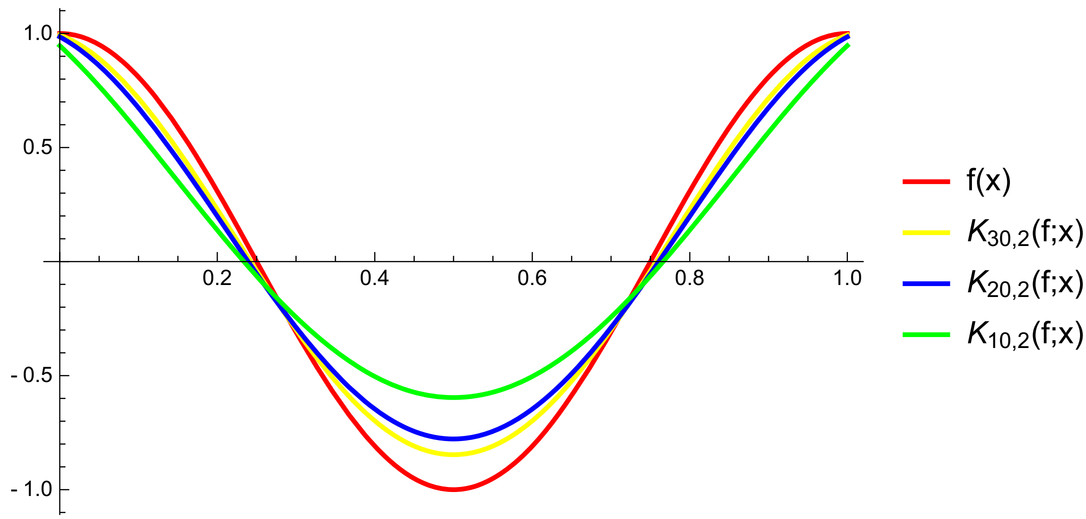

Figure 3: Approximation process of $K_{n,2}(f;\, x)$ for $n = 10, 20, 30$.

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