Research Article

Muhammad Amer Latif*

New extensions related to Fejér-type inequalities for *GA*-convex functions

https://doi.org/10.1515/dema-2024-0006
received May 31, 2023; accepted February 8, 2024

Abstract: In this study, some mappings related to the Fejér-type inequalities for *GA*-convex functions are defined over the interval $[0, 1]$. Some Fejér-type inequalities for *GA*-convex functions are proved using these mappings. Properties of these mappings are considered and consequently we obtain refinements of some known results.

Keywords: Hermite-Hadamard inequality, convex function, harmonic *GA*-convex function, Fejér inequality

MSC 2020: 26D15, 26D20, 26D07

1 Introduction

For convex functions, the following double inequality has great significance in the literature and is known as Hermite-Hadamard's inequality [1,2]:

Let $\varphi : I \longrightarrow \mathbb{R}, \varnothing \neq I \subseteq \mathbb{R}, \nu_1, \nu_2 \in I$ with $\nu_1 < \nu_2$ be a convex function, then

$$\varphi\left(\frac{\nu_1 + \nu_2}{2}\right) \leq \frac{1}{\nu_2 - \nu_1}\int_{\nu_1}^{\nu_2}\varphi(\xi)\mathrm{d}\xi \leq \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}, \tag{1.1}$$

the inequality (1.1) holds in reversed direction if φ is concave.

Fejér [3] established the following double inequality as a weighted generalization of (1.1):

$$\varphi\left(\frac{\nu_1 + \nu_2}{2}\right)\int_{\nu_1}^{\nu_2}\vartheta(\xi)\mathrm{d}\xi \leq \int_{\nu_1}^{\nu_2}\varphi(\xi)\vartheta(\xi)\mathrm{d}\xi \leq \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\vartheta(\xi)\mathrm{d}\xi, \tag{1.2}$$

where $\varphi : I \longrightarrow \mathbb{R}$ and $\varnothing \neq I \subseteq \mathbb{R}, \nu_1, \nu_2 \in I$ with $\nu_1 < \nu_2$ is any convex function and $\vartheta : [\nu_1, \nu_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\xi = \frac{\nu_1 + \nu_2}{2}$.

Inequalities (1.1) and (1.2) have many extensions and generalizations. We refer the readers to [4,5], the studies carried out by Ardic et al. which deal with the Ostrowski-type inequalities for *GG*-convex and *GA*-convex functions and some other important inequalities for inequalities via *GG*-convexity and *GA*-convexity, respectively. Dragomir and Latif discussed in their studies some important Fejér-type integral inequalities related to the geometrically-arithmetically convex functions and their applications in [6–17]. In [18,19], Kunt and İşcan have proven very interesting results of Hermite-Hadamard and Fejér-type inequalities for *GA*-convex for *GA*-*s*-convex functions. The author has provided results of Hermite-Hadamard-type and weighted Hermite-Hadamard-type using differentiable *GA*-convex, coordinated *GA*-convex, and geometrically-quasi-convex mappings, and the notion of geometrically symmetric mappings in [20–22]. Further research studies have been accomplished concomitant to the Hermite-Hadamard and Fejér-type results using

*** Corresponding author: Muhammad Amer Latif,** Department of Basic Sciences, Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia, e-mail: m_amer_latif@hotmail.com, mlatif@kfu.edu.sa

Open Access. © 2024 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.

the notions of geometrically-arithmetically, geometrically quasi convex functions, and a no-negative integrable geometrically symmetric function in [23–25]. Here we also mention the contributions of the mathematicians, for instance, Noor et al. [26] who proved inequalities for geometrically-arithmetically h-convex functions, Obeidat and Latif [27] obtained the results of weighted Hermite-Hadamard-type inequalities using geometrically quasi-convex functions and a no-negative integrable geometrically symmetric function, Qi and Xi [28] accomplished some new Hermite-Hadamard-type inequalities for geometrically quasi-convex functions, and Zhang et al. [29] who proved Hermite-Hadamard-type inequalities for differentiable GA-convex functions.

Many important results that characterize the properties of the mappings related to Inequalities (1.1) and (1.2), and inequalities that provide refinements of Inequalities (1.1) and (1.2) are discussed by a number of researchers. Dragomir et al. [30] considered inequalities of Hadamard's type for Lipschitzian mappings. Dragomir et al. [31] gave refinements of Hadamard's inequalities. In another study, Dragomir [32] proved Hadamard's inequality for convex functions by defining some functionals. Dragomir and Agarwal complemented the study carried out in [33] by defining new mappings associated with Hadamard's inequalities for convex functions and obtained some refinements of (1.1). Dragomir [34] further investigated mappings in connection to Hadamard's inequalities and obtained some more refinements to (1.1). Tseng et al. [35–37] generalized the results given in [30–34] and acquired new inequalities of Hermite-Hadamard-Fejér-type involving convex functions and a weight function that is non-negative integrable symmetric with respect to the mean of the closed interval. Yang and Hong [38] also proved some refinements of (1.1) by considering the properties of some functionals. The interested readers are referred to Yang and Tseng [39–42] for more results of properties of functionals in connection to (1.1) and (1.2), and results which refine and generalize the inequalities (1.1) and (1.2) [43,44].

One of the generalizations of the convex functions is geometrically-arithmetically convex functions also known as GA-convex functions which is stated as follows:

Definition 1. [45] A function $\varphi : I \subseteq (0, \infty) \to \mathbb{R}$ is considered to be GA-convex, if

$$\varphi(\xi^{\varrho}\lambda^{1-\varrho}) \le \varrho\varphi(\xi) + (1-\varrho)\varphi(\lambda) \tag{1.3}$$

for all $\xi, \lambda \in I$ and $\varrho \in [0, 1]$. A function $\varphi : I \to \mathbb{R}$ is concave if the inequality in (1.3) is reversed.

We state some important facts that relate GA-convex and convex functions and use them to prove the main results.

Theorem 1. [45] *If* $[\nu_1, \nu_2] \subset (0, \infty)$ *and the function* $\chi : [\ln\nu_1, \ln\nu_2] \to \mathbb{R}$ *is convex (concave) on* $[\ln\nu_1, \ln\nu_2]$, *then the function* $\varphi : [\nu_1, \nu_2] \to \mathbb{R}$, $\varphi(\varrho) = \chi(\ln\varrho)$ *is GA-convex (concave) on* $[\nu_1, \nu_2]$.

Remark 1. It is obvious from Theorem 1 that if $\varphi : [\nu_1, \nu_2] \to \mathbb{R}$ is GA-convex on $[\nu_1, \nu_2] \subset (0, \infty)$, then $\varphi \circ \exp$ is convex on $[\ln\nu_1, \ln\nu_2]$. It follows that $\varphi \circ \exp$ has finite lateral derivatives on $(\ln\nu_1, \ln\nu_2)$ and by gradient inequality for convex functions we have

$$\varphi \circ \exp(\xi) - \varphi \circ \exp(\lambda)(\xi - \lambda) \ge \varphi(\exp\lambda)\exp(\lambda),$$

where $\varphi(\exp\lambda) \in [\varphi'_-(\exp\lambda), \varphi'_+(\exp\lambda)]$ for any $\xi, \lambda \in (\ln\nu_1, \ln\nu_2)$.

Theorem 2. (Jensen's inequality for GA-convex functions) [45,46] *Let* $\varphi : I \subset (0, \infty) \to \mathbb{R}$ *be a GA-convex function and* $[k, K] \subset I^\circ$. *Assume also that* $h : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfying the bounds*

$$0 < k \le h(\varrho) \le K < \infty \quad \textit{for } \mu\text{–a.e. } \varrho \in \Omega$$

and $w \ge 0$ μ*-a.e. on* Ω *with* $\int_\Omega w \mathrm{d}\mu = 1$. *If* $\varphi \in \partial\varphi$ *the subdifferential of* φ *and* $\varphi \circ h, \ln h \in L_w(\Omega, \mu)$, *then*

$$\varphi \circ \exp\left(\int_\Omega w \ln h \mathrm{d}\mu\right) \le \int_\Omega (\varphi \circ h) w \mathrm{d}\mu. \tag{1.4}$$

The following inequality of Hermite-Hadamard-type for *GA*-convex functions holds ([26] for an extension for *GA h*-convex functions):

Theorem 3. [26] *Let* $\varphi : I \subseteq (0, \infty) \to \mathbb{R}$ *be a GA-convex function and* $\nu_1, \nu_2 \in I$ *with* $\nu_1 < \nu_2$. *If* $\varphi \in L([\nu_1, \nu_2])$, *then the following inequalities hold*:

$$\varphi(\sqrt{\nu_1\nu_2}) \leq \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_2}^{\nu_1}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi \leq \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}. \tag{1.5}$$

The notion of geometrically symmetric functions was introduced in [23].

Definition 2. [23] A function $\vartheta : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ is geometrically symmetric with respect to $\sqrt{\nu_1\nu_2}$ if

$$\vartheta(\xi) = \vartheta\left(\frac{\nu_1\nu_2}{\xi}\right)$$

holds for all $\xi \in [\nu_1, \nu_2]$.

Fejér-type inequalities using *GA*-convex functions and the notion of geometric symmetric functions were presented in Latif et al. [23].

Theorem 4. [23] *Let* $\varphi : I \subseteq (0, \infty) \to \mathbb{R}$ *be a GA-convex function and* $\nu_1, \nu_2 \in I$ *with* $\nu_1 < \nu_2$. *If* $\varphi \in L([\nu_1, \nu_2])$ *and* $\vartheta : [\nu_1, \nu_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *is non-negative, integrable, and geometrically symmetric with respect to* $\sqrt{\nu_1\nu_2}$, *then*

$$\varphi(\sqrt{\nu_1\nu_2})\int_{\nu_2}^{\nu_1}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \leq \int_{\nu_2}^{\nu_1}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi \leq \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}\int_{\nu_2}^{\nu_1}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{1.6}$$

Suppose that $\varphi : I \subseteq (0, \infty) \to \mathbb{R}$ is *GA*-convex on I and $\nu_1, \nu_2 \in I$, let $\chi, \mathcal{F}, \mathcal{V} : [0, 1] \to \mathbb{R}$ be defined by

$$\chi(\varrho) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho})\mathrm{d}\xi,$$

$$\mathcal{F}(\varrho) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\int_{\nu_1}^{\nu_2}\frac{1}{\xi\lambda}\varphi(\xi^{\varrho}\lambda^{1-\varrho})\mathrm{d}\xi\mathrm{d}\lambda,$$

and

$$\mathcal{V}(\varrho) = \frac{1}{2(\ln\nu_2 - \ln\nu_1)}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}\left[\varphi\left(\nu_2^{\frac{1+\varrho}{2}}\xi^{\frac{1-\varrho}{2}}\right) + \varphi\left(\nu_1^{\frac{1+\varrho}{2}}\xi^{\frac{1-\varrho}{2}}\right)\right]\mathrm{d}\xi.$$

Latif et al. [47] obtained the following refinements for Inequalities (1.5):

Theorem 5. [47] *A function* $\varphi : I \subseteq (0, \infty) \to \mathbb{R}$ *as above. Then,*

(*i*) χ *is GA-convex on* $(0, 1]$.

(*ii*) *We have*

$$\inf_{\varrho\in[0,1]}\chi(\varrho) = \chi(0) = \varphi(\sqrt{\nu_1\nu_2})$$

and

$$\sup_{\varrho\in[0,1]}\chi(\varrho) = \chi(1) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi.$$

(*iii*) χ *increases monotonically on* $[0, 1]$.

The following theorem holds:

Theorem 6. [47] *Let* $\varphi : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ *be as above. Then,*

(*i*) $\mathcal{F}(\varrho + \frac{1}{2}) = \mathcal{F}(\frac{1}{2} - \varrho)$ *for all* ϱ *in* $[0, \frac{1}{2}]$.

(*ii*) $\mathcal{F}$ *is GA-convex on* $(0, 1]$.

(*iii*) *We have*

$$\sup_{\varrho\in[0,1]} \mathcal{F}(\varrho) = \mathcal{F}(0) = \mathcal{F}(1) = \frac{1}{(\ln\nu_2 - \ln\nu_1)^2}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}\varphi(\xi)\mathrm{d}\xi$$

and

$$\inf_{\varrho\in[0,1]} \mathcal{F}(\varrho) = \mathcal{F}\left(\frac{1}{2}\right) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\int_{\nu_1}^{\nu_2}\frac{1}{\xi\lambda}\varphi(\sqrt{\xi\lambda})\mathrm{d}\xi\mathrm{d}\lambda.$$

(*iv*) *The following inequality is valid*:

$$\varphi(\sqrt{\xi\lambda}) \le \mathcal{F}\left(\frac{1}{2}\right).$$

(*v*) $\mathcal{F}$ *decreases monotonically on* $[0, \frac{1}{2}]$ *and increases monotonically on* $\left[\frac{1}{2}, 1\right]$.

(*vi*) *We have the inequality* $\chi(\varrho) \le \mathcal{F}(\varrho)$ *for all* $\varrho \in [0, 1]$.

Theorem 7. [47] *Let* $\mathcal{V} : [0, 1] \to \mathbb{R}$ *and* $\varphi : [\nu_1, \nu_2] \subset (0, \infty) \to \mathbb{R}$ *be as defined above. Then,*

(*i*) $\mathcal{V}$ *is GA-convex on* $(0, 1]$.

(*ii*) *The following hold*:

$$\inf_{\varrho\in[0,1]} \mathcal{V}(\varrho) = \mathcal{V}(0) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi$$

and

$$\sup_{\varrho\in[0,1]} \mathcal{V}(\varrho) = \mathcal{V}(1) = \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}.$$

(*iii*) $\mathcal{V}$ *increases monotonically on* $[0, 1]$.

Motivated by the studies conducted in [48,30–42], we define some new functionals involving *GA*-convex functions and a non-negative integrable symmetric weight functions which is with respect to the geometric mean of the end points of the closed interval in connection to Inequalities (1.5) and (1.6) to prove new Féjer-type inequalities which indeed provide refinement inequalities as well.

2 Main results

Let us now define some mappings on $[0, 1]$ related to (1.6) and prove some refinement inequalities.

$$\chi_1(\varrho) = \frac{1}{2}\left[\varphi\left(\nu_1^{\frac{1+\varrho}{2}}\nu_2^{\frac{1-\varrho}{2}}\right) + \varphi\left(\nu_1^{\frac{1-\varrho}{2}}\nu_2^{\frac{1+\varrho}{2}}\right)\right],$$

$$\chi(\varrho) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho})\mathrm{d}\xi,$$

$$\chi_\vartheta(\varrho) = \int_{\nu_1}^{\nu_2} \varphi(\xi^\varrho(\sqrt{\nu_1\nu_2})^{1-\varrho})\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi,$$

$$\mathcal{T}(\varrho) = \frac{1}{2(\ln\nu_2 - \ln\nu_1)}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}[\varphi(\nu_1^\varrho\xi^{1-\varrho}) + \varphi(\nu_2^\varrho\xi^{1-\varrho})]\mathrm{d}\xi,$$

and

$$\mathcal{T}_\vartheta(\varrho) = \frac{1}{2}\int_{\nu_1}^{\nu_2}[\varphi(\nu_1^\varrho\xi^{1-\varrho}) + \varphi(\nu_2^\varrho\xi^{1-\varrho})]\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi,$$

where $\varphi : [\nu_1, \nu_2] \to \mathbb{R}$ is a *GA*-convex function and $\vartheta : [\nu_1, \nu_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\xi = \sqrt{\nu_1\nu_2}$.

The following result is very important to establish the results of this section.

Lemma 1. [24] *Let $\varphi : [\nu_1, \nu_2] \to \mathbb{R}$ be a GA-convex function and let $\nu_1 \le \lambda_1 \le \xi_1 \le \xi_2 \le \lambda_2 \le \nu_2$ with $\xi_1\xi_2 = \lambda_1\lambda_2$. Then,*

$$\varphi(\xi_1) + \varphi(\xi_2) \le \varphi(\lambda_1) + \varphi(\lambda_2).$$

Proof. For $\lambda_1 = \lambda_2$, the result is obvious. We observe that

$$\xi_1 = (\lambda_1)^{\frac{\ln\lambda_2 - \ln\xi_1}{\ln\lambda_2 - \ln\lambda_1}}(\lambda_2)^{\frac{\ln\xi_1 - \ln\lambda_1}{\ln\lambda_2 - \ln\lambda_1}}$$

and

$$\xi_2 = (\lambda_1)^{\frac{\ln\lambda_2 - \ln\xi_2}{\ln\lambda_2 - \ln\lambda_1}}(\lambda_2)^{\frac{\ln\xi_2 - \ln\lambda_1}{\ln\lambda_2 - \ln\lambda_1}}$$

are in the interval $[\nu_1, \nu_2]$, and $\xi_1\xi_2 = \lambda_1\lambda_2$.

By applying the *GA*-convexity, we obtain

$$\begin{aligned}\varphi(\xi_1) + \varphi(\xi_2) &\le \left(\frac{\ln\lambda_2 - \ln\xi_1}{\ln\lambda_2 - \ln\lambda_1}\right)\varphi(\lambda_1) + \left(\frac{\ln\xi_1 - \ln\lambda_1}{\ln\lambda_2 - \ln\lambda_1}\right)\varphi(\lambda_2) + \left(\frac{\ln\lambda_2 - \ln\xi_2}{\ln\lambda_2 - \ln\lambda_1}\right)\varphi(\lambda_1) + \left(\frac{\ln\xi_2 - \ln\lambda_1}{\ln\lambda_2 - \ln\lambda_1}\right)\varphi(\lambda_2)\\ &= \varphi(\lambda_1) + \varphi(\lambda_2).\end{aligned}$$

□

We first prove a result similar to results proved in [39] for *GA*-convex functions which provide refinement inequalities for (1.6).

Theorem 8. *Let $\varphi : [\nu_1, \nu_2] \subset (0, \infty) \to \mathbb{R}$ be a GA-convex function, $0 < \alpha < 1$, $0 < \beta < 1$, $\sigma = \nu_1^\alpha\nu_2^{1-\alpha}$, $\nu_0 = \nu_2\nu_1^{-1}\min\left\{\frac{\alpha}{1-\beta}, \frac{1-\alpha}{\beta}\right\}$ and let $\vartheta : [\nu_1, \nu_2] \to \mathbb{R}$ be non-negative and integrable and $\vartheta(\sigma\varrho^{-\beta}) = \vartheta(\sigma\varrho^{1-\beta})$, $\varrho \in [0, \nu_0]$. Then,*

$$\begin{aligned}\varphi(\nu_1^\alpha\nu_2^{1-\alpha})\int_{\sigma\varrho^{-\beta}}^{\sigma\varrho^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi &\le \frac{1-\beta}{\beta}\int_{\sigma\varrho^{-\beta}}^{\sigma}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi + \frac{\beta}{1-\beta}\int_{\sigma}^{\sigma\varrho^{1-\beta}}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi\\ &\le [\alpha\varphi(\nu_2) + (1-\alpha)\varphi(\nu_1)]\int_{\sigma\varrho^{-\beta}}^{\sigma\varrho^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi.\end{aligned} \tag{2.1}$$

Proof. For every $\varrho \in [0, \nu_0]$, we have the identity

$$\begin{aligned}\int_{\sigma\varrho^{-\beta}}^{\sigma\varrho^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi &= \int_{\sigma\varrho^{-\beta}}^{\sigma\varrho^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi + \int_{\sigma}^{\sigma\varrho^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi = \beta\int_0^\varrho\frac{\vartheta(\sigma\varrho^{-\beta})}{\xi}\mathrm{d}\xi + (1-\beta)\int_0^\varrho\frac{\vartheta(\sigma\varrho^{-\beta})}{\xi}\mathrm{d}\xi\\ &= \int_0^\varrho\frac{\vartheta(\sigma\varrho^{-\beta})}{\xi}\mathrm{d}\xi.\end{aligned} \tag{2.2}$$

We now prove that the mapping $\mathcal{W}:[0,\nu_0]\to\mathbb{R}$ defined by

$$\mathcal{W}(\varrho)=(1-\beta)\varphi(\sigma\varrho^{-\beta})+\beta\varphi(\sigma\varrho^{1-\beta})$$

is GA-convex $(0,\nu_0]$ and monotonically increasing on $[0,\nu_0]$.

Since the sum of two GA-convex functions is a GA-convex, hence $\mathcal{W}$ is a GA-convex on $(0,\nu_0]$. Let $\varrho\in(0,\nu_0]$, it follows from the GA-convexity of φ that

$$\mathcal{W}(\varrho)=(1-\beta)\varphi(\sigma\varrho^{-\beta})+\beta\varphi(\sigma\varrho^{1-\beta})\geq\varphi(\sigma^{1-\beta}\varrho^{-\beta(1-\beta)}\sigma^{\beta}\varrho^{\beta(1-\beta)})=\varphi(\sigma)=\varphi(\nu_1^{\alpha}\nu_2^{1-\alpha}).\tag{2.3}$$

We observe that $0<\alpha\leq\alpha+\frac{\beta\ln\varrho}{\ln\nu_2-\ln\nu_1}\leq1$, $0\leq(1-\alpha)-\frac{\beta\ln\varrho}{\ln\nu_2-\ln\nu_1}\leq1-\alpha<1$, $0\leq\alpha\leq\alpha-\frac{(1-\beta)\ln\varrho}{\ln\nu_2-\ln\nu_1}\leq\alpha\leq1$, and $0<1-\alpha\leq(1-\alpha)+\frac{(1-\beta)\ln\varrho}{\ln\nu_2-\ln\nu_1}\leq1$. Thus, by Remark 1 $\varphi\circ\exp$ is convex on $[\ln\nu_0,\infty)$ and hence we obtain

$$\begin{aligned}\mathcal{W}\circ\exp(\ln\varrho)&=(1-\beta)\varphi\circ\exp(\ln\sigma-\beta\ln\varrho)+\beta\varphi(\ln\sigma+(1-\beta)\ln\varrho)\\&=(1-\beta)\varphi\circ\exp\left(\left(\alpha+\frac{\beta\ln\varrho}{\ln\nu_2-\ln\nu_1}\right)\ln\nu_1+\left((1-\alpha)-\frac{\beta\ln\varrho}{\ln\nu_2-\ln\nu_1}\right)\ln\nu_2\right)\\&\quad+\beta\varphi\left(\left(\alpha-\frac{(1-\beta)\ln\varrho}{\ln\nu_2-\ln\nu_1}\right)\ln\nu_1+\left((1-\alpha)+\frac{(1-\beta)\ln\varrho}{\ln\nu_2-\ln\nu_1}\right)\ln\nu_2\right)\\&\leq(1-\beta)\left(\alpha+\left(\frac{\beta}{\ln\nu_2-\ln\nu_1}\right)\ln\varrho\right)\varphi\circ\exp(\ln\nu_1)\\&\quad+(1-\beta)\left((1-\alpha)-\left(\frac{\beta}{\ln\nu_2-\ln\nu_1}\right)\ln\varrho\right)\varphi\circ\exp(\ln\nu_2)\\&\quad+\beta\left(\alpha-\left(\frac{1-\beta}{\ln\nu_2-\ln\nu_1}\right)\ln\varrho\right)\varphi\circ\exp(\ln\nu_1)\\&\quad+\beta\left((1-\alpha)+\left(\frac{1-\beta}{\ln\nu_2-\ln\nu_1}\right)\ln\varrho\right)\varphi\circ\exp(\ln\nu_2)=\alpha\varphi(\nu_1)+(1-\alpha)\varphi(\nu_2).\end{aligned}\tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\varphi(\nu_1^{\alpha}\nu_2^{1-\alpha})\leq\mathcal{W}(\varrho)\leq\alpha\varphi(\nu_1)+(1-\alpha)\varphi(\nu_2).\tag{2.5}$$

Finally, for $\varrho_1,\varrho_2\in(0,\nu_0]$, such that $0<\ln\varrho_1<\ln\varrho_2\leq\ln\nu_0$, since $\mathcal{W}\circ\exp(\ln\varrho)$ is convex, it follows from (2.3) that

$$\frac{\mathcal{W}\circ\exp(\ln\varrho_2)-\mathcal{W}\circ\exp(\ln\varrho_1)}{\ln\varrho_2-\ln\varrho_1}=\frac{\mathcal{W}(\varrho_2)-\mathcal{W}(\varrho_1)}{\ln\varrho_2-\ln\varrho_1}\geq0.$$

This shows that $\mathcal{W}$ is increasing on $(0,\nu_0]$.

Since ϑ is non-negative, multiplying (2.5) with $\frac{\vartheta(\sigma\varrho^{-\beta})}{\xi}$, integrating the resulting inequalities over $[0,\varrho]$, and using $\vartheta(\sigma\varrho^{-\beta})=\vartheta(\sigma\varrho^{1-\beta})$, we have

$$\begin{aligned}\varphi(\nu_1^{\alpha}\nu_2^{1-\alpha})\int_0^{\varrho}\frac{\vartheta(\sigma\varrho^{-\beta})}{\xi}\mathrm{d}\xi&\leq(1-\beta)\int_0^{\varrho}\frac{\varphi(\sigma\varrho^{-\beta})\vartheta(\sigma\varrho^{-\beta})}{\xi}\mathrm{d}\xi+\beta\int_0^{\varrho}\frac{\varphi(\sigma\varrho^{1-\beta})\vartheta(\sigma\varrho^{1-\beta})}{\xi}\mathrm{d}\xi\\&\leq[\alpha\varphi(\nu_1)+(1-\alpha)\varphi(\nu_2)]\int_0^{\varrho}\frac{\vartheta(\sigma\varrho^{-\beta})}{\xi}\mathrm{d}\xi.\end{aligned}\tag{2.6}$$

By using Identity (2.2) in (2.6), we obtain (2.1). □

Remark 2. If we choose $\alpha = \frac{\vartheta}{\vartheta+q}$, $\beta = \frac{1}{2}$, $\varrho = \lambda^2$ in Theorem 8, then

$$\begin{aligned}\varphi\left(v_1^{\frac{\vartheta}{\vartheta+q}} v_2^{\frac{q}{\vartheta+q}}\right)\int\limits_{\lambda^{-1}\sigma}^{\lambda\sigma}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi &\le \frac{1-\beta}{\beta}\int\limits_{\lambda^{-1}\sigma}^{\sigma}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi + \frac{\beta}{1-\beta}\int\limits_{\sigma}^{\lambda\sigma}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi\\ &\le [\alpha\varphi(v_1) + (1-\alpha)\varphi(v_2)]\int\limits_{\lambda^{-1}\sigma}^{\lambda\sigma}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi.\end{aligned} \tag{2.7}$$

Remark 3. If we choose $\alpha = \beta = \frac{1}{2}$, $\varrho = v_0 = v_2 v_1^{-1}$ in Theorem 8, then we obtain (1.6).

Remark 4. If we choose $\alpha = \beta = \frac{1}{2}$, $\varrho = v_0 = v_2 v_1^{-1}$ in Theorem 8, then we obtain (1.5).

Theorem 9. *Let φ, σ, and v_0 be defined as in Theorem 8, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \le 1$, and let $\mathcal{X}$ be defined on $[0, 1]$ as*

$$\mathcal{X}(\varrho) = \frac{1-\beta}{\alpha(\ln v_2 - \ln v_1)}\int\limits_{0}^{v_2^{\frac{\alpha}{1-\beta}} v_1^{-\frac{\alpha}{1-\beta}}}[(1-\beta)\varphi(\sigma\xi^{-\varrho\beta}) + \beta\varphi(\sigma\xi^{(1-\beta)\varrho})]\frac{\mathrm{d}\xi}{\xi}. \tag{2.8}$$

Then, $\mathcal{X}$ is GA-convex on $(0, 1]$, monotonically increasing on $[0, 1]$ and

$$\begin{aligned}\varphi(v_1^{\alpha}v_2^{1-\alpha}) &\le \mathcal{X}(\varrho) \le \mathcal{X}(1)\\ &= \frac{1-\beta}{\alpha(\ln v_2 - \ln v_1)}\int\limits_{0}^{v_2^{\frac{\alpha}{1-\beta}} v_1^{-\frac{\alpha}{1-\beta}}}\frac{1}{\xi}[(1-\beta)\varphi(\sigma\xi^{-\varrho\beta}) + \beta\varphi(\sigma\xi^{(1-\beta)\varrho})]\mathrm{d}\xi\\ &\le \alpha\varphi(v_1) + (1-\alpha)\varphi(v_2).\end{aligned}$$

Proof. Since φ is GA-convex on $[v_1, v_2]$ this prove the GA-convexity of $\mathcal{X}$ on $(0, v_0]$. By using the condition $\alpha + \beta \le 1$ implies that $v_0 = \frac{\alpha}{1-\beta}v_2 v_1^{-1}$. Since the mapping $\mathcal{W} : [0, v_0] \to \mathbb{R}$ defined by

$$\mathcal{W}(\varrho) = (1-\beta)\varphi(\sigma\varrho^{-\beta}) + \beta\varphi(\sigma\varrho^{1-\beta}) \tag{2.9}$$

has been proved to be monotonically increasing on $[0, v_0]$, thus the mapping $\mathcal{X}$ is also monotonically increasing on $[0, 1]$.

Inequality (2.8) follows from inequality (2.5) and the fact that $\mathcal{X}$ is monotonically increasing on $[0, 1]$. This accomplished the proof of the theorem. □

The next theorem can be proved similarly.

Theorem 10. *Let φ, χ, v_0, α, and β be defined as in Theorem 9. Let $\mathcal{X}_1$ be defined on $[0, 1]$ as*

$$\mathcal{X}_1(\varrho) = \frac{1-\beta}{\alpha(\ln v_2 - \ln v_1)}\int\limits_{0}^{v_2^{\frac{\alpha}{1-\beta}} v_1^{-\frac{\alpha}{1-\beta}}}\left[(1-\beta)\varphi\left(\sigma v_1^{\frac{\alpha\beta}{1-\beta}} v_2^{-\frac{\alpha\beta}{1-\beta}}\xi^{\beta(1-\varrho)}\right) + \beta\varphi(\sigma v_1^{-\alpha}v_2^{\alpha}\xi^{(1-\beta)(1-\varrho)})\right]\frac{\mathrm{d}\xi}{\xi}. \tag{2.10}$$

Then, $\mathcal{X}_1$ is GA-convex monotonically increasing on $[0, 1]$, and

$$\begin{aligned}&\frac{(1-\beta)^2}{\alpha\beta(\ln v_2 - \ln v_1)}\int\limits_{v_1^{\frac{\alpha\beta}{1-\beta}} v_2^{1-\frac{\alpha\beta}{1-\beta}}}^{\sigma}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi + \frac{\beta}{\alpha(\ln v_2 - \ln v_1)}\int\limits_{\sigma}^{v_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi\\ &\le \mathcal{X}_1(\varrho) \le \mathcal{X}_1(1) = (1-\beta)\varphi\left(v_1^{\frac{\alpha}{1-\beta}} v_2^{1-\frac{\alpha}{1-\beta}}\right) + \beta\varphi(v_1) \le \alpha\varphi(v_1) + (1-\alpha)\varphi(v_2).\end{aligned} \tag{2.11}$$

Remark 5. Taking $\alpha = \beta = \frac{1}{2}$ in inequality (2.8) reduces to

$$\chi(\varrho) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho})\mathrm{d}\xi.$$

Remark 6. Taking $\alpha = \beta = \frac{1}{2}$ in inequality (2.10) reduces to

$$\mathcal{V}(\varrho) = \frac{1}{2(\ln\nu_2 - \ln\nu_1)}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}\left[\varphi\left(\nu_2^{\frac{1+\varrho}{2}}\xi^{\frac{1-\varrho}{2}}\right) + \varphi\left(\nu_1^{\frac{1+\varrho}{2}}\xi^{\frac{1-\varrho}{2}}\right)\right]\mathrm{d}\xi. \tag{2.12}$$

Theorem 11. *Let φ, α, β, χ, ν_0 be defined as in Theorem 9 and let ϑ be defined as in Theorem 8. Let $\mathcal{Y}$ be a function defined on* $[0,1]$ *by*

$$\mathcal{Y}(\varrho) = \int_0^s[(1-\beta)\varphi(\sigma\varrho\xi^{-\beta\varrho})\vartheta(\sigma\xi^{-\beta}) + \beta\varphi(\sigma\xi^{(1-\beta)\varrho})\vartheta(\sigma\xi^{1-\beta})]\mathrm{d}\xi \tag{2.13}$$

for some $s \in [0,\nu_0]$. *Then,* $\mathcal{Y}$ *is GA-convex and monotonically increasing on* $[0,1]$ *and*

$$\varphi(\nu_1^{\alpha}\nu_2^{1-\alpha})\int_{\sigma s^{\beta}}^{\sigma s^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \le \mathcal{Y}(\varrho) \le \mathcal{Y}(1) = \frac{1-\beta}{\beta}\int_{\sigma s^{\beta}}^{\sigma}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi + \frac{\beta}{1-\beta}\int_{\sigma}^{\sigma s^{1-\beta}}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.14}$$

Proof. Since φ is GA-convex and ϑ is non-negative, we see that $\mathcal{Y}$ is GA-convex on $(0,1]$. Next for each $\xi \in [0,s]$, where $s \in [0,\nu_0]$, it follows from Theorem 8 that $h(\xi^{\varrho}) = (1-\beta)\varphi(\sigma\varrho\xi^{-\beta\varrho}) + \beta\varphi(\sigma\xi^{(1-\beta)\varrho})$ is increasing for $\varrho \in [0,1]$. Using the identity $\vartheta(\sigma\varrho^{\beta}) = \vartheta(\sigma\varrho^{1-\beta})$, we see that $\mathcal{Y}(\varrho)$ is increasing on $[0,1]$. Therefore, inequality (2.14) follows immediately. □

Theorem 12. *Let φ, α, β, σ, ν_0 be defined as in Theorem 11, and let ϑ be defined as in Theorem 8. Let $\mathcal{Y}_1$ be a function defined on* $[0,1]$ *by*

$$\mathcal{Y}_1(\varrho) = \int_0^s\frac{1}{\xi}[(1-\beta)\varphi(\sigma s^{-\beta}\xi^{\beta(1-\varrho)})\vartheta(\sigma s^{-\beta}\xi^{\beta}) + \beta\varphi(\sigma s^{1-\beta}\xi^{-(1-\beta)(1-\varrho)})\vartheta(\sigma s^{1-\beta}\xi^{-(1-\beta)})]\mathrm{d}\xi \tag{2.15}$$

for some $s \in [0,\nu_0]$. *Then,* $\mathcal{Y}_1$ *is GA-convex* $(0,1]$ *and monotonically increasing on* $[0,1]$, *and*

$$\begin{aligned}\frac{1-\beta}{\beta}&\int_{\sigma s^{-\beta}}^{\sigma}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi + \frac{\beta}{1-\beta}\int_{\sigma}^{\sigma s^{1-\beta}}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi\\ &\le \mathcal{Y}(\varrho) \le \mathcal{Y}(1) = [(1-\beta)\varphi(\sigma s^{-\beta}) + (1-\beta)\varphi(\sigma s^{1-\beta})]\\ &\quad\times\int_{\sigma s^{-\beta}}^{\sigma s^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \le [(1-\alpha)\varphi(\nu_1) + \alpha\varphi(\nu_2)]\int_{\sigma s^{-\beta}}^{\sigma s^{1-\beta}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi.\end{aligned} \tag{2.16}$$

Proof. Since φ is GA-convex and ϑ is non-negative, we see that $\mathcal{Y}$ is GA-convex on $(0,1]$. Next for each $\xi \in [0,\varrho]$, where $\varrho \in [0,\nu_0]$, it follows from Theorem 8 that $h(\varrho) = (1-\beta)\varphi(\sigma s^{-\beta}) + \beta\varphi(\sigma s^{1-\beta})$ and $k(\varrho) = s\xi^{-(1-\varrho)}$ are increasing on $[0,\nu_0]$ and $[0,1]$, respectively. Hence

$$h(k(\varrho)) = (1-\beta)\varphi(\sigma s^{-\beta}\xi^{\beta(1-\varrho)})\vartheta(\sigma s^{-\beta}\xi^{\beta}) + \beta\varphi(\sigma s^{1-\beta}\xi^{-(1-\beta)(1-\varrho)})\vartheta(\sigma s^{1-\beta}\xi^{-(1-\beta)})$$

is increasing on $[0,1]$. Using the identity $\vartheta(\sigma\varrho^{-\beta}) = \vartheta(\sigma\varrho^{1-\beta})$, we see that $\mathcal{Y}_1(\varrho)$ is increasing on $[0,1]$. Therefore, Inequalities (2.16) follows from

$$\varphi(\nu_1^{\alpha}\nu_2^{1-\alpha}) \le \mathcal{W}(k(\varrho)) \le (1-\alpha)\varphi(\nu_1) + \alpha\varphi(\nu_2)$$

and (2.16). □

Remark 7. Choose $\alpha = \beta = \frac{1}{2}$, $s = \nu_0 = \nu_1^{-1}\nu_2$ in Theorems 11 and 12. Then, Inequalities (2.14) and (2.16) reduce to

$$\begin{aligned}\varphi(\sqrt{\nu_1\nu_2})\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi &\le \mathcal{Y}(\varrho) \le \mathcal{Y}(1) = \int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi \\ &\le \mathcal{Y}_1(\varrho) \le \mathcal{Y}_1(1) = \frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi,\end{aligned} \tag{2.17}$$

where

$$\mathcal{Y}(\varrho) = \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho})\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi$$

and

$$\mathcal{Y}_1(\varrho) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\frac{1}{\xi}\left[\varphi\left(\nu_1^{\frac{1+\varrho}{2}}\xi^{\frac{1-\varrho}{2}}\right)\vartheta(\sqrt{\nu_1\xi}) + \varphi\left(\nu_2^{\frac{1+\varrho}{2}}\nu_1^{\frac{1-\varrho}{2}}\right)\vartheta(\sqrt{\xi\nu_2})\right]\mathrm{d}\xi. \tag{2.18}$$

Remark 8. Inequality (2.17) provides weighted generalizations of Theorems 5 and 7.

In the following results, we provide further refinements of Inequalities (1.5) and (1.6) for *GA*-convex functions by using Lemma 1.

Theorem 13. *Let φ, ϑ, χ_ϑ be defined as above. Then, the following Fejér-type inequalities hold*:

(*i*) *The following inequality holds*:

$$\begin{aligned}\varphi(\sqrt{\nu_1\nu_2})\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi &\le 2\int_{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}^{\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}}\varphi(\xi)\vartheta\left(\frac{\xi^2}{\sqrt{\nu_1\nu_2}}\right)\frac{\mathrm{d}\xi}{\xi} \\ &\le \int_0^1\chi_\vartheta(\varrho)\mathrm{d}\varrho \le \frac{1}{2}\left[\varphi(\sqrt{\nu_1\nu_2})\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi + \int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi\right].\end{aligned} \tag{2.19}$$

(*ii*) *If φ is differentiable on* $[\nu_1, \nu_2]$ *and ϑ is bounded on* $[\nu_1, \nu_2]$, *then for all* $\varrho \in [0, 1]$, *the inequality holds*:

$$\begin{aligned}0 &\le \int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi - \chi_\vartheta(\varrho) \\ &\le (1-\varrho)\left[(\ln\nu_2 - \ln\nu_1)\left[\frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\right] - \int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi\right]\|\vartheta\|_\infty,\end{aligned} \tag{2.20}$$

where $\|\vartheta\|_\infty = \sup_{\xi\in[\nu_1,\nu_2]}\vartheta(\xi)$.

(*iii*) *If φ is differentiable on* $[\nu_1, \nu_2]$, *then for all* $\varrho \in [0, 1]$, *we have the inequality*

$$0 \le \frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi - \chi_\vartheta(\varrho) \le \frac{(\ln\nu_1 - \ln\nu_2)(\nu_2\varphi'(\nu_2) - \nu_1\varphi'(\nu_1))}{4}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.21}$$

Proof. (i) Using techniques of integration and the hypothesis of ϑ, we have the following identities:

$$\varphi(\sqrt{\nu_1\nu_2})\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi^2}\mathrm{d}\xi = 4\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\int_0^{\frac{1}{2}}\varphi(\sqrt{\nu_1\nu_2})\frac{\vartheta(\xi)}{\xi}\mathrm{d}\varrho\mathrm{d}\xi, \tag{2.22}$$

$$2\int_{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}^{\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}}\varphi(\xi)\vartheta\left(\frac{\xi^2}{\sqrt{\nu_1\nu_2}}\right)\frac{d\xi}{\xi} = 2\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\int_0^{\frac{1}{2}}\left[\varphi\left(\xi^{\frac{1}{2}}\nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}\right) + \varphi\left(\xi^{-\frac{1}{2}}\nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}\right)\right]\frac{\vartheta(\xi)}{\xi}d\varrho d\xi, \tag{2.23}$$

$$\begin{aligned}\int_0^1\chi_\vartheta(\varrho)d\varrho &= \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\int_0^{\frac{1}{2}}[\varphi(\xi^{1-\varrho}(\sqrt{\nu_1\nu_2})^{\varrho}) + \varphi((\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}))]\frac{\vartheta(\xi)}{\xi}d\varrho d\xi \\ &+ \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\int_0^{\frac{1}{2}}[\varphi(\xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{1+\varrho}) + \varphi(\xi^{-(1-\varrho)}(\sqrt{\nu_1\nu_2})^{2-\varrho})]\frac{\vartheta(\xi)}{\xi}d\varrho d\xi,\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\varphi(\sqrt{\nu_1\nu_2})\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}d\xi + \int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}d\xi\right] \\ &= \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\int_0^{\frac{1}{2}}[\varphi(\xi) + \varphi(\sqrt{\nu_1\nu_2})]\frac{\vartheta(\xi)}{\xi}d\varrho d\xi + \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\int_0^{\frac{1}{2}}[\varphi(\sqrt{\nu_1\nu_2}) + \varphi(\nu_1\nu_2\xi^{-1})]\frac{\vartheta(\xi)}{\xi}d\varrho d\xi.\end{aligned} \tag{2.25}$$

By using Lemma 1, we observe that the following inequalities hold for all $\varrho \in \left[0, \frac{1}{2}\right]$ and $\xi \in [\nu_1, \sqrt{\nu_1\nu_2}]$:

The inequality

$$4\varphi(\sqrt{\nu_1\nu_2}) \le 2\left[\varphi\left(\xi^{\frac{1}{2}}\nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}\right) + \varphi\left(\xi^{-\frac{1}{2}}\nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}\right)\right] \tag{2.26}$$

holds for $\xi_1 = \xi_2 = \sqrt{\nu_1\nu_2}$, $\lambda_1 = \xi^{\frac{1}{2}}\nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}$, $\lambda_2 = \xi^{-\frac{1}{2}}\nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}$.

The inequality

$$2\varphi\left(\xi^{\frac{1}{2}}\nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}\right) \le \varphi(\xi^{1-\varrho}(\sqrt{\nu_1\nu_2})^{\varrho}) + \varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}) \tag{2.27}$$

holds for $\xi_1 = \xi_2 = \xi^{\frac{1}{2}}\nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}$, $\lambda_1 = \xi^{1-\varrho}(\sqrt{\nu_1\nu_2})^{\varrho}$, $\lambda_2 = \xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}$.

The inequality

$$2\varphi\left(\xi^{-\frac{1}{2}}\nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}\right) \le \varphi(\xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{1+\varrho}) + \varphi(\xi^{-(1-\varrho)}(\sqrt{\nu_1\nu_2})^{2-\varrho}) \tag{2.28}$$

holds for $\xi_1 = \xi_2 = \xi^{-\frac{1}{2}}\nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}$, $\lambda_1 = \xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{1+\varrho}$, $\lambda_2 = \xi^{-(1-\varrho)}(\sqrt{\nu_1\nu_2})^{2-\varrho}$.

The inequality

$$\varphi(\xi^{1-\varrho}(\sqrt{\nu_1\nu_2})^{\varrho}) + \varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}) \le \varphi(\xi) + \varphi(\sqrt{\nu_1\nu_2}) \tag{2.29}$$

holds for $\xi_1 = \xi^{1-\varrho}(\sqrt{\nu_1\nu_2})^{\varrho}$, $\xi_2 = \xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}$, $\lambda_1 = \xi$, $\lambda_2 = \sqrt{\nu_1\nu_2}$.

Finally, the inequality

$$\varphi(\xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{1+\varrho}) + \varphi(\xi^{-(1-\varrho)}(\sqrt{\nu_1\nu_2})^{2-\varrho}) \le \varphi(\sqrt{\nu_1\nu_2}) + \varphi(\nu_1\nu_2\xi^{-1}). \tag{2.30}$$

Multiplying Inequalities (2.26)–(2.30) with $\frac{\vartheta(\xi)}{\xi}$ and integrating them over ϱ on $\left[0, \frac{1}{2}\right]$, over ξ on $[\nu_1, \sqrt{\nu_1\nu_2}]$, and using identities (2.22)–(2.25), we derive (2.19).

(ii) Since $\varphi:[\nu_1,\nu_2]\to\mathbb{R}$ is GA-convex on $[\nu_1,\nu_2]$, hence $g:[\ln\nu_1,\ln\nu_2]\to\mathbb{R}$ defined by $g(\xi)=\varphi\circ\exp(\xi)$ is convex on $[\ln\nu_1,\ln\nu_2]$. Thus, by integration by parts, we obtain the following identity:

$$\begin{aligned}&\int_{\ln\nu_1}^{\frac{\ln\nu_1+\ln\nu_2}{2}}\left(\frac{\ln\nu_1+\ln\nu_2}{2}-\xi\right)[g'(\ln\nu_1+\ln\nu_2-\xi)-g'(\xi)]\mathrm{d}\xi\\&=\left(\frac{\ln\nu_2-\ln\nu_1}{2}\right)[g(\ln\nu_1)+g(\ln\nu_2)]-\int_{\ln\nu_1}^{\frac{\ln\nu_1+\ln\nu_2}{2}}[g(\ln\nu_1+\ln\nu_2-\xi)+g(\xi)]\mathrm{d}\xi.\end{aligned}\tag{2.31}$$

The equality (2.31) is equivalent to the following equality:

$$\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\frac{1}{\xi}\left(\frac{\ln\nu_1+\ln\nu_2}{2}-\ln\xi\right)[\nu_1\nu_2\xi^{-1}\varphi'(\nu_1\nu_2\xi^{-1})-\xi\varphi'(\xi)]\mathrm{d}\xi=(\ln\nu_2-\ln\nu_1)\left[\frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\right]-\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi.\tag{2.32}$$

Using substitution rules for integration and the hypothesis of ϑ, we have the following identities:

$$\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi=\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}[\varphi(\xi)+\varphi(\nu_1\nu_2\xi^{-1})]\frac{\vartheta(\xi)}{\xi}\tag{2.33}$$

and

$$\chi_\vartheta(\varrho)=\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}[\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho})+\varphi(\xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{1+\varrho})]\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi.\tag{2.34}$$

Now, using the convexity of $g(\xi)=\varphi\circ\exp(\xi)$ on $[\ln\nu_1,\ln\nu_2]$ and the hypothesis of ϑ, the following inequality holds for all $\varrho\in[0,1]$ and $\xi\in\left[\ln\nu_1,\frac{\ln\nu_1+\ln\nu_2}{2}\right]$:

$$\begin{aligned}&\left[g(\xi)-g\left(\varrho\xi+(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)\right)\right]\vartheta\circ\exp(\xi)\\&\quad+\left[g(\ln\nu_1+\ln\nu_2-\xi)-g\left(\varrho(\ln\nu_1+\ln\nu_2-\xi)+(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)\right)\right]\vartheta\circ\exp(\xi)\\&\le(1-\varrho)\left(\xi-\frac{\ln\nu_1+\ln\nu_2}{2}\right)g'(\xi)\vartheta\circ\exp(\xi)+(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}-\xi\right)g'(\ln\nu_1+\ln\nu_2-\xi)\vartheta\circ\exp(\xi)\\&=(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}-\xi\right)[g'(\ln\nu_1+\ln\nu_2-\xi)-g'(\xi)]\vartheta\circ\exp(\xi),\end{aligned}\tag{2.35}$$

which is equivalent to

$$\begin{aligned}&\left[\varphi\circ\exp(\xi)-\varphi\circ\exp\left(\varrho\xi+(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)\right)\right]\vartheta\circ\exp(\xi)\\&\quad+\left[\varphi\circ\exp(\ln\nu_1+\ln\nu_2-\xi)-\varphi\circ\exp\left(\varrho(\ln\nu_1+\ln-\xi)+(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)\right)\right]\vartheta\circ\exp(\xi)\\&\le(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}-\xi\right)[\exp(\xi)\varphi'(\exp(\xi))\\&\quad+\nu_1\nu_2\exp(\xi^{-1})(1-\varrho)\varphi'(\nu_1\nu_2\exp(\xi^{-1}))]\sup_{\xi\in[\ln\nu_1,\ln\nu_2]}|\vartheta\circ\exp(\xi)|.\end{aligned}\tag{2.36}$$

Integrating the above inequalities over ξ on $\left[\ln\nu_1,\frac{\ln\nu_1+\ln\nu_2}{2}\right]$, we obtain

$$\begin{aligned}&\int_{\ln\nu_1}^{\frac{\ln\nu_1+\ln\nu_2}{2}}\left[\varphi\circ\exp(\xi)-\varphi\circ\exp\left(\varrho\xi+(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)\right)\right]\vartheta\circ\exp(\xi)\mathrm{d}\xi\\&+\int_{\ln\nu_1}^{\frac{\ln\nu_1+\ln\nu_2}{2}}\left[\varphi\circ\exp(\ln\nu_1+\ln\nu_2-\xi)-\varphi\circ\exp\left(\varrho(\ln\nu_1+\ln-\xi)+(1-\varrho)\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)\right)\right]\vartheta\circ\exp(\xi)\mathrm{d}\xi\\&\leq(1-\varrho)\sup_{\xi\in[\ln\nu_1,\ln\nu_2]}|\vartheta\circ\exp(\xi)|\int_{\ln\nu_1}^{\frac{\ln\nu_1+\ln\nu_2}{2}}\left(\frac{\ln\nu_1+\ln\nu_2}{2}-\xi\right)\\&\quad\times[\nu_1\nu_2\exp(\xi^{-1})\varphi'(\nu_1\nu_2\exp(\xi^{-1}))-\exp(\xi)\varphi'(\exp(\xi))]\mathrm{d}\xi.\end{aligned}\tag{2.37}$$

After making use of suitable substitution, inequality (2.37) takes the form:

$$\begin{aligned}&\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\frac{1}{\xi}[\varphi(\xi)-\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho})]\vartheta(\xi)\mathrm{d}\xi\\&\quad+\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\frac{1}{\xi}\left[\varphi(\nu_1\nu_2\xi^{-1})-\varphi\left(\xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{\frac{1+\varrho}{2}}\right)\right]\vartheta(\xi)\mathrm{d}\xi\leq\|\vartheta\|_\infty(1-\varrho)\\&\quad\times\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\left(\frac{\ln\nu_1+\ln\nu_2}{2}-\ln\xi\right)[\nu_1\nu_2\xi^{-1}\varphi'(\nu_1\nu_2\xi^{-1})-\xi\varphi'(\xi)]\frac{\mathrm{d}\xi}{\xi}.\end{aligned}\tag{2.38}$$

Inequality (2.20) follows from (2.31), (2.32), (2.33), (2.34), and (2.38).
(iii) We use the fact that $\varphi:[\nu_1,\nu_2]\to\mathbb{R}$ is GA-convex on $[\nu_1,\nu_2]$, hence $g:[\ln\nu_1,\ln\nu_2]\to\mathbb{R}$ defined by $g(\xi)=\varphi\circ\exp(\xi)$ is convex on $[\ln\nu_1,\ln\nu_2]$. Thus,

$$\frac{g(\ln\nu_1)-g\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)}{2}\leq\frac{\ln\nu_1-\ln\nu_2}{4}g'(\ln\nu_1)$$

and

$$\frac{g(\ln\nu_2)-g\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)}{2}\leq\frac{\ln\nu_2-\ln\nu_1}{4}g'(\ln\nu_2).$$

Adding the above inequalities

$$\frac{g(\ln\nu_1)+g(\ln\nu_2)}{2}-g\left(\frac{\ln\nu_1+\ln\nu_2}{2}\right)\leq\frac{(\ln\nu_2-\ln\nu_1)(g'(\ln\nu_2)-g'(\ln\nu_1))}{4}.\tag{2.39}$$

Inequality (2.39) becomes

$$\frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}-\varphi(\sqrt{\nu_1\nu_2})\leq\frac{(\ln\nu_2-\ln\nu_1)(\nu_2\varphi'(\nu_2)-\nu_1\varphi'(\nu_1))}{4}.\tag{2.40}$$

Multiplying both right-hand side and left-hand side of (2.40) with $\frac{\vartheta(\xi)}{\xi^2}$ and integrating over $[\nu_1,\nu_2]$, we obtain

$$\frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi-\varphi(\sqrt{\nu_1\nu_2})\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi\leq\frac{(\ln\nu_2-\ln\nu_1)(\nu_2\varphi'(\nu_2)-\nu_1\varphi'(\nu_1))}{4}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi.\tag{2.41}$$

From (2.17) and (2.41) we obtain (2.21). □

Corollary 1. *Suppose that the assumption of Theorem 13 is satisfied and* $\vartheta(\xi)=\frac{1}{\ln\nu_2-\ln\nu_1}$, $\xi\in[\nu_1,\nu_2]$, *then*
(*i*) *The following inequalities hold*:

$$\varphi(\sqrt{\nu_1\nu_2}) \leq 2\int_{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}^{\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}} \frac{\varphi(\xi)}{\xi} \leq \int_0^1 \chi(\varrho)\mathrm{d}\varrho \leq \frac{1}{2}\left[\varphi(\sqrt{\nu_1\nu_2}) + \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi\right]. \tag{2.42}$$

(*ii*) *The following inequalities hold for all* $\varrho \in [0,1]$:

$$0 \leq \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi - \chi(\varrho) \leq (1-\varrho)\left[\frac{\varphi(\nu_1) + \varphi(\nu_2)}{2} - \frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)}{\xi}\mathrm{d}\xi\right]. \tag{2.43}$$

(*iii*) *The following inequalities are valid for all* $\varrho \in [0,1]$:

$$0 \leq \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2} - \chi(\varrho) \leq \frac{(\ln\nu_2 - \ln\nu_1)(\nu_2\varphi'(\nu_2) - \nu_1\varphi'(\nu_1))}{4}. \tag{2.44}$$

In the following theorems, we discuss inequalities for the functions χ, χ_ϑ, χ_1, $\mathcal{T}$ and $\mathcal{T}_\vartheta$ as considered above:

Theorem 14. *Let* φ, ϑ, χ_1, χ_ϑ *be defined as above. Then, we have the following Fejér-type inequalities*:

(*i*) *The following inequality holds for all* $\varrho \in [0,1]$:

$$\chi_\vartheta(\varrho) \leq \chi_1(\varrho)\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.45}$$

(*ii*) *The following inequalities hold*:

$$\begin{aligned} 2\int_{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}^{\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}} \varphi(\xi)\vartheta\left(\frac{\xi^2}{\sqrt{\nu_1\nu_2}}\right)\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi &\leq \frac{1}{2}\left[\varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}\right) + \varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right)\right]\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \\ &\leq (\ln\nu_2 - \ln\nu_1)\int_0^1 \chi_1(\varrho)\vartheta(\nu_1^{1-\varrho}\nu_2^{\varrho})\mathrm{d}\varrho \\ &\leq \frac{1}{2}\left[\varphi(\sqrt{\nu_1\nu_2}) + \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}\right]\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \end{aligned} \tag{2.46}$$

(*iii*) *If* φ *is differentiable on* $[\nu_1, \nu_2]$ *and* ϑ *is bounded on* $[\nu_1, \nu_2]$, *then, for all* $\varrho \in [0,1]$, *we have the following inequality*:

$$0 \leq \chi_\vartheta(\varrho) - \varphi(\sqrt{\nu_1\nu_2})\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \leq (\ln\nu_2 - \ln\nu_1)[\chi_1(\varrho) - \chi(\varrho)]\|\vartheta\|_\infty, \tag{2.47}$$

where $\|\vartheta\|_\infty = \sup_{\xi\in[\nu_1,\nu_2]}\vartheta(\xi)$.

Proof. (i) Using techniques of integration and the hypothesis of ϑ, we have that the following identity holds on $[0,1]$:

$$\chi_1(\varrho)\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi = \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\left[\varphi\left(\nu_1^{\frac{1+\varrho}{2}}\nu_2^{\frac{1-\varrho}{2}}\right) + \varphi\left(\nu_1^{\frac{1-\varrho}{2}}\nu_2^{\frac{1+\varrho}{2}}\right)\right]\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.48}$$

By Lemma 1, the following inequality

$$\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}) + \varphi(\xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{1+\varrho}) \leq \varphi\left(\nu_1^{\frac{1+\varrho}{2}}\nu_2^{\frac{1-\varrho}{2}}\right) + \varphi\left(\nu_1^{\frac{1-\varrho}{2}}\nu_2^{\frac{1+\varrho}{2}}\right) \tag{2.49}$$

holds for all $\xi \in [\nu_1, \sqrt{\nu_1\nu_2}]$ with

$$\xi_1 = \xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho},\ \xi_2 = \xi^{-\varrho}(\sqrt{\nu_1\nu_2})^{1+\varrho},$$
$$\lambda_1 = \nu_1^{\frac{1+\varrho}{2}}\nu_2^{\frac{1-\varrho}{2}} \quad \text{and} \quad \lambda_2 = \nu_1^{\frac{1-\varrho}{2}}\nu_2^{\frac{1+\varrho}{2}}$$

Multiplying both right-hand side and left-hand side of (2.49) with $\frac{\vartheta(\xi)}{\xi}$, integrating over $[\nu_1, \sqrt{\nu_1\nu_2}]$, and using (2.34) and (2.49), we obtain (2.45).

(ii) We can observe that

$$\frac{1}{2}\left[\varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}\right) + \varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right)\right]\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi = \left[\varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}\right) + \varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right)\right]\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.50}$$

By using GA-symmetric assumption on ϑ, we obtain

$$2\int_{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}^{\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}}\varphi(\xi)\vartheta\left(\frac{\xi^2}{\sqrt{\nu_1\nu_2}}\right)\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi = \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\left[\varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}\xi^{\frac{\xi}{2}}\right) + \varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}\xi^{-\frac{\xi}{2}}\right)\right]\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.51}$$

We can also see that the following identity holds:

$$\begin{aligned}
&(\ln\nu_2 - \ln\nu_1)\int_0^1\chi_1(\varrho)\vartheta(\nu_1^{1-\varrho}\nu_2^{\varrho})\mathrm{d}\varrho \\
&= (\ln\nu_2 - \ln\nu_1)\left[\int_{\frac{1}{2}}^{1}\varphi\left(\nu_1^{\frac{1+\varrho}{2}}\nu_2^{\frac{1-\varrho}{2}}\right)\vartheta(\nu_1^{1-\varrho}\nu_2^{\varrho})\mathrm{d}\varrho + \int_0^{\frac{1}{2}}\varphi\left(\nu_1^{\frac{1+\varrho}{2}}\nu_2^{\frac{1-\varrho}{2}}\right)\vartheta(\nu_1^{1-\varrho}\nu_2^{\varrho})\mathrm{d}\varrho\right. \\
&\quad \left. + \int_0^{\frac{1}{2}}\varphi\left(\nu_1^{\frac{1-\varrho}{2}}\nu_2^{\frac{1+\varrho}{2}}\right)\vartheta(\nu_1^{1-\varrho}\nu_2^{\varrho})\mathrm{d}\varrho + \int_{\frac{1}{2}}^{1}\varphi\left(\nu_1^{\frac{1-\varrho}{2}}\nu_2^{\frac{1+\varrho}{2}}\right)\vartheta(\nu_1^{1-\varrho}\nu_2^{\varrho})\mathrm{d}\varrho\right] \\
&= \frac{1}{2}\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\left[\varphi(\sqrt{\nu_1\xi}) + \varphi\left(\nu_1\nu_2^{\frac{1}{2}}\xi^{-\frac{1}{2}}\right) + \varphi(\sqrt{\xi\nu_2}) + \varphi\left(\nu_2\nu_1^{\frac{1}{2}}\xi^{-\frac{1}{2}}\right)\right]\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi.
\end{aligned} \tag{2.52}$$

Finally, we also have

$$\frac{1}{2}\left[\varphi(\sqrt{\nu_1\nu_2}) + \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}\right]\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi = \left[\varphi(\sqrt{\nu_1\nu_2}) + \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}\right]\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.53}$$

By Lemma 1, the following inequalities hold for all $\xi \in [\nu_1, \sqrt{\nu_1\nu_2}]$:

The inequality

$$\varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}\xi^{\frac{1}{2}}\right) + \varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}\xi^{-\frac{1}{2}}\right) \le \varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}\right) + \varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right) \tag{2.54}$$

holds with the choices of $\xi_1 = \nu_1^{\frac{1}{4}}\nu_2^{\frac{1}{4}}\xi^{\frac{1}{2}}$, $\xi_2 = \nu_1^{\frac{3}{4}}\nu_2^{\frac{3}{4}}\xi^{-\frac{1}{2}}$, $\lambda_1 = \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}$, and $\lambda_2 = \nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}$.

The inequality

$$\varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}\right) \le \frac{1}{2}\left[\varphi\left(\nu_1\nu_2^{\frac{1}{2}}\xi^{-\frac{1}{2}}\right) + \varphi(\sqrt{\nu_1\xi})\right] \tag{2.55}$$

holds with the choices of $\xi_1 = \xi_2 = \nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}$, $\lambda_1 = \sqrt{\nu_1\xi}$, and $\lambda_2 = \nu_1\nu_2^{\frac{1}{2}}\xi^{-\frac{1}{2}}$.

The inequality

$$\varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right) \le \frac{1}{2}\left[\varphi\left(\nu_2\nu_1^{\frac{1}{2}}\xi^{-\frac{1}{2}}\right) + \varphi(\sqrt{\xi\nu_2})\right] \tag{2.56}$$

holds with the choices of $\xi_1 = \xi_2 = \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}$, $\lambda_1 = \sqrt{\xi\nu_2}$, $\lambda_2 = \nu_2\nu_1^{\frac{1}{2}}\xi^{-\frac{1}{2}}$.

The inequality

$$\varphi\left(\nu_1\nu_2^{\frac{1}{2}}\xi^{-\frac{1}{2}}\right) + \varphi(\sqrt{\nu_1\xi}) \leq \varphi(\nu_1) + \varphi(\sqrt{\nu_1\nu_2}) \tag{2.57}$$

holds with the choices of $\xi_1 = \sqrt{\nu_1\xi}$, $\xi_2 = \nu_1\nu_2^{\frac{1}{2}}\xi^{-\frac{1}{2}}$, $\lambda_1 = \nu_1$, $\lambda_2 = \sqrt{\nu_1\nu_2}$.

The inequality

$$\varphi\left(\nu_1^{\frac{1}{2}}\nu_2\xi^{-\frac{1}{2}}\right) + \varphi\left(\frac{2\nu_2\xi}{\nu_2 + \xi}\right) \leq \varphi(\sqrt{\nu_1\nu_2}) + \varphi(\nu_2) \tag{2.58}$$

holds with the choices of $\xi_1 = \sqrt{\xi\nu_2}$, $\xi_2 = \nu_1^{\frac{1}{2}}\nu_2\xi^{-\frac{1}{2}}$, $\lambda_1 = \sqrt{\nu_1\nu_2}$, $\lambda_2 = \nu_2$.

Multiplying (2.54)–(2.58) with $\frac{\vartheta(\xi)}{\xi}$, integrating them over $[\nu_1, \sqrt{\nu_1\nu_2}]$, and using (2.50)–(2.53), we obtain (2.46).

(iii) By integration by parts, we obtain

$$\begin{aligned}
&\varrho \int_{\ln\nu_1}^{\frac{\ln\nu_1+\ln\nu_2}{2}} \left[\left(\xi - \frac{\ln\nu_1 + \ln\nu_2}{2}\right)g'\left(\varrho\xi + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\right. \\
&\quad \left. + \left(\frac{\ln\nu_1 + \ln\nu_2}{2} - \xi\right)g'\left(\varrho(\ln\nu_1 + \ln\nu_2 - \xi) + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\right]\mathrm{d}\xi \\
&= \varrho \int_{\ln\nu_1}^{\ln\nu_2} \left(\xi - \frac{\ln\nu_1 + \ln\nu_2}{2}\right)g'\left(\varrho\xi + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\mathrm{d}\xi \\
&= \frac{\ln\nu_2 - \ln\nu_1}{2}\left[\varphi\left(\nu_1^{\frac{1+\varrho}{2}}\nu_2^{\frac{1-\varrho}{2}}\right) + \varphi\left(\nu_1^{\frac{1-\varrho}{2}}\nu_2^{\frac{1+\varrho}{2}}\right)\right] - \int_{\nu_1}^{\nu_2}\frac{1}{\xi}\varphi(\xi^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho})\mathrm{d}\xi \\
&= (\ln\nu_2 - \ln\nu_1)[\chi_1(\varrho) - \chi(\varrho)].
\end{aligned} \tag{2.59}$$

Using the convexity of g and the hypothesis of ϑ, the following inequality holds for all $\varrho \in [0,1]$ and $\xi \in \left[\ln\nu_1, \frac{\ln\nu_1 + \ln\nu_2}{2}\right]$:

$$\begin{aligned}
&\left[g\left(\varrho\xi + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right) - g\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right]\vartheta \circ \exp(\xi) \\
&\quad + \left[g\left(\varrho(\ln\nu_1 + \ln\nu_2 - \xi) + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right) - g\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right]\vartheta \circ \exp(\xi) \\
&\leq \varrho\left(\xi - \frac{\ln\nu_1 + \ln\nu_2}{2}\right)g'\left(\varrho\xi + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\vartheta \circ \exp(\xi) \\
&\quad + \varrho\left(\frac{\ln\nu_1 + \ln\nu_2}{2} - \xi\right)g'\left(\varrho(\ln\nu_1 + \ln\nu_2 - \xi) + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\vartheta \circ \exp(\xi) \\
&= \varrho\left(\frac{\ln\nu_1 + \ln\nu_2}{2} - \xi\right)\left[g'\left(\varrho(\ln\nu_1 + \ln\nu_2 - \xi) + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\right. \\
&\quad \left. - g'\left(\varrho\xi + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\right]\vartheta \circ \exp(\xi) \\
&\leq \varrho\left(\frac{\ln\nu_1 + \ln\nu_2}{2} - \xi\right)\left[g'\left(\varrho(\ln\nu_1 + \ln\nu_2 - \xi) + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\right. \\
&\quad \left. - g'\left(\varrho\xi + (1-\varrho)\left(\frac{\ln\nu_1 + \ln\nu_2}{2}\right)\right)\right] \|\vartheta\|_\infty.
\end{aligned} \tag{2.60}$$

Integrating (2.60), using (2.59) and (2.17), we obtain (2.47). □

Corollary 2. *According to the assumptions of Theorem* 14 *with* $\vartheta(\xi) = \frac{1}{\ln\nu_2 - \ln\nu_1}$, $\xi \in [\nu_1, \nu_2]$, *the following inequalities hold*:

(*i*) *The following inequality holds for all* $\varrho \in [0,1]$:

$$\chi(\varrho) \le \chi_1(\varrho).$$

(*ii*) *The following inequality holds*:

$$\begin{aligned} 2\int_{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}^{\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}} \frac{\varphi(\xi)}{\xi}\mathrm{d}\xi &\le \frac{1}{2}\left[\varphi\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}\right) + \varphi\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right)\right] \\ &\le \int_0^1 \chi_1(\varrho)\mathrm{d}\varrho \le \frac{1}{2}\left[\varphi(\sqrt{\nu_1\nu_2}) + \frac{\varphi(\nu_1) + \varphi(\nu_2)}{2}\right]. \end{aligned} \tag{2.61}$$

(*iii*) *The inequality*

$$0 \le \chi(\varrho) - \varphi(\sqrt{\nu_1\nu_2}) \le \chi_1(\varrho) - \chi(\varrho) \tag{2.62}$$

holds for all $\varrho \in [0,1]$.

Theorem 15. *Let* φ, ϑ, χ_1, χ_ϑ, $\mathcal{T}_\vartheta$ *be defined as above. Then, we have the following results*:

(*i*) $\mathcal{T}_\vartheta$ *is GA-convex on* $(0,1]$.

(*ii*) *The following inequalities hold for all* $\varrho \in [0,1]$:

$$\begin{aligned} \chi_1(\varrho)\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \le \mathcal{T}_\vartheta(\varrho) &\le (1-\varrho)\int_{\nu_1}^{\nu_2}\frac{\varphi(\xi)\vartheta(\xi)}{\xi}\mathrm{d}\xi + \varrho\cdot\frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \\ &\le \frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi, \end{aligned} \tag{2.63}$$

$$\chi_\vartheta(1-\varrho) \le \mathcal{T}_\vartheta(\varrho), \tag{2.64}$$

and

$$\frac{\chi_\vartheta(\varrho) + \chi_\vartheta(1-\varrho)}{2} \le \mathcal{T}_\vartheta(\varrho). \tag{2.65}$$

(*iii*) *The following bound is true*:

$$\sup_{\varrho\in[0,1]} \mathcal{T}_\vartheta(\varrho) = \frac{\varphi(\nu_1)+\varphi(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.66}$$

Proof. (i) Since φ is *GA*-convex and ϑ is non-negative, we see that $\mathcal{T}_\vartheta$ is *GA*-convex on $(0,1]$.
(ii) We observe that the following identity holds on $[0,1]$:

$$\mathcal{T}_\vartheta(\varrho) = \frac{1}{2}\int_{\nu_1}^{\sqrt{\nu_1\nu_2}} [\varphi(\nu_1^\varrho\xi^{1-\varrho}) + \varphi(\nu_1^\varrho(\nu_1\nu_2\xi^{-1})^{1-\varrho}) + \varphi(\nu_2^\varrho\xi^{1-\varrho}) + \varphi(\nu_2^\varrho(\nu_1\nu_2\xi^{-1})^{1-\varrho})]\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi. \tag{2.67}$$

By Lemma 1, the following inequalities hold for all $\xi \in [\nu_1, \sqrt{\nu_1\nu_2}]$:

$$2\varphi(\nu_1^\varrho(\sqrt{\nu_1\nu_2})^{1-\varrho}) \le \varphi(\nu_1^\varrho\xi^{1-\varrho}) + \varphi(\nu_1^\varrho(\nu_1\nu_2\xi^{-1})^{1-\varrho}) \tag{2.68}$$

with

$$\xi_1 = \xi_2 = \nu_1^\varrho(\sqrt{\nu_1\nu_2})^{1-\varrho}, \quad \lambda_1 = \nu_1^\varrho\xi^{1-\varrho}, \quad \text{and} \quad \lambda_2 = \nu_1^\varrho(\nu_1\nu_2\xi^{-1})^{1-\varrho}.$$

$$2\varphi(\nu_2^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}) \leq \varphi(\nu_2^{\varrho}\xi^{1-\varrho}) + \varphi(\nu_2^{\varrho}(\nu_1\nu_2\xi^{-1})^{1-\varrho}) \tag{2.69}$$

with

$$\xi_1 = \xi_2 = \nu_2^{\varrho}(\sqrt{\nu_1\nu_2})^{1-\varrho}, \quad \lambda_1 = \varphi(\nu_2^{\varrho}\xi^{1-\varrho}) \quad \text{and} \quad \lambda_2 = \nu_2^{\varrho}(\nu_1\nu_2\xi^{-1})^{1-\varrho}.$$

Multiplying Inequalities (2.68) and (2.69) with $\frac{\vartheta(\xi)}{\xi}$, integrating them over ξ on $[\nu_1, \sqrt{\nu_1\nu_2}]$, and using identities (2.48) and (2.67), we derive the first inequality of (2.63). Using the *GA*-convexity of φ and inequality (2.17), the last part of (2.63) holds. Using again the *GA*-convexity of φ, we obtain

$$\begin{aligned}\chi_{\vartheta}(1-\varrho) &= \int_{\nu_1}^{\nu_2}\varphi(\xi^{1-\varrho}(\sqrt{\nu_1\nu_2})^{\varrho})\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \\ &= \int_{\nu_1}^{\nu_2}\varphi\left(\nu_1^{\frac{\varrho}{2}}\xi^{\frac{1-\varrho}{2}}\nu_1^{\frac{\varrho}{2}}\xi^{\frac{1-\varrho}{2}}\right)\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi \\ &\leq \frac{1}{2}\int_{\nu_1}^{\nu_2}[\varphi(\nu_1^{\varrho}\xi^{1-\varrho}) + \varphi(\nu_2^{\varrho}\xi^{1-\varrho})]\frac{\vartheta(\xi)}{\xi}\mathrm{d}\xi = \mathcal{T}_{\vartheta}(\varrho).\end{aligned} \tag{2.70}$$

From (2.45), (2.63), and (2.70), we obtain (2.65).
(iii) (2.66) holds due to inequality (2.63). The theorem is thus accomplished. □

3 Conclusion

The topic of mathematical inequalities has become an emerging topic since the past four decades and lot of research has been produced by a number of mathematicians with novel results. This topic has lot of applications in applied mathematics, pure mathematics, and other applied sciences. A number of novel results have been established using convexity and its generalizations with applications in numerical analysis, fixed point theory, differential equations, and optimization. In this study, we have used the GA-convexity as a generalization of convexity to obtain new Fejér-type inequalities with the help of some mappings defined over the interval $[0, 1]$. We have discussed some very interesting properties of those mappings and as a consequence, we obtain refinements of number of results previously obtained in this topic.

Acknowledgments: The author would be very thankful to all the anonymous referees for their very useful and constructive comments in order to improve the article.

Funding information: This work is supported by the Deanship of Scientific Research, King Faisal University under the Ambitious Researcher Track (Research Project Number GrantA141).

Author contributions: I am the only author for conceptualization and writing up this manuscript.

Conflict of interest: The author declares no conflict of interest.

Data availability statement: No data have been used in the manuscript.

References

[1] C. Hermite, *Sur deux limites d'une intégrale dé finie*, Mathesis **3** (1893), 82.
[2] J. Hadamard, *Étude sur les propriétés des fonctions entiéres en particulier daune function considéré par Riemann*, J. Math. Pures Appl. **58** (1893), 171–215.

[3] L. Fejér, *Über die Fourierreihen, II*, Math. Naturwiss. Anz Ungar. Akad. Wiss. **24** (1906), 369–390 (In Hungarian).

[4] M. A. Ardic, A. O. Akdemir, and E. Set, *New Ostrowski like inequalities for GG-convex and GA-convex functions*, Math. Ineq. Appl. **19** (2016), 1159–1168.

[5] M. A. Ardic, A. O. Akdemir, and K. Yildiz, *On some new inequalities via GG-convexity and GA-convexity*, Filomat **32** (2018), 5707–5717.

[6] H. Budak, *On Fejér-type inequalities for convex mappings utilizing fractional integrals of a function with respect to another function*, Results Math. **74** (2019), no. 1, 29.

[7] S. I. Butt, J. Nasir, M. A. Dokuyucu, A. O. Akdemir, and E. Set, *Some Ostrowski-Mercer type inequalities for differentiable convex functions via fractional integral operators with strong kernels*, Appl. Math. Comput. **21** (2022), no. 3, 329–348.

[8] H. Budak and Y. Bakiş, *On Fejér-type inequalities for products two convex functions*, Note di Mat. **40** (2020), no. 1, 27–43.

[9] S. S. Dragomir, M. A. Latif, and E. Momoniat, *Fejér-type integral inequalities related with geometrically-arithmetically convex functions with applications*, Acta Comment. Univ. Tartu. Math. **23** (2019), no. 1, 51–64.

[10] S. S. Dragomir, *Inequalities of Jensen type for GA-convex functions*, RGMIA Res. Rep. Collect. **18** (2015), 1–26.

[11] S. S. Dragomir, *On some integral inequalities for convex functions*, Zb.-Rad. (Kragujevac), **18** (1996), 21–25.

[12] H. Kara, H. Budak, and M. A. Ali, *On new generalization of Fejer type inequalities for double integrals*, Int. J. Nonlinear Anal. Appl. **14** (2023), 1, 375–391.

[13] S. Kızıl and M. A. Ardıc, *Inequalities for strongly convex functions via Atangana-Baleanu integral operators*, Turkish J. Sci. **6** (2021), no. 2, 96–109.

[14] J. B. Liu, S. I., Butt, J. Nasir, A. Aslam, A. Fahad, and J. Soontharanon, *Jensen-Mercer variant of Hermite-Hadamard-type inequalities via Atangana-Baleanu fractional operator*, AIMS Math. **7** (2022), no. 2, 2123–2141.

[15] C. P. Niculescu, *Convexity according to the geometric mean*, Math. Inequal. Appl. **3** (2000), no. 2, 155–167.

[16] M. A. Noor and K. I. Noor, *Some new classes of strongly generalized preinvex functions*, TWMS J. Pure Appl. Math. **12** (2021), no. 2, 181–192.

[17] M. E. Ozdemir, *New refinements of Hadamard integral Inequality via k-fractional integrals for p-convex function*, Turkish J. Sci. **6** (2021), no. 1, 1–5.

[18] M. Kunt and İ. İşcan, *Fractional Hermite-Hadamard-Fejér-type inequalities for GA-convex functions*, Turk. J. Ineq. **2** (2018), 1–20.

[19] I. İissscan, *Hermite-Hadamard-type inequalities for GA-s-convex functions*, Le Matematiche **19** (2014), 129–146.

[20] M. A. Latif, *New Hermite-Hadamard-type integral inequalities for GA-convex functions with applications*, Analysis **34** (2014), 379–389.

[21] M. A. Latif, *Hermite-Hadamard-type inequalities for GA-convex functions on the co-ordinates with applications*, Proc. Pak. Acad. Sci. **52** (2015), no. 4, 367–379.

[22] M. A. Latif, *Weighted Hermite-Hadamard-type inequalities for differentiable GA-convex and geometrically quasi-convex mappings*, Rocky Mountain J. Math. **51** (2022), no. 6, 1899–1908.

[23] M. A. Latif, S. S. Dragomir, and E. Momoniat, *Some Fejér-type integral inequalities for geometrically-arithmetically-convex functions with applications*, Filomat **32** (2018), no. 6, 2193–2206.

[24] M. A. Latif, *Extensions of Fejér-type inequalities for GA-convex functions and related results*, Filomat **37** (2023), no. 24, 8041–8055.

[25] M. A. Latif, S. S. Dragomir, and E. Momoniat, *Some estimates on the Hermite-Hadamard inequality through geometrically quasi-convex functions*, Miskolc Math. Notes **18** (2017), no. 2, 933–946. DOI: https://doi.org/10.18514/MMN.2017.1819.

[26] M. A. Noor, K. I. Noor, and M. U. Awan, *Some inequalities for geometrically-arithmetically h-convex functions*, Creative Math. Inform **23** (2014), 91–98.

[27] S. Obeidat and M. A. Latif, *Weighted version of Hermite-Hadamard-type inequalities for geometrically quasi-convex functions and their applications*, J. Inequal. Appl. **2018** (2018), no. 1, 307.

[28] F. Qi and B.-Y. Xi, *Some Hermite-Hadamard-type inequalities for geometrically quasi-convex functions*, Proc. Indian Acad. Sci. Math. Sci. **124** (2014), no. 3, 333–342.

[29] X.-M. Zhang, Y.-M Chu, and X.-H. Zhang, *The Hermite-Hadamard-type inequality of GA-convex functions and its application*, J. Inequal. Appl. **2010** (2010), 507560.

[30] S. S. Dragomir, Y. J. Cho, and S. S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. Math. Anal. Appl. **245** (2000), 489–501.

[31] S. S. Dragomir, D. S. Milosevic, and J. Sandor, *On some refinements of Hadamard's inequalities and applications*, Univ. Belgrad. Publ. Elek. Fak. Sci. Math. **4** (1993), 3–10.

[32] S. S. Dragomir, *On Hadamard's inequality for convex functions*, Mat. Balk. **6** (1992), 215–222.

[33] S. S. Dragomir and R. P. Agarwal, *Two new mappings associated with Hadamard's inequalities for convex functions*, Appl. Math. Lett. **11** (1998), no. 3, 33–38.

[34] S. S. Dragomir, *Two mappings in connection to Hadamardís inequalities*, J. Math. Anal. Appl. **167** (1992), 49–56.

[35] K. L. Tseng, S. R. Hwang, and S. S. Dragomir, *On some new inequalities of Hermite-Hadamard-Fejér-type involving convex functions*, Demonstratio Math. **XL** (2007), no. 1, 51–64.

[36] K. L. Tseng, S. R. Hwang, and S. S. Dragomir, *Fejér-type inequalities (I)*, J. Inequal Appl **2010** (2010), 531976.

[37] K. L. Tseng, S. R. Hwang, and S. S. Dragomir, *Fejér-type Inequalities (II)*, Math. Slovaca **67** (2017), 109–120.

[38] G. S. Yang and M. C. Hong, *A note on Hadamard's inequality*, Tamkang. J. Math. **28** (1997), no. 1, 33–37.

[39] G. S. Yang and K. L. Tseng, *On certain integral inequalities related to Hermite-Hadamard inequalities*, J. Math. Anal. Appl. **239** (1999), 180–187.

[40] G. S. Yang and K. L. Tseng, *Inequalities of Hadamard's type for Lipschitzian mappings*, J. Math. Anal. Appl. **260** (2001), 230–238.

[41] G. S. Yang and K. L. Tseng, *On certain multiple integral inequalities related to Hermite-Hadamard inequalities*, Utilitas Math. **62** (2002), 131–142.

[42] G. S. Yang and K. L. Tseng, *Inequalities of Hermite-Hadamard-Fejér-type for convex functions and Lipschitzian functions*, Taiwanese J. Math **7** (2003), no. 3, 433–440.

[43] R. Turker and H. O. Kavurmacı, *Generalized inequalities for quasi-convex functions via generalized Riemann-Liouville fractional integrals*, Turkish J. Sci. **7** (2022), no. 3, 219–230.

[44] R. Xiang, *Refinements of Hermite-Hadamard-type inequalities for convex functions via fractional integrals*, J. Appl. Math. Informatics **33** (2015), no. 1–2, 119–125. DOI: http://dx.doi.org/10.14317/jami.2015.119.

[45] S. S. Dragomir, *Some new inequalities of Hermite-Hadamard-type for GA-convex functions*, Ann. Univers. Mariae Curie-Sklodowska Lublin-Polonia **LXXII** (2018), Sectio A, no. 1, 55–68.

[46] S. S. Dragomir, *Inequalities of Hermite-Hadamard-type for GA-convex functions*, Ann. Math. Sil. **32** (2018), 145–168.

[47] M. A. Latif, H. Kalsoom, Z. A. Khan, and A. A. Al-moneef, *Refinement mappings related to Hermite-Hadamard-type inequalities for GA-convex sunction*, Mathematics **10** (2022), 1398, DOI: https://doi.org/10.3390/math10091398.

[48] S. S. Dragomir, *On Hadamard's inequality for the convex mappings defined on a ball in the space and applications*, Math. Ineq. Appl. **3** (2000), 177–187.